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ITERATION OF λ-COMPLETE FORCING NOTIONS NOT COLLAPSING λ^+

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ABSTRACT. We look for a parallel to the notion of "proper forcing" among λ-complete forcing notions not collapsing λ^+. We suggest such a definition and prove that it is preserved by suitable iterations.

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1. Introduction. This work follows [8, 10] (and see history there), but we do not rely on those papers. Our goal in this and the previous papers is to develop a theory parallel to "properness in CS iterations" for iterations with larger supports. In [8, 10] we have presented parallels to [11, 13], whereas here we try to have parallels to [12], [14, Chapters III and V, Sections 5 and 7] and hopefully [15, Chapters VI and XVIII].

It seems too much to hope for a notion fully parallel to "proper" among λ-complete forcing notions as even for "λ^+-c.c. λ-complete," there are problems. We should also remember the ZFC limitations for possible iteration theorems. For example, if in the definition of the forcing notion $\mathbb{Q}^*$ in Section 4 we demand $h^p \restriction e_\delta \subseteq h_\delta$, then the proof fails. This may seem a drawback, but one should look at [15, Appendix, page 985, Theorem 3.6(2) and page 990, Theorem 3.9]. By those results, if $\mathcal{S}^* = \mathcal{S}_\lambda^{\lambda^+}$, A_δ, h_δ are as in Context 4.6, and we ask a success on a club, then for some $\langle h_\delta : \delta \in \mathcal{S}_\lambda^{\lambda^+}\rangle$ we fail. Now, if we allow only $h_\delta : A_\delta \to 2$ and we ask for "success of the uniformization" on an end segment of A_δ (for all such $\langle A_\delta : \delta \in \mathcal{S}_\lambda^{\lambda^+}\rangle$), then we also fail as we may code colourings with values in λ.

In Section 2 we formulate our definitions (including *properness over* λ, see Definition 2.3). We believe that our main Definition 2.3 is quite reasonable and applicable. One may also define a version of it where the diamond is "spread out." Section 3 is devoted to the proof of the preservation theorem, and Section 4 gives three (relatively easy) examples of forcing notions fitting our scheme. We conclude the paper with the discussion of applications and variants.

Notation 1.1. Our notation is rather standard and compatible with that of classical textbooks (see Jech [2]). In forcing we keep the older (Cohen's) convention that *a stronger condition is the larger one.*

(1) For a filter D on λ, the family of all D-positive subsets of λ is called D^+. (So $A \in D^+$ if and only if $A \subseteq \lambda$ and $A \cap B \neq \emptyset$ for all $B \in D$.)

(2) Every forcing notion $\mathbb{P}$ under considerations is assumed to have the weakest condition $\emptyset_{\mathbb{P}}$, that is, $(\forall p \in \mathbb{P})(\emptyset_{\mathbb{P}} \leq_{\mathbb{P}} p)$. Also we assume $* \notin \mathbb{P}$ is a fixed object belonging to all the N's we consider.

(3) A tilde indicates that we are dealing with a name for an object in a forcing extension (like $\underset{\sim}{x}$). The canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic filter over $\mathbf{V}$ is denoted by $\underset{\sim}{G}_{\mathbb{P}}$. In iterations, if $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$ and $p \in \lim(\bar{\mathbb{Q}})$, then we keep the convention that $p(\alpha) = \underset{\sim}{\emptyset}_{\mathbb{Q}_\alpha}$ for $\alpha \in \zeta^* \setminus \mathrm{Dom}(p)$.

(4) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet $(\alpha, \beta, \gamma, \dots)$ and also by i, j (with possible sub- and superscripts).

(5) A bar above a letter denotes that the object considered is a sequence; usually $\bar{X}$ will be $\langle X_i : i < \zeta \rangle$, where ζ denotes the length of $\bar{X}$. Often our sequences will be indexed by a set of ordinals, say $\mathcal{S} \subseteq \lambda$, and then $\bar{X}$ will typically be $\langle X_\delta : \delta \in \mathcal{S} \rangle$. Semi-diamond sequences will be called $\bar{F}$ (with possible superscripts).

In our definitions (and proofs) we will use somewhat special diamond-like sequences (see Definition 2.1(2)). The difference between them and classical diamonds is quite minor, so we remember the following.

Definition 1.2. (1) Let D be a filter on λ. We say that $\bar{F} = \langle F_\delta : \delta \in \mathcal{S} \rangle$ is a D-diamond sequence if $\mathcal{S} \in D^+$, $F_\delta \in {}^\delta\delta$ for $\delta \in \mathcal{S}$, and

$$(\forall f \in {}^\lambda\lambda)(\{\delta \in \mathcal{S} : F_\delta \subseteq f\} \in D^+). \tag{1.1}$$

We may also call such $\bar{F}$ *a* $(D, \mathcal{S})$*-diamond sequence.*

(2) We say that $(D, \mathcal{S})$ *has diamonds* if there is a $(D, \mathcal{S})$-diamond. We say that D *has diamonds* if D is a normal filter on λ and for every $\mathcal{S} \in D^+$ there is a $(D, \mathcal{S})$-diamond.

Definition 1.3. A forcing notion $\mathbb{P}$ is λ*-complete* if every $\le_{\mathbb{P}}$-increasing chain of length less than λ has an upper bound in $\mathbb{P}$. It is λ*-lub-complete* if every $\le_{\mathbb{P}}$-increasing chain of length less than λ has a least upper bound in $\mathbb{P}$.

Proposition 1.4. (1) *If D is a filter on λ including all co-bounded subsets of λ, then the family of all diagonal intersections of members of D constitutes a normal filter (but in general not necessarily proper). We call this family the normal filter generated by D.*

(2) *If $\mathbb{P}$ is a λ-complete forcing notion and D is a normal filter on λ, then in $\mathbf{V}^{\mathbb{P}}$ the filter D generates a proper normal filter on λ. (By abusing the notation, we will denote this filter also by D or, if we want to stress that we work in the forcing extension, by $D^{\mathbf{V}[G_{\mathbb{P}}]}$.)*

Moreover, by the λ-completeness of $\mathbb{P}$, if $X \in D^+ \cap \mathbf{V}$, then $\Vdash_{\mathbb{P}} X \in D^+$, and if $X \in \mathbf{V}$, $p \Vdash_{\mathbb{P}} X \in D^{\mathbf{V}^{\mathbb{P}}}$, then $X \in D$.

(3) *If $\mathbb{P}$ is a λ-complete forcing notion and $\bar{F} = \langle F_\delta : \delta \in \mathcal{S} \rangle$ is a D-diamond sequence, then*

$$\Vdash_{\mathbb{P}} \text{“}\bar{F} \text{ is a } D\text{-diamond sequence”}. \tag{1.2}$$

Definition 1.5 and Proposition 1.6 below are not central for us, but they may be used to get somewhat stronger results (see [9]).

Definition 1.5. Let pr be a definable pairing function on λ, for example $\mathrm{pr}(\alpha, \beta) = \omega^{\alpha+\beta} + \beta$, and let $\bar{F} = \langle F_\delta : \delta \in \mathcal{S} \rangle$ be a D-diamond sequence.

For an ordinal $\alpha < \lambda$ we let $\bar{F}^{[\alpha]} = \langle F^{[\alpha]}_\delta : \delta \in \mathcal{S} \rangle$, where each $F^{[\alpha]}_\delta$ is a function with domain δ and such that

$$F_\delta^{[\alpha]}(\beta) = \begin{cases} F_\delta(\mathrm{pr}(\alpha,\beta)) & \text{if well defined,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Proposition 1.6. *If $\bar{F}$ is a D-diamond sequence, then for every $\alpha < \lambda$, $\bar{F}^{[\alpha]}$ is also a D-diamond sequence.*

Throughout we will assume the following.

Context 1.7. (a) λ is an uncountable cardinal, $\lambda = \lambda^{<\lambda}$, and

(b) D is a normal filter on λ (usually D is the club filter $\mathcal{D}_\lambda$ on λ),

(c) $\mathcal{S} \in D^+$ contains all successor ordinals below λ, $0 \notin \mathcal{S}$, and $\mathcal{S}' = \lambda \setminus \mathcal{S}$ is unbounded in λ,

(d) there is a $(D,\mathcal{S})$-diamond sequence.

2. The definitions. In this section, we define a special genericity game, properness over $(D,\mathcal{S})$-semi-diamonds and the class of forcing notions we are interested in.

Definition 2.1. Let $\mathbb{P}$ be a forcing notion and let $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be such that $\|N\| = \lambda$, $N^{<\lambda} \subseteq N$ and $\{\lambda, \mathbb{P}, D, \mathcal{S}\} \in N$. Let $h : \lambda \to N$ be such that the range $\mathrm{Rang}(h)$ of the function h includes $\mathbb{P} \cap N$.

(1) We say that $\bar{F} = \langle F_\delta : \delta \in \mathcal{S}\rangle$ is *a $(D,\mathcal{S})$-semi-diamond sequence* if $F_\delta \in {}^\delta\delta$ for $\delta \in \mathcal{S}$ and

($*$) for every $\le_{\mathbb{P}}$-increasing sequence $\bar{p} = \langle p_\alpha : \alpha < \lambda\rangle \subseteq \mathbb{P} \cap N$, we have

$$\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h \circ F_\delta(\alpha) = p_\alpha)\} \in D^+. \tag{2.1}$$

(2) Let $\bar{F}$ be a $(D,\mathcal{S})$-semi-diamond. A sequence $\bar{q} = \langle q_\delta : \delta \in \mathcal{S}\rangle \subseteq N \cap \mathbb{P}$ is called *an $(N,h,\mathbb{P})$-candidate over $\bar{F}$* (or *$(N,h,\mathbb{P},\bar{F})$-candidate*) whenever

(α) for every open dense subset $\mathcal{I} \in N$ of $\mathbb{P}$

$$\{\delta \in \mathcal{S} : q_\delta \in \mathcal{I}\} = \mathcal{S} \mod D, \tag{2.2}$$

(β) if $\delta \in \mathcal{S}$ is a limit ordinal and $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is a $\le_{\mathbb{P}}$-increasing sequence of members of $\mathbb{P} \cap N$, then q_δ is its upper bound in $\mathbb{P}$.

(3) Let $\bar{q}$ be an $(N,h,\mathbb{P},\bar{F})$-candidate and $r \in \mathbb{P}$. We define a game $\eth(r,N,h,\mathbb{P},\bar{F},\bar{q})$ of two players, the *generic player* and the *anti-generic player*, as follows. A play lasts λ moves, in the ith move conditions $r_i^-, r_i \in \mathbb{P}$ and a set $C_i \in D$ are chosen such that

- $r_i^- \in N$, $r_i^- \le r_i$, $r \le r_i$,
- $(\forall j < i)(r_j \le r_i$ and $r_j^- \le r_i^-)$, and
- the generic player chooses r_i^-, r_i, C_i if $i \in \mathcal{S}$, and the anti-generic player chooses r_i^-, r_i, C_i if $i \in \mathcal{S}'$.

If at some moment during the play there is no legal move for one of the players, then the anti-generic player wins. If the play lasted λ moves, then the generic player wins the play whenever

($\circledast$) if $\delta \in \mathcal{S} \cap \bigcap_{i<\delta} C_i$ is a limit ordinal, and $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle = \langle r^-_\alpha : \alpha < \delta\rangle$, then $q_\delta \le r_\delta$.

(4) Let $\bar{q}$ be an $(N,h,\mathbb{P},\bar{F})$-candidate, $\bar{F}$ a $(D,\mathcal{S})$-semi-diamond. A condition $r \in \mathbb{P}$ is *$(N,h,\mathbb{P})$-generic for $\bar{q}$ over $\bar{F}$* if the generic player has a winning strategy in the game $\Game(r,N,h,\mathbb{P},\bar{F},\bar{q})$.

Observation 2.2. (1) In the game $\Game(r,N,h,\mathbb{P},\bar{F},\bar{q})$, for each of the players, if it increases conditions r^-_i, r_i, its choice can only improve its situation. Making sets C_i (for $i \in \mathcal{S}$) smaller can only help the generic player.

(2) If forcing with $\mathbb{P}$ does not add new subsets to λ, then the game in Definition 2.1(5) degenerates as, without loss of generality, r forces a value to $\underset{\sim}{G}_{\mathbb{P}} \cap N$; the condition does not degenerate, in fact this condition (which implies adding no new λ-sequences) is preserved by $(<\lambda^+)$-support iterations (see [8]).

