Remarks on the numbers of ideals of Boolean algebra and open sets of a topology

Abstract: We prove that the cardinals μ which may be the number of ideals of an infinite Boolean algebras are restricted: $\mu = \mu^{\aleph_0}$ and if $\kappa \leq \mu$ is strong limit then $\mu^{<\kappa} = \mu$. Similar results hold for the number of open sets of a compact space (we need $w(x)^{<\hat{s}(x)} = 2^{<\hat{s}(x)}$). We also prove that if $\mu \geq \beth_2$ is the number of open subsets of a Hausdorff space $X, \mu < \mu^{\aleph_0}$ *then* $0^{\#}$ exists, (in fact, the consequences of the covering lemma on cardinal arithmetic are violated). We also prove that if the spread μ of a Hausdorff space X satisfies $\mu > \beth_2(cf\ \mu)$ that the sup is obtained. For regular spaces $\mu > 2^{cf\ \mu}$ is enough.

Similarly for $3(X)$ and $h(X)$.

§0 Introduction.

We deal with some problems on Boolean algebras and their parallel on topological spaces. The problems are: what can be the number of ideals [open sets], and is the spread (and related cardinals) necessarily obtained (remember it is defined as a supremum.) Compare with the well known result that the cellularity (= first κ for which the κ- chain condition holds) is regular. We shall use freely the duality between a Boolean algebra and its space of ultrafilters. Recall

0.1 Definition : For a topological space X:

1) $s(X) = \sup\{|A|: A$ is a discrete subspace$\} + \aleph_0$ (note that A is a discrete subspace if $A = \{y_i: i < \alpha\}$ and for some open subsets $u_i (i < \alpha)$, $y_i \in u_j \iff i = j$).

2) $z(X) = \sup\{|A|: A = \{y_i: i < \alpha\}$, and for some open u_i $(i < \alpha)$, $i = j \Longrightarrow y_i \in u_j \Longrightarrow i \geq j\} + \aleph_0$.

3) $h(X) = \sup\{|A|: A = \{y_i: i < \alpha\}$ for some open $u_i(i < \alpha)$, $i = j \Longrightarrow y_i \in u_j \Longrightarrow i \leq j\} + \aleph_0$.

4) $\hat{s}(X), \hat{z}(X), \hat{h}(X)$ are defined similarly with $|A|^+$ instead $|A|$.

5) For a Boolean algebra B ,$\varphi(B)$ is $\varphi(X)$ where X is the space of ultrafilters of B.

On the problem of the attainment of the supremum when the cofinality is $\aleph_0$ see Hajnal and Juhasz [HJ 1], Juhasz [J1], Shelah [Sh 3] 1.1 (p. 252) and then Kunen and Roitman [KR].

On a counterexample for higher cofinalities see Roitman [R] and lately Juhaz and Shelah [JSh]. On the number of open subsets see Hajnal and Juhasz [HJ2] and Juhasz [J2]; the author observed in fall 1977 (see [Sh 6] for the main consequence) that by having a specific cardinal exponentiation function we can get from counterexample to the attainment of the spread when the cofinality is κ, a Hausdorff space X with $o(X)^\kappa > o(X)$ (this extra demand on the set theory has caused no trouble). This connected our two problems. The author had withdrawn another announcement of [Sh 6]: $o(X) = o(X)^{\aleph_0}$ for X a Lindelof space.

This work is written in the order it was conceived.

§1 The numbers of ideals of a Boolean Algebra

1.1 Theorem: Let B be an infinite Boolean Algebra, $Id(B)$ the set of ideals of B, $id(B)$ its power. Then $id(B) = id(B)^{\aleph_0}$.

Proof: Suppose not, $\lambda = Min\{\kappa : \kappa^{\aleph_0} \geq id(B)\}$, so $cf\ \lambda = \aleph_0$, $\lambda \leq id(B) < \lambda^{\aleph_0}$. Now $\lambda > 2^{\aleph_0}$ as $id(B) \geq 2^{\aleph_0}$, so $\lambda = \sum_n \lambda_n$, $\lambda_n < \lambda_{n+1} < \lambda$, $\lambda_n = \lambda_n^{\aleph_0}$. We define by induction on $n, a_n \in B$, $a_n \cap a_\ell = 0$ for $\ell < n$, $id(B \restriction a_n) \geq \lambda_n$, $id(B \restriction (1 - \bigcup_{\ell<n} a_\ell) \geq \lambda$. We should fail for some n, so w.l.o.g. for

no $a \in B$, $id(B \restriction a) \geq \lambda_n, id(B \restriction (1-a) \geq \lambda$. W.l.o.g. $n = 0$, so $\mathcal{J} = \{a \in B{:}id(B \restriction a) \leq \lambda_0\}$ is a maximal ideal. Now $|B| < \lambda$ (otherwise $|\mathcal{J}| \geq \lambda$, each countable subset of $\mathcal{J}$ generates an ideal, there are $\geq \lambda^{\aleph_0} > id(B)$ such countable subsets, and each ideal of B of this form has power $\leq \lambda_0$ hence has at most $\lambda_0^{\aleph_0} = \lambda_0 < \lambda$ countable subsets. Contradiction). So W.l.o.g. $|B| < \lambda_0$. Now $Id^0(B) = \{I \in Id(B){:}I \not\subseteq \mathcal{J}\} \subseteq \bigcup_{a \in \mathcal{J}} \{I \in Id(B){:}1-a \in I\}$ has power $\leq \sum_{a \in \mathcal{J}} id(B \restriction a) \leq |B| + \lambda_0$

So $Id^0(B)$ has power $\leq \lambda_0$. Also $Id^1(B) = \{I \in Id(B){:}I \subseteq \mathcal{J}$ but for some $a \in B-I$ there is no $b < a, b \in \mathcal{J}-I\}$ has power $\leq \lambda_0$ (for each such a,I $I \cap (B \restriction a) = \mathcal{J} \cap (B \restriction a)$, and for $I \cap (B \restriction (1-a))$ we have $\leq id(B \restriction (1-a)) \leq \lambda_0$ possibilities. So $Id^2(B) \stackrel{def}{=} Id(B) - Id^0(B) - Id^1(B)$ has cardinality $id(B)$. For each $I \in Id^2(B)$ choose by induction on $i, a_i \in \mathcal{J}-I$ such that $a_i \cap a_j \in I$ for $j < \alpha$, and let $\bar{a}^I = \langle a_i{:}i < \alpha \rangle$ be the resulting maximal sequence. Note that:

$\hat{s}(B) = Min\{\mu$: there are no $a_i \in B (i < \mu)$, a_i not in the ideal generated by $\{a_j{:}j \neq i\}\}$,

and let

$\kappa = Min\{\mu$: there are no μ pairwise disjoint non zero elements of $B\}$.

Clearly $\kappa \leq \hat{s}(B)$, and for $\mu < \hat{s}(B)$, $2^\mu \leq id(B)$ so $2^{<\hat{s}(B)} \leq id(B)$. It is known that $cf\ \hat{s}(B) > \aleph_0$, so $(2^{<\hat{s}(B)})^{\aleph_0} = 2^{<\hat{s}(B)}$ hence $2^{<\hat{s}(B)} < \lambda$ and w.l.o.g. $2^{<\hat{s}(B)} < \lambda_0$. Now easily if $\bar{a}^I = \bar{a}^J = \langle a_i{:}i < \alpha \rangle$, $I \cap (B \restriction a_i) = J \cap (B \restriction a_i)$ for $i < \alpha$, then $I = J$ (if e.g. $I \not\subseteq J$, choose $x \in I-J$, then x is a good candidate as a_α for J). We shall prove for each $\bar{a}$ that $\{I{:}I \in Id^2(B), \bar{a}^I = \bar{a}\} \leq \lambda^*$ for fixed $\lambda^* < \lambda$. By the argument above this is equal to $|\{\langle I \restriction (B \restriction a_i){:}i \rangle{:}I \in Id^{(2)}, \bar{a}^I = \bar{a}\}|$ which is $\leq |\{\langle J_i{:}i < \alpha \rangle{:}J_i \subseteq B \restriction a_i$ an ideal, $a_j \cap a_i \in J_i$ for $j \neq i\}|$. Let $\mu_i = |\{J{:}J \subset B \restriction a_i$ an ideal (so $a_i \not\in J$) and for $j \neq i$, $a_j \cap a_i \in J\}|$. So the number is $\leq \prod_{i<\alpha} \mu_i$. Easily $\prod_{i<\alpha} \mu_i \leq id(B)$, and $\mu_i < \lambda_0$ but by cardinal arithmetic $(\prod_{i<\alpha} \mu_i)^{\aleph_0} = \prod_{i<\alpha} \mu_i$ (or $\prod_{i<\alpha} \mu_i \leq \lambda_0$) [you can see in 2.11], so $\prod_{i<\alpha} \mu_i < \lambda$. By more cardinal arithmetic (see 2.11) there is a bound λ^* as required.

So necessarily $|\{\bar{a}^I : I \in Id^2(B)\}| \geq \lambda$. Now each $\bar{a}^I$ has length $< \hat{s}(B)$ so $\lambda \leq |B|^{<\hat{s}(B)}$, and as $cf\ \hat{s}(B) > \aleph_0$, $cf\ \lambda = \aleph_0$ clearly there is $\mu < \hat{s}(B)$, $|B|^\mu \geq \lambda$. Let $\vartheta = Max\{\kappa, \mu^+\}$. So ϑ is regular $\vartheta \leq \hat{s}(B)$, B satisfies the ϑ-c.c. and $|B|^{<\vartheta} \geq \lambda$ and $2^{<\vartheta} \leq 2^{<\hat{s}(B)} \leq \lambda_0$. So $|B|^{<\vartheta} > 2^{<\vartheta}$. Let $\chi = Min\{\chi : \chi^{<\vartheta} \geq |B|\}$, then $\chi > 2^{<\vartheta}$, $\chi^{<\vartheta} = |B|^{<\vartheta} \geq \lambda$ and $(\forall \mu < \chi)\mu^{<\vartheta} < \chi$. By [Sh 1] 4.4 B has a subset of power χ no one in the ideal generated by the others. So $\chi < \hat{s}(B)$ so $2^\chi \leq id(B)$, but $2^\chi \geq \chi^{<\vartheta} \geq \lambda$ so $2^\chi \geq \lambda^{\aleph_0} > id(B)$ contradiction.

§2 On the number of open sets

2.1 Notation: 1) X is an infinite Hausdorff space, τ the family of open subsets of X, any $Y \subset X$ is equipped with the induced topology i.e $\tau^Y = \tau(Y) = \{U \cap Y : U \in X\}$. $\underset{\sim}{B}$ will denote a base of X.

2) Let $o(X) = |\tau|$, (and for $Y \subseteq X$, $o(Y) = |\{U \cap Y : U \in \tau\}|$.

3) $\hat{s}(X) = \{|A|^+ :$ A a discrete subspace of X, (i.e. (A, τ^A) is a discrete space $\}$.

4) $\underset{\sim}{B}$ is a strong base of X if for every $y \in X$, there is v, such that $y \in v \in \tau$, and $[y \in u \subseteq v, u \in \tau \Longrightarrow v \in \underset{\sim}{B}]$.

We shall assume in 2.3, 2.4:

2.2 Hypothesis: We assume λ is an infinite cardinal, $cf\ \lambda = \aleph_0$,

$(\forall \mu)(\aleph_0 \leq \mu < \lambda \rightarrow \mu^{\aleph_0} < \lambda)$ and at least one of the following holds:

(I) $\chi \leq o(X) < \lambda^{\aleph_0}$, $\chi = \lambda$

(II) $\chi \leq o(X) < \lambda^{\aleph_0}$, $\chi = \lambda^+$,

(III) $\chi \leq o(X) < \lambda^{\aleph_0}$, $\chi = \lambda$, and X is strongly Hausdorff (which means: for every infinite $A \subset X$ there are $p_n \in A$ and pairwise disjoint

$u_n \in \tau, p_n \in U_n$).

2.2A Explanation: We shall want to get a contradiction or at least get information on how an example like that looks like.

So we allow to replace X by X^* if $\chi \le o(X^*) < \lambda^{\aleph_0}$ is still satisfied; but we shall use this for open X^* only.

2.3 Claim: Assume 2.2.

1) $\lambda > 2^{\aleph_0}$ and we can find λ_n, $\lambda_n = \lambda_n^{\aleph_0} < \lambda_{n+1} < \lambda$, $\lambda = \sum_{n<\omega} \lambda_n$.

2) W.l.o.g. there are no disjoint open sets $u, v (\in \tau)$ such that $o(u) \ge \chi$, $o(v) \ge \lambda$. (and even no open disjoint u, v such that $o(u) \ge \chi$, $o(v) \ge \lambda_0$) [and even no open u, v such that $o(u - v) \ge \chi$, $o(v - u) \ge \lambda_0$, but then we pass to a non-open subspace.]

3) W.l.o.g. every point y has an open number u_y (so $y \in u_y \in \tau$) such that $o(u_y) < \lambda$.

4) $o(X) \ge 2^{<\hat{s}(X)}$; hence if $cf\ \hat{s}(X) > \aleph_0$ then $\lambda > 2^{<\hat{s}(X)}$ and w.l.o.g. $\lambda_0 > 2^{<\hat{s}(X)}$.

5) if $|X| \ge \beth_2$ then $|X| < \lambda$ (and w.l.o.g. $|X| < \lambda_0$; similarly $|X| \ge 2^{2^\kappa} \Longrightarrow |X|^\kappa \le o(X)$).

Proof : 1) If every $y \in X$ is isolated, X has $2^{|X|}$ open subsets, but X is infinite so $o(X) \ge 2^{\aleph_0}$. If $y^* \in X$ is not isolated we define by induction on $n, u_n, v_n \in \tau$ and y_n such that : $y^* \in u_n$, $y_n \in v_n$, $u_n \cap v_n = \phi$, and $v_{n+1} \subseteq u_n$, $u_{n+1} \subseteq u_n$. (choose $y_0 \in X, y_0 \ne y^*$ then choose v_0, u_0; if u_n is defined, choose $y_n \in u_n - \{y^*\}$ and then u_{n+1}, v_{n+1} using "X is Hausdorff".) So $\{u_n : n < \omega\}$ are open non empty pairwise disjoint hence $o(X) \ge |\{ \bigcup_{n \in S} u_n : S \subseteq w\}| = 2^{\aleph_0}$.

In any case $o(X) \ge 2^{\aleph_0}$ but $\lambda \le o(X)^{\aleph_0} > o(X)$ hence $o(X) > 2^{\aleph_0}$, but $o(X) < \lambda^{\aleph_0}$, so $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$.

so $(\forall\mu<\lambda)(\mu+\aleph_0<\lambda)$ hence $(\forall\mu<\lambda)\mu^{\aleph_0}<\lambda$ hence we can find λ_n as required.

