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A WEAK VERSION OF $\diamond$ WHICH FOLLOWS FROM $2^{\aleph_0} < 2^{\aleph_1}$

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ABSTRACT

We prove that if CH holds (or even if $2^{\aleph_0} < 2^{\aleph_1}$), then a weak version of $\diamond$ holds. This weak version of $\diamond$ is a $\diamond$-like principle, and is strong enough to yield some of the known consequences of $\diamond$.

§1. Introduction

The combinatorial principle $\diamond$ says that there are functions $f_\alpha : \alpha \to 2 = \{0,1\}$, $\alpha < \omega_1$, such that for every function $f: \omega_1 \to 2$, the set $\{\alpha < \omega_1 \mid f \restriction \alpha = f_\alpha\}$ is a stationary subset of ω_1. The principle was first formulated by Jensen, who proved that it holds if we assume $V = L$, that it implies CH (but not conversely), and that it implies the negation of the Souslin hypothesis. For further details we refer the reader to [1] and [2].

Let Φ denote the following assertion:

> For each $F: 2^{\underset{\smile}{\omega_1}} \to 2$ there is a $g \in 2^{\omega_1}$ such that for any $f \in 2^{\omega_1}$, the set $\{\alpha \in \omega_1 \mid F(f \restriction \alpha) = g(\alpha)\}$ is stationary.

(By 2^λ we mean the set $\{f \mid f: \lambda \to 2\}$. We set $2^{\underset{\smile}{\lambda}} = \bigcup_{\alpha<\lambda} 2^\alpha$.)

Of course, for particular F the existence of a function g as in Φ may not be at all problematical (e.g. if F is constant). But as we shall indicate, Φ itself is quite a strong assumption. It is easily seen to be a consequence of $\diamond$. Indeed, if $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamond$-sequence, then given F we set $g(\alpha) = F(f_\alpha)$ to verify Φ. This indicates why we refer to Φ as a "weak version of $\diamond$".

The main result of this paper is that $2^{\aleph_0} < 2^{\aleph_1}$ implies Φ. We also prove that Φ yields some known consequences of $\diamond$.

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A generalisation of Φ is suggested by generalisations of $\diamondsuit$. Jensen, in fact, proved not only that $\diamondsuit$ follows from $V = L$, but the more general principle $\diamondsuit(S)$, where S is any stationary subset of ω_1, and where $\diamondsuit(S)$ is the same as $\diamondsuit$ except that the f_α's are only defined for $\alpha \in S$. (Clearly, $\diamondsuit$ is a consequence of any instance of $\diamondsuit(S)$.)

If $S \subseteq \omega_1$, we denote by $\Phi(S)$ the assertion that for any $F = 2^{\omega_1>} \to 2$ there is a $g \in 2^{\omega_1}$ such that for any $f \in 2^{\omega_1}$, the set $\{\alpha \in S \mid F(f \restriction \alpha) = g(\alpha)\}$ is stationary.

Clearly, if $\Phi(S)$ holds, then S must be stationary. Let us call a subset S of ω_1 *small* if $\Phi(S)$ fails. We prove that the small sets form a normal ideal. Even assuming CH, however, we cannot prove that every stationary set is not small. We refer the reader to [6] for details on this point.

We work in ZFC set theory and use the usual notation and conventions. In particular, an ordinal number is the same as the set of all smaller ordinals, and a cardinal number is an initial ordinal. We reserve lower case Greek letters for ordinals. The sequence of infinite initial ordinals commences thus: $\omega, \omega_1, \omega_2, \cdots$, and $\aleph_\alpha$ denotes ω_α considered as a cardinal. The meanings of the terms *closed unbounded* ("club") and *stationary* applied to subsets of ω_1 is assumed known. (See, e.g., [1].)

§2. The evolution of Φ

It is perhaps illuminating to present a brief account of the evolution of the principle Φ.