(3) Also if $\mathcal{S}_1 \subseteq \mathcal{S} \mod D$, $\mathcal{S}_1 \in D^+$, then in Definition 2.1 we can replace $\mathcal{S}$ by $\mathcal{S}_1$. (Again, the generic player can guarantee $C_i \cap \mathcal{S}_1 \subseteq \mathcal{S}$.)

(4) If $\mathbb{P}$ is λ-complete and r is $(N,\mathbb{P})$-generic (in the usual sense, that is, $r \Vdash_{\mathbb{P}}$ "$N[\underset{\sim}{G}_{\mathbb{P}}] \cap \mathbf{V} = N$"), then both players have always legal moves in the game $\Game(r,N,h,\mathbb{P},\bar{F},\bar{q})$.

Also if the forcing notion $\mathbb{P}$ is λ-lub-complete, then both players have always legal moves in the game $\Game(r,N,h,\mathbb{P},\bar{F},\bar{q})$ (for any r).

Definition 2.3. (1) Let $\mathcal{S} \in D^+$. We say that a forcing notion $\mathbb{P}$ is *proper over $(D,\mathcal{S})$-semi-diamonds* whenever (there is a $(D,\mathcal{S})$-diamond and):

(a) $\mathbb{P}$ is λ-complete, and

(b) if χ is large enough, $p \in \mathbb{P}$ and $N \prec (\mathcal{H}(\chi),\in,<^*_\chi)$, $\|N\| = \lambda$, $N^{<\lambda} \subseteq N$ and $\{\lambda,p,\mathbb{P},D,\mathcal{S},\ldots\} \in N$, and $h : \lambda \to N$ satisfies $\mathbb{P} \cap N \subseteq \mathrm{Rang}(h)$, and $\bar{F}$ is a $(D,\mathcal{S})$-semi-diamond for $(N,h,\mathbb{P})$, and $\bar{q} = \langle q_\delta : \delta \in \mathcal{S}\rangle$ is an $(N,h,\mathbb{P},\bar{F})$-candidate, then there is $r \in \mathbb{P}$ stronger than p and such that r is $(N,h,\mathbb{P})$-generic for $\bar{q}$ over $\bar{F}$.

(2) $\mathbb{P}$ is said to be *proper over D-semi-diamonds* if it is proper over $(D,\mathcal{S})$-semi-diamonds for every $\mathcal{S} \in D^+$ (so D has diamonds). The family of forcing notions proper over D-semi-diamonds is denoted K^1_D.

(3) A forcing notion $\mathbb{P}$ is *proper over λ* if it is proper over D-semi-diamonds for every normal filter D on λ which has diamonds.

Remark 2.4. Does D matter? Yes, as we may use some "large D" and be interested in preserving its largeness properties.

Proposition 2.5. *If $\mathbb{P}$ is a λ^+-complete forcing notion, then $\mathbb{P}$ is proper over λ.*

Proof. The proof is straightforward. $\square$

Proposition 2.6. (1) *If N, $\mathbb{P}$, h are as in Definition 2.1, $\mathbb{P}$ is λ-complete, and $\bar{F}$ is a $(D,\mathcal{S})$-semi-diamond, then there is an $(N,h,\mathbb{P},\bar{F})$-candidate. Furthermore,*

($+$) *if $\mathcal{I} \in N$ is an open dense subset of $\mathbb{P}$, then $q_\delta \in \mathcal{I}$ for every large enough δ.*

(2) *Let r be $(N,h,\mathbb{P})$-generic over $\bar{F}$ for some $(N,h,\mathbb{P},\bar{F})$-candidate $\bar{q}$. Then*

(a) *if $\langle r^-_i, r_i, C_i : i < \lambda\rangle$ is a result of a play of $\Game(r,N,h,\mathbb{P},\bar{F},\bar{q})$ in which the* *generic player uses its winning strategy, then*

$$G' = \{p \in \mathbb{P} \cap N : (\exists i < \lambda)(p \leq r_i)\} \tag{2.3}$$

is a subset of $\mathbb{P} \cap N$ *generic over* N*, and*

(b) r *is* $(N, \mathbb{P})$*-generic (in the usual sense).*

(3) *If* $\mathbb{P}$ *is proper over* $(D, \mathcal{S})$*-semi-diamonds,* $\mu \geq \lambda$*,* $Y \subseteq [\mu]^{\leq\lambda}$*,* $Y \in \mathbf{V}$*, then:*

(a) *forcing with* $\mathbb{P}$ *does not collapse* λ^+*,*

(b) *forcing with* $\mathbb{P}$ *preserves the following two properties:*

(i) Y *is a cofinal subset of* $[\mu]^{\leq\lambda}$ *(under inclusion),*

(ii) *for every large enough* χ *and* $x \in \mathcal{H}(\chi)$*, there is* $N \prec (\mathcal{H}(\chi), \in)$ *such that* $\|N\| = \lambda$*,* $N \cap \lambda^+ \in \lambda^+$*,* $N^{<\lambda} \subseteq N$*,* $N \cap \mu \in Y$ *(i.e., the stationarity of* Y *under the relevant filter).*

Proof. (1) The proof is immediate (by the λ-completeness of $\mathbb{P}$).

(2) Clause (a) should be clear (remember Definition 2.1(2)(α)). For clause (b) note that $0 \in \mathcal{S}'$, so in the game $\eth(r, N, h, \mathbb{P}, \bar{F}, \bar{q})$ the condition r_0 is chosen by the anti-generic player. So if the conclusion fails, then for some $\mathbb{P}$-name $\underset{\sim}{\alpha} \in N$ for an ordinal we have $r \nVdash$ "$\underset{\sim}{\alpha} \in N$". Thus the anti-generic player can choose r_0 so that $r_0 \Vdash$ "$\underset{\sim}{\alpha} = \alpha_0$" for some ordinal $\alpha_0 \notin N$, what guarantees it to win the play.

(3) The proof is straightforward from (2). $\square$

Very often checking properness over D-semi-diamonds (for particular examples of forcing notions) we get somewhat stronger properties that motivate the following definition.

Definition 2.7. We say that a condition $r \in \mathbb{P}$ is *N-generic for D-semi-diamonds* if it is $(N, h, \mathbb{P})$-generic for $\bar{q}$ over $\bar{F}$ whenever h, $\bar{q}$, $\bar{F}$ are as in Definition 2.1. Omitting D we mean "for every normal filter D with diamonds."

The following notion is not of main interest in this paper, but surely it is interesting from the point of view of general theory.

Definition 2.8. Let $0 < \alpha < \lambda^+$.

(1) Let $\mathcal{S} \in D^+$. We say that a forcing notion $\mathbb{P}$ is *α-proper over $(D, \mathcal{S})$-semi-diamonds* whenever

(a) $\mathbb{P}$ is λ-complete, and

(b) if χ is large enough, $p \in \mathbb{P}$ and

- $\bar{N} = \langle N_\beta : \beta < \alpha\rangle$ is an increasing sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ such that $\|N_\beta\| = \lambda$, $N_\beta^{<\lambda} \subseteq N_\beta$, $\{\lambda, p, \mathbb{P}, \bar{N} \restriction \beta\} \in N_\beta$, and
- $\bar{F}^\beta = \langle F^\beta_\delta : \delta \in \mathcal{S}\rangle$, $F^\beta_\delta \in {}^\delta\delta$ (for $\beta < \alpha$),
- $h_\beta : \lambda \to N_\beta$, $\mathbb{P} \cap N_\beta \subseteq \mathrm{Rang}(h_\beta)$ and $\langle h_\gamma, \bar{F}^\gamma : \gamma < \beta\rangle \in N_\beta$, and
- $\bar{F}^\beta$ is a $(D, \mathcal{S})$-semi-diamond sequence for $(N_\beta, h_\beta, \mathbb{P})$, and
- $\bar{q}^\beta = \langle q^\beta_\delta : \delta \in \mathcal{S}\rangle$ is an $(N_\beta, h_\beta, \mathbb{P})$-candidate over $\bar{F}^\beta$, and $\langle \bar{q}^\gamma : \gamma < \beta\rangle \in N_\beta$,

then there is $r \in \mathbb{P}$ above p which is $(N_\beta, h_\beta, \mathbb{P})$-generic for $\bar{q}^\beta$ over $\bar{F}^\beta$ for each $\beta < \alpha$.

(2) We define "$\mathbb{P}$ *is α-proper over D-semi-diamonds*" (and K^α_D) and "$\mathbb{P}$ *is α-proper over* λ" in a way parallel to Definition 2.3(2), (3).

Remark 2.9. Note that for $\alpha = 1$ (in Definition 2.8) we get the same notions as in Definition 2.3.

3. The preservation theorem.

In Theorem 3.7 below we prove a preservation theorem for our forcing notions. It immediately gives the consistency of the suitable forcing axiom (see Section 5.1). Also the proof actually specifies which semi-diamond sequences $\bar{F}$ are used.

First, recall the following result.

Proposition 3.1. *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \zeta^* \rangle$ is a $(<\lambda^+)$-support iteration such that for each $\alpha < \zeta^*$*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underset{\sim}{\mathbb{Q}}_\alpha \text{ is } \lambda\text{-complete”.} \tag{3.1}$$

Then the forcing $\mathbb{P}_{\zeta^*}$ is λ-complete.

Before we engage in the proof of the preservation theorem, we prove some facts of more general nature than the one of our main context. If, for example, all iterands are λ-lub-complete, then Proposition 3.3 below is obvious.

Temporary context 3.2. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \zeta^* \rangle$ be a $(<\lambda^+)$-support iteration of λ-complete forcing notions. We also suppose that N is a model as in Definition 2.1, $\bar{\mathbb{Q}}, \ldots \in N$.

Proposition 3.3. *Suppose that $\zeta \in (\zeta^*+1) \cap N$ is a limit ordinal of cofinality $\mathrm{cf}(\zeta) < \lambda$ and $r \in \mathbb{P}_\zeta$ is such that*

$$(\forall \varepsilon \in \zeta \cap N)(r \restriction \varepsilon \text{ is } (N, \mathbb{P}_\varepsilon)\text{-generic}). \tag{3.2}$$

Assume that conditions $s_\beta \in N \cap \mathbb{P}_\zeta$ (for $\beta < \delta$, $\delta < \lambda$) are such that

$$(\forall \beta' < \beta < \delta)(s_{\beta'} \leq s_\beta \leq r). \tag{3.3}$$

Then there are conditions $s \in N \cap \mathbb{P}_\zeta$ and $r^+ \in \mathbb{P}_\zeta$ such that $s \leq r^+$, $r \leq r^+$ and $(\forall \beta < \delta)(s_\beta \leq s)$.

Proof. Let $\langle i_\gamma : \gamma < \mathrm{cf}(\zeta) \rangle \subseteq N \cap \zeta$ be a strictly increasing continuous sequence cofinal in ζ. By induction on γ choose r^-_γ, r_γ such that

(α) $r^-_\gamma \in \mathbb{P}_{i_\gamma} \cap N$ is above (in $\mathbb{P}_{i_\gamma}$) of all $s_\beta \restriction i_\gamma$ for $\beta < \delta$,

(β) $r_\gamma \in \mathbb{P}_{i_\gamma}$, $r^-_\gamma \leq_{\mathbb{P}_{i_\gamma}} r_\gamma$, and $r \restriction i_\gamma \leq r_\gamma$,

(γ) if $\gamma < \varepsilon < \mathrm{cf}(\zeta)$ then $r^-_\gamma \leq r^-_\varepsilon$ and $r_\gamma \leq r_\varepsilon$.

(The choice is clearly possible as $r \restriction i_\gamma$ is $(N, \mathbb{P}_{i_\gamma})$-generic.)

Let $r^+ \in \mathbb{P}_\zeta$ be an upper bound of $\langle r_\gamma : \gamma < \mathrm{cf}(\zeta) \rangle$ (remember clause (γ) above); then also $r \leq r^+$. Now we are going to define a condition $s \in \mathbb{P}_\zeta \cap N$. We let $\mathrm{Dom}(s) = \bigcup \{ \mathrm{Dom}(r^-_{\gamma+1}) \cap [i_\gamma, i_{\gamma+1}) : \gamma < \mathrm{cf}(\zeta) \}$, and for $\xi \in \mathrm{Dom}(s)$, $i_\gamma \leq \xi < i_{\gamma+1}$, we let $s(\xi)$ be a $\mathbb{P}_\xi$-name for the following object in $\mathbf{V}[G_{\mathbb{P}_\xi}]$ (for a generic filter $G_{\mathbb{P}_\xi} \subseteq \mathbb{P}_\xi$ over $\mathbf{V}$):

(i) If $r^-_{\gamma+1}(\xi)[G_{\mathbb{P}_\xi}]$ is an upper bound of $\{ s_\beta(\xi)[G_{\mathbb{P}_\xi}] : \beta < \delta \}$ in $\underset{\sim}{\mathbb{Q}}_\xi[G_{\mathbb{P}_\xi}]$, then $s(\xi)[G_{\mathbb{P}_\xi}] = r^-_{\gamma+1}(\xi)[G_{\mathbb{P}_\xi}]$.