2) Let $u_0=X$, define by induction on n, $1\le n<\omega$, u_n,v_n such that

(i) $u_n\in\tau, v_n\in\tau$; usually we demand they are disjoint.

(ii) $v_{n+1}\subseteq u_n, u_{n+1}\subseteq u_n$

(iii) $o(v_n-u_n-\bigcup_{\ell<n}v_\ell)\ge\lambda_n$

(iv) $o(u_n-v_n-\bigcup_{\ell<n}v_\ell)\ge\chi$

If we succeed, then v_ℓ are open, $v_n-\bigcup_{\ell\ne n}v_\ell\subseteq(v_n-u_n-\bigcup_{\ell<n}v_\ell)$ hence $o(v_n-\bigcup_{\ell\ne n}v_\ell)\ge\lambda_n$, so by Fact 2.3A below $o(X)\ge\prod_{n<\omega}\lambda_n=\lambda^{\aleph_0}>o(X)$ contradiction.

2.3A Fact: i) If $v_n\in\tau$ then $o(X)\ge\prod_{n<\omega}o(v_n-\bigcup_{\ell\ne n}v_\ell)$.

ii) If $v_i\in\tau(i<\alpha)$ then $o(X)\ge\prod_{i<\alpha}o(v_i-\bigcup_{j\ne i}v_j)$.

Proof : i) Let $\mu_n=o(v_n-\bigcup_{\ell\ne n}v_\ell)$ and let $v_i^n\in\tau(i<\mu_n)$ be such that $\{v_i^n\cap(v_n-\bigcup_{\ell\ne n}v_\ell):i<\mu_n\}$ are pairwise distinct. If $\rho\in\prod_{n<\omega}\mu_n$ let $v_\rho=\bigcup_{n<\omega}(v^n_{\rho(n)}\cap v_\ell)$. Clearly $v_\rho\in\tau$ and if $\rho\ne\nu\in\prod_{n<\omega}\mu_n$, then for some $k,\rho(k)\ne\nu(k)$, hence $v_\rho\cap(v_k-\bigcup_{\ell\ne k}v_\ell)=v^k_{\rho(k)}\cap(v_k-\bigcup_{\ell\ne k}v_\ell)$ $\ne v^k_{\nu(k)}\cap(v_k-\bigcup_{\ell\ne k}v_\ell)=v_\nu\cap(v_k-\bigcup_{\ell\ne k}v_\ell)$ hence $v_\rho\ne v_\nu$. So $o(X)=|\tau|\ge\prod_{n<\omega}\mu_n$ as required.

(ii) Similarly.

We return to the proof of 2.3.

(3) Let $Y=\bigcup\{v\in\tau:o(v)<\lambda\}$. If in $X-Y$ there is a non isolated point y^*, then the proof is as in 1) (with $y_n\in X-Y$). If every point of $X-Y$ is

isolated then: $o(X-Y) = 2^{|X-Y|}$. As $o(X)$ is infinite easily $o(X) = o(Y)$ or $o(X) = o(X-Y)$. The latter is impossible as $(2^{|X-Y|})^{\aleph_0} = 2^{|X-Y|}$ because it is infinite.

(4) If $y_i \in v_i \in \tau$, $y_i \notin v_j$, for $i < \alpha$, $i \neq j < \alpha$, then $\{\bigcup_{i \in S} v_i : S \subseteq \alpha\}$ is a family of $2^{|\alpha|}$ distinct open subsets of X, so $o(X) \geq 2^{|\alpha|}$. By the definition of $\hat{s}(X)$, $o(X) \geq 2^{<\hat{s}(X)}$. The second phrase is by cardinal arithmetic.

(5) Assume $|X| \geq \lambda$. For any countable $A \subseteq X$, the closure of A is a closed subset of X of power $\leq \beth_2$. The number of A is $|X|^{\aleph_0} > |X| \geq \beth_2$, and for any such A; $|\{B : B \subseteq X$ countable, the closure of B is the closure of $A\}|$ has power $\leq \beth_2$, so we finish.

2.4 Claim: Assume 2.2. 1) W.l.o.g.

(*) for every $y \in X$ for some $v_y \in \tau, y \in v_y$, $o(v_y) \leq \lambda_0$,

except possibly when: Hypothesis (I), holds (and not II or III) and $(\exists n)\ \lambda_n^{\beth_1} > \lambda$ (hence $\lambda_n^{\beth_1} \geq o(X)$).

2) $|X| < \lambda$ so w.l.o.g. $|X| < \lambda_0$ so X has strong base of power $< \lambda_0$.

Remark: So if $\lambda \geq \beth_2$, then (w.l.o.g.) $\lambda_0 > \beth_2$, $\lambda_0 = \lambda_0^{\aleph_0}$, $\lambda_0^{\beth_1} > \lambda_0^{+\omega}$, so $o^{\#}$ exist so the conclusion of [J2, 4.7, p. 97] holds.

Proof : 1) Let $Y_n = \bigcup\{v \in \tau : o(v) \leq \lambda_n\}$. By 2.3(3) $X = \bigcup Y_n$. If for some n $o(Y_n) \geq \chi$ we can replace X by Y_n. So assume $o(Y_n) < \chi$. Hence $Y_n \neq X$. If X is strongly Hausdorff choose $y_n \in X - Y_n$. As $X = \bigcup_{n<\omega} Y_n$, $Y_n \subseteq Y_{n+1}$, $\{y_n : n < \omega\}$ is infinite. By the definition of strongly Hausdorff applied to $\{y_n : n < \omega\}$ there are distinct $n(k) < \omega$, and $u_k \in \tau, y_{n(k)} \in u_k$, $\langle u_k : k < \omega \rangle$ pairwise disjoint. So $o(u_k) \geq \lambda_{n(k)}$, (as $y_{n(k)} \in u_k$) and $o(X) \geq \prod_k o(u_k) \geq \prod_{k<\omega} \lambda_{n(k)} = \lambda^{\aleph_0} > o(X)$ contr.

So we have dealt with Hypothesis III.

Next assume Hypothesis II, so

$$\sum_{n<\omega} o(Y_n) \le \sum_{n<\omega} \lambda = \lambda < \chi$$

So the following fact is sufficient.

2.4A Fact: If $Z_n \subset X$ is open (for $n < \omega$) $\sum_n o(Z_n) + \aleph_0 < o(\bigcup_n Z_n)$ then

$$o(\bigcup_n Z_n)^{\aleph_0} = o(\bigcup_n Z_n) = (\aleph_0 + \sum o(Z_n))^{\aleph_0}$$

Proof : Let $\vartheta = \aleph_0 + \sum o(Z_n)$.

We define a tree T with ω levels. Now T_n, the n'th level, will be $\{(u,n): u \subset \bigcup_{\ell<n} Z_\ell : u \in \tau\}$; the order will be: $(u,n) \le (v,m)$ iff $n \le m, u = v \cap (\bigcup_{\ell<n} Z_\ell)$. As $\bigcup_{\ell<n} Z_\ell$ is open (as well as $\bigcup_{\ell<\omega} Z_\ell$). $|T_\ell| = o(\bigcup_{\ell<n} Z_\ell) \le \sum_{\ell<n} o(Z_\ell) \le \vartheta$, and $o(\bigcup_{\ell<\omega} Z_\ell)$ is the number of ω-branches of T, so it is $> \vartheta \ge \sum |T_n|$. But in that case it is well known that the number of ω-branches of T is $\vartheta^{\aleph_0}$, as required. So we have proved 2.4A.

We are left with case I, and assume that for each n, $\lambda_n^{\beth_1} < \lambda$; let $C = \{(\vartheta^{\beth_1})^+ : \vartheta < \lambda\}$, $\varphi(Y) = o(Y)$, and apply 2.5A below, we get a contradiction.

Proof of 2.4(2): Let for $y \in X$ $v_y \in \tau, y \in v_y, o(v_y) < \lambda_0$. Suppose $|X| > \lambda_0^+$. Clearly $o(v_\tau) \ge |v_\tau|$ so $|v_\tau| < \lambda_0$. By Hajnal free subset theorem (see [J1]) there is $Y \subseteq X, |Y| = |X|$ such that $(\forall y \ne z \in Y)(y \notin v_z)$. So $|Y| < \hat{s}(X)$, so $o(X) \ge 2^{|Y|} = 2^{|X|}$ contradiction. So $|X| \le \lambda_0^+$, then $\{u \cap v_y : u \in \tau, y \in X\}$ is a strong basis of X of power $< \lambda_0^+ + \lambda_0$. Renaming we finish.

We can abstract from the proof of Kunen and Roitman [KJ] (or see [J2]), the following theorem. See 4.4(2), or 3.2A(2) for a simpler proof of 2.5(1)) even weakening (iv) to: $X \ne \bigcup_{\varphi(u)<\lambda_n} u$ for each n.

2.5 Lemma : 1) Suppose $cf\ \lambda = \aleph_0 < \lambda$, $\lambda = \sum_{n<\omega} \lambda_n, \lambda_n < \lambda$, X a topological space, and φ is a function from subsets of X to cardinals, satisfying:

(i) $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

(ii) $\varphi(X) \geq \lambda$

(iii) for an unbounded family C of cardinals $< \lambda$:

$\oplus$ if $\vartheta \in C$, $A_i \subseteq X (i < 2^{\aleph_0})$ and $\varphi(A_i) < \vartheta$ then $\varphi(\bigcup_i A_i) < \vartheta$.

(iv) $\varphi(\bigcup_{\varphi(u)<\lambda_n} u) < \lambda$.

Then there are open sets $u_n \subseteq X$ such that $\varphi(u_n - \bigcup_{\substack{\ell \neq n \\ \ell < \omega}} u_\ell) \geq \lambda_n$ for $n < \omega$.

2) We can replace $\oplus$ by $\oplus_a + \oplus_b$ where:

$\oplus_a$ if $\langle A_\eta : \eta \in {}^\omega 2 \rangle$ is a partition of X, and $\bigcup \{A_\eta : \eta(k) = 0\}$ is open for each $k < \omega, \vartheta \in C$ and $B \subseteq X, \varphi(B) \geq \vartheta$, *then* for some no-where dense set $K \subseteq {}^\omega 2$, $\varphi(B \cap \bigcup_{\eta \in K} A_\eta) \geq \vartheta$

and

$\oplus_b$ if $A_n \subseteq X$, $\vartheta \in C, \varphi(A_n) < \vartheta$ then $\varphi(\bigcup_{n<\omega} A_n) < \vartheta$.

3) If X is strongly Hausdorff, (i), (ii) suffice.

Proof : 1) We shall use (i) freely.

Case I: $\varphi(Y) < \lambda$ where $Y = \bigcup \{v : v \in \tau, \varphi(v) < \lambda\}$.

So $\varphi(X-Y) = \lambda$: if $X-Y$ has a non isolated point y^*, then we can define distinct $y_n \in X - Y_n - \{y^*\}$ and pairwise disjoint $u_n, y_n \in u_n \in \tau_n$, y^* not in the closure of u_n. So as $y_n \in Y$, $\varphi(u_n) \geq \lambda > \lambda_n$ and $u_n = u_n - \bigcup_{\ell<n} u_\ell$. So the u_n's are as required. So $X-Y$ is a discrete space hence $o(X-Y) = 2^{|X-Y|}$, but $o(X-Y) = \lambda$, contradiction.

So we can assume $\varphi(Y) \geq \lambda$, so w.l.o.g. $X = Y$ i.e.,

(*) for each $y \in X$ for some $v, y \in v \in \tau, \varphi(v) < \lambda$.

Case II: For every open $Y \subset X, \varphi(Y) \geq \lambda$ and $\vartheta < \lambda$, and $\langle v_y : y \in Y \rangle$ satisfying $y \in v_y \in \tau$ *there are* $p \in Y$, open $u, p \in u \subset v_p$ and open $Z \subseteq Y$, $\varphi(Z) \geq \lambda$ and v_z^0, a neighborhood of z, for $z \in Z$ such that: for every $z_m \in Z$, $\varphi(u - \bigcup_{n<\omega} v_{z_n}^0) \geq \vartheta$.

We define by induction on $n, 1 \leq n < \omega$, $p_n, u_n, y_n, \vartheta_n$ and $\langle v_y^n : y \in Y_n \rangle$ such that

(1) $Y_n \subseteq X, \varphi(Y_n) \geq \lambda$, $Y_{n+1} \subseteq Y_n$, Y is open.

(2) for $y \in Y_n$, v_y^n is an open neighborhood of y, $v_y^{n+1} \subseteq v_y^n$.

(3) $\vartheta_n \geq \lambda_n$ $\vartheta_{n+1} > \vartheta_n$.

(4) $p_n \in u_n \in \tau$, $\vartheta_n \leq \varphi(u_n) < \vartheta_{n+1}$, $u_n \subseteq v_{p_n}^n$

(5) for every $z_\ell \in Y_n (\ell < \omega)$ $\varphi(u_n - \bigcup_{\ell<\omega} v_{z_\ell}^n) \geq \vartheta_n$.

For $n = 0$ we stipulate $Y_0 = X$, $v_y^n (y \in Y_0)$ an open number of y with $\varphi(v_y^n)$ minimal and $\vartheta_0 = \beth_1 + \lambda_0$.

Suppose $Y_n, \langle v_y^n : y \in Y_n \rangle$ as defined. Choose $\vartheta_{n+1} < \lambda$ such that $\vartheta_{n+1} > \lambda_n$, $\vartheta_{n+1} > \vartheta_\ell$, $\varphi(u_\ell)$ when $0 < \ell < n+1$. Next apply the hypothesis of the case to Y_n, and ϑ_n and $\langle v_y^n : y \in Y_n \rangle$, so there are $p = p_{n+1} \in Y_n$, $u = u_{n+1}$, $Y = Y_{n+1}$, and $\langle v_z^{n,0} : z \in Y_{n+1} \rangle$ such that:

$Y_{n+1} \subseteq Y_n$, $\varphi(Y_{n+1}) \geq \lambda$, $p_{n+1} \in u_{n+1} \subseteq v_{p_{n+1}}^n$, $z \in v_z^{n,0} \in \tau$, and for $z_\ell \in Y_{n+1} (\ell < \omega)$, $\varphi(u_{n+1} - \bigcup_{\ell<\omega} v_{z_\ell}^{n,0}) \geq \vartheta_{n+1}$.

We let $v_z^{n+1} = v_z^{n,0} \cap v_z^n$.