One of the consequences of $\diamondsuit$ is the result, **W**, that every Whitehead (abelian) group of order $\aleph_1$ is free. (See [4], or the presentation in (3). Also, [5] considers the case of groups of order greater than $\aleph_1$.) Against this is the result that if we assume Martin's Axiom together with $2^{\aleph_0} > \aleph_1$, then there is a non-free Whitehead group of order $\aleph_1$. (See [4].) Naturally, it was hoped that **W** was not a consequence of CH alone. And in trying to establish this fact, Shelah noticed that **W** fails if $C(S)$ holds for all stationary sets $S \subseteq \omega_1$, where $C(S)$ is the following principle (to be considered later in §5 — where we also attempt to explain its meaning!):

$C(S)$: if for each limit ordinal $\delta \in S$ there is an increasing ω-sequence η_δ converging to δ, and if $k_\delta \in 2^\omega$, then for some $k \in 2^{\omega_1}$ it is the case that for all $\delta \in S$, $k(\eta_\delta(n)) = k_\delta(n)$ for all but finitely many n.

We shall see later that $C(\omega_1)$ is a consequence of Martin's axiom plus $2^{\aleph_0} > \aleph_1$. And it is easily seen that $\neg C(S)$ follows from $\diamondsuit(S)$. Shelah conjectured that

$C(\omega_1)$ was consistent with ZFC + GCH. Devlin refuted this conjecture by showing that CH implies $\neg C(\omega_1)$. (In fact he proved that CH implies $\neg C(B)$ for any club set $B \subseteq \omega_1$; moreover, the functions k_δ, k were allowed to map into ω and not just 2.)

Devlin's original proof used metamathematical techniques (precisely, inner models of set theory). Devlin, Jensen and Shelah all independently observed that the proof could be modified to eliminate the use of inner models, and that the assumption of CH could then be weakened to $2^{\aleph_0} < 2^{\aleph_1}$. Shelah took this a step further by "extracting" from the proof the principle Φ (this extraction was so trickly, that it is somewhat misleading to use the world "extract" at all[†]), and obtaining further consequences of Φ, together with the results of §3, §6, §7.

§3. Small sets

Let $\mathcal{F}$ be the filter of subsets of ω_1 generated by the club subsets of ω_1, $\mathcal{I}$ the dual ideal. (Thus $\mathcal{I}$ is the ideal of non-stationary subsets of ω_1.) It is well known that $\mathcal{F}$ and $\mathcal{I}$ are normal (i.e. in the case of $\mathcal{I}$, if I_ν, $\nu < \omega_1$, are in $\mathcal{I}$, so is $I = \{\tau \in \omega_1 \mid (\exists \nu < \tau)(\tau \in I_\nu)\}$). In particular, both $\mathcal{F}$ and $\mathcal{I}$ are countably complete.

We say a set $S \subseteq \omega_1$ is *small* if there is $F: 2^{\underset{\smile}{\omega_1}} \to 2$ such that for all $g \in 2^{\omega_1}$ there is $f \in 2^{\omega_1}$ such that $\{\alpha \in S \mid F(f \restriction \alpha) = g(\alpha)\} \in \mathcal{I}$. Let $\mathcal{S}$ denote the collection of all small subsets of ω_1. Clearly, Φ is equivalent to the assertion $\omega_1 \notin \mathcal{S}$.

3.1. THEOREM. *$\mathcal{S}$ is a normal ideal on ω_1.*

PROOF. Clearly, if $S' \subseteq S \in \mathcal{S}$, then $S' \in \mathcal{S}$. It therefore suffices to show that if $\{S_\nu \mid \nu < \omega_1\} \subseteq \mathcal{S}$, then $S = \{\alpha \in \omega_1 \mid (\exists \nu < \alpha)(\alpha \in S_\nu)\} \in \mathcal{S}$. Let F_ν testify the smallness of S_ν, each ν. Let $h: \omega_1 \times \omega_1 \leftrightarrow \omega_1$, and let $C = \{\alpha \in \omega_1 \mid h''\alpha \times \alpha = \alpha\}$. Notice that C is club in ω_1.

We define $F: 2^{\underset{\smile}{\omega_1}} \to 2$ as follows. Let $f \in 2^\alpha$, $\alpha < \omega_1$. If $\alpha \in C$ and there is $\nu < \alpha$ with $\alpha \in S_\nu$, pick the least such ν and set $F(f) = F_\nu(f^*)$, where $f^* \in 2^\alpha$ is defined by $f^*(\tau) = f(h(\nu, \tau))$. Otherwise set $F(f) = 0$.