(ii) If not (i), but $\{ s_\beta(\xi)[G_{\mathbb{P}_\xi}] : \beta < \delta \}$ has an upper bound in $\underset{\sim}{\mathbb{Q}}_\xi[G_{\mathbb{P}_\xi}]$, then $s(\xi)[G_{\mathbb{P}_\xi}]$ is the $<^*_\chi$-first such upper bound.

(iii) If neither (i) nor (ii), then $s(\xi)[G_{\mathbb{P}_\xi}] = s_0(\xi)[G_{\mathbb{P}_\xi}]$.

It should be clear that $s \in \mathbb{P}_\zeta \cap N$. Now,

• $s \le r^+$.

Why? By induction on $\xi \in \zeta \cap N$ we show that $s \restriction \xi \le r^+ \restriction \xi$. Steps "$\xi = 0$" and "$\xi$ limit" are clear, so suppose that we have proved $s \restriction \xi \le r^+ \restriction \xi$, $i_\gamma \le \xi < i_{\gamma+1}$ (and we are interested in the restrictions to $\xi+1$). Assume that $G_{\mathbb{P}_\xi} \subseteq \mathbb{P}_\xi$ is a generic filter over $\mathbf{V}$ such that $r^+ \restriction \xi \in G_{\mathbb{P}_\xi}$. Since $s_\beta \restriction i_{\gamma+1} \le r^-_{\gamma+1} \le r_{\gamma+1} \le r^+$, we also have $\{s_\beta \restriction \xi : \beta < \delta\} \subseteq G_{\mathbb{P}_\xi}$ and $r^-_{\gamma+1} \restriction \xi \in G_{\mathbb{P}_\xi}$. Hence $r^-_{\gamma+1}(\xi)[G_{\mathbb{P}_\xi}]$ is an upper bound of $\{s_\beta(\xi)[G_{\mathbb{P}_\xi}] : \beta < \delta\}$. Therefore, $s(\xi)[G_{\mathbb{P}_\xi}] = r^-_{\gamma+1}(\xi)[G_{\mathbb{P}_\xi}] \le r_{\gamma+1}(\xi)[G_{\mathbb{P}_\xi}] \le r^+(\xi)[G_{\mathbb{P}_\xi}]$ (see (i) above) and we are done.

The proof of the proposition will be finished once we show

• $(\forall \beta < \delta)(s_\beta \le s)$.

Why does this hold? By induction on $\xi \in \zeta \cap N$ we show that $s_\beta \restriction \xi \le s \restriction \xi$ for all $\beta < \delta$. Steps "$\xi = 0$" and "ξ limit" are as usual clear, so suppose that we have proved $s_\beta \restriction \xi \le s \restriction \xi$ (for $\beta < \delta$), $i_\gamma \le \xi < i_{\gamma+1}$ (and we are interested in the restrictions to $\xi+1$). Assume that $G_{\mathbb{P}_\xi} \subseteq \mathbb{P}_\xi$ is a generic filter over $\mathbf{V}$ such that $s \restriction \xi \in G_{\mathbb{P}_\xi}$. Then also (by the inductive hypothesis) $\{s_\beta \restriction \xi : \beta < \delta\} \subseteq G_{\mathbb{P}_\xi}$ and therefore $\langle s_\beta(\xi)[G_{\mathbb{P}_\xi}] : \beta < \delta \rangle$ is an increasing sequence of conditions from the (λ-complete) forcing $\underset{\sim}{\mathbb{Q}}_\xi[G_{\mathbb{P}_\xi}]$. Thus this sequence has an upper bound, and $s(\xi)[G_{\mathbb{P}_\xi}]$ is such an upper bound (see (i) and (ii) above), as required. □

In the proof of the preservation theorem we will (like in the proof of the preservation of properness [15, Chapter III, Section 3.3]) have to deal with *names* for conditions in the iteration. This motivates the following definition (which is in the spirit of [15, Chapter X], so this is why "RS").

Definition 3.4. (1) An RS-condition in $\mathbb{P}_{\zeta^*}$ is a pair (p, w) such that $w \in [(\zeta^*+1)]^{<\lambda}$ is a closed set, $0, \zeta^* \in w$, p is a function with domain $\mathrm{Dom}(p) \subseteq \zeta^*$, and

$(\otimes)_1$ for every two successive members $\varepsilon' < \varepsilon''$ of the set w, $p \restriction [\varepsilon', \varepsilon'')$ is a $\mathbb{P}_{\varepsilon'}$-name of an element of $\mathbb{P}_{\varepsilon''}$ whose domain is included in the interval $[\varepsilon', \varepsilon'')$.

The family of all RS-conditions in $\mathbb{P}_{\zeta^*}$ is denoted by $\mathbb{P}^{\mathrm{RS}}_{\zeta^*}$.

(2) If $(p, w) \in \mathbb{P}^{\mathrm{RS}}_{\zeta^*}$ and $G_{\mathbb{P}_{\zeta^*}} \subseteq \mathbb{P}_{\zeta^*}$ is a generic filter over $\mathbf{V}$, then we write $(p, w) \in' G_{\mathbb{P}_{\zeta^*}}$ whenever

$(\otimes)_2$ for every two successive members $\varepsilon' < \varepsilon''$ of the set w,

$$(p \restriction [\varepsilon', \varepsilon''))[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\varepsilon'}] \in G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\varepsilon''}. \tag{3.4}$$

(3) If $(p_1, w_1), (p_2, w_2) \in \mathbb{P}^{\mathrm{RS}}_{\zeta^*}$, then we write $(p_1, w_1) \le' (p_2, w_2)$ whenever

$(\otimes)_3$ for every generic $G_{\mathbb{P}_{\zeta^*}} \subseteq \mathbb{P}_{\zeta^*}$ over $\mathbf{V}$, if $(p_2, w_2) \in' G_{\mathbb{P}_{\zeta^*}}$ then $(p_1, w_1) \in' G_{\mathbb{P}_{\zeta^*}}$ and for each two successive members $\varepsilon' < \varepsilon''$ of the set $w_1 \cup w_2$ we have

$$(p_1 \restriction [\varepsilon', \varepsilon''))[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\varepsilon'}] \le_{\mathbb{P}_{\varepsilon''}} (p_2 \restriction [\varepsilon', \varepsilon''))[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\varepsilon'}]. \tag{3.5}$$

Remark 3.5. If $(p, w) \in \mathbb{P}^{\mathrm{RS}}_{\zeta^*}$, $\varepsilon' \le \xi < \varepsilon''$, $\varepsilon', \varepsilon''$ are successive members of w, then $p(\xi)$ is a $\mathbb{P}_{\varepsilon'}$-name for a $\mathbb{P}_\xi$-name of a member of $\underset{\sim}{\mathbb{Q}}_\xi$. One may look at this name as a

$\mathbb{P}_\xi$-name. However, note that if we apply this approach to each ξ, we may not end up with a condition in $\mathbb{P}_{\zeta^*}$ because of the support!

Proposition 3.6. (1) *For each $(p,w) \in \mathbb{P}^{\rm RS}_{\zeta^*}$ there is $q \in \mathbb{P}_{\zeta^*}$ such that $(p,w) \leq' (q,\{0,\zeta^*\})$.*

(2) *If $(p,w) \in \mathbb{P}^{\rm RS}_{\zeta^*}$ and $q \in \mathbb{P}_{\zeta^*}$, then there is $q^* \in \mathbb{P}_{\zeta^*}$ stronger than q and such that for each successive members $\varepsilon' < \varepsilon''$ of w the condition $q^* \restriction \varepsilon'$ decides $p \restriction [\varepsilon',\varepsilon'')$ (i.e., $q \restriction \varepsilon' \Vdash$ "$p \restriction [\varepsilon',\varepsilon'') = p_{\varepsilon',\varepsilon''}$" for some $p_{\varepsilon',\varepsilon''} \in \mathbb{P}_{\zeta^*}$).*

(3) *Let $(p_i,w_i) \in \mathbb{P}^{\rm RS}_{\zeta^*} \cap N$ (for $i < \delta < \lambda$), and $s \in \mathbb{P}_{\zeta^*} \cap N$, $r \in \mathbb{P}_{\zeta^*}$ be such that*

$$s \leq r, \quad (\forall j < i < \delta)\ ((p_j,w_j) \leq' (p_i,w_i) \leq' (r,\{0,\zeta^*\})). \tag{3.6}$$

Assume that either r is $(N,\mathbb{P}_{\zeta^*})$-generic, or ζ^* is a limit ordinal of cofinality ${\rm cf}(\zeta^*) < \lambda$ and for every $\zeta < \zeta^*$ the condition $r \restriction \zeta$ is $(N,\mathbb{P}_\zeta)$-generic. Then there are conditions $s' \in N \cap \mathbb{P}_{\zeta^*}$ and $r' \in \mathbb{P}_{\zeta^*}$ such that $s \leq s' \leq r'$, $r \leq r'$ and $(\forall i < \delta)((p_i,w_i) \leq' (s',\{0,\zeta^*\}))$.

Proof. (1), (2) The proof is straightforward (use the λ-completeness of $\mathbb{P}_{\zeta^*}$).

(3) If r is $(N,\mathbb{P}_{\zeta^*})$-generic, then our assertion is clear (remember clause (2)). So suppose that we are in the second case (so $\aleph_0 \leq {\rm cf}(\zeta^*) < \lambda$). Let $\langle i_\gamma : \gamma < {\rm cf}(\zeta^*)\rangle \subseteq N \cap \zeta$ be a strictly increasing continuous sequence cofinal in ζ^*. For $\gamma < {\rm cf}(\zeta^*)$ and $i < \delta$ let $p^\gamma_i = p_i \restriction i_\gamma$, $w^\gamma_i = (w_i \cap i_\gamma) \cup \{i_\gamma\}$ (clearly $(p^\gamma_i,w^\gamma_i) \in \mathbb{P}^{\rm RS}_{i_\gamma}$). Since $r \restriction i_\gamma$ is $(N,\mathbb{P}_{i_\gamma})$-generic, we may inductively pick conditions s_γ, r_γ (for $\gamma < {\rm cf}(\zeta^*)$) such that

- $s \restriction i_\gamma \leq s_\gamma \in \mathbb{P}_{i_\gamma} \cap N$, $r \leq r_\gamma \in \mathbb{P}_{\zeta^*}$,
- $(\forall i < \delta)((p^\gamma_i,w^\gamma_i) \leq' (s_\gamma,\{0,i_\gamma\}))$, $s_\gamma \leq r_\gamma \restriction i_\gamma$,
- if $\beta < \gamma < {\rm cf}(\zeta^*)$ then $s \restriction i_\beta \leq s_\beta \leq s_\gamma$ and $r_\beta \leq r_\gamma$.

Let $r^* \in \mathbb{P}_{\zeta^*}$ be stronger than all r_γ's. Now apply Proposition 3.3. □

Now we may state and prove our main result.

Theorem 3.7. *Let D, $\mathcal{S}$, $\mathcal{S}'$ be as in Context 1.7. Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \zeta^*\rangle$ is a $(<\lambda^+)$-support iteration such that for each $\alpha < \zeta^*$*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underset{\sim}{\mathbb{Q}}_\alpha \text{ is proper for } D\text{-semi-diamonds”.} \tag{3.7}$$

Then $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ is proper for D-semi-diamonds.

Proof. By Proposition 3.1, the forcing notion $\mathbb{P}_{\zeta^*}$ is λ-complete, so we have to concentrate on showing clause Definition 2.3(1)(b) for it.