Easily everything is o.k. Now in the end, as $u_\ell \subseteq v_{p_\ell}^n$ for $\ell < n$, and by (5) for n

$$\varphi(u_n - \bigcup_{\ell>n} u_\ell) \geq \varphi(u_n - \bigcup_{\ell>n} v_{p_\ell}^n) \geq \vartheta_n$$

As for $\ell < n, \varphi(u_\ell) < \vartheta_n$ clearly

$\varphi(u_n - \bigcup_{\ell \neq n} u_\ell) \geq \vartheta_n$, as required.

Case III: Not Cases I,II.

So (*) holds, and there are open $Y \subseteq X, \varphi(Y) \geq \lambda$, $\vartheta < \lambda$ and $\langle v_y : y \in Y \rangle, y \in v_y \in \tau$, witnessing the failure of Case II. W.l.o.g. $X = Y$, $\vartheta \in C$, $y \in u \subseteq v_y (u \in \tau) \Longrightarrow \varphi(u) \geq \varphi(v_y)$. If $\varphi(v_p) \geq \vartheta$, by (iii) $\oplus$:

(**) if $p \in u \in \tau$, $u \subseteq v_p$, *then* $\varphi(\{z \in Y:$ for some $v \in T, z \in v, \varphi(v \cap u) < \vartheta\}) < \lambda$ [if this fails $p, u, Z = \{v: \varphi(v \cap u) < \lambda\}$ and $\langle v_z^0 : z \in Z \rangle$ where $z \in v_z^0$, $\varphi(v_z^0 \cap u) < \lambda$, exemplify Z, ϑ do not witness the failure the assumption of Case II].

Define by induction on n, $p_n^\ell \in Y, u_n^\ell \in \tau$, for $\ell = 1,2$ and ϑ_n such that:

(1) $p_n^\ell \in u_n^\ell, u_n^1 \cap u_n^2 = \phi$, $u_n^\ell \subseteq v_{p_n^\ell}$.

(2) $\vartheta < \vartheta_n \in C, \vartheta_n \geq \lambda_n, \vartheta_{n+1} > \vartheta_n, \vartheta$.

(3) $\vartheta_n \leq \varphi(u_n^1), \varphi(u_n^2) < \vartheta_{n+1}$.

(4) for every open neighborhood v of p_n^k, if $m < n$:

$$\varphi(u_m^\ell \cap v) \geq \vartheta_\ell$$

For $n = 0$ choose $\vartheta_0 \in C, \vartheta_0 > \lambda_0 + \vartheta$ then choose $p^1 \neq p^2$ in Y such that $\varphi(v_{p_0^\ell}) \geq \vartheta_0$ (possible by assumption (iv)) and then choose $u_0^\ell \in \tau, p_0^\ell \in u_0^\ell \subseteq v_{p_0^\ell}, u_0^1 \cap u_0^2 = \phi$. For $n+1$, choose first $\vartheta_{n+1} \in C, \vartheta_{n+1}$ larger than $\vartheta_n, \lambda_{n+1}, \varphi(u_0^\ell), \ldots, \varphi(u_n^\ell)$ for $\ell = 1,2$ (remember (*)). Now we should choose p_{n+1}^1, p_{n+1}^2, such that $\varphi(v_{p_{n+1}^\ell}) \geq \vartheta_{n+1}$, and for each $\ell \leq n$, (4) holds. Each demand excludes a set in $\{A : \varphi(A) < \lambda\}$, (note that $\bigcup \{v_p : o(v_p) < \vartheta\}$ satisfies this by assumption (iv)) so there are distinct p_{n+1}^1, p_{n+1}^2 as required, and now choose disjoint u_{n+1}^1, u_{n+1}^2, such that $p_{n+1}^\ell \in u_{n+1}^\ell \subseteq v_{p_{n+1}^\ell}$.

Define for $\eta \in {}^\omega 2$, $A_\eta = \bigcap_{\eta(n)=0} u_n^1 \cap \bigcap_{\eta(n)=1} (X - u_\eta^1)$.

We define by induction on $n < \omega$, η_n, k_n, m_n, such that

(a) $\eta_n \in {}^\omega 2$

(b) $n \leq k_n < m_n < k_{n+1} < m_{n+1}$

(c) for $\ell < n, \eta_\ell(k_n) = \eta_\ell(m_n)$

(d) $\varphi(u^1_{k_n} \cap u^2_{m_n} \cap A_{\eta_n}) \geq \vartheta_{k_n}$.

For n=0 let $k_n = 0, m_n = 1$, now $\varphi(u^1_{k_n} \cap u^2_{m_n}) \geq \vartheta_{k_n}$ by condition (4) above. Then there is η_0 is required in (4) by $\oplus$. For $n > 0$ we first can find k_n m_n as required in (b),(c) and then η_n as above.

Now let $u_n = u^1_{k_n} \cap u^2_{m_n}$. So now by (c) $u_\ell \cap A_{\eta_m} = \phi$ for $\ell > m$, so $u_n - \bigcup_{\ell > n} u^1_{k_n} \cap u^2_{m_n} \supseteq A_{\eta_n}$ hence

$\varphi(u_n - \bigcup_{\ell > n} u_\ell) \geq \vartheta_n$; as $\varphi(u_\ell) < \vartheta_n$ for $\ell < n$, $\varphi(u_n - \bigcup_{\ell \neq n} u_\ell) \geq \vartheta_n$ so we finish.

2) Similar proof - instead $u^1_{k_n} \cap u^2_{m_n}$ we use finite such intersection and strengthen (4) accordingly (and $\{\eta_n\}$ is replaced by a no where dense set.)

Remark: If in 2.5(1) we weaken (iv) to $\varphi(X - \bigcup\{u: \varphi(u) < \lambda_n\}) \geq \lambda$, by changing φ so to satisfy (iv).

2.6 Lemma: 1) Suppose X is a Hausdorff space, $\underset{\sim}{B}$ a basis for X and $o(v) \leq \lambda_0$ for $v \in \underset{\sim}{B}$. Suppose further that $2^{<\hat{s}(X)} < o(X)$, $\lambda_0 < o(X)$ and for no $\kappa < \hat{s}(X)$, $(\lambda_0)^\kappa = o(X)$. Then $|\underset{\sim}{B}|^{<\hat{s}(X)} \geq o(X)$.

2) Under Hypothesis 2.2, if (*) of 2.4 holds, cf $\hat{s}(X) > \aleph_0$, and $\underset{\sim}{B}$ is a basis for X *then* $|\underset{\sim}{B}|^{<\hat{s}(X)} \geq o(X)$ (so for some χ and ϑ : $\chi^{<\vartheta} > o(X) \geq (\chi + 2^{<\vartheta})^{+\omega}$).

3) If X is a Hausdorff space $\beth_2 \leq o(X) < o(X)^{\aleph_0}$ *then* for some χ, ϑ : $(\chi + 2^{<\vartheta})^{+\omega} \leq o(X) < \chi^{<\vartheta}$.

2.6A Remark: The conclusion in 2.6(3) implies $0^\#$ exists by the covering lemma, and similarly much more.

* * *

We first prove some facts, where $\underset{\sim}{B}$ is a base of a Hausdorff space X and $o(v) \le \lambda_0$ for $v \in B_0$.

2.7 Definition : 1) We say $\bar{v} = \langle v_i : i < \alpha \rangle$ is good for u (where $u, v_i \in \tau$) if

(i) $v_i - u \neq \phi$

(ii) $v_i \in \underset{\sim}{B}$ (hence $v_i \in \tau$)

(iii) for $i \neq j < \alpha, v_i \cap v_j \subseteq u$.

2) We say $\bar{v}$ is maximally good for u if $\bar{v}$ is good for u but for no $v \in \underset{\sim}{B}$ is $\bar{v}\char`^\langle v \rangle$ good for u.

2.8 Observation: 1) For every $u \in \tau$ there $\bar{v}$ maximally good for it.

2) If $\langle v_i : i < \alpha \rangle$ is good for u, then $\alpha < \hat{s}(X)$.

Proof : 1) Immediate.

2) By (i) of Definition 2.7(1)) there is $y_i \in v_i - u$. Now $y_i \in v_i - u \in \tau$, and $i \neq j \Longrightarrow y_i \not\in v_j$ (as then $y_i \in v_j \cap v_i - u$.)

2.9 Fact: Let $G = \{\langle v_i : i < \alpha \rangle : v_i \in \underset{\sim}{B},\ v_i \not\subseteq \bigcup\{v_j : j < \alpha, j \neq i\}\ \}$.

1) If $\bar{v}$ is good for some u then $\bar{v} \in G$.

2) For each $\bar{v} = \langle v_i : i < \alpha \rangle \in G$ the following two sets has the same power:

$P_{\bar{v}} = \{u : \bar{v}$ is maximaly good for $u\}$.

$Q_{\bar{v}} = \{\langle J_i : i < \alpha \rangle : \bigcup_{j \neq i} (v_i \cap v_j) \subseteq J_i \subset v_i$, (so $J_i \neq v_i$) and J_i is open $\}$.

Proof : 1) Immediate.

2) We define H, a function with domain $P_{\bar{v}} : H(u) = \langle v_i \cap u : i < \alpha \rangle$.

Clearly $H(u) \in Q_{\bar v}$. Now H is one to one: if $H(u_1) = H(u_2)$ but $u_1 \neq u_2$ then w.l.o.g. $u_1 \not\subseteq u_2$, choose $y \in u_1 - u_2$, then choose $v \in \underset{\sim}{B}$, $y \in v \subseteq u_1$. So y witness $v \not\subseteq u_2$; and for $i < \alpha, v \cap v_i \subseteq u_1$ (as $v \subseteq u_1$) but $v \cap v_i \subseteq v_i$, $u_1 \cap v_i = u_2 \cap v_i$ so also $v \cap v_i \subseteq u_2$. We conclude that v contradicts the maximality of $\bar v$ (as good for u_2). So H is one to one.

Now for any $\langle J_i : i < \alpha \rangle \in Q_{\bar v}$, $J \overset{def}{=} \bigcup_{i<\alpha} J_i$ is an open set and easily $v_i \cap v_j \subseteq J_i \subseteq J$ for $i \neq j, J \cap v_i = J_i$ and $v_i \not\subseteq J$. So $\bar v$ is good for J. Let $u^* = \bigcup \{u : \bar v \text{ is good for } u, u \cap v_i = J_i\}$. Easily $\bar v$ is maximally good for u^* and $H(u^*) = \langle J_i : i < \alpha \rangle$.

2.10 Fact: For $\bar v \in G$, for some $\mu^i_{\bar v}$, $|Q_{\bar v}| = \prod_{i < \ell(\bar v)} \mu^i_{\bar v}$, and $\mu^i_{\bar v} \le \lambda_0$.

Proof: Let $\mu^i_{\bar v} = |\{J \in \tau : \bigcup_{j \neq i} (v_j \cap v_i) \subseteq J \subset v_i\}|$. Clearly $\mu^i_{\bar v} \le o(v_i)$, but $v_i \in \underset{\sim}{B}$ so $\mu^i_{\bar v} \le \lambda_0$. By the definition of $Q_{\bar v}$, $|Q_{\bar v}| = \prod_i \mu^i_{\bar v}$.

2.11 Observation: By cardinal arithmetic:

1) If $\mu = \prod_{i<\alpha} \mu_i$ then $\mu = \prod_{\ell=1}^{n} (\chi_\ell)^{\kappa(\ell)}$, where $n < \omega$,, $\chi_\ell \le \sup\{\mu_i : i < \alpha\}$, $\sum_{\ell=1}^{n} \kappa(\ell) = |\alpha|$. Also $(\forall i < \alpha)[\chi_\ell > \mu_i > \chi_{\ell+1} \rightarrow \kappa(\ell) \ge cf\ \chi_\ell]$ and $\kappa(\ell) = |\{i : \mu_i \le \chi_\ell, \text{ and } (\forall m)[\chi_m < \chi_\ell \Longrightarrow \chi_m < \mu]\}|$

2) In 1) if $\mu > \mu_i$ for each i, μ infinite then $\mu^{\aleph_0} = \mu$; in fact $\mu = \chi^{\kappa}$ for some $\chi \le \sum_{i<\alpha} \mu_i$, $\aleph_0 \le \kappa \le |\alpha|$.

3) Suppose $\chi \ge 2^{<s}$, then $\{\prod_{i<\alpha} \mu_i : \alpha < s, \mu_i \le \chi$ for each $i < \alpha$ but $\prod_{i<\alpha} \mu_i > \chi\}$ is finite.

4) If $\chi \ge 2^{<s} (s \ge \aleph_0)$ then for some $\vartheta < s$: $\chi^{\vartheta} = \chi^{<s}$.

Remark: In particular, in 3) $\{\lambda^\sigma : 2^\sigma < \lambda\}$ is finite. When I visited Budapest (in April 84) I learned that this already appeared explicitly in the Hungarian book of Hajnal on Set Theory.

Proof: 1) We define χ_ℓ by induction on ℓ, $\chi_1 > \chi_2 > \cdots$. Let $\chi_1 = \sup_{i<\alpha} \mu_i$.

If χ_ℓ is defined and is a successor cardinal, let $\chi_\ell = (\chi_{\ell+1})^+$. If χ_ℓ is defined, $\chi_\ell = 1$ let $n = \ell$.
If $\chi_\ell > 0$ is a limit cardinal, let $\chi_{\ell+1}$ be the minimal $\chi < \chi_\ell, \chi \geq 1$ such that for every χ^*, if $\chi < \chi^* < \chi_\ell$ then

(*) $|\{i<\alpha : \chi<\mu_i \leq \chi_\ell\}| = |\{i < \alpha : \chi^* < \mu_i \leq \chi_\ell\}|$.

Now χ exists as $\langle |\{i < \alpha : \chi < \mu_i \leq \chi_i\}| : \chi < \chi_\ell \rangle$ is a decreasing sequence.

Clearly for some ℓ $\chi_\ell = 1$, so $\ell = n$. Now $\prod_{i<\alpha} \mu_i = \prod_{\ell=1}^{n} \Pi\{\mu_i : \chi_{\ell+1} < \mu_i \leq \chi_\ell\}$ (remember $\mu_i \neq 0$, and we can ignore $\mu_i = 1$).

By (*), $\Pi\{\mu_i : \chi_{\ell+1} < \mu_i \leq \chi_\ell\} = \chi_\ell^{\kappa(\ell)}$, where $\kappa(\ell) = |\{i : \chi_{\ell+1} < \mu_i \leq \chi_\ell\}|$.

The last phrase is easy too.

2) Easy.