Let $g \in 2^{\omega_1}$ be given. We construct an $f \in 2^{\omega_1}$ for which $\{\alpha \in S \mid F(f \restriction \alpha) = g(\alpha)\} \in \mathcal{I}$, thereby showing that $S \in \mathcal{S}$. For each $\nu < \omega_1$, S_ν is small by F_ν, so we can find $f_\nu \in 2^{\omega_1}$ such that $N_\nu = \{\alpha \in S_\nu \mid F_\nu(f_\nu \restriction \alpha) = g(\alpha)\} \in \mathcal{I}$. Since $\mathcal{I}$ is normal, $N = \{\alpha \in \omega_1 \mid (\exists \nu < \alpha)(\alpha \in N_\nu)\} \in \mathcal{I}$. Define $f \in 2^{\omega_1}$ by setting $f(h(\nu, \tau)) = f_\nu(\tau)$, each ν, τ. Suppose that $\{\alpha \in S \mid F(f \restriction \alpha) = g(\alpha)\} \notin \mathcal{I}$. Thus, as C is club, $E = \{\alpha \in S \cap C \mid F(f \restriction \alpha) = g(\alpha)\} \notin \mathcal{I}$. But suppose $\alpha \in E$. Since

[†] This remark is due to Devlin alone.

$\alpha \in S$ we can find $\nu < \alpha$ with $\alpha \in S_\nu$. Let ν be the least such. Then, by definition, $F(f \restriction \alpha) = F_\nu((f \restriction \alpha)^*)$. So as $\alpha \in E$, $F_\nu((f \restriction \alpha)^*) = g(\alpha)$. But for all $\tau < \alpha$, $(f | \alpha)^*(\tau) = (f \restriction \alpha)(h(\nu, \tau)) = f(h(\nu, \tau)) = f_\nu(\tau)$. Hence $(f \restriction \alpha)^* = f_\nu \restriction \alpha$, giving $F_\nu(f_\nu \restriction \alpha) = g(\alpha)$. Thus $\alpha \in N_\nu$. We have therefore shown that $\alpha \in E \rightarrow (\exists \nu < \alpha)(\alpha \in N_\nu)$. In other words, $E \subseteq N$. Hence $E \in \mathcal{I}$, which is absurd. This proves that $S \in \mathcal{S}$. $\square$

3.2. COROLLARY. *Φ holds iff $\mathcal{S}$ is a non-trivial normal ideal on ω_1.* $\square$

§4. $2^{\aleph_0} < 2^{\aleph_1} \rightarrow \Phi$

4.1. THEOREM. (1) *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Then Φ.*

(2) *Assume $\lambda^{\aleph_0} < 2^{\aleph_1}$. Then for every $F: \lambda^{\underset{\smile}{\omega}} \rightarrow 2$ there is $g \in 2^{\omega_1}$ such that for every $f \in \lambda^{\omega_1}$, $\{\alpha < \omega_1 | g(\alpha) = F(f \restriction \alpha)\}$ is stationary.*

PROOF. We prove (1). The proof of (2) is similar.

Well, suppose Φ fails. Then $\omega_1 \in \mathcal{S}$, so we can find $F: 2^{\underset{\smile}{\omega}} \rightarrow 2$ such that for all $g \in 2^{\omega_1}$ there is $f \in 2^{\omega_1}$ with $(\alpha \in \omega_1 | F(f \restriction \alpha) = g(\alpha)\} \in \mathcal{F}$. (Given any g, let f be related to $1 - g$ as in definition of $\mathcal{S}$ to get this.)

Fix some one-one correspondence H between the set of all sequences of the form $(\alpha, g_0, f_0, \cdots, g_\nu, f_\nu, \cdots)_{\nu<\beta}$, where $\alpha, \beta < \omega_1$ and $g_\nu, f_\nu \in 2^\alpha$ for all $\nu < \beta$, and the set 2^ω.