So suppose that χ is large enough, $p \in \mathbb{P}_{\zeta^*}$ and $N \prec (\mathcal{H}(\chi),\in,<^*_\chi)$, $\|N\| = \lambda$, $N^{<\lambda} \subseteq N$ and $\{\lambda,p,\bar{\mathbb{Q}},\mathbb{P}_{\zeta^*},D,\mathcal{S},\ldots\} \in N$, and $h : \lambda \to N$ satisfies $\mathbb{P}_{\zeta^*} \cap N \subseteq {\rm Rang}(h)$. Furthermore, suppose that $\bar{F} = \langle F_\delta : \delta \in \mathcal{S}\rangle$ is a $(D,\mathcal{S})$-semi-diamond and $\bar{q} = \langle q_\delta : \delta \in \mathcal{S}\rangle$ is an $(N,h,\mathbb{P}_{\zeta^*},\bar{F})$-candidate. We may assume that for each $\delta \in \mathcal{S}$

($\odot$) if $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is not a $\leq_{\mathbb{P}_{\zeta^*}}$-increasing sequence of members of $\mathbb{P}_{\zeta^*} \cap N$, then $h \circ F_\delta(\alpha) = *$ for all $\alpha < \delta$.

(Just suitably modify F_δ whenever the assumption of ($\odot$) holds—note that the modification does not change the notion of a candidate, the game from Definition 2.1(3), etc.)

Before we define a generic condition $r \in \mathbb{P}_{\zeta^*}$ for $\bar{q}$ over $\bar{F}$, we introduce the notation used later and give two important facts.

Let $i \in N \cap (\zeta^* + 1)$ and let $G_{\mathbb{P}_i} \subseteq \mathbb{P}_i$ be generic over $\mathbf{V}$. We define

- $h^{\langle i \rangle}[G_{\mathbb{P}_i}] : \lambda \to N[G_{\mathbb{P}_i}]$ is such that if $h(\gamma)$ is a function, $i \in \mathrm{Dom}(h(\gamma))$ and $(h(\gamma))(i)$ is a $\mathbb{P}_i$-name, then $(h^{\langle i \rangle}[G_{\mathbb{P}_i}])(\gamma) = (h(\gamma))(i)[G_{\mathbb{P}_i}]$, otherwise it is $*$;
- $h^{[i]} : \lambda \to N$ is defined by $h^{[i]}(\gamma) = (h(\gamma)) \restriction i$ provided $h(\gamma)$ is a function, and $*$ otherwise;
- $\mathcal{S}^{\langle i \rangle}[G_{\mathbb{P}_i}] = \{\delta \in \mathcal{S} : \text{if } \delta \text{ is limit, then } q_\delta \restriction i \in G_{\mathbb{P}_i}\}$;
- $\bar{q}^{\langle i \rangle}[G_{\mathbb{P}_i}]$ is $\langle q_\delta(i)[G_{\mathbb{P}_i}] : \delta \in \mathcal{S}^{\langle i \rangle}[G_{\mathbb{P}_i}]\rangle$;
- $\bar{q}^{[i]} = \langle q_\delta \restriction i : \delta \in \mathcal{S}\rangle$;
- $\bar{F}^{\langle i \rangle}[G_{\mathbb{P}_i}]$ is $\langle F_\delta : \delta \in \mathcal{S}^{\langle i \rangle}[G_{\mathbb{P}_i}]\rangle$.

Observe that $h^{[i]} : \lambda \to N$ is such that $\mathbb{P}_i \cap N \subseteq \mathrm{Rang}(h^{[i]})$ and $h^{\langle i \rangle}[G_{\mathbb{P}_i}]$ is such that $N[G_{\mathbb{P}_i}] \cap \underset{\sim}{\mathbb{Q}}_i[G_{\mathbb{P}_i}] \subseteq \mathrm{Rang}(h^{\langle i \rangle}[G_{\mathbb{P}_i}])$.

The following claim is an immediate consequence of $(\odot)$.

Claim 3.8. Assume that $i \in N \cap (\zeta^* + 1)$. Then $\bar{F}$ is a $(D, \mathcal{S})$-semi-diamond sequence for $(N, h^{[i]}, \mathbb{P}_i)$ and $\bar{q}^{[i]}$ is an $(N, h^{[i]}, \mathbb{P}_i, \bar{F})$-candidate.

Claim 3.9. Assume that $i \in N \cap (\zeta^* + 1)$ and $r \in \mathbb{P}_i$ is $(N, h^{[i]}, \mathbb{P}_i)$-generic for $q^{[i]}$ over $\bar{F}$. Let $G_{\mathbb{P}_i} \subseteq \mathbb{P}_i$ be a generic filter over $\mathbf{V}$, $r \in G_{\mathbb{P}_i}$. Then in $\mathbf{V}[G_{\mathbb{P}_i}]$:

(1) $\mathcal{S}^{\langle i \rangle}[G_{\mathbb{P}_i}] \in D^+$,
(2) $\bar{F}^{\langle i \rangle}[G_{\mathbb{P}_i}]$ is a $(D, \mathcal{S}^{\langle i \rangle}[G_{\mathbb{P}_i}])$-semi-diamond for $(N[G_{\mathbb{P}_i}], h^{\langle i \rangle}[G_{\mathbb{P}_i}], \underset{\sim}{\mathbb{Q}}_i[G_{\mathbb{P}_i}])$, and
(3) $\bar{q}^{\langle i \rangle}[G_{\mathbb{P}_i}]$ is an $(N[G_{\mathbb{P}_i}], h^{\langle i \rangle}[G_{\mathbb{P}_i}], \underset{\sim}{\mathbb{Q}}_i[G_{\mathbb{P}_i}], \bar{F}^{\langle i \rangle}[G_{\mathbb{P}_i}])$-candidate.

Proof of Claim 3.9. (1) Follows from (2).

(2) Assume that this fails. Then we can find a condition $r^* \in \mathbb{P}_i$, a $\mathbb{P}_i$-name $\underset{\sim}{\bar{q}}' = \langle \underset{\sim}{q}'_\alpha : \alpha < \lambda\rangle \subseteq N$ for an increasing sequence of conditions from $\underset{\sim}{\mathbb{Q}}_i$, and $\mathbb{P}_i$-names $\underset{\sim}{A}_\xi$ for members of $D \cap \mathbf{V}$ such that $r \le_{\mathbb{P}_i} r^* \in G_{\mathbb{P}_i}$ and

$$r^* \Vdash_{\mathbb{P}_i} \text{“}(\forall \delta \in \mathcal{S}^{\langle i \rangle} \cap \triangle_{\xi<\lambda} \underset{\sim}{A}_\xi)(\langle h^{\langle i \rangle} \circ F_\delta(\alpha) : \alpha < \delta\rangle \neq \underset{\sim}{\bar{q}}' \restriction \delta)\text{”}. \tag{3.8}$$

Consider a play $\langle r_j^-, r_j, C_j : i < \lambda\rangle \subseteq \mathbb{P}_i$ of the game $\eth(r, N, h^{[i]}, \mathbb{P}_i, \bar{F}, \bar{q}^{[i]})$ in which the generic player uses its winning strategy and the anti-generic player plays as follows. In addition to keeping the rules of the game, it makes sure that at stage $j \in \mathcal{S}'$:

- $r_j \ge r^*$ (so $r_0 \ge r^*$; remember the anti-generic player plays at 0),
- r_j decides the values of all $\underset{\sim}{A}_\xi$ for $\xi < j$.

Let $A_\xi \in D \cap \mathbf{V}$ be such that $r_j \Vdash$ “$\underset{\sim}{A}_\xi = A_\xi$” for sufficiently large $j \in \mathcal{S}'$.

Note that the sequence $\langle r_j^{-\frown}\langle \underset{\sim}{q}'_j\rangle : j < \delta\rangle$ is $\le_{\mathbb{P}_{i+1}}$-increasing. So, as D is normal and $A_\xi, C_j \in D$ and $\bar{F}$ is a semi-diamond for $(N, h^{[i+1]}, \mathbb{P}_{i+1})$ (by Claim 3.8), we may find a limit ordinal $\delta \in \mathcal{S} \cap \triangle_{\xi<\lambda} A_\xi \cap \triangle_{j<\lambda} C_j$ such that $\langle h^{[i+1]} \circ F_\delta(j) : j < \delta\rangle = \langle r_j^{-\frown}\langle \underset{\sim}{q}'_j\rangle : j < \delta\rangle$. Then also $\langle h^{[i]} \circ F_\delta(j) : j < \delta\rangle = \langle r_j^- : j < \delta\rangle$, and since the play is won by the generic player, we conclude that $q_\delta \restriction i \le r_\delta$. But then taking sufficiently large $j \in \mathcal{S}'$ we have

$$r_j \Vdash \text{“}\delta \in \mathcal{S}^{\langle i \rangle} \cap \triangle_{\xi<\lambda} \underset{\sim}{A}_\xi, \qquad \langle h^{\langle i \rangle} \circ F_\delta(\alpha) : \alpha < \delta\rangle = \underset{\sim}{\bar{q}}' \restriction \delta\text{”}, \tag{3.9}$$

a contradiction.

(3) Should be clear. □

Fix a bijection $\Upsilon : \zeta^* \cap N \to \gamma^* \le \lambda$. Also let $\langle(\underset{\sim}{\tau}_i, \zeta_i) : i < \lambda\rangle$ list all pairs $(\underset{\sim}{\tau}, \zeta) \in N$ such that $\zeta \le \zeta^*$, $\mathrm{cf}(\zeta) \ge \lambda$ and $\underset{\sim}{\tau}$ is a $\mathbb{P}_\zeta$-name for an ordinal.

Next, by induction, we choose a sequence $\langle(p_i, w_i) : i < \lambda\rangle \subseteq \mathbb{P}^{\mathrm{RS}}_{\zeta^*} \cap N$ such that

(i) $(p, \{0, \zeta^*\}) \le' (p_i, w_i) \le' (p_j, w_j)$, for $i < j < \lambda$,

(ii) if $i < j < \lambda$ and $\Upsilon(\varepsilon) \le i$, then $\varepsilon \in \mathrm{Dom}(p_i)$ and $p_i(\varepsilon) = p_j(\varepsilon)$,

(iii) if $i < \lambda$ is a limit ordinal, then w_i is the closure of $\bigcup_{j<i} w_j$, and if, additionally, $\varepsilon \in \mathrm{Dom}(q_i)$ is such that $\Upsilon(\varepsilon) \ge i$ (and $i \in \mathcal{S}$, of course), then $\varepsilon \in \mathrm{Dom}(p_i)$ and $p_i(\varepsilon)$ is such that

($\otimes$) for every generic $G_{\mathbb{P}_{\zeta^*}} \subseteq \mathbb{P}_{\zeta^*}$ over $\mathbf{V}$ such that $(p_i, w_i) \in' G_{\mathbb{P}_{\zeta^*}}$, and two successive members ε', ε'' of the set w_i such that $\varepsilon' \le \varepsilon < \varepsilon''$ we have that if $\{p_j(\varepsilon)[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\varepsilon'}][G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_\varepsilon] : j < i\} \cup \{q_i(\varepsilon)[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_\varepsilon]\}$ has an upper bound in $\underset{\sim}{\mathbb{Q}}_\varepsilon[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_\varepsilon]$, then $p_i(\varepsilon)[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\varepsilon'}][G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_\varepsilon]$ is such an upper bound,

(iv) for each $i < \lambda$, for some $\xi \in N \cap \zeta_i$ and a $\mathbb{P}_\xi$-name $\underset{\sim}{\tau} \in N$ we have that $\sup(\{\varepsilon < \zeta_i : \Upsilon(\varepsilon) \le i\} \cup (w_i \cap \zeta_i)) < \xi$, $w_{i+1} = w_i \cup \{\xi\}$, $p_{i+1} \restriction \xi = p_i \restriction \xi$ and if $G_{\mathbb{P}_{\zeta^*}} \subseteq \mathbb{P}_{\zeta^*}$ is generic over $\mathbf{V}$ and $(p_{i+1}, w_{i+1}) \in' G_{\mathbb{P}_{\zeta^*}}$, then $\underset{\sim}{\tau}_i[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_{\zeta_i}] = \underset{\sim}{\tau}[G_{\mathbb{P}_{\zeta^*}} \cap \mathbb{P}_\xi]$

(It should be clear that there are no problems in the induction and it is possible to pick (p_i, w_i) as above.) From now on we will treat each $p_i(\xi)$ as a $\mathbb{P}_\xi$-name for a member of $\underset{\sim}{\mathbb{Q}}_\xi$.