3) By 2) if $\prod_{i<\alpha} \mu_i \geq \chi$, $\mu_i \leq \chi, \alpha < s$ then for some $\vartheta \leq \chi, \kappa \leq |\alpha|$, $\vartheta^\kappa = \prod_{i<\alpha} \mu_i$, so $\vartheta^\kappa \leq \chi^\kappa \leq (\prod_{i<\alpha} \mu_i)^\kappa = (\vartheta^\kappa)^\kappa = \vartheta^\kappa$, hence $\prod_{i<\alpha} \mu_i = \chi^\kappa$ where $\kappa \leq |\alpha|$. So it suffices to prove $\{\chi^\kappa : \kappa < s\}$ is finite. Suppose $\chi^{\kappa(n)}$ are distinct for $n < \omega$, where for each n $\kappa(n) < s$. W.l.o.g. $\kappa(n) < \kappa(n+1)$. Let $\chi_n = Min\{\mu : \mu^{\kappa(n)} \geq \chi\}$, so easily:

(i) for each n, $\chi_n \geq \chi_{n+1}$.

(ii) $\chi_n^{\kappa(n)} = \chi^{\kappa(n)}$.

By (i) w.l.o.g. $\langle \chi_n : n < \omega \rangle$ is constant; as we have assumed $\{\chi^{\kappa(n)} : n < \omega\}$ are distinct, by (ii) $\{\chi_n^{\kappa(n)} : n < \omega\}$ are distinct.

But $(\forall \sigma < \chi_n) \sigma^{\kappa(n)} < \chi_n$, hence $(\forall \sigma < \chi_0)(\forall n < \omega)$ $(\sigma^{\kappa(n)} < \chi_0)$, and clearly $cf(\chi_n) \leq \kappa(n)$, so $\chi_n^{\kappa(n)} = \chi_n^{cf(\chi_n)} = \chi_0^{cf(\chi_0)}$. But $\chi_n^{\kappa(n)} = \chi^{\kappa(n)}$ are distinct, contradiction.

4) Follows from 3).

Proof of 2.6(1): Suppose $|\underset{\sim}{B}|^{<\hat{s}(X)} < o(X)$. By 2.8(1), 2.9(1), $\tau = \bigcup\{P_{\bar{v}}:\bar{v} \in G\}$, hence $o(X) = |\tau| \le \sum_{\bar{v}\in G} |P_{\bar{v}}|$. By 2.9(2) $o(X) \le \sum_{\bar{v}\in G} |Q_{\bar{v}}|$, and by 2.8(2), $|G| \le |\underset{\sim}{B}|^{<\hat{s}(X)}$. So to get a contradiction it suffices to prove that $\sup\{|Q_{\bar{v}}| : \bar{v} \in G\} < o(X)$. By 2.10, $|Q_{\bar{v}}| = \prod_{i<\ell(\bar{v})} \mu^i_{\bar{v}}$ where $\mu^i_{\bar{v}} \le \lambda_0$ (as $v_i \in \underset{\sim}{B}$ by an assumption) and $\ell(\bar{v}) < \hat{s}(X)$ (by 2.8(2).) W.l.o.g. $(\forall i)(\mu^i_{\bar{v}} > 1)$.

Now by 2.11, for some natural number of $n(\bar{v})$ and cardinals $\mu_{\bar{v},\ell} \le \lambda_0$ and $\kappa(\bar{v},\ell) \le \ell(\bar{v}) < \hat{s}(X)$, for $(\ell < n)$:

$$|Q_{\bar{v}}| = \prod_{\ell=1}^{n(\bar{v})} (\mu_{\bar{v},\ell})^{\kappa(\bar{v},\ell)}$$

so if $Q_{\bar{v}}$ is infinite, $Q_{\bar{v}} = \underset{\ell = 1,n}{Max}\,(\mu^{\kappa(\bar{v},\ell)}_{\bar{v},\ell})$.

But $(\mu_{\bar{v},\ell})^{\kappa(\bar{v},\ell)} \ge \lambda_0$ implies $(\mu_{\bar{v},\ell})^{\kappa(\bar{v},\ell)} = \lambda_0^{\kappa(\bar{v},\ell)}$ so $|Q_{\bar{v}}| \ge \lambda_0$, implies that for some $\kappa(\bar{v}) \le \ell(\bar{v})$, $|Q_{\bar{v}}| = \lambda_0^{\kappa(\bar{v})}$. But $\ell(\bar{v}) < \hat{s}(X)$.

So we have proved: if $|Q_{\bar{v}}| \ge \lambda_0$ then $|Q_{\bar{v}}| = \lambda_0^{\kappa(\bar{v})}$ where $\kappa(\bar{v}) < \hat{s}(X)$. But we have assumed $(\lambda_0)^{\kappa(\bar{v})} \ne o(X)$ and we know $|Q_{\bar{v}}| = |P_{\bar{v}}| \le o(X)$, so necessarily $|Q_{\bar{v}}| \ge \lambda_0 \Longrightarrow |Q_{\bar{v}}| < o(X)$. But $\lambda_0 < o(X)$ so $|Q_{\bar{v}}| < o(X)$. The same argument gives, $\sup\{|Q_{\bar{v}}| : \bar{v} \in G\} \le \sup[\{\lambda_0\} \cup \{\lambda_0^\kappa : \kappa < \hat{s}(X), \lambda_0^\kappa < o(X)\}]$ but by 2.11 this is $\lambda_0^{\kappa(0)}$, for some $\kappa(0) < \hat{s}(X)$ hence this supremum is $< o(X)$, which we have shown is enough for 2.6(2).

Proof of 2.6(2): We use freely 2.3, 2.4. So (w.l.o.g.) $|X| < \lambda_0, X$ has a strong base $\underset{\sim}{B}, |\underset{\sim}{B}| < \lambda_0$, $o(v) < \lambda_0$ for $v \in \underset{\sim}{B}$, and $2^{<\hat{s}(X)} \le o(X)$. As $cf\ \hat{s}(X) > \aleph_0$, $(2^{<\hat{s}(X)})^{\aleph_0} = 2^{<\hat{s}(X)}$ hence $2^{<\hat{s}(X)} < \lambda$ hence w.l.o.g. $2^{<\hat{s}(X)} < \lambda_0$. So all the assumptions of 2.6(1) hold, hence $|\underset{\sim}{B}|^{<\hat{s}(X)} \ge o(X)$ as required. The last phrase holds if we choose $\chi = |\underset{\sim}{B}|$, $\vartheta = \hat{s}(X)$. Note $(\chi + 2^{<\vartheta})^{+\omega} = (|\underset{\sim}{B}| + 2^{<\hat{s}(X)})^{+\omega} \le \lambda_0^{+\omega} \le o(X)$ (as $\lambda_0^{\aleph_0} = \lambda_0$ also $(\lambda_0^{+n})^{\aleph_0} = \lambda_0^{+n}$) and $o(X) \le |\underset{\sim}{B}|^{<\hat{s}(X)}$.

Proof of 2.6(3): Now X satisfy I from Hypothesis 2.2. If (*) of 2.4 holds we finish by 2.6(2). Otherwise by 2.4 for some n $\lambda_n^{\beth_1} > \lambda$, hence $\lambda_n^{\beth_1} > o(X)$, hence $\lambda_n > \beth_2$ (as $o(X) \geq \beth_2$). Remember $\lambda_n^{\aleph_0} = \lambda_n$. Let $\chi = \lambda_n$, $\vartheta = \beth_1^+$, they satisfy the required conclusion.

A corollary of [Sh 1] 4.4 is

2.12 Observation: If B is in infinite Boolean algebra then $|B|^{<\hat{s}(X)} \leq 2^{<\hat{s}(B)}$.

Proof: Let κ be the cellularity of B, so κ is regular, $> \aleph_0, \kappa \leq \hat{s}(B)$, and let $\lambda = Min\{\lambda : \lambda^{<\kappa} \geq |B|\}$; as κ is regular $(\lambda^{<\kappa})^{<\kappa} = \lambda^{<\kappa}$. If $\lambda > 2^{<\kappa}$ then $(\forall \mu < \lambda)\ \mu^{\kappa} < \lambda$, and by [Sh 1] 4.4, $\lambda < \hat{s}(B)$ so $2^{\lambda} \geq \lambda^{<\kappa} \geq |B|$, hence

$$|B|^{<\hat{s}(X)} \leq (|B|^{\lambda})^{<\hat{s}(B)} \leq ((2^{\lambda})^{\lambda})^{<\hat{s}(B)} = 2^{<\hat{s}(B)}$$

If $\lambda \leq 2^{<\kappa}$, then $|B| \leq 2^{<\kappa}$; remember $\kappa \leq \hat{s}(B)$ now if $\kappa = \hat{s}(B)$, then $|B|^{<\hat{s}(B)} = 2^{<\hat{s}(B)}$ as $\kappa = \hat{s}(B)$ is regular; and if $\kappa < \hat{s}(B)$, $|B|^{<\hat{s}(B)} \leq (2^{\kappa})^{<\hat{s}(B)} = 2^{<\hat{s}(B)}$.

2.13 Conclusion: 1) If B is a Boolean algebra, $id(B)^{\aleph_0} = id(B)$.

2) If X is locally compact Hausdorff space then $o(X)^{\aleph_0} = o(X)$.

Proof : 1) Let X be the space of ultrafilters of B, considering B as a basis. So $id(B) = o(X)$. By 2.6(2) (note X is strongly Hausdorff) $o(X) < o(X)^{\aleph_0}$ implies $|B|^{<\hat{s}(B)} > o(X)$, but $o(X) \geq 2^{<\hat{s}(X)} = 2^{<\hat{s}(B)}$ contradicting 2.12.

2) We need the parallel of 2.12, which is proved by translating the proof of [Sh] 4.2, 4.4 to topology, which is done in 2.14 below.

2.14 Lemma : Let X be a locally compact Hausdorff compact space with cellularity κ.

1) If $(\forall \vartheta < \mu)(\vartheta^{<\kappa} < \mu)$ (so $2^{<\kappa} < \mu$) and every basis of X consisting of regular open sets has power $\geq \mu$ then $\hat{s}(X) \geq \mu$.

2) If μ is regular, X has a subspace Y whose topology is a refinement of

λ_2.

Note: Theorem 2.14 was prooved by F. Argyros and A. Tsarpaleas independently of [Sh].

Proof: The proof are like [Sh] 4,2 , 4.4; we concentrate on 2.14(2), so μ is regular (anyhow we shall use only this part). Here $\bar{u}$ denote the closure of u. Really it is a repetition of [Sh 1] with one change; use of compactness for a family of sets $u^2_\alpha - \overline{u^1_\beta}$.

2) Let $\underset{\sim}{B}$ be such a base. W.l.o.g. $(\forall u \in \underset{\sim}{B})[\bar{u}$ is compact] (otherwise replace $\underset{\sim}{B}$ by $\{u \in \underset{\sim}{B} : \bar{u}$ is compact$\}$). Let $\chi = (2^{2^{|X|}})^+$,$H(\chi)$ the family of sets of hereditary power $< \chi$. We define by induction on $i < \mu$, $N_i \prec (H(\chi),\in)$, such that $\underset{\sim}{B} \in N_i$, $\|N_i\| < \mu$, $\langle N_j : j \leq i \rangle \in N_{i+1}, N_j \prec N_i$ for $j < i$ and every sequence of $< \kappa$ member of N_i belong to N_i when i is a successor ordinal. (hence when $cf\ i \geq \kappa$). For each $i < \mu$, let $\underset{\sim i}{B} = \{u \in N_i : u$ regular open, $\bar{u}$ compact$\}$.

As $|\underset{\sim i}{B}| < \mu$ by a hypothesis it is not a basis of X, hence there are in N_{i+1} $p_i \in X, u_i^0 \in B$, $p_i \in u_i^0$, such that for no $v \in \underset{\sim i}{B}$, $p_i \in v \subset u_i^0$. We can find for $\ell < 3$, $u_i^\ell \in \underset{\sim i+1}{B}$, such that $p_i \in u_i^\ell$, $\overline{u_i^{\ell+1}} \subset u_i^\ell$. Restrict ourselves to case $cf\ i \geq \kappa$.

Let $J_i^2[I_i^\ell]$ be a maximal family of pairwise disjoint open sets $u \in \underset{\sim i}{B}, u \subset u_i^\ell$ $[u \cap u_i^\ell = \phi]$. So J_i^ℓ, I_i^ℓ, are subsets of N_i of power $< \kappa$ (as κ is the cellularity of X) hence $J_i^\ell, I_i^\ell \in N_i$. Let $A_i^\ell = X - \overline{\bigcup I_\ell^i}$, so A_i^ℓ is open, belongs to N_i (non empty) $u_i^\ell \subset A_i^\ell$ (as $X - u_i^\ell$ is closed, $\bigcup I_i^\ell \subset X - u_i^\ell$) and there is no open (non empty) $v \subset A_i^\ell - u_i^\ell$, $v \in N_i$. Also $A_i^\ell \in N_i$. Let $B_i^\ell = \bigcup J_i^\ell$, so $B_i^\ell \subset u_i^\ell, B_i^\ell$ is open belongs to N_i and there is no open $v \subset u_i^\ell - B_i^\ell$, $v \in N_i$. By Fodour's Lemma there are A^ℓ, B^ℓ such that $S = \{i : i < \mu, cf\ i < \kappa.$ $A_i^\ell = A^\ell, B_i^\ell = B^\ell$ for $\ell = 0,1,2\}$ is stationary. It is enough to prove

(*) for disjoint finite $w(1), w(2) \subset S$,

$$\bigcap_{\alpha \in w(1)} u^2_\alpha \not\subset \bigcup_{\beta \in w(2)} u^1_\beta$$

As then for any non empty $w \subset S$, $\{\overline{u^2_\alpha} - u^1_\beta : \alpha \in w, \beta \in S - w\}$ is a family of closed sets, the intersection of any finitely many is non empty and $\overline{u^2_\alpha}$ is compact for $\alpha \in w$, so there is q_w in the intersection. So $\{((q_{\{\alpha\}}, u^2_\alpha) : \alpha \in s\}$ exemplify $\hat{s}(X) > \mu$, and if $S = \{\xi_i : i < \mu\}$, let $H: {}^{\lambda}2 \to X$ be define by $H(\eta) = q_{\{\xi_1 + i:\, \eta(i) = 0\} \cup \{\xi_0\}}$, then $Y = \{H(\eta): \eta \in {}^{\lambda}2\}$ is as required.

Let $RO(X)$ be the Boolean Algebra of regular open subsets of X. So in $RO(X)$ we identify $u \in \tau(X)$ with $int(\overline{u})$ (and so the operations are changed accordingly). So $RO(X)$ is complete, in $RO(X)$ $\bigcup_{i<\alpha} A_i = int(\overline{\bigcup_{i<\alpha} A_i})$ i.e. the interior of $\bigcup_{i<\alpha} A_i$; $\bigcap_{i<\alpha} A_i = int(\bigcap_{i<\alpha} A_i)$. So $RO(X)$ satisfies the κ-chain condition and $RO(X) \cap N_i$ is a complete subalgebra.