Let $g \in 2^{\omega_1}$ be given. Pick $f \in 2^{\omega_1}$ such that $\{\alpha \in \omega_1 | F(f \restriction \alpha) = g(\alpha)\} \in \mathcal{F}$, and let $C \subseteq \omega_1$ be club with $\alpha \in C \rightarrow F(f \restriction \alpha) = g(\alpha)$. By induction on $n \in \omega$ we define functions $g_\nu, f_\nu \in 2^{\omega_1}$, $\nu < \omega . n$, and club sets $C_n \subseteq \omega_1$, so that whenever $\nu < \omega n$, $C_n \subseteq \{\alpha \in \omega_1 | F(f_\nu \restriction \alpha) = g_\nu(\alpha)\}$.

Stage 1. $(n = 1)$. For each $\nu < \omega$, let $g_\nu = g$, $f_\nu = f$, $C_\nu = C$.

Stage $n+1$. $(n \geqq 1)$. For each $\alpha < \omega_1$, let $\beta_{\alpha,n}$ be the least member of C_n greater than α and set:

$$\langle g_{\omega n+k}(\alpha) | k < \omega \rangle = H(\beta_{\alpha,n}, g_0 \restriction \beta_{\alpha,n}, f_0 \restriction \beta_{\alpha,n}, \cdots, g_\nu \restriction \beta_{\alpha,n} f_\nu \restriction \beta_{\alpha,n}, \cdots)_{\nu<\omega.n}.$$

This defines $g_{\omega n+k} \in 2^{\omega_1}$ for all $k \in \omega$. By hypothesis there are functions $f_{\omega n+k} \in 2^{\omega_1}$ such that $A_{\omega n+k} = \{\alpha \in \omega_1 | F(f_{\omega n+k} \restriction \alpha) = g_{\omega n+k}(\alpha)\} \in \mathcal{F}$, each k. Let $C_{n+1} \subseteq C_n \cap \bigcap_{k<\omega} A_{\omega n+k}$ be any club set now.

Clearly, for each $g \in 2^{\omega_1}$ we may carry out such a definition. Let g^g_ν, f^g_ν, $\nu < \omega . \omega$, and C^g_n, $n < \omega$, be the sequences so defined when we commence with g.

Define an equivalence relation E on 2^{ω_1} now by: gEg' iff:

(i) $\min(\bigcap_{n<\omega} C^g_n) = \min(\bigcap_{n<\omega} C^{g'}_n) = \gamma$ (say);

(ii) $g_\nu^g \restriction \gamma = g_\nu^{g'} \restriction \gamma$ and $f_\nu^g \restriction \gamma = f_\nu^{g'} \restriction \gamma$ for all $\nu < \omega . \omega$.

Now, the equivalence relation E clearly has at most $2^{\aleph_0}$ equivalence classes. But $2^{\aleph_0} < 2^{\aleph_1}$ and there are $2^{\aleph_1}$ possible functions g. Hence we can find functions g, g' such that $g \neq g'$ and gEg'.

From now on we write g_ν, f_ν, C_n for g_ν^g, f_ν^g, C_n^g, and g'_ν, f'_ν, C'_n for $g_\nu^{g'}$, $f_\nu^{g'}$, $C_n^{g'}$. And we set $C = \bigcap_{n<\omega} C_n$, $C' = \bigcap_{n<\omega} C'_n$. Let $\langle \gamma_\rho \mid \rho < \omega_1 \rangle$ be the canonical enumeration of C, $\langle \gamma'_\rho \mid \rho < \omega_1 \rangle$ that of C'.

We prove by induction on ρ that $\gamma_\rho = \gamma'_\rho$, and that for all $\nu < \omega . \omega$, $g_\nu \restriction \gamma_\rho = g'_\nu \restriction \gamma_\rho$, $f_\nu \restriction \gamma_\rho = f'_\nu \restriction \gamma_\rho$. This will, of course, yield the desired contradiction, since, in particular, we shall have $g = g_0 = g'_0 = g'$, contrary to $g \neq g'$.