Now we are going to define an $(N, h, \mathbb{P}_{\zeta^*})$-generic condition $r \in \mathbb{P}$ for $\bar{q}$ over $\bar{F}$ in the most natural way. Its domain is $\mathrm{Dom}(r) = \zeta^* \cap N$ and for each $i \in \zeta^* \cap N$

$$r \restriction i \Vdash \text{“}r(i) \ge p_{\Upsilon(i)}(i) \text{ is } (N[\underset{\sim}{G}_{\mathbb{P}_i}], h^{\langle i\rangle}, \underset{\sim}{\mathbb{Q}}_i)\text{-generic for } \bar{q}^{\langle i\rangle} \text{ over } \bar{F}^{\langle i\rangle}\text{”}. \tag{3.10}$$

Main Claim 3.10. For every $\zeta \in (\zeta^* + 1) \cap N$, the generic player has a winning strategy in the game $\eth(r \restriction \zeta, N, h^{[\zeta]}, \mathbb{P}_\zeta, \bar{F}, \bar{q}^{[\zeta]})$.

Proof of Claim 3.10. We prove the claim by induction on $\zeta \in (\zeta^* + 1) \cap N$. For $\zeta \in \zeta^* \cap N$ this implies that $r(\zeta)$ is well defined (remember Claim 3.9). Of course for $\zeta = \zeta^*$ we finish the proof of the theorem.

Suppose that $\zeta \in (\zeta^* + 1) \cap N$ and we know that $r \restriction \zeta'$ is $(N, h^{[\zeta']}, \mathbb{P}_{\zeta'})$-generic for $\bar{q}^{[\zeta']}$ over $\bar{F}$ for all $\zeta' \in N \cap \zeta$. We are going to describe a winning strategy for the generic player in the game $\eth(r \restriction \zeta, N, h^{[\zeta]}, \mathbb{P}_\zeta, \bar{F}, \bar{q}^{[\zeta]})$. The inductive hypothesis is not used in the full strength in the definition of the strategy, but we need it in several places, for example, to know that r is well defined as well as that we have the $\underset{\sim}{\mathrm{st}}_i$'s below. Also note that it implies that $(p^\zeta_i, w^\zeta_i) \le' (r \restriction \zeta, \{0, \zeta\})$ for all $i < \lambda$, where $p^\zeta_i = p_i \restriction \zeta$ and $w^\zeta_i = (w_i \cap \zeta) \cup \{\zeta\}$. Moreover, during the play, both players will always have legal moves. Why? By the inductive hypothesis we know that $r \restriction \zeta'$ is $(N, \mathbb{P}_{\zeta'})$-generic for all $\zeta' \in \zeta \cap N$. Therefore, if ζ is a successor or a limit ordinal of cofinality $\ge \lambda$, then we immediately get that $r \restriction \zeta$ is $(N, \mathbb{P}_\zeta)$-generic (remember clause (iv) of the choice of the p_i's!), and thus Observation 2.2(4) applies. If ζ is a limit ordinal of cofinality $\mathrm{cf}(\zeta) < \lambda$, then we may use Proposition 3.3.

Let $\underset{\sim}{\mathrm{st}}_i$ be a $\mathbb{P}_i$-name for the winning strategy of the generic player in $\eth(r(i), N[\underset{\sim}{G}_{\mathbb{P}_i}], h^{\langle i\rangle}, \underset{\sim}{\mathbb{Q}}_i, \bar{F}^{\langle i\rangle}, \bar{q}^{\langle i\rangle})$, and let

$$E_0 \stackrel{\mathrm{def}}{=} \{\delta < \lambda : \delta \text{ is a limit of points from } \mathcal{S}'\}. \tag{3.11}$$

Plainly, E_0 is a club of λ.

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Let the generic player play as follows. As an aside, it will construct sequences $\langle \underset{\sim}{r}^{\ominus}_{j'}(\varepsilon), \underset{\sim}{r}^{\oplus}_{j'}(\varepsilon) : j' < \lambda,\ \varepsilon \in \zeta \cap N\rangle$ and $\langle \underset{\sim}{C}^{\xi}_{j'}(\varepsilon) : j', \xi < \lambda,\ \varepsilon \in \zeta \cap N\rangle$ so that

- $\underset{\sim}{r}^{\ominus}_{j'}(\varepsilon)$ is a $\mathbb{P}_\varepsilon$-name for a member of $\underset{\sim}{\mathbb{Q}}_\varepsilon \cap N[\underset{\sim}{G}_{\mathbb{P}_\varepsilon}]$, $\underset{\sim}{r}^{\oplus}_{j'}(\varepsilon)$ is a $\mathbb{P}_\varepsilon$-name for a member of $\underset{\sim}{\mathbb{Q}}_\varepsilon$, $\underset{\sim}{C}^{\xi}_{j'}(\varepsilon)$ is a $\mathbb{P}_\varepsilon$-name for a member of $D \cap \mathbf{V}$, and
- if $j \in \mathcal{S}$, $j' \le j$, and $\Upsilon(\varepsilon) \le j$, then after the jth move (which is a move of the generic player) the terms $\langle \underset{\sim}{C}^{\xi}_{j'}(\varepsilon) : \xi < \lambda\rangle$, $\underset{\sim}{r}^{\ominus}_{j'}(\varepsilon)$, and $\underset{\sim}{r}^{\oplus}_{j'}(\varepsilon)$ are defined.

So suppose that $j^* \in \mathcal{S}$ and $\langle r^-_j, r_j, C_j : j < j^*\rangle$ is the result of the play so far. To clearly describe the answer of the generic player we will consider two (only slightly different) cases in the order in which they appear in the game. (Remember (r_0, C_0) is chosen by the anti-generic player and that all successor moves are done by the generic player.)

Case 3.11. Consider ordinals j' such that for some $j_0, j_1 \in \mathcal{S}'$ we have

$$j_0 < j' < \min\left(\mathcal{S}' \setminus (j_0+1)\right) = j_1. \tag{3.12}$$

First the generic player picks conditions $s^-, s \in \mathbb{P}_\zeta$, $s^- \in N$ such that $r^-_{j_0} \le s^- \le s$, $r_{j_0} \le s$ and for each $\xi \in \zeta \cap N$ we have

$$s^- \restriction \xi \Vdash \text{“}(\forall i < j_0)(p_i(\xi) \le s^-(\xi))\text{”}. \tag{3.13}$$

(Why possible? By Proposition 3.6(3).)

Now the generic player looks at $\varepsilon_\gamma < \zeta$ such that $\Upsilon(\varepsilon_\gamma) = \gamma < j_1$. It picks $\mathbb{P}_{\varepsilon_\gamma}$-names $\underset{\sim}{r}^{\ominus}_{j'}(\varepsilon_\gamma)$, $\underset{\sim}{r}^{\oplus}_{j'}(\varepsilon_\gamma)$, $\underset{\sim}{C}^{\xi}_{j'}(\varepsilon_\gamma)$ so that $s \restriction \varepsilon_\gamma$ forces that

$$\langle \underset{\sim}{r}^{\ominus}_{j'}(\varepsilon_\gamma), \underset{\sim}{r}^{\oplus}_{j'}(\varepsilon_\gamma), \triangle_{\xi<\lambda}\underset{\sim}{C}^{\xi}_{j'}(\varepsilon_\gamma) : j' < j_1\rangle \tag{3.14}$$

is a play according to $\underset{\sim}{\mathrm{st}}_{\varepsilon_\gamma}$ in which the moves of the anti-generic player are determined as follows. First, it keeps the convention that if $j' \in \mathcal{S} \setminus \mathcal{S}^{\langle \varepsilon_\gamma\rangle}$, then $(\underset{\sim}{r}^{\ominus}_{j'}(\varepsilon_\gamma), \underset{\sim}{r}^{\oplus}_{j'}(\varepsilon_\gamma), \triangle_{\xi<\lambda}\underset{\sim}{C}^{\xi}_{j'}(\varepsilon_\gamma))$ is (a name for) the $<^*_\chi$-first legal answer to the play so far. Now, if $\gamma < j_0$, then we have already the play up to j_0 (it easily follows from the inductive construction that $s \restriction \varepsilon_\gamma$ indeed forces that it is a “legal” play). The j_0th move of the anti-generic player is stipulated as $\underset{\sim}{r}^{\ominus}_{j_0}(\varepsilon_\gamma) = s^-(\varepsilon_\gamma)$, $\underset{\sim}{r}^{\oplus}_{j_0}(\varepsilon_\gamma) = s(\varepsilon_\gamma)$, $\underset{\sim}{C}^{\xi}_{j_0}(\varepsilon_\gamma) = \bigcap_{j \le j_0} C_j$, and next we continue up to j_1 keeping our convention. If $j_0 \le \gamma < j_1$, then the generic player lets $\underset{\sim}{r}^{\ominus}_0(\varepsilon_\gamma) = s^-(\varepsilon_\gamma), \underset{\sim}{r}^{\oplus}_0(\varepsilon_\gamma) = s(\varepsilon_\gamma)$, $\underset{\sim}{C}^{\xi}_0(\varepsilon_\gamma) = \bigcap_{j \le j_0} C_j$ and then it “plays” the game according to $\underset{\sim}{\mathrm{st}}_{\varepsilon_\gamma}$ up to j_1 keeping our convention for all $j' \notin \mathcal{S}^{\langle \varepsilon_\gamma\rangle}$.

Next, the generic player picks a condition $r^* \in \mathbb{P}_\zeta$ and $\mathbb{P}_{\varepsilon_\gamma}$-names $\underset{\sim}{\tau}_{j'}(\varepsilon_\gamma) \in N$ (for $\gamma < j_1$, $\varepsilon_\gamma < \zeta$, $j_0 < j' < j_1$) such that

- $r^* \ge s$, and for every γ, $j' < j_1$ we have

$$r^* \restriction \varepsilon_\gamma \Vdash_{\mathbb{P}_{\varepsilon_\gamma}} \quad \text{“}\underset{\sim}{r}^{\oplus}_{j'}(\varepsilon_\gamma) \le r^*(\varepsilon_\gamma), \qquad \underset{\sim}{r}^{\ominus}_{j'}(\varepsilon_\gamma) = \underset{\sim}{\tau}_{j'}(\varepsilon_\gamma)\text{”}, \tag{3.15}$$

- for every j', $\xi < j_1$ and $\gamma < j_1$ with $\varepsilon_\gamma < \zeta$, the condition $r^* \restriction \varepsilon_\gamma$ decides the value of $\underset{\sim}{C}^{\xi}_{j'}(\varepsilon_\gamma)$, and

$$r^* \restriction \varepsilon_\gamma \Vdash \text{“}\underset{\sim}{C}^{\xi}_{j'}(\varepsilon_\gamma) \setminus (\xi+1) = C^{\xi}_{j'}(\varepsilon_\gamma)\text{”}, \tag{3.16}$$

  where $C^{\xi}_{j'}(\varepsilon_\gamma) \in D \cap \mathbf{V}$.

Then it lets $r^-_{j'} \in N \cap \mathbb{P}_\zeta$ (for $j' \in (j_0, j_1)$) be conditions such that

$$\mathrm{Dom}\,(r^-_{j'}) = \mathrm{Dom}\,(s^-) \cup \{\varepsilon_\gamma : \gamma < j_1\ \varepsilon_\gamma < \zeta\}, \tag{3.17}$$

and for $\xi \in \mathrm{Dom}(r^-_{j'})$

$$r^-_{j'} \restriction \xi \Vdash \begin{array}{l}\text{“if } \Upsilon(\xi) < j_1 \text{ and } \langle \underset{\sim}{\tau}_j(\xi) : j_0 < j < j_1\rangle \text{ is an increasing} \\ \text{sequence of conditions stronger than } s^-(\xi), \\ \text{then } r^-_{j'}(\xi) = \underset{\sim}{\tau}_{j'}(\xi), \text{ otherwise } r^-_{j'}(\xi) = s^-(\xi)\text{”.}\end{array} \tag{3.18}$$

Finally, for $j' \in (j_0, j_1)$ it plays $r^-_{j'}, r^*, \bigcap\{C^\xi_{j'}(\varepsilon_\gamma) : j', \gamma, \xi < j_1,\ \varepsilon_\gamma < \zeta\} \cap E_0$.

Case 3.12. Consider ordinals j' such that for some $j_0 \in \mathcal{S}$ and $j_1 \in \mathcal{S}'$ we have

$$\sup\{i \in \mathcal{S}' : i < j'\} = j_0 \le j' < \min\,(\mathcal{S}' \setminus j_0) = j_1. \tag{3.19}$$

The generic player proceeds as above, the difference is that now j_0 “belongs to” the generic player, and that it is a limit of moves of the anti-generic player. Again, we look at $\varepsilon_\gamma < \zeta$ such that $\Upsilon(\varepsilon_\gamma) = \gamma < j_1$.