So in $RO(X)$, A_i^ℓ is minimal such that $A_i^\ell \in N_i$, $u_i^\ell \subseteq A_i^\ell$ and B_i^ℓ is maximal such that $B_i^\ell \subseteq u_i^\ell$, $B_i^\ell \in N_i$.

Proof of (*): We shall work in $RO(X)$ and prove by induction on $n = |w(1)| + |w(2)|$;.

$(*)^+$ $RO(X) \models \bigcap_{\alpha \in w(1)} u_\alpha^2 \not\subseteq \bigcup_{\alpha \in w(2)} u_\alpha^1 \cup B^1$.

When n is zero the statement is obvious. Let $\alpha = Max((w(1) \cup w(2))$ and $Max(w(1) \cup w(2) - \{\alpha\}) \leq \beta < \alpha$.

By the induction hypothesis $v = \bigcap_{\substack{\gamma \in w(1)\\ \gamma \neq \alpha}} u_\gamma^2 - \bigcup_{\substack{\gamma \in w(2)\\ \gamma \neq \alpha}} \overline{u_\gamma^1} \cup B^1$ is $\neq 0$ (in $RO(X)$). Clearly $v \in B_{\sim\alpha}$, and if (*) fails then $v \subseteq B_\alpha^1 = B^1$ (if $\alpha \in w(2)$) or $\phi = v \cap A_\alpha^2 = v \cap A^2$ (if $\alpha \in w(1)$). In both cases a contradiction follows.

2.18 Conclusion: For locally compact X, $w(X)^{<\hat{s}(X)} \leq 2^{<\hat{s}(X)}$.

Proof : Suppose $w(X)^{<\hat{s}(X)} > 2^{<\hat{s}(X)}$, let $\mu = Min\{\mu^{<\kappa} \geq w(X)\}$, where κ is the cellularity of X. Clearly $\kappa \leq \hat{s}(X), \mu \leq w(X)$, and $(\forall \chi < \mu)(\chi^{<\kappa} < \mu)$ (as $(\chi^{<\kappa})^{<\kappa} = \chi^{<\kappa}$, κ being regular). So by 2.17 $\mu < \hat{s}(X)$ but $|w(X)|^{<\hat{s}(X)} \leq (\mu^{<\kappa})^{<\hat{s}(X)} \leq \mu^{<\hat{s}(X)} \leq (2^{<\mu})^{<\hat{s}(X)} \leq 2^{<\hat{s}(X)}$ contradiction. [if we want to use only the part of 2.17 actually prove, note that

a) $\mu = \hat{s}(X)$ is singular (by the previous argument).

b) if μ is not strong limit, let $\vartheta < \mu \leq 2^{\vartheta}$, so

$\models w(X)^{<\hat{s}(X)} \le (\mu^{<\kappa})^{<\hat{s}(X)} = \mu^{<\hat{s}(X)} \le (2^{\vartheta})^{<\hat{s}(X)} = 2^{<\hat{s}(X)}$ contradiction;

c) if μ is strong limit singular $\hat{s}(X) = \mu$ is impossible (see [J2] or 3.4.]).

§3 Nice cardinal functions on a topological space.

3.1 Definition : 1) φ is nice for X if φ is a function from subsets of the topological space X to cardinals satisfying

(i) $\varphi(A) \le \varphi(A \cup B) \le \varphi(A) + \varphi(B) + \aleph_0$ (i.e. monotonicity and subadditivity)

2) We call φ (χ,μ)-complete provided that if $A_i \subset X$, $\varphi(A_i) < \chi$ for $i < \mu$ then $\varphi(\bigcup_{i<\mu} A_i) < \chi$.

Let $C(\varphi,\mu) = \{\chi : \varphi \text{ is } (\chi,\mu)\text{-complete}\}$.

3) We call φ $(<\lambda,\mu)$-complete, if for arbitrarily large $\chi < \lambda, \varphi$ is (χ,μ)-complete.

4) Let Ch_φ be the function from X to cardinals

$$Ch_\varphi(y) = Min\{\varphi(u) : y \in u \in \tau(X)\}$$

3.1A Remark: 1) We can replace $i < \mu$ by $i < \alpha < \mu$ and made suitable changes later.

2) In our applications we can restrict the domain of φ to the Boolean Algebra generated by $\tau(X)$ and even more, e.g. in 3.2 to simple combinations of the $u_{i,\xi,\zeta}$.

3) We can change the definition of $(<\lambda,\mu)$-complete to

(*) if $A_i \subset X(i < \mu)$, $\underset{i<\mu}{Sup}\, \varphi(A_i) < \lambda$ then $\varphi(\bigcup_{i<\mu} A_i) < \lambda$

without changing our subsequence use. [we then will use: if $\varphi(A_\alpha) < \chi_i$ for $\alpha < \mu$ then $\varphi(\bigcup_{\alpha<\mu} A_\alpha) < \chi_{i+1}$].

3.2 Lemma : Suppose λ is singular of cofinality ϑ, $\lambda = \sum_{i<\vartheta} \chi_i$, $\chi_i < \lambda$,

$\vartheta < \lambda$ and $\mu = \beth_5(\vartheta)^+$ or even $\mu = \beth_2(\beth_2(\vartheta)^+)^+$. If

(i) φ is nice for X.

(iii) $X_{\chi_i} = \{y \in X: Ch_\varphi(y) \geq \chi_i\}$ has cardinality $\geq \mu$ for $i < \vartheta$.

(iii) φ is $(<\lambda,\mu)$-complete.

Then there are open $u_i \subseteq X (i < \vartheta)$ such that

$$\varphi(u_i - \bigcup_{j \neq i} u_j) \geq \chi_i$$

Remark: If $|\{y \in X: Ch_\varphi(y) \geq \chi_i\}| < \mu$ it essentially follows from (χ_i,μ)-completeness that $\varphi(X_{\chi_i}) \geq \lambda$ where $X_\chi = \bigcup\{v \in \tau(X): \varphi(v) < \chi\}$. Otherwise $\varphi(X - X_{\chi_i}) \geq \lambda$ by additivity, but $\varphi(X - X_{\chi_i}) \leq \Pi\{\varphi(\{y\}): y \in X - X_{\chi_i}\}$ so by (χ_i,μ)-completeness for some $y \in X$, $\varphi(\{y\}) \geq \chi_i$ which is impossible for the instances which interest us.

Proof: W.l.o.g. $\chi_i \in C$, $C \overset{def}{=} C(\varphi,\mu) \cap \lambda$. Choose dinstinct $y_{i,\xi} \in X - X_{\chi_i}$ for $i < \vartheta$, $\xi < \mu$.

Let $u^\alpha_{i,\xi,\zeta} (i < \vartheta, \xi \neq \zeta < \mu)$ be open sets such that $y_{i,\xi} \in u_{i,\xi,\zeta}$ and $u_{i,\xi,\zeta} \cap u_{i,\zeta,\xi} = \phi$. Now

(*) for every $i < \vartheta, \xi(0) < \xi(1) < \xi(2) < \mu$, there is $x = x_{i,\xi(0),\xi(1),\xi(2)}$ such that :

(a) $x \in u_{i,\xi(1),\xi(0)} \cap u_{i,\xi(1),\xi(2)}$.

(b) if $\mathcal{P} \subseteq \Gamma \overset{def}{=} \{u_{j,\xi,\zeta}, X - u_{j,\xi,\zeta} : j < \vartheta, \xi \neq \zeta < \mu\}$,

$$|\mathcal{P}| \leq \vartheta, \text{ and } x \in \bigcap_{A \in \mathcal{P}} A \quad \text{then} \quad \varphi(\bigcap_{A \in \mathcal{P}} A) \geq \chi_i$$

If (*) fail, (for $i,\xi(0),\xi(1),\xi(2)$) then for every $x \in u_{i,\xi(1),\xi(0)} \cap u_{i,\xi(1),\xi(2)}$ some $\mathcal{P}$ contradicts (b). So there are $\mathcal{P}_i \subseteq \Gamma$ $(i < \alpha)$, $|\mathcal{P}_i| \leq \vartheta$, $\varphi(\bigcap_{A \in \mathcal{P}_i} A) < \chi_i$, and $\bigcup_{i<\alpha} \bigcap_{A \in \mathcal{P}_i} A \supseteq u_{i,\xi(1),\xi(0)} \cap u_{i,\xi(1),\xi(2)}$. As $\alpha \leq |\Gamma|^\vartheta \leq \mu^\vartheta = \mu$, by the (χ_i,μ)-completeness (as $\varphi(\bigcap_{A \in \mathcal{P}_i} A) < \chi_i$):

$$\varphi(u_{i,\xi(1),\xi(0)} \cap u_{i,\xi(1),\xi(2)}) \le \prod_{i<\alpha} \varphi(\bigcap_{A \in p_i} A) < \chi_i$$

But $y_{i,\xi(1)} \in u_{i,\xi(1),\xi(0)} \cap u_{i,\xi(1),\xi(2)}$, $y_{i,\xi(1)} \notin X_{\chi_i}$, contradiction. So (*) holds and let $x_{i,\xi(0),\xi(1),\xi(2)}$ exemplify it. Now define a five place function F on $\{y_{i,\xi}: i < \vartheta, \xi < \mu\}$, if $i \neq j < \vartheta$, $\xi(0) < \xi(1) < \xi(2) < \mu$, $\zeta(0) < \zeta(1) < \mu$:

$$F(y_{i,\xi(0)}, y_{i,\xi(1)}, y_{i,\xi(2)}, y_{j,\zeta(0)}, y_{j,\zeta(1)})$$

is 0 if $x_{i,\xi(0),\xi(1),\xi(2)} \in u_{j,\zeta(0),\zeta(1)}$ and is 1 otherwise.

By Erdos Rado if $\mu = \beth_5(\vartheta)^+$ and [Sh 2] if $\mu = \beth_2(\beth_2(\vartheta)^+)^+$ (see remark 3.18 below) there are $\xi(i,\ell)$ ($i < \vartheta$, $\ell < 3$) such that for $i \neq j < \vartheta$:

$$F(y_{i,\xi(i,0)}, y_{i,\xi(i,1)}, y_{i,\xi(i,2)}, y_{j,\xi(j,0)}, y_{j,\xi(j,1)}) =$$
$$F(y_{i,\xi(i,0)}, y_{i,\xi(i,1)}, y_{i,\xi(i,2)}, y_{j,\xi(j,1)}, y_{j,\xi(j,2)})$$

(and $\xi(i,0) < \xi(i,1) < \xi(i,2)$)

We can conclude that

$$x_{i,\xi(i,0),\xi(i,1),\xi(i,2)} \notin u_{j,\xi(j,1),\xi(j,0)} \cap u_{j,\xi(j,1),\xi(j,2)}$$

(because $u_{i,\xi,\zeta} \cap u_{i,\zeta,\xi} = \phi$ for $\xi \neq \zeta$).

Let $u_i = u_{i,\xi(i,1),\xi(i,0)} \cap u_{i,\xi(i,1),\xi(i,2)}$. So $x_{i,\xi(i,0),\xi(i,1),\xi(i,2)} \in u_i - \bigcup_{j \neq i} u_j$, and by the choice of $x_{i,\xi(i,0),\xi(i,1),\xi(i,2)}$, u_i clearly $\varphi(u_i - \bigcup_{j \neq i} u_j) \ge \chi_i$, as required.

3.2A Remark: 1) The demand on μ is (see [Sh 2] Definition 1) to be able to use that $\langle (\mu)_\vartheta \rangle$ have $\langle (3)_\vartheta \rangle$-cannonization for $\{<2;3>^5_2, <3;2>^5_2\}$, but really $\{<2;3>^5_2, <3;2>^2_2\}$.

Really we can define F for any five tuples from $\{y_{i,\xi}: i < \vartheta, \xi < \mu\}$, and it is enough to find $\xi(i,\ell) < \mu$, $\alpha(i,\ell) < \vartheta$ (for $i < \vartheta$, $\ell < 3$) such that $\vartheta = \operatorname{Sup}_{i<\vartheta} (\operatorname{Min}_{\ell<3} \alpha(i,\ell))$, $[k \neq m \Longrightarrow \xi(i,k) \neq \xi(i,m)]$ and for $i < j < \vartheta$,

$$F(y_{\alpha(i,0),\xi(i,0)}, y_{\alpha(i,1),\xi(i,1)}, y_{\alpha(i,2),\xi(i,2)}, y_{\alpha(j,0),\xi(j,0)}, y_{\alpha(j,1),\xi(j,1)})$$
$$= F(y_{\alpha(i,0),\xi(i,0)}, y_{\alpha(i,1),\xi(i,1)}, y_{\alpha(i,2),\xi(i,2)}, y_{\alpha(j,1),\xi(j,1)}, y_{\alpha(j,2),\xi(j,2)}.)$$

2) If $\vartheta > \aleph_0$ is weakly compact, $\mu = 2^\vartheta$ is o.k.; in fact we can use just $\{y_{i,0}: i < \vartheta\}$ by 3.2A(1).

3.2B Remark: How do we apply [Sh 2] in the proof of 3.2? By the compositioin claim [Sh 2, Claim 5, p. 349] it is enough to prove that:

(a) $\langle (\beth_2(\beth_2(\vartheta)^+)^+)_\vartheta \rangle$ has a $\langle (\beth_2(\vartheta)^{++}))_\vartheta \rangle$-canonization for $\{(2;3)^2_2, (3;2)^2_2\}$.

(b) $\langle (\beth_2(\vartheta)^{++})_\vartheta \rangle$ has a $\langle ((2^\vartheta)^+)_\vartheta \rangle$-canonization for $\{(2;1)^2_2\}$ really even for $\{(2;1)^2_{\beth_2(\vartheta)}\}$.

(c) $\langle ((2^\vartheta)^+)_\vartheta \rangle$ has a $\langle (3)_\vartheta \rangle$-canonization for $\{(1)^1_{2^\vartheta}\}$.

Now (c) is trivial, and (a) we get by e.g. applying [Sh 2, 6(B), p. 249] twice; Now to get (b) (and even for $\{(2;1)^2_{\beth_2(\vartheta)}\}$) we apply [Sh 2, 6(F)] with $S = \vartheta$, $\lambda_\xi = \beth_2(\vartheta)^{++}$, $\kappa_\xi = (2^\vartheta)^+$, and check the condition.

3.3 Theorem : 1) If $\mu = \beth_5(cf\ \lambda)^+ < \lambda$, or $\mu = \beth_2(\beth_2(cf\ \lambda)^+)^+ < \lambda$, X is a Hausdorff space, with spread λ, then the supremum is obtained, i.e., $\hat{s}(X) \neq \lambda$.