For $\rho = 0$, the desired equalities hold because gEg'. And for limit ρ, the induction step is trivial because $\gamma_\rho = \sup_{\sigma<\rho} \gamma_\sigma$, $\gamma'_\rho = \sup_{\sigma<\rho} \gamma'_\sigma$. So assume now the equalities for ρ. We prove them for $\rho + 1$.

For each n and each $\alpha < \omega_1$, let $M_{\alpha,n} = (\beta_{\alpha,n}, g_0 \restriction \beta_{\alpha,n}, f_0 \restriction \beta_{\alpha,n}, \cdots, g_\nu \restriction \beta_{\alpha,n}, f_\nu \restriction \beta_{\alpha,n}, \cdots)_{\nu<\omega n}$, with $\beta_{\alpha,n}$, etc. as above, and define $M'_{\alpha,n}$ similarly for g'. By definition, $H(M_{\gamma_\rho,n}) = \langle g_{\omega n+k}(\gamma_\rho) \mid k < \omega \rangle$. But $\gamma_\rho \in C$, so this implies that $H(M_{\gamma_\rho,n}) = \langle F(f_{\omega n+k} \restriction \gamma_\rho) \mid k < \omega \rangle$. So, by induction hypothesis, we get $H(M_{\gamma_\rho,n}) = \langle F(f'_{\omega n+k} \restriction \gamma_\rho \mid k < \omega \rangle$, and reversing the above implications for the g' situation yields $H(M_{\gamma_\rho,n}) = H(M'_{\gamma_\rho,n})$. Hence as H is one-one, we have $M_{\gamma_\rho,n} = M'_{\gamma_\rho,n}$. In particular, $\beta_{\gamma_\rho,n} = \beta'_{\gamma_\rho,n}$. But this holds for all n, and we clearly have $\gamma_{\rho+1} = \sup_{n<\omega} \beta_{\gamma_\rho,n}$ and $\gamma'_{\rho+1} = \sup_{n<\omega} \beta'_{\gamma_\rho,n}$. Hence $\gamma_{\rho+1} = \gamma'_{\rho+1}$. Moreover, since $M_{\gamma_\rho,n} = M'_{\gamma_\rho,n}$ we have $g_\nu \restriction \beta_{\gamma_\rho,n} = g'_\nu \restriction \beta_{\gamma_\rho,n}$ for all n, so $g_\nu \restriction \gamma_{\rho+1} = g'_\nu \restriction \gamma_{\rho+1}$, all $\nu < \omega . \omega$, and likewise for f_ν, f'_ν. So we are done. $\square$

§5. Colouring ladder systems on ω_1

As a first application of Φ we consider the following problem. Let Ω denote the set of limit ordinals in ω_1. If $\delta \in \Omega$, a *ladder* on δ is a strictly increasing ω-sequence cofinal in δ. A *ladder system* on Ω is a sequence $\langle \eta_\delta \mid \delta \in \Omega \rangle$ such that η_δ is a ladder on δ, each δ. If $\eta = \langle \eta_\delta \mid \delta \in \Omega \rangle$ is a ladder system, by a *colouring* of η we mean a sequence $\langle k_\delta \mid \delta \in \Omega \rangle$ such that $k_\delta \in 2^\omega$, each δ. (The idea is that we think of $k_\delta(n)$ as colouring the point $\eta_\delta(n)$ either black or white. Notice that the same ordinal can be coloured both ways at the same time if it lies on two different ladders.) A *uniformisation* of a colouring k of η is a function $f \in 2^{\omega_1}$ such that for all $\delta \in \Omega$ there exists an $n \in \omega$ such that $m \geqq n \rightarrow k_\delta(m) = f(\eta_\delta(m))$. (So f colours the countable ordinals in such a way as to agree with the colouring k_δ of η_δ on all but finitely many points.) It is easily seen that to demand that the above n be 0 always would mean that only "trivial" colourings would have "uniformisations". The basic question is: given a ladder system on ω_1, is every colouring uniformisable?