If $\gamma < j_0$, then every condition in $\mathbb{P}_{\varepsilon_\gamma}$ stronger than all $r_j \restriction \varepsilon_\gamma$ (for $j < j_0$) forces that

$$\langle \underset{\sim}{r}^\ominus_{j'}(\varepsilon_\gamma), \underset{\sim}{r}^\oplus_{j'}(\varepsilon_\gamma), \triangle_{\xi<\lambda} \underset{\sim}{C}^\xi_{j'}(\varepsilon_\gamma) : j' < j_0\rangle \tag{3.20}$$

is a legal play in which the generic player uses $\underset{\sim}{\mathrm{st}}_{\varepsilon_\gamma}$. The generic player determines $\underset{\sim}{r}^\ominus_{j'}(\varepsilon_\gamma)$, $\underset{\sim}{r}^\oplus_{j'}(\varepsilon_\gamma)$, and $\underset{\sim}{C}^\xi_{j'}(\varepsilon_\gamma)$ for $j' \in [j_0, j_1)$ “playing the game” as earlier (with the same convention that if $j' \in \mathcal{S} \setminus \mathcal{S}^{\langle \varepsilon_\gamma\rangle}$, then the j'th move of the anti-generic player is stipulated as the $<^*_\chi$-first legal move).

If $j_0 \le \gamma < j_1$, then (any condition stronger than all $r_j \restriction \varepsilon_\gamma$ for $j < j_0$ forces that) $\langle r^-_j(\varepsilon) : j < j_0\rangle$, $\langle r_j(\varepsilon) : j < j_0\rangle$ are increasing, and $r^-_j(\varepsilon_\gamma) \le r_j(\varepsilon_\gamma)$ and $r(\varepsilon_\gamma)$ is $(N[\underset{\sim}{G}_{\mathbb{P}_{\varepsilon_\gamma}}], \underset{\sim}{\mathbb{Q}}_{\varepsilon_\gamma})$-generic. So, by Observation 2.2(4), the generic player may let $(r^\ominus_0(\varepsilon_\gamma)$, $r^\oplus_0(\varepsilon_\gamma))$ be the $<^*_\chi$-first such that for all $j < j_0$ we have $r^-_j(\varepsilon) \le r^\ominus_0(\varepsilon_\gamma) \in N[\underset{\sim}{G}_{\mathbb{P}_{\varepsilon_\gamma}}]$, $r_j(\varepsilon_\gamma) \le r^\oplus_0(\varepsilon_\gamma)$. It also lets $C^\xi_0(\varepsilon_\gamma) = \bigcap_{j<j_0} C_j$. Then the generic player chooses $\underset{\sim}{r}^\ominus_{j'}(\varepsilon_\gamma)$, $\underset{\sim}{r}^\oplus_{j'}(\varepsilon_\gamma)$, and $\underset{\sim}{C}^\xi_{j'}(\varepsilon_\gamma)$ for $0 < j' < j_1$ “playing the game” with the strategy $\underset{\sim}{\mathrm{st}}_{\varepsilon_\gamma}$ (and keeping the old convention for $j' \notin \mathcal{S}^{\langle\varepsilon_\gamma\rangle}$).

Next the generic player picks a condition $r^* \in \mathbb{P}_\zeta$ (stronger than all r_j for $j < j_0$), $\mathbb{P}_{\varepsilon_\gamma}$-names $\underset{\sim}{\tau}_{j'}(\varepsilon_\gamma) \in N$ and sets $C^\xi_{j'}(\varepsilon_\gamma) \in D \cap \mathbf{V}$ (for $j', \gamma, \xi < j_1$) as in the previous case. Then it chooses conditions $s^- \in N \cap \mathbb{P}_\zeta$ and $r^+ \in \mathbb{P}_\zeta$ such that $r^* \le r^+$ and $(\forall j < j_0)(r^-_j \le s^- \le r^+)$. (Why possible? If ζ is limit of cofinality $\mathrm{cf}(\xi) < \lambda$, use Proposition 3.3; otherwise we know that r is $(N, \mathbb{P}_\zeta)$-generic.) Next it defines conditions $r^-_{j'} \in N \cap \mathbb{P}_\zeta$ (for $j_0 \le j' < j_1$) so that

$$\mathrm{Dom}\,(r^-_{j'}) = \mathrm{Dom}(s^-) \cup \{\varepsilon_\gamma : \gamma < j_1 \text{ and } \varepsilon_\gamma < \zeta\}, \tag{3.21}$$

and for $\xi \in \mathrm{Dom}(r^-_{j'})$

$$r^-_{j'} \restriction \xi \Vdash \begin{array}{l}\text{“if } \Upsilon(\xi) < j_1 \text{ and } \langle \underset{\sim}{\tau}_j(\xi) : j_0 \le j < j_1\rangle \text{ is an increasing} \\ \text{sequence of conditions above all } r^-_j(\xi) \text{ for } j < j_0, \\ \text{then } r^-_{j'}(\xi) = \underset{\sim}{\tau}_{j'}(\xi), \text{ otherwise } r^-_{j'}(\xi) = s^-(\xi)\text{”.}\end{array} \tag{3.22}$$

Finally, for $j_0 \le j' < j_1$ it plays $r^-_{j'}, r^+, \bigcap\{C^\xi_{j'}(\varepsilon_\gamma) : j', \gamma, \xi < j_1,\ \varepsilon_\gamma < \zeta\}$.

Why does the strategy described above work? Suppose that $\langle r^-_j, r_j, C_j : j < \lambda\rangle$ is a play of the game $\Game(r \restriction \zeta, N, h^{[\zeta]}, \mathbb{P}_\zeta, \bar{F}, \bar{q}^{[\zeta]})$ in which the generic player used this strategy and let $\langle \underset{\sim}{r}'_{j'}(\varepsilon) : j' < \lambda,\ \varepsilon \in \zeta \cap N\rangle$ and $\langle \underset{\sim}{C}^{\xi}_{j'}(\varepsilon) : j', \xi < \lambda,\ \varepsilon \in \zeta \cap N\rangle$ be the sequences it constructed aside. (As we said earlier, the game surely lasted λ steps and thus the sequences described above have length λ.)

We argue that condition Definition 2.1(3)($\circledast$) holds.

Assume that a limit ordinal $\delta \in \mathcal{S} \cap \bigcap_{j<\delta} C_j$ (so in particular $\delta \in E_0$) is such that $(*)_\delta$ $\langle h^{[\zeta]} \circ F_\delta(\alpha) : \alpha < \delta\rangle = \langle r^-_\alpha : \alpha < \delta\rangle$.

We are going to show that $q_\delta \le r_\delta$ and for this we prove by induction on $\varepsilon \in (\zeta + 1) \cap N$ that $q_\delta \restriction \varepsilon \le r_\delta \restriction \varepsilon$. For $\varepsilon = \zeta$ this is the desired conclusion.

For $\varepsilon = 0$ this is trivial, and for a limit ε it follows from the definition of the order (and the inductive hypothesis).

So assume that we have proved $q_\delta \restriction \varepsilon \le r_\delta \restriction \varepsilon$, $\varepsilon < \zeta$, and consider the restrictions to $\varepsilon + 1$. If $\Upsilon(\varepsilon) \ge \delta$ then by the choice of conditions s, s^- in Case 3.11, we know that

$$r_\delta \restriction \varepsilon \Vdash \text{“}(\forall i < \delta)(\exists j' < \delta)(p_i(\varepsilon) \le r^-_{j'}(\varepsilon))\text{”}. \tag{3.23}$$

Now look at the clause (iii) of the choice of the p_δ at the beginning: what we have just stated (and $(*)_\delta$) implies that

$$r_\delta \restriction \varepsilon \Vdash \text{“}p_\delta(\varepsilon) \text{ is an upper bound to } \{q_\delta(\varepsilon)\} \cup \{p_i(\varepsilon) : i < \delta\}\text{”}. \tag{3.24}$$

Thus, $r_\delta \restriction \varepsilon \Vdash$ “$q_\delta(\varepsilon) \le p_\delta(\varepsilon) \le r_\delta(\varepsilon)$”, so we are done. Suppose now that $\Upsilon(\varepsilon) < \delta$ and let $j_1 = \min(\mathcal{S}' \setminus \delta)$. Look at what the generic player has written aside: $r_\delta \restriction \varepsilon$ forces that $\langle r^{\ominus}_{j'}(\varepsilon), r^{\oplus}_{j'}(\varepsilon), \triangle_{\xi<\lambda} \underset{\sim}{C}^{\xi}_{j}(\varepsilon) : j < j_1\rangle$ is a play according to $\underset{\sim}{\mathrm{st}}_\varepsilon$ and $\delta \in \bigcap_{j,\xi<\delta} \underset{\sim}{C}^{\xi}_{j}(\varepsilon) \cap \mathcal{S}^{\langle\varepsilon\rangle}$, so we are clearly done in this case too (remember the choice of r^*), and the proof of Claim 3.10 is completed. □

Applying Main Claim 3.10 to $\zeta = \zeta^*$ we may conclude the proof of Theorem 3.7. □

Remark 3.13. Note that if the iterands $\underset{\sim}{\mathbb{Q}}_\xi$ are (forced to be) λ-lub-closed, then the proof of Theorem 3.7 substantially simplifies.

4. Examples. Our first example of a proper over λ forcing notion is a relative of the forcing introduced by Baumgartner for adding a club to $\aleph_1$. Its variants were also studied in Abraham and Shelah [1]; (see also [15, Chapter III]).

The forcing notion $\mathbb{P}^*$ is defined as follows:

a condition in $\mathbb{P}^*$ is a function p such that

(a) $\mathrm{Dom}(p) \subseteq \lambda^+$, $\mathrm{Rang}(p) \subseteq \lambda^+$, $|\mathrm{Dom}(p)| < \lambda$, and

(b) if $\alpha_1 < \alpha_2$ are both from $\mathrm{Dom}(p)$, then $p(\alpha_1) < \alpha_2$;

the order $\le$ *of* $\mathbb{P}^*$ is the inclusion $\subseteq$.

Clearly, we have the following result.

Proposition 4.1. *The forcing notion $\mathbb{P}^*$ is λ-lub-complete and $|\mathbb{P}^*| = \lambda^+$.*

But also, the following holds true.

Proposition 4.2. *The forcing notion $\mathbb{P}^*$ is proper over λ.*

Proof. Assume that $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is as in Definition 2.1, and $p \in \mathbb{P}^* \cap N$. Put $j^* = N \cap \lambda^+$ and $r = p \cup \{\langle j^*, j^*\rangle\}$.

Claim 4.3. (1) If $r' \in \mathbb{P}$, $r \leq r'$, then $r' \restriction j^* \in N \cap \mathbb{P}^*$ and $r' \restriction j^* \leq r'$.

(2) If $r' \in \mathbb{P}$, $r \leq r'$, and $r' \restriction j^* \leq r'' \in N \cap \mathbb{P}^*$, then $r' \cup r'' \in \mathbb{P}^*$ is stronger than both r' and r''.

(3) If $\bar{p} = \langle p_\xi : \xi < \zeta^*\rangle \subseteq \mathbb{P}^*$ is $\leq$-increasing and $\zeta^* < \lambda$, then $\bar{p}$ has a least upper bound $q \in \mathbb{P}^*$, and $q \restriction j^*$ is a least upper bound of $\langle p_\xi \restriction j^* : \xi < \zeta^*\rangle$.

Proof of Claim 4.3. (1) By the definition of $\mathbb{P}^*$,

$$\alpha \in \mathrm{Dom}(r') \cap j^* \Longrightarrow r'(\alpha) < j^* \Longrightarrow r'(\alpha) \in N. \tag{4.1}$$

(2), (3) should be clear. $\square$

Claim 4.4. The condition r is N-generic for semi-diamonds (see Definition 2.7).

Proof of Claim 4.4. Suppose that D is a normal filter on λ, $\mathcal{S} \in D^+$. Let $h : \lambda \to N$ be such that $N \cap \mathbb{P}^* \subseteq \mathrm{Rang}(h)$, $\bar{F} = \langle F_\delta : \delta \in \mathcal{S}\rangle$ be a $(D, \mathcal{S})$-semi-diamond, and let $\bar{q} = \langle q_\delta : \delta \in \mathcal{S}\rangle$ be an $(N, h, \mathbb{P}^*, \bar{F})$-candidate.