2) The same apply to $h(Y), z(X)$.

Proof : Suppose X is a Hausdorff space, $\hat{s}(X) \geq \lambda$. Let $\lambda = \sum_{i<cf\ \lambda} \chi_i$, $\chi_i < \lambda$, $\vartheta \stackrel{def}{=} cf\ \lambda$, let $|A_i| = \chi_i$, A_i discrete w.l.o.g. $X = \bigcup_{i<\vartheta} A_i$ and let $\varphi(A) = |A|$, and let C be the family of regular cardinals $< \lambda$ but $> \mu$. Now (i), (iii) are immediate. If (ii) fail for χ, by Hajnal free subset theorem the spread is λ. Otherwise we can find by lemma 3.2 open $u_i (i < cf\ \lambda)$, $|u_i - \bigcup_{j \neq i} u_j| \geq \chi_i$ w.l.o.g. each χ_i is regular $> cf\ \lambda$, so for each i for some $\alpha_i < cf\ \lambda$, $(u_i - \bigcup_{j \neq i} u_j) \cap A_{\alpha_i}$ has power χ_i. The rest is easy too.

3.4 Lemma: Suppose κ is a strong limit cardinal, X an infinite Hausdorff space, $o(X) \geq \kappa$. If $o(X)^{<\kappa} > o(X)$ then for some $Y \subseteq X$ and χ, $|X| \leq \chi = \chi^{<\kappa} < o(X)$, $|X - Y| < \kappa$, Y open, $o(Y) = o(X)$, $Y = \bigcup \{v \in \tau : o(v) < \chi\}$, so Y has a strong base of power χ.

Proof: For $\kappa = \aleph_0$, this is trivial; if κ is strongly inaccessible then κ is the limit of strong limit singular cardinals, and it suffice to prove it for each of them [let for $\sigma < \chi$

$\chi_\sigma = Min\{\chi\colon X - \bigcup\{u \in \tau\colon o(u) < \chi\}$ has cardinality $< \sigma\}$.

$Z_\sigma = \{y \in X : Ch_o(y) \geq \chi_\sigma\}$ $(= X - \bigcup\{u \in \tau : o(u) < \chi\})$

so when σ increases χ_0 decrease, (and χ_σ is well defined: $\chi_\sigma \leq o(X)$); so for some $\sigma(0) < \kappa$, $\chi_\sigma = \chi_{\sigma(0)}$ hence $Z_\sigma = Z_{\sigma(0)}$ whenever $\sigma(0) \leq \sigma < \kappa$. W.l.o.g. $\sigma(0)$ is strong limit singular; checking the definition of $\chi_\sigma, \chi_\kappa = \chi_{\sigma(0)}$. (as $Ch_o(z) \geq \kappa$ for $z \in Z_{\sigma(0)}$) For every strong limit singular σ, $\sigma(0) < \sigma < \kappa$, as 3.4 is assumed to be proved for it, there are χ, Y as required; clearly (by the "Min" in the definition of χ_σ) $\chi_\kappa = \chi_\sigma \leq \chi = \chi^{<\sigma}$, so $\chi_\kappa^{<\sigma} < o(X)$. As $o(X) \geq \kappa > \sigma$, κ strong limit regular, clearly $o(X) \geq \kappa \geq 2^{<\kappa}$, hence $o(X) > 2^{<\kappa}$, so either $\chi_\kappa^{<\kappa} \leq (2^{<\kappa})^{<\kappa} = 2^{<\kappa} < o(X)$ or by 2.11. $\chi_\kappa^{<\kappa} = \chi_\kappa^\sigma$ for some $\sigma < \kappa$, hence $\chi_\kappa^{<\kappa} < o(X)$. Now $\chi = \chi_\kappa^{<\kappa}$ is as required (if $|X| > \chi$ use Hajnal free subset theorem.)]

So w.l.o.g. κ is a strong limit singular cardinal. Let X be a counterexample, i.e. $o(X)^{<\kappa} > o(X)$.

Let $\lambda = Min\{\lambda : \lambda^\kappa \geq o(X)\}$, so $\lambda^\kappa = o(X)^\kappa > o(X)$, and $\lambda \leq o(X)$. Also $[\sigma < \kappa, \chi < \lambda \Longrightarrow \chi^\sigma < \lambda]$ and $cf\ \lambda \leq \kappa$. Let $\vartheta = cf\ \lambda$, so $\vartheta \leq \kappa$ but ϑ is regular so $\vartheta < \kappa$, and also $\mu \overset{def}{=} \beth_5(\vartheta)^+$ is $< \kappa$, hence $(\forall \sigma < \kappa)\sigma^\mu < \lambda$.

We define the function φ:

$\varphi(A) = |\{u \cap A : u$ is an open subset of $X\}|$.

The family C of cardinals will be $\{(\chi^\mu)^+ : \chi < \lambda\}$.

Now we want to apply the lemma 3.2. Its conclusion clearly suffice by 2.3A (ii) . Now "φ is nice for X" and "φ is $(< \lambda, \mu)$-complete" are immediate. So (ii) necessarily fail for some $\chi < \lambda$. So $Y = \bigcup\{v : o(v) < \chi\}$ satisfies $|X - Y| < \mu$, hence $o(Y) = o(X)$ [as $o(X - Y) \leq 2^\mu < \kappa \leq o(X)$]. Also $|Y| < \lambda$ [otherwise by Hajnal free subset theorem , $\hat{s}(X) \geq \hat{s}(Y) > \lambda$, hence $o(X) \geq 2^\lambda$, but $2^\lambda \geq o(X)$ so $o(X) = 2^\lambda$, hence $o(X)^\kappa = o(X)$ contr]. So Y (as a subspace) has a strong base $\underset{\sim}{B}$ of power $\leq \chi + |X| < \lambda$.

3.5 Conclusion: If X is Hausdorff space, κ strong limit cardinal $o(X) \geq \kappa$,

$o(X)^{<\kappa} > o(X)$, *then* for every base $\underset{\sim}{B}$ of X $|\underset{\sim}{B}|^{<\hat{s}(X)} \geq o(X)$.

Proof : See 3.3, and apply 2.6 to the space Y.

3.6 Conclusion: 1) If B is a Boolean Algebra , κ strong limit and $|B| \geq \kappa$ then $id(B)^{<\kappa} = id(B)$.

2) If X is locally compact Hausdorff space, κ strong limit, then $o(X)^{<\kappa} = o(X)$.

Proof : 1) By 3.5 applied to the space of ultrafilters of B, $|B|^{<\hat{s}(B)} \geq o(X)$. By 2.12 $|B|^{<\hat{s}(B)} = 2^{<\hat{s}(B)}$, and clearly $2^{<\hat{s}(B)} \leq o(X)$, so $|B|^{<\hat{s}(B)} = 2^{<\hat{s}(B)} = o(X)$. Now $cf\ \hat{s}(X) \geq \kappa$ by 3.4 (as $\hat{s}(X) \geq \beth_5(\mu)^+$ whenever $\mu < \kappa$), hence $(2^{<\hat{s}(X)})^{<\kappa} = 2^{<\hat{s}(X)}$. As $id(B) = o(X)$ we finish.

2) By 3.5 $|\underset{\sim}{B}|^{<\hat{s}(X)} \geq o(X)$ for every base $\underset{\sim}{B}$, but by 2.18 $w(X)^{<\hat{s}(X)} \leq 2^{<\hat{s}(X)}$. As $2^{<\hat{s}(X)} \leq o(X)$ we get $2^{<\hat{s}(X)} = o(X)$, as $\hat{s}(X) \geq \kappa$ (remember κ strong limit, $o(X) \geq \kappa$) by 3. 4 $cf\ \hat{s}(X) \geq \kappa$ hence $(2^{<\hat{s}(X)})^{<\kappa} = 2^{<\hat{s}(X)}$.

Remark: If you want to apply only the part of 2.18, 2.17 actually proved, separate the case λ is strong limit in 3.4.

§4 Further consequences.

4.1 Claim: Let B be a Boolean Algebra , χ a cardinal, and we define by induction on i, ideals $I_i = I_i^{\chi}(B)$ increasing continuous: $I_0 = \{0\}$, $I_{i+1} = \{x \in B : id((B/I_i) \restriction (x/I_i)) < \chi_i\}$ where χ_i is choose as a minimal cardinal $< \chi$ such that $I_{i+1} \neq I_i$.

1) For some $\gamma = \gamma(*) = i_{\chi}(B) < |B|^+, I_{\gamma(*)}$ is defined but not $\chi_{\gamma(*)}$ (nor $I_{\gamma(*)+1}$).

2) $B = I_{\gamma(*)}$ or for every $x \in B - I_{\gamma(*)}$, $id((B/I_{\gamma(*)}) \restriction (b/I_{\gamma(*)})) \geq \chi$.

3) The number of ideals $J \subseteq I_{\gamma(*)}$ of B has the form $\sum_{\alpha<\beta} \mu_\alpha^{\kappa(\alpha)}$ where $\beta \leq |B|^{<\hat{s}(B)}$, $\mu_\alpha < \chi$, $\kappa(\alpha) < \hat{s}(B)$.

This follows from:

4.2 Claim: For a Hausdorff space X with a base $\underset{\sim}{B}$ and cardinal χ, define by induction on i $u_i = u_i^{\chi}(X)$:

$u_0 = \phi$

$u_{i+1} = u_i \bigcup \{v : v \in \underset{\sim}{B}\ , o(v - u_i) < \chi_i\}$ where $\chi_i < \lambda$ is minimal such that $u_{i+1} \neq u_i$ $u_\delta = \bigcup_{i<\delta} u_i$ (so u_i is increasing continuous.)

1) For some $\gamma(*) = \gamma^{\chi}(X) < |X|^+$, (and $\gamma(*) < |w(X)|^+$) $u_{\gamma(*)}$ is defined but not $u_{\gamma(*)+1}$ and for every $y \in X - u_{\gamma(*)}, (\forall v)(y \in v \in \tau \rightarrow o(v - u_{\gamma(*)}) \geq \chi.)$

2) $o(u_{\gamma(*)})$ if $> |\underset{\sim}{B}|^{<\hat{s}(X)}$, has the form $\sum_{\alpha<\beta} \mu_\alpha^{\kappa(\alpha)}$ where $\beta \leq |\underset{\sim}{B}|^{<\hat{s}(B)}$, $[\mu_\alpha < \chi$, or $\mu_\alpha = \chi, \kappa(\alpha) \geq cf\ \chi]$ and $\kappa(\alpha) < \hat{s}(X)$.

Proof : Like 2.6.

For every $u \subseteq u_{\gamma(*)}$ choose by induction on i, v_i, such that:

(i) $v_j \bigcap v_i \subseteq u$ for $j < i$.

(ii) $v_i \not\subseteq u$, $v_i \in \underset{\sim}{B}$.

(iii) $v_i \subseteq u_{\alpha(i)}$ for some $\alpha(i) \leq \gamma(*)$ but for no $\beta < \alpha(i)$ and $v' \subseteq v_i$, is $v' \not\subseteq u$, $v' \subseteq u_\beta$ and $v' \in \tau$.

So let β be first such that v_β is not defined. By (iii) for each $i < \beta$ $\alpha(i)$ is successor ordinal and $u_{\alpha(i)-1} \bigcap v_i \subseteq u$. As in 2.6 $\bar{v} = \langle v_j : j < \beta \rangle$, $u \bigcap v_j$ determine u, the number of u corresponding to $\bar{v}$ is $\prod_{j<\beta} o(v_i - u_{\alpha(i)-1} - \bigcup_{j \neq i} v_j)$ each multiplicant is $\leq o(v_i) \leq \chi_i < \chi$, $\beta < \hat{s}(X)$ and the number of $\bar{v}$ is $\leq |B|^{<\hat{s}(X)}$.

4.3 Remark: At least for compact spaces, this gives heavy restrictions on the relevant cardinals.

Let $\aleph_0 \le \kappa_0 < \cdots < \kappa_n$ list the cardinals κ such that $2^\kappa < o(X)$, and for some $\lambda = \lambda[\kappa]$, $\kappa = cf\ \lambda$, and $\lambda^\kappa > o(X) > \lambda$ but $(\forall \chi < \lambda)$ $[\chi^\kappa < \lambda]$ so $o(X)^{<\kappa_0} = o(X) < o(X)^{\kappa_0}$ (if there is no such κ we have no problem). As $\lambda[\kappa_a] = \lambda[\kappa_b]$ implies $\kappa_a = \kappa_b$, and $[\kappa_a < \kappa_b \Longrightarrow \lambda([\kappa_a]) > \lambda([\kappa_b])]$, clearly n is finite and trivially each κ_ℓ is regular and let for $\ell = 1,n$, $\lambda_\ell = Min\{\lambda\colon \lambda^{\kappa_\ell} \ge o(X)\}$; but $\lambda[\kappa_\ell] \ge \lambda_\ell$ (as $\lambda[\kappa_\ell]^\kappa \ge o(X))$) and $\lambda[\kappa_\ell] \le \lambda_\ell$ (as $(\forall \chi < \kappa[\kappa_\ell])$ $[\chi^\kappa < \lambda[\kappa_\ell]]$), so $\lambda[\kappa_\ell] = \lambda_\ell$. Hence $cf\ \lambda_\ell = \kappa_\ell$, $\lambda_0 > \lambda_1 > \cdots > \lambda_n$, $(\forall \chi < \lambda_\ell)[\chi^{\kappa_\ell} < \lambda_\ell]$. Moreover (for $\ell < n$) $(\forall \chi < \lambda_\ell)(\chi^{<\kappa_{\ell+1}} < \lambda_\ell)$ [first suppose $\chi < \lambda$, $\kappa_\ell \le \vartheta < \kappa_{\ell+1}$, if $\chi^\vartheta \ge \lambda_\ell$ then $\chi^\vartheta \ge \lambda^\vartheta \ge \lambda^{\kappa_\ell} \ge o(X)$, w.l.o.g. χ is minimal with this property, so $\chi^\vartheta \ge o(X) > 2^{\kappa_{\ell+1}} \ge 2^\vartheta$ hence $\chi > 2^\vartheta$. Clearly $(\forall \mu < \chi)(\mu^\vartheta < o(X))$ hence $(\forall \mu < \chi)(\mu^\vartheta < \chi)$, and $cf\,(\chi) \le \vartheta$ (otherwise $\chi^\vartheta = \sum_{\alpha<\chi} |\alpha|^\vartheta \le \chi < \lambda_\ell \le o(X)$ contr.). So $cf\ \chi \le \vartheta < \kappa_{\ell+1}$ and by χ's minimality $(\forall\ \mu < \chi)(\mu^{cf\ \chi} \le \mu^\vartheta < \chi)$. Lastly $cf\ \chi > \kappa_\ell$ [otherwise $\chi^\vartheta = \chi^{cf\ \chi} \le \chi^{\kappa_\ell} < \lambda_\ell$ contradicting the assumption of ϑ]. So $\vartheta \in \{\kappa_0, \ldots, \kappa_n\}$, contr. Secondly suppose $\chi^{<\kappa_{\ell+1}} \ge \lambda_\ell$, for some $\chi < \lambda_\ell$, as $\vartheta < \kappa_{\ell+1} \Longrightarrow 2^\vartheta < \lambda_\ell$, by 2.11 for some $\vartheta < \kappa_{\ell+1}$, $\chi^\vartheta = \chi^{<\kappa_{\ell+1}}$ and we get the first case].