5.1. THEOREM. *Assume Φ. Let η be a ladder system on ω_1. Then there is a colouring k of η which cannot be uniformised.*

PROOF. If $f \in 2^\alpha$, set

$$F(f) = \begin{cases} 0, & \text{if } (\exists n)(\forall m \geqq n)[f(\eta_\alpha(m)) = 0]; \\ 1, & \text{otherwise.} \end{cases}$$

This defines $F: 2^{\underset{\smile}{\omega}} \to 2$. Let $g \in 2^{\underset{\smile}{\omega}}$ be as in Φ. For each $\delta \in \Omega$, define $k_\delta \in 2^\omega$ by $k_\delta(n) = 1 - g(\delta)$. Suppose $f \in 2^{\omega_1}$ were to uniformise $\langle k_\delta \mid \delta \in \Omega \rangle$. Then for all $\delta \in \Omega$ there is $n < \omega$ such that $m \geqq n \to k_\delta(m) = f(\eta_\delta(m))$; i.e. there is, for each $\delta \in \Omega$ an $n < \omega$ such that $m \geqq n \to 1 - g(\delta) = f(\eta_\delta(m))$. But by Φ there is $\delta \in \Omega$ such that $F(f \restriction \delta) = g(\delta)$. Fixing this δ, therefore, we have $g(\delta) = 0 \leftrightarrow F(f \restriction \delta) = 0 \leftrightarrow (\exists n)(\forall m \geqq n)[f(\eta_\delta(m)) = 0] \leftrightarrow (\exists n)(\forall m \geqq n)(1 - g(\delta) = 0) \leftrightarrow g(\delta) = 1$, a contradiction. Hence $\langle k_\delta \mid \delta \in \Omega \rangle$ has no uniformisation. □

REMARK. Examination of the above proof will show that the colouring k cannot even be uniformised on a club subset of Ω.

If follows from 5.1 that if $2^{\aleph_0} = \aleph_1$ (say), then any ladder system on ω_1 has a non-uniformisable colouring. We cannot, however, avoid all use of extra assumptions as our next result shows.

By MA (Martin's Axiom) we mean the following assertion: if **P** is a poset satisfying c.c.c., and if $\mathscr{F}$ is a collection of at most fewer than $2^{\aleph_0}$ dense subsets of $\boldsymbol{P}$, then $\boldsymbol{P}$ has an $\mathscr{F}$-generic subset. It is known that $2^{\aleph_0} = \aleph_1 \to \mathrm{MA}$ but that $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ is consistent with ZFC.

5.2. THEOREM. *Assume $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$. Let $\eta = \langle \eta_\alpha \mid \alpha \in \Omega \rangle$ be a ladder system. Then every colouring of η is uniformisable.*

PROOF. Let $k = \langle k_\alpha \mid \alpha \in \Omega \rangle$ be a colouring of η. Let **P** consist of the set of all pairs $\langle X, h \rangle$ such that:

(i) X is a finite subset of Ω;
(ii) $h: \bigcup_{\alpha \in X} \mathrm{ran}(\eta_\alpha) \to \omega$;
(iii) $(\forall \alpha \in X)(\exists n \in \omega)(\forall m > n)[h(\eta_\alpha(m)) = k_\alpha(m)]$.

Regard **P** as a poset under the ordering $\langle X', h' \rangle \leqq \langle X, h \rangle \leftrightarrow X' \supseteq X \,\&\, h' \supseteq h$.

CLAIM. **P** *satisfies the c.c.c.*

Proof of Claim. Let $A = \{\langle X_\nu, h_\nu \rangle \mid \nu < \omega_1\}$ be a set of distinct elements of **P**. We show that A contains a pair of distinct, compatible elements, and hence that **P** satisfies the c.c.c.

We may assume that for all $\nu < \omega_1$, $|X_\nu| = p$. Let $\langle \delta^1_\nu, \cdots, \delta^p_\nu \rangle$ be the canonical enumeration of X_ν. Set $\delta^0_\nu = 0$.

Let $H_1(\nu) = \max(X_\nu \cap \nu)$. Then H_1 is regressive on $\omega_1 - \omega$, so on some stationary set $S_1 \subseteq \omega_1$, H_1 is constant with value, say, α_1. We may assume that for some q $(1 \leqq q \leqq p)$, $\delta^q_\nu = H_1(\nu) = \alpha_1$ for all $\nu \in S_1$. Let $S_1 = \{\nu(\gamma) \mid \gamma < \omega_1\}$.