We have to show that the condition r is $(N, h, \mathbb{P}^*)$-generic for $\bar{q}$ over $\bar{F}$, and for this we have to show that the generic player has a winning strategy in the game $\Game(r, N, h, \mathbb{P}^*, \bar{F}, \bar{q})$. Note that the set

$$E_0 \stackrel{\mathrm{def}}{=} \{\delta < \lambda : \delta \text{ is a limit of members of } \mathcal{S}\} \tag{4.2}$$

is a club of λ (so $E_0 \in D$). Now, the strategy that works for the generic player is the following one:

At stage $\delta \in \mathcal{S}$ of the play, when a sequence $\langle r_i^-, r_i, C_i : i < \delta\rangle$ has been already constructed, the generic player lets $C_\delta = E_0 \setminus (\delta + 1)$ and it asks:

$(*)$ Is there a common upper bound to $\{r_i : i < \delta\} \cup \{q_\delta\}$?

If the answer to $(*)$ is "yes," then the generic player puts $Y = \{r_i : i < \delta\} \cup \{q_\delta\}$; otherwise it lets $Y = \{r_i : i < \delta\}$. Now it chooses r_δ to be the $<^*_\chi$-first element of $\mathbb{P}^*$ stronger than all members of Y and $r_\delta^- = r_\delta \restriction j^* \in N$.

Why the strategy described above is the winning one? Let $\langle r_i^-, r_i, C_i : i < \lambda\rangle$ be a play according to this strategy. Suppose that $\delta \in \mathcal{S} \cap \triangle_{i<\lambda} C_i$ is a limit ordinal such that $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle = \langle r_\alpha^- : \alpha < \delta\rangle$. So, q_δ is stronger than all r_α^- (for $\alpha < \delta$), and for cofinally many $\alpha < \delta$ we have $r_\alpha^- = r_\alpha \restriction j^*$. Therefore, $q_\delta \geq r_\alpha \restriction j^*$ and (by Claim 4.3) $\{r_\alpha : \alpha < \delta\} \cup \{q_\delta\}$ has an upper bound. Now look at the choice of r_δ. $\square$

The proposition follows immediately from Claim 4.4. $\square$

Proposition 4.5. (1) *$\mathbb{P}^*$ is α-proper over λ if and only if $\alpha < \lambda$.*

(2) *If D is a normal filter on λ, $\mathcal{S} \in D^+$, and $\bar{F}$ is a $(D, \mathcal{S})$-diamond, $0 < \alpha < \lambda^+$, then $\mathbb{P}^* \in K_D^{\alpha,s}[\bar{F}]$ if and only if $\alpha < \lambda$.*

Proof. (1) Follows from (2).

(2) Assume $\alpha < \lambda$.

Let $\bar{N} = \langle N_\beta : \beta < \alpha\rangle$, $h_\beta : \lambda \to N_\beta$ and $\bar{q}^\beta$ be as in Definition 2.8(1)(b), $p \in \mathbb{P}^* \cap N_0$. Let $j_\beta^* = N_\beta \cap \lambda^+$ (for $\beta < \alpha$) and put $r = p \cup \{(j_\beta^*, j_\beta^*) : \beta < \alpha\}$. Clearly, $r \in \mathbb{P}^*$ and

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$r \restriction j^*_\beta \in N_\beta$ for each $\beta < \alpha$ (remember $\bar{N} \restriction \beta \in N_\beta$). By the proof of Proposition 4.2, the condition $r \restriction j^*_{\beta+1}$ is $(N_\beta, h_\beta, \mathbb{P}^*)$-generic for $\bar{q}^\beta$ over $\bar{F}$.

To show that $\mathbb{P}^* \notin K^{\alpha,s}_D[\bar{F}]$ for $\alpha \geq \lambda$, it is enough to do this for $\alpha = \lambda$. So, pick any $\bar{N} = \langle N_\beta : \beta < \lambda \rangle$, $h_\beta : \lambda \to N_\beta$ and $\bar{q}^\beta$ as in Definition 2.8(1)(b), and let $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$.

Let $\underset{\sim}{\varphi}$ be a $\mathbb{P}^*$-name for the generic partial function from λ^+ to λ^+, that is, $\Vdash_{\mathbb{P}^*} \underset{\sim}{\varphi} = \bigcup \underset{\sim}{G}_{\mathbb{P}^*}$. We claim that

($\circledast$) $\Vdash_{\mathbb{P}^*}$ "$(\exists \beta < \lambda)(\exists i \in \mathrm{Dom}(\underset{\sim}{\varphi}) \cap N_\beta)(\underset{\sim}{\varphi}(i) \notin N_\beta)$".

Why? Let $p \in \mathbb{P}^*$. Take $\beta_0 < \lambda$ such that $\mathrm{Dom}(p) \cap N_\lambda \subseteq N_{\beta_0}$ (remember $|p| < \lambda$). If for some $i \in \mathrm{Dom}(p) \cap N_{\beta_0}$ we have $p(i) \notin N_{\beta_0}$, then

$$p \Vdash \text{"}(\exists i \in \mathrm{Dom}(\underset{\sim}{\varphi}) \cap N_{\beta_0})(\underset{\sim}{\varphi}(i) \notin N_{\beta_0})\text{"}. \tag{4.3}$$

Otherwise, we let $\delta^* = N_{\beta_0} \cap \lambda^+$ and $\delta^{**} = N_{\beta_0+1} \cap \lambda^+$, and we put $q = p \cup \{(\delta^*, \delta^{**})\}$. Then clearly $q \in \mathbb{P}^*$ is a condition stronger than p and

$$q \Vdash \text{"}(\exists i \in \mathrm{Dom}(\underset{\sim}{\varphi}) \cap N_{\beta_0+1})(\underset{\sim}{\varphi}(i) \notin N_{\beta_0+1}). \tag{4.4}$$

It should be clear that ($\circledast$) implies that there is no condition $r \in \mathbb{P}^*$ which is $(N_\beta, h_\beta, \mathbb{P}^*)$-generic for $\bar{q}^\beta$ for all $\beta < \alpha$ (remember Proposition 2.6(2)). □

For the second example we assume the following.

Context 4.6. (a) Assume λ, D, $\mathcal{S}$, $\mathcal{S}'$ are as in Context 1.7,

(b) $\mathcal{S}^* \subseteq \mathcal{S}^{\lambda^+}_\lambda \stackrel{\text{def}}{=} \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$,

(c) $\langle A_\delta, h_\delta : \delta \in \mathcal{S}^* \rangle$ is such that for each $\delta \in \mathcal{S}^*$ we have:

(d) $A_\delta \subseteq \delta$, $\mathrm{otp}(A_\delta) = \lambda$ and A_δ is a club of δ, and

(e) $h_\delta : A_\delta \to \lambda$.

The forcing notion $\mathbb{Q}^*$ is defined as follows:

a condition in $\mathbb{Q}^*$ is a tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ such that

(a) $u^p \in [\lambda^+]^{<\lambda}$, $v^p \in [\mathcal{S}^*]^{<\lambda}$,

(b) $\bar{e}^p = \langle e^p_\delta : \delta \in v^p \rangle$, where each e^p_δ is a closed bounded subset of A_δ, and $e^p_\delta \subseteq u^p$,

(c) if $\delta_1 < \delta_2$ are from v^p, then

$$\sup(e_{\delta_2}) > \delta_1, \qquad \sup(e_{\delta_1}) > \sup(A_{\delta_2} \cap \delta_1), \tag{4.5}$$

(d) $h^p : u^p \to \lambda$ is such that for each $\delta \in v^p$ we have

$$h^p \restriction \{\alpha \in e_\delta : \mathrm{otp}(\alpha \cap e_\delta) \in \mathcal{S}'\} \subseteq h_\delta; \tag{4.6}$$

the order $\leq$ *of* $\mathbb{Q}^*$ is such that $p \leq q$ if and only if $u^p \subseteq u^q$, $h^p \subseteq h^q$, $v^p \subseteq v^q$, and for each $\delta \in v^p$ the set e^q_δ is an end-extension of e^p_δ.

A tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ satisfying clauses (a), (b), and (d) above will be called *a pre-condition*. Note that every pre-condition can be extended to a condition in $\mathbb{Q}^*$.

One easily verifies the following proposition.

Proposition 4.7. *The forcing notion* $\mathbb{Q}^*$ *is* λ*-lub-complete. Also* $\mathbb{Q}^*$ *satisfies the* λ^+*-chain condition.*

Proposition 4.8. *The forcing notion $\mathbb{Q}^*$ is proper over λ.*

Proof. Assume that $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is as in Definition 2.1, $\langle A_\delta, h_\delta : \delta \in \mathcal{S}^* \rangle \in N$ and $p \in \mathbb{Q}^* \cap N$. We are going to show that the condition p is N-generic for semi-diamonds.

So suppose that h, $\bar{q}$ and $\bar{F}$ are as in Definition 2.1. For $r \in \mathbb{Q}^*$, let $r \restriction N$ be such that $u^{r\restriction N} = u \cap N$, $v^{r\restriction N} = v \cap N$, $\bar{e}^{r\restriction N} = \bar{e}^r \restriction N$, $h^{r\restriction N} = h^r \restriction N$. Note that $r \restriction N \in N$. We describe the winning strategy of the generic player in the game $\eth(p, N, h, \mathbb{P}^*, \bar{F}, \bar{q})$. For this we first fix a list $\{j_i : i < \lambda\}$ of $N \cap \mathcal{S}^*$, and we let $E_0 = \{\delta < \lambda : \delta$ is a limit of members of $\mathcal{S}\}$.

Suppose that we arrive to a stage $\delta \in \mathcal{S}$ and $\langle r^-_i, r_i, C_i : i < \delta \rangle$ is the sequence played so far. The generic player first picks a condition r'_δ stronger than all r_i's played so far and, if possible, stronger than q_δ. Then it plays a condition r_δ above r'_δ such that

- if $\gamma \in v^{r_\delta}$, then $\mathrm{otp}(e^{r_\delta}_\gamma) > \delta$, and
- $\{j_i : i < \delta\} \subseteq v^{r_\delta}$,

and $r^-_\delta = r_\delta \restriction N$. The set C_δ played at this stage is $[\alpha, \lambda) \cap E_0$, where α is the first ordinal such that $v^{r_\delta} \cap N \subseteq \{j_i : i < \alpha\}$ and $\mathrm{otp}(A_\gamma \cap (\max(e^{r_\delta}_\gamma) + 1)) < \alpha$ for all $\gamma \in v^{r_\delta}$.

Why is this a winning strategy? Let $\langle r^-_i, r_i, C_i : i < \lambda \rangle$ be a play according to this strategy, and suppose that $\delta \in \mathcal{S} \cap \triangle_{i<\lambda} C_i$ is a limit ordinal such that

$$\langle h \circ F_\delta(\alpha) : \alpha < \delta \rangle = \langle r^-_\alpha : \alpha < \delta \rangle. \tag{4.7}$$

Note that then

(i) if $\gamma \in \bigcup_{i<\delta} v^{r_i}$ then $\bigcup_{i<\delta} e^{r_i}_\gamma$ is an unbounded subset of $\{\varepsilon \in A_\gamma : \mathrm{otp}(\varepsilon \cap A_\gamma) < \delta\}$, and

(ii) $\bigcup_{i<\delta} v^{r_i} \cap N = \{j_i : i < \delta\}$.

We want to show that there is a common upper bound to $\{r_i : i < \delta\} \cup \{q_\delta\}$ (which, by the definition of our strategy, will finish the proof). First we choose a pre-condition $r = (u^r, v^r, \bar{e}^r, h^r)$ such that

- $v^r = v^{q_\delta} \cup \bigcup_{i<\delta} v^{r_i}$,
- if $\gamma \in v^{q_\delta}$, then we let $e^r_\gamma = e^{q_\delta}_\gamma$, if $\gamma \in \bigcup_{i<\delta} v^{r_i} \setminus v^{q_\delta}$, then
$$e^r_\gamma = \bigcup \{e^{r_i}_\gamma : i < \delta,\ \gamma \in v^{r_i}\} \cup \{\text{the } \delta\text{th member of } A_\gamma\}, \tag{4.8}$$
- $u^r = u^{q_\delta} \cup \bigcup_{i<\delta} u^{r_i} \cup \{\text{the } \delta\text{th member of } A_\gamma : \gamma \in v^r \setminus v^{q_\delta}\}$,
- $h^r \supseteq h^{q_\delta} \cup \bigcup_{i<\delta} h^{r_i}$.

Why is the choice possible? As $\delta \notin \mathcal{S}'$! Now we may extend r to a condition in $\mathbb{Q}^*$ picking for each $\gamma \in v^r$ large enough $\beta_\gamma \in A_\gamma$ and adding β_γ to e^r_γ (and extending u^r, h^r suitably). □

Our last example is a natural generalization of the forcing notion $\mathbb{D}_\omega$ from Newelski and Rosłanowski [3]. We work in the context of Context 1.7.