Let $\lambda_{n+1} = Min\{\chi\colon 2^\chi \ge o(X)\}$ and $\kappa_{n+1} = cf\ \lambda_{n+1}$; so $\lambda_{n+1} \le \lambda_n$, hence, as above) $(\forall \chi < \lambda_n)(\forall \vartheta < \lambda_{n+1})[\chi^\vartheta < \lambda_n]$. By the proof of 3.4 $\beth_5(\kappa_\ell)^+ \ge \kappa_{\ell+1}$ (for $\ell < n$), otherwise using $\lambda_n, \kappa_\ell, \mu = \beth_5(\kappa_\ell)^+$ we get contradiction. If λ_{n+1} is singular, $\langle 2^\chi\colon \chi < \lambda_{n+1}\rangle$ is not eventually constant [as then $(\exists \chi < \lambda_{n+1}) 2^\chi = 2^{\chi_{n+1}}$], $2^{<\lambda_{n+1}} \le o(X)$, $(2^{<\lambda_{n+1}})^{\kappa_{n+1}} = 2^{\lambda_n} > o(X)$, so $\lambda[\kappa_{n+1}] = 2^{<\lambda_{n+1}}$, so $\lambda_n = \lambda_{n+1}$ hence $\beth_{6(n+1)}(\kappa_0) \ge o(X)$, $o(X)^{<\kappa_0} = o(X)$. If λ_{n+1} is regular, then $(\forall \vartheta < \lambda_{n+1})$ $(\forall \chi < \lambda_n)[\chi^\vartheta < \lambda_n]$ hence $\beth_5(\kappa_n)^+ \ge \lambda_{n+1}$, so we get the same conclusion.

4.4 Lemma: Suppose X is a Hausdorff space, λ a singular cardinal, $\vartheta = cf\ \lambda$, $\lambda = \sum_{i<\vartheta} \chi_i, \chi_i < \lambda$, $\mu < \lambda$ and (i), (ii), (iii) of 3.2 holds (for φ).

1) If $\mu = \beth_2(\vartheta)^+$ (or even $\sum_{\sigma<\vartheta} \beth_2(\sigma)^+$) *then* there are open sets $u_i (i < \vartheta)$ such that $\varphi(u_i - \bigcup_{j>i} u_j) \ge \chi_i$.

2) If $X = \bigcup\{u\colon o(u) < \lambda\}$, μ as in 1) then there are open sets u_i such

that $\varphi(u_i - \bigcup_{j\neq i} u_j) \geq \chi_i$.

3) If $\mu \geq \beth_3(2^{<\vartheta})^+$, φ is $(<\chi_0,\mu)$-complete, then there are $u_i (i < \vartheta)$ such that $\varphi(u_i - \bigcup_{j\neq i} u_j) \geq \chi_0$ (so λ, $\chi_i (0 < i < \vartheta)$, are irrelevant).

Remarks: 1) Part 1) of the lemma is suitable to deal with Boolean algebras, part 2) with existence of $\{x_\alpha : \alpha < \lambda\}$ such that for every $\alpha < \lambda$ for some u, $x_\alpha \in u \cap \{x_\beta : \beta < \lambda\} \subseteq \{x_\beta : \beta \leq \alpha\}$.

Proof: 1) We repeat the proof of 3.2, for $\mu = \beth_2(\vartheta)^+$, but cannot use the partition relation used there, but we can use a weaker one. We choose by induction on $j < \vartheta$, $\xi(j,0) < \xi(j,2) < \xi(j,2) < \mu$ such that for $i < j$:

$$F(y_{i,\xi(i,0)}, y_{i,\xi(i,1)}, y_{i,\xi(i,2)}, y_{j,\xi(j,0)}, y_{j,\xi(j,1)}) =$$
$$F(y_{i,\xi(i,0)}, y_{i,\xi(i,1)}, y_{i,\xi(i,2)}, y_{j,\xi(j,1)}, y_{j,\xi(j,2)})$$

This is clearly possible by the assumption on μ.

We can conclude that, letting $u_i = u_{i,\xi(i,1),\xi(i,0)} \cap u_{i,\xi(i,1),\xi(i,2)}$ then $x_{i,\xi(i,0),\xi(i,1),\xi(i,2)} \in u_i - \bigcup_{j>i} u_j$, so we can get the desired conclusion.

2) In the proof of 1) we can take care that for every $i < \vartheta$, $\xi \neq \zeta < \mu$, $u_{i,\xi,\zeta}$ satisfies $o(u_{i,\xi,\zeta}) < \lambda$; hence we shall get $o(u_i) < \lambda$. So by thinning the sequence $\langle u_i : i < \vartheta \rangle$, as $o(u_i) \geq \chi_i$, $\lambda = \sup_{i<\vartheta} \chi_i$ we can assume:

$$[i < j \Longrightarrow o(u_i) < u_j].$$

As φ is $(<\chi_i,\mu)$-complete, $\vartheta \leq \mu$, necessarily $o(\bigcup_{i<j} u_i) < \chi_j$. Hence

$$o(u_i - \bigcup_{j\neq i} u_j) = o((u_i - \bigcup_{j<i} u_j) - \bigcup_{j<i} u_j) \geq \chi_j$$

as required.

3) Really the proof is as in 3.2, but we use (for $\sigma = 2$, κ finite large enough, note $\mu = \beth_2(\sigma^{<\vartheta})^+$; is O.K. in 4.5):

4.5 Observation: If F is a 5-place function from μ to σ, $\sigma \geq 2, \vartheta \geq \aleph_0$ $\mu \to (\kappa)^3_\psi$, $\psi = 2^{(\sigma^{<\vartheta})+\kappa}$ $\kappa \to (3)^2_\sigma$ [e.g. $\mu > \beth_1(\psi^{2^{<\sigma}}) = \beth_3(\sigma^{<\vartheta}+\kappa)$ $\kappa = (2^{<\sigma})^+ + \aleph_0$],

$\kappa \to (3)^2_\vartheta$ [e.g. $\mu > \beth_1(2^{\psi+2^{<\vartheta}}) = \beth_3(\sigma^{<(\vartheta+\kappa^+)})$, $\kappa = (2^{<\vartheta})^+$, or κ is finite large enough] then, there are distinct $\xi(i,\ell)(i < \vartheta, \ell > 3)$ such that, for $i \neq j$:

$$F(\xi(i,0),\xi(i,1),\xi(i,2),\xi(j,0),\xi(j,1)) =$$
$$F(\xi(i,0),\xi(i,1),\xi(i,2),\xi(j,1),\xi(j,2))$$

Remark: We can get of course more general theorem.

Proof : We choose by induction on $i < \vartheta$, $Y_i \subseteq \mu$, $|Y_i| \le \sigma^{|i|+\kappa} + \aleph_0$, Y_i increasing and all "types" of cardinality $< |i|^+ + \kappa^+$ realized in μ are realized in Y_{i+1}. Let $Y = \bigcup_{i<\vartheta} Y_i$. Now we can find distinct $\xi^*(\ell) \in \mu - Y$ for $\ell < \kappa$ such that for every $\xi_0,\xi_1,\xi_2 \in \bigcup_{i<\vartheta} Y_i$ there are $c_1(\xi_0,\xi_1,\xi_2), c_2(\xi_0,\xi_1)$ such that

$(*)_a$ for every $\ell < m < \kappa$ $F(\xi_0,\xi_1,\xi_2,\xi^*(\ell),\xi^*(m)) = c_1(\xi_0,\xi_1,\xi_2)$

$(*)_b$ and for every $\ell < m < n < \kappa$ $F(\xi^*(\ell),\xi^*(m),\xi^*(k),\xi_0,\xi_1) = c_2(\xi_0,\xi_1)$.

Why we can do this? We want to apply the partition relation $\mu \to (\kappa)^3_\psi$, for this we have to check what is the number of " colours", clearly it is $\le 2^{(\kappa^2|Y|^3 + \kappa^3|Y|^2)} \le 2^{\aleph_0+\kappa+(\sigma^{<(\vartheta+\kappa^+)})}) = \psi$. Now we choose by induction on $i < \vartheta$, $\xi(i,\ell), \ell < \kappa$ such that :

(i) $\xi(i,0),\xi(i,2),\xi(i,2)$ are distinct.

(ii) $\xi(i,\ell) \in Y_{i+1} - Y_i$.

(iii) $F(\xi(j,0),\xi(j,1),\xi(j,2),\xi(i,\ell),\xi(i,m)) = F(\xi(j,0),\xi(j,1),\xi(j,2),\xi^*(\ell),\xi^*(m))$, when $j < i$, and $\ell, m < \kappa$.

(iv) $F(\xi(i,\ell_1),\xi(i,\ell_2),\xi(i,\ell_3),\xi(j,\ell_4),\xi(j,\ell_5)) =$

$$F(\xi^*(\ell_1),\xi^*(\ell_2),\xi^*(\ell_3),\xi(j,\ell_4),\xi(j,\ell_5))$$

when $j < i$, $\ell_1 < \cdots < k$.

There is no problem in doing this:

For each $i < \vartheta$, as $\kappa \to (3)^2_\sigma$ there are $\ell_0(i) < \ell_1(i) < \ell_2(i) < \kappa$ such that:

$$F(\xi^*(0),\xi^*(1),\xi^*(2),\xi(i,\ell_0(i)),\xi(i,\ell_1(i))) =$$
$$F(\xi^*(0),\xi^*(1),\xi^*(2),\xi(i,\ell_1(i)),\xi(i,\ell_2(i)))$$

Now $\xi'(i,m) = \xi(i,\ell_m(i))$ $(i < \vartheta, m < 3)$ are as reequired.

4.4A Remark: Assume (i), (ii), (iii) of 3.2. We try to decrease μ. Let $Z_i = \{y \in X\colon Ch_\varphi(y) \geq \chi_i\}$, so $|Z_i| \geq \mu$, and let $X_{<\lambda} = \bigcup\{u : \varphi(u) < \lambda\}$. If $|X - X_{<\chi}| < \mu$ then necessarily $|Z_i \cap X_{<\chi}| \geq \mu$, so we can continue as in 4.4(2). So we assume $|X - X_{<\chi}| \geq \mu$ and let $y_\xi \in X - X_{<\lambda}$ $(\xi < \mu)$ be distinct. Choose for $\xi < \zeta$, open disjoint sets $u_{\xi,\zeta}, u_{\zeta,\xi}$ such that $y_\xi \in u_{\xi,\zeta}$, $y_\zeta \in u_{\zeta,\xi}$. As in 3.2's proof we can choose for distinct $\xi(0),\xi(1),\xi(2) < \mu$, $x_{i,\xi(0),\xi(1),\xi(2)} \in u_{\xi(1),\xi(0)} \cap u_{\xi(1),\xi(2)}$ such that: for every $\mathcal{P} \subseteq \{u_{\xi,\zeta}, x - u_{\xi,\zeta}\colon \xi,\zeta < \mu\}$, $|\mathcal{P}| \leq \vartheta$,

$$[x_{i,\xi(0),\xi(1),\xi(2)} \in \bigcap_{a \in \mathcal{P}} a \Longrightarrow \varphi(\bigcap_{a \in \mathcal{P}} a) \geq \chi_i]$$

We need the parallel of 4.5 for ϑ functions simultaneously or, what is equivalent, the range of F has cardinality 2^ϑ, so $\sigma = 2^\vartheta$, and we get $\mu \geq \beth_5(\vartheta)^+$ but this is not interesting.

§5 When the spread is obtained and how helpful is regularity of the space

5.1 Lemma : 1) Suppose X is a regular (i.e. T_3) topological space, $\underset{\sim}{B}$ a base of X, $\lambda = \sum_{i<\vartheta} \chi_i$, $\vartheta < \chi_i < \lambda$, $\mu = (2^\vartheta)^+$ and

(i) φ is nice for X,

(ii) for every (closed) $Y \subseteq X$ with $\varphi(Y) \geq \lambda$ and $i < \vartheta$, there are $y_\alpha \in Y$ $(\alpha < \mu)$, $Ch_{\varphi \restriction Y}(y_\alpha) \geq \chi_i$ and $\{y_\alpha : \alpha < \mu\}$ is a discrete set,

(iii) φ is $(<\lambda,\mu)$-complete.

Then for some $u_i \in \underset{\sim}{B}$ $(i < \vartheta)$, $\varphi(u_i - \bigcup_{j \neq i} u_j) \geq \chi_i$.

2) Instead $\mu = (2^\vartheta)^+$ it suffices that $\mu = \mu^\vartheta > 2^\vartheta$ (and (i), (ii), (iii)).

3) We can replace (ii) above by

(ii)' for each $i < \vartheta$ there are $u_\alpha \in \underset{\sim}{B}(\alpha<\mu)$ such that:

$(\forall g:\mu \to 2^\vartheta)(\exists \alpha \neq \beta)[g(\alpha) = g(\beta) \wedge \varphi[(u_\alpha - \bar{u}_\beta) \cap Y) \geq \chi_i]$.

or

(ii)'' there are $u_\alpha, y_\alpha \in u_\alpha \in \underset{\sim}{B}$, such that: $Ch_\varphi(y_\alpha) \geq \chi_i$ and

$(\forall g;\mu \to 2^\vartheta)(\exists \alpha \neq \beta)[g(\alpha) = g(\beta) \wedge y_\alpha \notin \overline{u_\beta}]$.

Proof : 1) W.l.o.g. φ is (χ_i,μ)-complete for $i < \vartheta$. We first try to choose a family K of open subsets of X, (or even $\subset \underset{\sim}{B}$), and a $Y \subset X$ such that:

(A) $|K| = |Y| = (2^\vartheta)^+$.