Let $H_2(\nu(\gamma)) = \sup[\nu(\gamma) \cap \{\eta_{\delta^l_{\nu(\gamma+1)}}(m) \mid l = 1, \cdots, p; m < \omega\}$. Then H_2 is regressive on S_1 so there is $S_2 \subseteq S_1$ such that S_2 is stationary and H_2 is constant on S_2, say with value α_2.

Clearly, there is a stationary set $S_3 \subseteq S_2$ such that $h_{\nu(\gamma+1)} \restriction \alpha$ is independent of $\gamma \in S_3$.

Let $C = \{\gamma \mid \beta < \gamma \rightarrow [\nu(\beta) < \gamma \;\&\; \max X_{\nu(\beta)} < \gamma]\}$. Clearly C is club in ω_1. Let $\beta, \gamma \in C \cap S_3$ be limit ordinals. Then $\langle X_{\nu(\beta+1)}, h_{\nu(\beta+1)} \rangle$ and $\langle X_{\nu(\gamma+1)}, h_{\nu(\gamma+1)} \rangle$ are compatible. The claim is proved.

For $\alpha \in \Omega$ now, set $D_\alpha = \{\langle X, h \rangle \in \mathbf{P} \mid \alpha \in X\}$.

Claim. *Each D_α is dense in* **P**.

Proof of Claim. Let $\langle X, h \rangle \in \mathbf{P}$. We show that $\langle X, h \rangle$ has an extension in D_α. If $\alpha \in X$ there is nothing to prove, so we shall assume otherwise. Since η_α is cofinal in α and X is finite, there is $n \in \omega$ such that $(\forall m > n)[\eta_\alpha(m) \notin \bigcup_{\delta \in X} \operatorname{ran}(\eta_\delta)]$. Let $X' = X \cup \{\alpha\}$, and define $h' : \bigcup_{\delta \in X'} \operatorname{ran}(\eta_\delta) \rightarrow \omega$ by:

$$h'(\sigma) = \begin{cases} h(\sigma), & \text{if } \sigma \in \operatorname{dom}(h), \\ k_\alpha(m), & \text{if } \sigma \notin \operatorname{dom}(h) \text{ and } \sigma = \eta_\alpha(m). \end{cases}$$

Clearly, by our above remark, $\langle X', h' \rangle \in \mathbf{P}$. Moreover $\langle X', h' \rangle \leqq \langle X, h \rangle$ and $\langle X', h' \rangle \in D_\alpha$. This proves the claim.

By MA, let G be $\{D_\alpha \mid \alpha \in \Omega\}$-generic on **P**. Set $h = \bigcup \{h' \mid (\exists X')[\langle X', h' \rangle \in G]\}$. Clearly, h is a function from a subset of ω_1 into ω. Moreover,

$$(\forall \alpha \in \Omega)\{\operatorname{ran}(\eta_\alpha) \subseteq \operatorname{dom}(h) \;\&\; (\exists n \in \omega)(\forall m > n)[h(\eta_\alpha(m)) = k_\alpha(m)]\}.$$

Let $\bar{h} : \omega_1 \rightarrow \omega$ extend h. Then $\bar{h}$ is a uniformisation of k. The theorem is proved. □

5.3. Corollary. *Assume* MA $+ 2^{\aleph_0} > \aleph_1$. *Then* Φ *fails.* $\square$

§6. Whitehead groups

For a background to the *Whitehead Problem*, we refer the reader to [3]. The undecidability of the problem was proved by Shelah in [4]. In order to give more indication of the motivation leading to the formulation of Φ, we sketch (without full proofs or definitions being given) a result related to this problem.