Definition 4.9. (1) A set $T \subseteq {}^{<\lambda}\lambda$ is a complete λ-tree if

(a) $(\forall \eta \in T)(\exists \nu \in T)(\eta \vartriangleleft \nu)$, and T has the $\vartriangleleft$-smallest element called $\mathrm{root}(T)$,

(b) $(\forall \eta, \nu \in {}^{<\lambda}\lambda)(\mathrm{root}(T) \trianglelefteq \eta \vartriangleleft \nu \in T \Rightarrow \eta \in T)$,

(c) if $\langle \eta_i : i < \delta \rangle \subseteq T$ is a $\vartriangleleft$-increasing chain, $\delta < \lambda$, then there is $\eta \in T$ such that $\eta_i \vartriangleleft \eta$ for all $i < \delta$.

Let $T \subseteq {}^{<\lambda}\lambda$ be a complete λ-tree.

(2) For $\eta \in T$ we let $\mathrm{succ}_T(\eta) = \{\alpha < \lambda : \eta^\frown\langle\alpha\rangle \in T\}$.

(3) We let $\mathrm{split}(T) = \{\eta \in T : |\mathrm{succ}_T(\eta)| > 1\}$.

(4) A sequence $\rho \in {}^\lambda\lambda$ is a *λ-branch through T* if

$$(\forall \alpha < \lambda)(\mathrm{lh}(\mathrm{root}(T)) \leq \alpha \Rightarrow \rho \restriction \alpha \in T). \tag{4.9}$$

The set of all λ-branches through T is called $\lim_\lambda(T)$.

(5) A subset $\mathbf{F}$ of the λ-tree T is *a front* in T if no two distinct members of $\mathbf{F}$ are $\triangleleft$-comparable and

$$(\forall \eta \in \lim_\lambda(T))(\exists \alpha < \lambda)(\eta \restriction \alpha \in \mathbf{F}). \tag{4.10}$$

(6) For $\eta \in T$ we let $(T)^{[\eta]} = \{\nu \in T : \eta \trianglelefteq \nu\}$.

Now we define a forcing notion $\mathbb{D}_\lambda$:

A condition in $\mathbb{D}_\lambda$ is a complete λ-tree T such that

(a) $\mathrm{root}(T) \in \mathrm{split}(T)$ and $(\forall \eta \in \mathrm{split}(T))(\mathrm{succ}_T(\eta) = \lambda)$,

(b) $(\forall \eta \in T)(\exists \nu \in T)(\eta \triangleleft \nu \in \mathrm{split}(T))$,

(c) if $\delta < \lambda$ is limit and a sequence $\langle \eta_i : i < \delta\rangle \subseteq \mathrm{split}(T)$ is $\triangleleft$-increasing, then $\eta = \bigcup_{i<\delta} \eta_i \in \mathrm{split}(T)$.

The order of $\mathbb{D}_\lambda$ is the reverse inclusion.

Proposition 4.10. *The forcing notion $\mathbb{D}_\lambda$ is proper over λ.*

Proof. First we argue that $\mathbb{D}_\lambda$ is λ-lub-complete. So suppose that $T_\alpha \in \mathbb{D}_\lambda$ are such that $(\forall \alpha < \beta < \delta)(T_\beta \subseteq T_\alpha)$, $\delta < \lambda$. We claim that $T \stackrel{\mathrm{def}}{=} \bigcap_{\alpha<\delta} T_\alpha$ is a condition in $\mathbb{D}_\lambda$. Clearly T is a tree, and $\mathrm{root}(T) = \bigcup_{\alpha<\delta} \mathrm{root}(T_\alpha)$. By clause (c) (for T_α's) we see that $\mathrm{succ}_T(\mathrm{root}(T)) = \lambda$, and in a similar way we justify that T satisfies other demands as well.

Now suppose that D, $\mathcal{S}$, N, h, $\bar{F}$ and $\bar{q}$ are as in Definition 2.1, $T \in \mathbb{D}_\lambda \cap N$. Choose inductively complete λ-trees $T_\alpha \in \mathbb{D}_\lambda \cap N$ and fronts $\mathbf{F}_\alpha \subseteq T_\alpha$ (of T_α) such that

(i) $\mathrm{root}(T_\alpha) = \mathrm{root}(T)$,

(ii) if $\alpha \leq \beta < \lambda$, then $T_\beta \subseteq T_\alpha \subseteq T$ and $\mathbf{F}_\alpha \subseteq \mathrm{split}(T_\beta)$, and

(iii) if $\eta \in \mathbf{F}_\alpha$ then $\mathrm{otp}(\{i < \mathrm{lh}(\eta) : \eta \restriction i \in \mathrm{split}(T_\alpha)\}) = \alpha$,

(iv) if δ is limit, then $T_\delta = \bigcap_{\alpha<\delta} T_\alpha$,

(v) if $\delta \in \mathcal{S}$ is limit and $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is an increasing sequence of conditions from $\mathbb{D}_\lambda \cap N$ and $\bigcap_{\alpha<\delta} h \circ F_\delta(\alpha) \subseteq T_\delta$, and $\eta = \bigcup_{\alpha<\delta} \mathrm{root}(h \circ F_\delta(\alpha)) \in \mathbf{F}_\delta$, then for some $\nu \in T_\delta$ we have

$$\eta \triangleleft \nu \in \mathbf{F}_{\delta+1}, \qquad q_\delta \leq (T_{\delta+1})^{[\nu]}. \tag{4.11}$$

Now we let $r = \bigcap_{\alpha<\lambda} T_\alpha$. It should be clear that $r \in \mathbb{D}_\lambda$.

Claim 4.11. The condition r is $(N, h, \mathbb{D}_\lambda)$-generic for $\bar{q}$ over $\bar{F}$.

Proof of Claim 4.11. We have to describe a winning strategy of the generic player in the game $\Game(r, N, h, \mathbb{D}_\lambda, \bar{F}, \bar{q})$. Let E_0 be the club of limits of members of $\mathcal{S}'$. Let the generic player play as follows.

Assume we have arrived to stage $i \in \mathcal{S}$ of the play when $\langle r_j^-, r_j, C_j : j < i\rangle$ has been already constructed. If $i \notin E_0$ then the generic player chooses r_i^-, $r_i \in \mathbb{D}_\lambda$ such that

(A) $r_i \subseteq \bigcap_{j<i} r_j$, $r_i^- \subseteq \bigcap_{j<i} r_j^- \cap \bigcap_{j<i} T_j$, and $r_i^- \in N$, $r_i^- \leq r_i$,

(B) $\mathrm{root}(r_i^-) = \mathrm{root}(r_i) \in \mathbf{F}_{\alpha(i)}$ for some $\alpha(i) > i$,

and lets $C_i = E_0 \setminus (\alpha(i)+1)$. If $i \in E_0$ then the generic player picks r_i, r_i^- satisfying (A) and (B) and such that

(C) if possible, then $q_\delta \leq r_i^-$ and it takes C_i as earlier.

Why is this a winning strategy? First, as $\mathbb{D}_\lambda$ is λ-lub-complete, the play really lasts λ moves. Suppose that $\delta \in \mathcal{S} \cap \bigcap_{i<\delta} C_i$ is such that

$$\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle = \langle r_\alpha^- : \alpha < \delta\rangle. \tag{4.12}$$

Let $\eta = \bigcup_{\alpha<\delta} \mathrm{root}(r_\alpha^-)$. Note that (as $\delta \in E_0$ and by (B)) we have $\eta \in \mathbf{F}_\delta$ and (by (A)) $\bigcap_{\alpha<\delta} r_\alpha^-$ is included in T_δ. Therefore, by clause (v) of the choice of the T_δ, for some $\nu \in T_\delta$ we have $\eta \vartriangleleft \nu \in \mathbf{F}_{\delta+1}$ and $q_\delta \leq (T_{\delta+1})^{[\nu]}$. But this immediately implies that it was possible to choose r_i^- stronger than q_δ in (C) (remember $r = \bigcap_{\alpha<\delta} T_\alpha$). □

This complete the proof of Proposition 4.10. □

5. Discussion

5.1. The axiom. We can derive forcing axioms as usual, see [15, Chapter VII and VIII]. For example, if κ is a supercompact cardinal larger than λ, then we can find a κ-cc λ-complete, proper over D-semi-diamonds forcing notion $\mathbb{P}$ of cardinality κ such that

- $\Vdash_{\mathbb{P}} 2^\lambda = \kappa$
- $\mathbb{P}$ collapses every $\mu \in (\lambda^+, \kappa)$, no other cardinal is collapsed,
- in $\mathbf{V}^{\mathbb{P}}$: if $\mathbb{Q}$ is a forcing notion proper over D-semi-diamonds, $\mathcal{I}_\alpha$ are open dense subsets of $\mathbb{Q}$ for $\alpha < \lambda^+$, then there is a directed set $G \subseteq \mathbb{Q}$ intersecting every $\mathcal{I}_\alpha$ (for $\alpha < \lambda^+$).

If we restrict ourselves to $|\mathbb{Q}| = \kappa$, it is enough that κ is indescribable enough.

In ZFC, we have to be more careful concerning $\mathbb{Q}$.

5.2. Future applications. Real applications of the technology developed here will be given in a forthcoming paper [4], where we will present more examples of proper for λ forcing notions (concentrating on the case of inaccessible λ). We start there developing a theory parallel to that of [5, 6, 7] aiming at generalizing many of the cardinal characteristics of the continuum to larger cardinals.

5.3. Why our definitions? The main reason why our definitions are (perhaps) somewhat complicated is that, in addition to ZFC limitations, we wanted to cover some examples with "large creatures" (to be presented in [4]). We also wanted to have a real preservation theorem: the (limit of the) iteration is of the same type as the iterands (though for many applications the existence of $(N, \mathbb{P}_\zeta)$-generic conditions could be enough).

Why do we have the sets C_i in the game, and not just say that "the set of good δ's is in D"? It is caused by the fact that already if we want to deal with the composition of two forcing notions (the successor step), the respective set from D would have appeared only *after* the play, and there would be simply too many possible sets to consider.

With the current definition the generic player discovers during the play which $\delta \in \mathcal{S}$ are active.

Why semi-diamonds (and not just diamonds)? As we want that $\bar{q}^{\langle i \rangle}$, $\bar{F}^{\langle i \rangle}$ are as claimed in Claim 3.9 (for the respective parameters).

5.4. Strategic completeness. We may replace "λ-complete" by (a variant of) "strategically λ-complete." This requires some changes in our definitions (and proofs) and it will be treated in [9].

5.5. Relation to [8]. There is a drawback in the approach presented in this paper: we do not include the one from [8], say when $\mathcal{S} \subseteq \mathcal{S}_\lambda^{\lambda^+}$ is stationary and $\mathcal{S}_\lambda^{\lambda^+} \setminus \mathcal{S}$ is also stationary.

One of the possible modifications of the present definitions for the case of inaccessible λ, can be sketched as follows. We have $\langle \lambda_\delta : \delta \in \mathcal{S} \rangle$, $\lambda_\delta = (\lambda_\delta)^{|\delta|}$; $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \rangle$ is replaced by $\bar{q} = \langle q_{\delta,t} : \delta \in \mathcal{S},\ t \in \mathrm{Par}_{\delta^*,\delta} \rangle$ (where $\delta^* = N \cap \lambda^+$), and $\bar{\mathrm{Par}} = \langle \mathrm{Par}_{\delta^*,\delta} : \delta \in \mathcal{S} \rangle \in \mathbf{V}$ is constant for the iteration (like D).

In the forcing $\mathbb{P}$: for $\bar{p} = \langle p_j : j < \delta \rangle$, $\delta \in \mathcal{S}$, $t \in \mathrm{Par}_{\delta^*,\delta}$, there is an upper bound $q[\bar{p}, t]$ of $\bar{p}$ (this is a part of $\mathbb{P}$).

For each δ, each $\mathrm{Par}_{\delta^*, N_\delta \cap \gamma}$, $\prod_{i \in N_\delta} \mathrm{Par}_{\delta^*,\delta}$ has cardinality $\lambda_\delta = (\lambda_\delta)^{|\delta|}$ (N_δ is of cardinality $|\delta|$; γ is the length of the iteration). Having $\langle p_j : j < \delta \rangle \subseteq N_\delta$ we can find $\langle q_t^\delta : t \in \mathrm{Par}_{\delta, N_\delta \cap \gamma} \rangle$ as in [8].

Several (more complex) variants of properness over semi-diamonds will be presented in [9] and Rosłanowski and Shelah [4].

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