(B) *if* u is the union of $< \vartheta$ members of K, $\varphi(X-u) \geq \lambda$ and $i < \vartheta$ *then* there is a sequence $\langle y_\alpha, v^0_\alpha, v^1_\alpha : \alpha < (2^\vartheta)^+\rangle$ such that: $y_\alpha \in Y-u$, $[y_\alpha \in v^1_\beta \Longleftrightarrow \alpha = \beta]$, $v^0_\alpha, v^1_\alpha \in K$, $y_\alpha \in v^0_\alpha \subseteq \bar{v}^0_\alpha \subseteq v^1_\alpha$, and $(\forall v \in \tau(X))$ $[y_\alpha \in v \to \varphi(v-u) \geq \chi_i]$.

It is easy to find such K, Y (by (ii)). Let for $i < \vartheta$,

$Z_i(K) \overset{def}{=} \{z \in X$: if $u_\alpha \in K(\alpha < \vartheta)$, and $u^*_\alpha \in \{u_\alpha, X-u_\alpha\}$ and $z \in u^*_\alpha$ for each $\alpha < \vartheta$ then $\varphi(\bigcap_{\alpha<\vartheta} u^{t(*)}_\alpha) \geq \chi_i\}$.

By the proof of 3.2 for each $i < \vartheta$ there is $z_i \in Z_i(K)$. Now we choose by induction on i, x_i, u_i such that:

(a) $u_i \in K$, $x_i \in Z_i(K)$,

(b) $x_i \in u_i, (\forall \varepsilon < i)(x_\varepsilon \notin u_i \wedge x_i \notin u_\varepsilon)$,

(c) $z_\varepsilon \notin u_i$ when $i < \varepsilon < \vartheta$.

Suppose x_j, u_j are defined for $j < i$. We want to apply (B) to $\bigcup_{j<i} u_j$, now for each ε, if $i \leq \varepsilon < \vartheta$ then $\varphi(X - \bigcup_{j<i} u_j) \geq \chi_\varepsilon$ as $\{u_j : j < i\} \subseteq K$, $z_\varepsilon \notin \bigcup_{j<i} u_j$ and $z_\varepsilon \in Z_\varepsilon(K)$. Hence $\varphi(X - \bigcup_{j<i} u_j) \geq \lambda$. So by (B) above there is $\langle y, v^0_\alpha, v^1_\alpha : \alpha < (2^\vartheta)^+\rangle$ as mentioned there. By cardinality consideration, for some $\alpha \neq \beta$,

$$v_\alpha^0 \cap (\{z_j : j < \vartheta\} \cup \{y_j : j < i\}) = v_\beta^0 \cap (\{z_j : j < \vartheta\} \cup \{y_j : j < i\})$$

So $u_i \overset{def}{=} v_\alpha^0 - \overline{v_\beta^0}$ is open, is disjoint to $\{z_j : j < \vartheta\} \cup \{y_j : j < i\}$, and y_α belongs to it (as $y_\alpha \notin v_\beta^1, \overline{v_\beta^0} \subseteq v_\beta^1$). As (by (B)) $(\forall v \in \tau(X))[y_\alpha \in v \to \varphi(v - \bigcup_{j<i} u_j) \geq \chi_i]$, clearly $\varphi(u_i - \bigcup_{j<i} u_j) \geq \chi_i$, hence (as in 3.2) there is $x_i \in Z_i(K) \cap (u_i - \bigcup_{j<i} u_j)$. So we succeed in the induction. In the end as $u_i \in K$, $x_i \in Z_i(K) \cap (u_i - \bigcup_{j \neq i} u_j)$ clearly $\varphi(u_i - \bigcup_{j \neq i} u_j) \geq \chi_i$, so we finish.

2),3) Similar.

5.2 Lemma : Suppose X is a Hausdorff space, $\lambda = \sum_{i<\vartheta} \chi_i$, $\chi_i < \lambda$ and $\mu = \beth_2(\vartheta)^+$, $\underset{\sim}{B}$ a base for X, and

(i) φ is nice for X.

(ii) for every (closed) $Y \subseteq X, \varphi(Y) \geq \lambda$, and $i < \vartheta$ there are at least μ points $y \in Y$ with $Ch_{\varphi \restriction Y}(y) \geq \chi_i$.

(iii) φ is $(<\lambda, \mu)$-complete.

Then for some $u_i \in \underset{\sim}{B}, (i < \vartheta)$ $\varphi(u_i - \bigcup_{j \neq i} u_j) \geq \chi_i$.

Proof : Like the previous one, replacing (B) by (B)', (C)' (D)':

(B)' if u is the union of $< \vartheta$ members of $K, \varphi(X - u) \geq \lambda$ and $i < \vartheta$ then there are $\beth_2(\vartheta)^+$ points $y \in Y - u$ such that $(\forall v \in \tau(X))(y \in v \to \varphi(v - u) \geq \chi_i]$.

(C)' if $y_1 \neq y_2 \in Y$ then for some $u, v \in K$, $y_1 \in u, y_2 \in v, u \cap v = \phi$.

(D)' K is closed under finite intersections.

Then having defined $u_j, x_j (j < i)$ and shown $\varphi(X - \bigcup_{j<i} u_j) \geq \lambda$, we can find distinct $y_\alpha \in Y - \bigcup_{j<i} u_j (\alpha < \beth_2(\vartheta)^+)$ such that $Ch_{X - \bigcup_{j<i} u_j}(y_\alpha) \geq \chi_i$. We let $A = \{z_j : j < \vartheta\} \cup \{x_j : j < i\}$, $I_\alpha = \{v \cap A : y_\alpha \in v \in K\}$, so for some $\alpha \neq \beta < \beth_2(\vartheta)^+$,

$I_\alpha = I_\beta$, and using (C)' there is $u_i \in K$, such that $y_\alpha \in u_i$, obviously $A \cap u_i = \phi$. As $y_\alpha \in u_i$ $\varphi(u_i - \bigcup_{j<i} u_j) \geq \chi_i$, hence there is $x_i \in Z_i(K) \cap (u_i - \bigcup_{J<i} u_j)$.

We may remember:

5.3 Fact : 1) Suppose $\kappa = \kappa^{<\kappa}$, $\chi = \sum_{i<\vartheta} \chi_i$, χ_i increasing continuous $\kappa < \vartheta < \chi_i$.

Then for some forcing notion P:

a) P is κ-complete satisfying the κ^+-chain condition.

b) In V^P there is a topological space X with a basis of clopen sets such that $\hat{h}(X) = \hat{z}(X) = \hat{s}(X) = \chi$, $o(X) = \sum_{i<\vartheta} 2^{\chi_i}$ and $|X| = \chi$.

2) In fact we can get that X is the dual of a Boolean algebra and there is no set of pairwise incomparable members of the Boolean algebra, of cardinality χ.

Proof: Let $p \in P$ be a set of $< \kappa$ atomic conditions with no two contradictory ones, where an atomic condition is $\alpha \in u_\beta$ or $\alpha \notin u_\beta$, where $\alpha, \beta < \chi$, and $\alpha \in [\chi_i, \chi_{i+1}) \Longrightarrow \beta < \chi_i \vee \beta = \alpha \vee \beta \geq \chi_{i+1}$.

Two conditions are contradictory if they have the form $\alpha \in u_\beta, \alpha \notin u_\beta$. The order is inclusion.

Now (a) is obvious.

In V^P we define:

$u_\beta^P = \{\alpha < \lambda : \alpha \in u_\beta$ belong to some $p \in \underset{\sim}{G}_P\}$

On χ we define a topological space X: by having $\{u_\beta^P : \beta < \chi\}$ be a basis of clopen sets.

The rest is easy too.

2) Similar (just as in [Sh 9] 4.4) . i.e. let $P = \{(B, W)$: B a Boolean algebra of cardinality $< \kappa$ generated by $\{x_i : i \in W\}$, W a subset of χ of cardinality $< \kappa$, and if $\alpha_0, \ldots, \alpha_n$ are distinct members of $W \cap [\chi_i, \chi_i^+)$ then $B \models x_{\alpha_0} \not\subseteq \bigcup_{\ell=1}^{n} x_{\alpha_\ell}\}$.

5.4 Conclusion: 1) If X is Hausdorff $\hat{s}(X)$ is singular of cofinality ϑ then $cf\ (\ \hat{s}(X)) < 2^{2^{\vartheta}}$. [repeat the proof of 3.3 but instead of 3.2 use 5.1 remembering $cf\,(2^{\kappa}) > \kappa$].

2) If X is regular (i.e. T_3) $\hat{s}(X)$ singular of cofinality ϑ then $cf\,(\hat{s}(X)) < 2^{\vartheta}$. [repeat the proof of 2.3 but instead of 3.2 use 5.2 remembering $cf\,(2^{\vartheta}) > \vartheta$].

3) Both results are best possible in the sense of complementary consistency results. (see [JSh] and 5.3).

4) We can replace above s by z or h.

5.5 Lemma : Suppose λ is singular of cofinality ϑ, $\lambda = \sum_{i<\vartheta} \chi_i$, $\chi_i < \lambda$, and $\mu \geq 0$. Assume further (for a topological space X and function φ):

(i) φ is nice for X.

(ii) $\{y \in X$: $Ch_{\varphi}(y) \geq \chi_i\}$ has power $\geq \mu_1$ for $i < \vartheta$.

(iii) φ is $(<\lambda, \mu_0)$-complete.

1) If X is Hausdorff, $\mu_0 = \mu_1 = \sum_{\kappa<\vartheta} \beth_2(\kappa)^+$, then for some $u_i \in \tau(X)$ (for $i < \vartheta$) for each $i, \varphi(u_i - \bigcup_{j<i} u_j) \geq \chi_i$.

2) If X is regular, $\mu_0 = \mu_1 = \sum_{\kappa<\vartheta} (2^{\kappa})^+$ *then* for some $u_i \in \tau(X)$ (for $i < \vartheta$) for each i $\varphi(u_i - \bigcup_{j<i} u_j) \geq \chi_i$.

Remark: The proofs are similar to those of 5.1, 5.2.

Proof : 1) W.l.o.g. φ is (χ_i,μ_0)-complete for each i. We define K,Y:

(A) K is a family of open subsets of X of power $\leq \mu_0$.

(B) Y is a subset of X of power $\leq \mu_1$.

(C) there are μ_0 distinct $y \in Y$ such that $Ch_\varphi(y) \geq \chi_i$.

(D) for any distinct $y_1,y_2 \in Y$ for some disjoint $u_1,u_2 \in K$, $y_1 \in u_1$ and $y_2 \in u_2$.

(E) K is closed under finite unions of intersections

There is no problem to carry this definition. Let $Z_i(K) = \{z \in X$: if for $j < \vartheta$ $a_j \subseteq X$, $a_j \in K \vee X-a_j \in K$, and $z \in a_j$ *then* $\varphi(\bigcap_{j<\vartheta} a_j) \geq \chi_i\}$. Now we choose by induction on $i < \vartheta$, x_i and u_i such that :

(a) $u_i \in K$, $x_i \in Z_i(K)$.

(b) $x_i \in u_i$,$(\forall j < i)$ $(x_i \notin u_j)$.

Suppose we have defined x_j,u_j for $j < i$.

By (C) above there are distinct $y^i_\alpha \in Y$ for $\alpha < \mu_0$, with $Ch_\varphi(y^i_\alpha) = \chi_i$. By (E) above there are, for $\alpha \neq \beta$ $u_{\alpha,\beta} \in K_{\zeta+1}$, such that $y^i_\alpha \in u^i_{\alpha,\beta}$, and $u^i_{\alpha,\beta} \cap u^i_{\beta,\alpha} = \phi$. Now as $\mu_0 \rightarrow (3)^2_{2^{\aleph_0}}$ for some $\alpha < \beta < \gamma < \mu_0$:

$$u^i_{\alpha,\beta} \cap \{x_j : j < i\} = u^i_{\beta,\gamma} \cap (\{x_j : j < i\})$$

As $u^i_{\beta,\alpha} \cap u^i_{\alpha,\beta} = \phi$, clearly $u_i = u^i_{\beta,\alpha} \cap u^i_{\beta,\gamma}$ is disjoint to $\{x_j : j < i\}$. Also $y^i_\beta \in u^i_{\beta,\alpha} \cap u^i_{\beta,\gamma}$, so $\varphi(u_i) \geq \chi_i$, hence as in the proof of 3.2 there is $x_i \in u_i \cap Z_i(K)$. In the end x_i witnesses $\varphi(u_i - \bigcup_{j>i} u_j) \geq \chi_i$ as $x_i \in u_i$, $(\forall j > i)(x_i \notin u_j)$.

2) Similarly (remembering the proof of 5.2).

References

[HJ1]

A. Hajnal and I. Juhasz, Some remarks on a property of topological cardinal functions, *Acta Math. Acad. Sci. Hungar.* 20 (1969), 25-37.

[HJ2]

A. Hajnal and I. Juhasz, On the number of open sets, *Ann. Univ. Sci. Budapest*, 16 (1973), 99-102.

[J1] I. Juhasz, Cardinal functions in topology, *Math. Center Tracts.* Amsterdam, 1971.

[J2] I. Juhasz, Cardinal functions in topology - ten year later, *Math. Center Tracts.* Amsterdan, 1980.

[JSh]

I. Juhasz and S. Shelah, How large can a hereditary separable or hereditarily Lindelof space by? *Israel J. of Math*, submitted.

[KR] K. Kunen and J. Roitman, Attaining the spread of cardinals of cofinality ω, *Pacific J. Math.* 70 (1977), 199-205.

[R] J. Roitman, Attaining the spread at cardinals which are not strong limit, *Pacific J. Math.* 57 (1975), 545-551.

[Sh1]

S. Shelah, Remarks on Boolean algebra, *Algebra Universalis*, 11 (1980), 77-89.

[Sh2]

----------, Canonization theorems and applications, *J. of Symb. Logic*, 46 (1981), 345-353.

[Sh3]

--------, On cardinal invariants in topology, *General Topology and its applications*, 7 (1977), 251-259.

[Sh4]

---------, On some problem in general topology, a preprint, Jan. 1978.

[Sh5]

--------, If $\diamondsuit_{\aleph_1}$ + "there is an $\aleph_1$-Kurepa tree with κ-branches" then some B.A. of power $\aleph_1$ has λ filters and $\lambda^{\aleph_0}$-ultrafilters. Mimeograph Notes from Madison, Fall 77.

[Sh6]

--------, On P-points, $\beta(\omega)$ and other results in general topology, *Notices of A.M.S. (1984)* 25 (1978), A-365.

[Sh7]

--------, Number of open sets and Boolean algebras with few endomorphisms. ***Abstracts of*** *A.M.S. 5(1984)*

[Sh8]

--------, Boolean algebras, General topology and independence results, *Abstracts of A.M.S.* 5(1984).

[Sh9]

----------, Constructions of many complicated uncountable structures and Boolean Algebra, *Israel J. Math.* 45 (1983), 100-146.