6.1. First we prove that $2^{\aleph_0} < 2^{\aleph_1} \rightarrow \Theta$, where Θ says: if $\langle f_\eta \mid \eta \in 2^{\omega_1} \rangle$ is such that $f_\eta : \omega_1 \rightarrow 2^\omega$, then there is $\eta \in 2^{\omega_1}$ such that the set $\{\delta \in \omega_1 \mid (\exists \rho \in 2^{\omega_1})\ [f_\rho \restriction \delta = f_\eta \restriction \delta \ \&\ \rho \restriction \delta = \eta \restriction \delta \ \&\ \rho(\delta) \neq \eta(\delta)\}$ is stationary. Briefly, the idea is this. For $\delta < \omega_1$, $\eta \in 2^\delta$, $h : \delta \rightarrow 2^\omega$, let $F(\eta, h) = 0$ if there is $\rho \in 2^{\omega_1}$ such that $\eta \subseteq \rho$, $f_\rho \restriction \delta = h$, and $\rho(\delta) = 0$, and set $F(\eta, h) = 1$ otherwise. Since we may regard any such pair $\langle \eta, h \rangle$ as a single function mapping δ into $2 \times 2^\omega$, we may apply 4.1 (2) to get a $\rho \in 2^{\omega_1}$ such that for every $\eta \in 2^{\omega_1}$ and every $h : \omega_1 \rightarrow 2^\omega$, $\{\delta \in \omega_1 \mid F(\eta \restriction \delta, h \restriction \delta) = \rho(\delta)\}$ is stationary. Now set $\eta(\alpha) = 1 - \rho(\alpha)$, and let $h = f_\eta$ in the above to see that η is as required.

6.2. The result on W-groups we wish to indicate is the following. Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let $G = \bigcup_{\nu < \omega_1} G_\nu$, where $\{G_\nu\}$ is an increasing, continuous sequence of countable abelian groups. If $\{\nu \in \omega_1 \mid G_{\nu+1}/G_\nu$ is not free$\} \in \mathcal{F}$, then G is not a W-group. (This was shown by *Shelah* to follow from $\diamond$.) Briefly, the idea is to define, for $\eta \in 2^{<\omega_1}$, a group H_η and an epimorphism $h_\eta : H_\eta \rightarrow G_{\mathrm{dom}(\eta)}$ with $\mathrm{Ker}(h_\eta) = \mathbf{Z}$, so that $\eta < \rho \rightarrow H_\eta \lhd H_\rho \ \&\ h_\eta = h_\rho \restriction H_\eta$. The definition is by induction on $\mathrm{dom}(\eta)$. For each η we can define $H_{\eta\frown\langle i\rangle}, h_{\eta\frown\langle i\rangle}$, $i = 0, 1$, so that if $g_i : G_{\mathrm{dom}(\eta)+1} \rightarrow H_{\eta\frown\langle i\rangle}$ is a homomorphism with $h_{\eta\frown\langle i\rangle} \circ g_i = 1$, then $g_0 \restriction G_{\mathrm{dom}(\eta)} \neq g_1 \restriction G_{\mathrm{dom}(\eta)}$. (See [4].) If now G was a W-group, then for every $\eta \in 2^{\omega_1}$ there would be $g_\eta : G \rightarrow H_\eta$ with $h_\eta \circ g_\eta = 1$. By Θ, for some $\eta \in 2^\delta$, $\delta < \omega_1$, $\eta\frown\langle i\rangle \subseteq \eta_i$, $g_{\eta_0} \restriction G_\delta = g_{\eta_1} \restriction G_\delta$, contrary to the construction.

§7. Further remarks

7.1. Generalising 4.1 (2) we can prove that if μ is regular, and for some $\theta < \mu, 2^\theta = \lambda^{<\mu} < 2^\mu$, then the conclusion of 4.1 (2) holds with μ in place of ω_1 (in the proof, θ takes the place of ω).

7.2. Connected with 5.1 we may also prove the following. For $\delta \in \Omega$, let D_δ be a non-principal ultrafilter on ω. A $\langle D_\delta \mid \delta \in \Omega \rangle$-uniformisation of a colouring k of a ladder system η is a function $f \in 2^{\omega_1}$ such that for each $\delta \in \Omega$, $\{n \in$

$\omega \mid f(\eta_\delta(n)) = k_\delta(n)\} \in D_\delta$. Using Φ we can prove that every ladder system has a colouring which is not $\langle D_\delta \mid \delta \in \Omega \rangle$-uniformisable.

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