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Author(s): Saharon Shelah and Martin Ziegler
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ALGEBRAICALLY CLOSED GROUPS OF LARGE CARDINALITY

SAHARON SHELAH AND MARTIN ZIEGLER

§0. Introduction. Let M be a countable algebraically closed group, κ an uncountable cardinal. We will prove in this paper the following theorems.

THEOREM 1. *There is an algebraically closed group N of cardinality κ which is $\infty - \omega$-equivalent to M.*

THEOREM 2. *There is an algebraically closed group N of cardinality κ which is $\infty - \omega$-equivalent to M, and contains a free abelian group of cardinality κ.*

THEOREM 3. *There are 2^κ nonisomorphic algebraically closed groups of cardinality κ which are $\infty - \omega$-equivalent to M.*

THEOREM 4. *There is an algebraically closed group N of cardinality κ which is $\infty - \omega$-equivalent to M and satisfies: Every subgroup of N of uncountable reqular cardinality contains a free subgroup of the same cardinality.*

Theorems 2 and 4 illustrate Theorem 3 by exhibiting two groups $N \equiv_{\infty\omega} M$ of cardinality κ which are nonisomorphic by obvious reasons. We state and prove Theorem 1 separately in order to give an easy example of our principal tool: the use of automorphisms instead of indiscernibles (see §2).

This method is due to the second author, who constructed two nonisomorphic algebraically closed groups $N \equiv_{\infty\omega} M$, $|N| = \kappa$ in his *Habilitationsschrift* (1976). Subsequently the first author used an improved version of the method to prove Theorem 3 (1976) and Theorem 4 (1977). Some improvements in our proof of Theorem 4 are due to the second author.

Theorem 1 answers a question of A. Macintyre, who proved it for $\kappa = \aleph_1$ [1] and—assuming Martin's axiom—for all $\kappa < 2^{\aleph_0}$ [2]. Note that Theorem 1 implies the existence of finite-generic groups of cardinality κ.

Macintyre and Shelah [8] dealt with similar problems for universal locally finite groups. Macintyre [7] dealt with similar problems (including algebraically closed groups) in cardinality $\aleph_1$, assuming $\diamondsuit_{\aleph_1}$, proving there is N, $\infty - \omega$-equivalent to M, with no uncountable abelian subgroup. Hickin [6] improves those results for universal locally finite groups (eliminate diamond, get more properties, but the cardinality was $\aleph_1$). Shelah (later than Ziegler's work) in [9] (assuming CH, for cardinality $\aleph_1$) and [10] (for cardinality $2^{\aleph_0}$) improve the results of [7]. He also constructed (assuming CH), N $\infty - \omega$-equivalent to M of cardinality $\aleph_1$, with no uncountable free group. The parallel of this for any cardinality remains open.

Addendum. By a slight modification of our method one can sharpen Theorems 2 and 4. In both cases there are 2^κ nonisomorphic groups N; if κ is regular, there are 2^κ groups N (satisfying all the respective conditions) which are mutually nonembeddable. This answers questions of the referee.

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§1. Prerequisites from group theory. A nontrivial group N is *algebraically closed*, if every finite system of equations with coefficients from N which is solvable in a supergroup of N has a solution in N.

(1.1) LEMMA ([1]). *Let N be a group.*

(1) *$N =_{\infty\omega} M$ iff N is algebraically closed and has—up to isomorphy—the same finitely generated subgroups as M.*

(2) *$N \cong M$ iff N is countable, algebraically closed and contains the same finitely generated subgroups as M.*

We sketch the proof. (1) (Only if) The class of all algebrically closed groups is definable by a sentence of $L_{\omega_1\omega}$. For every finitely generated F the class of all groups which embed F is definable by a sentence of $L_{\omega_1\omega}$.

(If) The theorem of Higman-Neumann-Neumann ("an isomorphism of subgroups can be extended to an inner automorphism of a supergroup") implies the *ω-homogeneity* of algebraically closed groups:

Every isomorphism of finitely generated subgroups can be extended to an (inner) automorphism.

Two ω-homogeneous models having the same finitely generated submodels are $\infty - \omega$-equivalent. For the family of all isomorphisms between finitely generated submodels has the back-and-forth property.

(2) follows from (1). Countable models which are $\infty - \omega$-equivalent are isomorphic.

(1.2) LEMMA. *A subgroup $L \subset M$ is isomorphic to M iff for all finitely generated $H \subset L$ and all $h \in M$ there is an $x \in C_M(H)$ s.t. $xhx^{-1} \in L$.*

($C_M(H)$ denotes the centralizer of H in M.)

PROOF. (Only if) ($\langle H, h\rangle$ denotes the group generated by $H \cup \{h\}$.) There is $y \in M$ s.t. $y\langle H, h\rangle y^{-1} \subset L$, by ω-homogeneity of M. The ω-homogeneity of L yields an element z of L s.t. $ywy^{-1} = zwz^{-1}$ for all $w \in H$. Put $x = z^{-1}y$.

(If) Let $H \subset L$ be finitely generated. By induction for all $h_1, \ldots, h_n \in M$ there is $x \in C_M(H)$ s.t. $xh_1x^{-1}, \ldots, xh_nx^{-1} \in L$. For, if $y \in C_M(H)$, $yh_1y^{-1}, \ldots, yh_{n-1}y^{-1} \in L$, choose $z \in C_M(\langle H, yh_1y^{-1}, \ldots, yh_{n-1}y^{-1}\rangle)$ s.t. $z(yh_ny^{-1})z^{-1} \in L$, and put $x = zy$.

We prove $L \cong M$ by 1.1.2. Let S be a finite system of equations with coefficients in $H \subset L$, which is solvable in a supergroup of L. S is also solvable in the amalgamated product of M and this supergroup, thus there must be a solution $h_1, \ldots, h_n$ of S in M. Let $x \in C_M(H)$, $xh_1x^{-1}, \ldots, xh_nx^{-1} \in L$. The last sequence is a solution of S in L. This proves that L is algebraically closed. On the other hand every finitely generated subgroup $\langle h_1, \ldots, h_n\rangle$ of M is embeddable in L.

Let $x_1, \ldots, x_m \in M$. An *implication* with coefficients $x_1, \ldots, x_m$ is a formula of the form

$$\bigwedge_{i<k} W_i(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m, v_1, \ldots, v_n) \doteq V_i(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m, v_1, \ldots, v_n) \\ \to W(\underset{\sim}{x}_1, \ldots, v_n) \doteq V(x_1, \ldots, v_n)$$

($\underset{\sim}{x}_i$ is a name for x_i, v_i a variable).

We quote from [3].

(1.3) LEMMA. *Let $x_1, \ldots, x_m \in M$. A recursively enumerable system of implications with coefficients $x_1, \ldots, x_m$ which is satisfiable in a supergroup of M is satisfiable in M.*

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Let $(f_i \mid i \in \boldsymbol{N})$ be an enumeration of the group F. If $F \subset M$, we call the enumeration *compatible* if there are $x_1, \dots, x_m \in M$ and a recursive sequence $W_i(v_1, \dots, v_m)$, $i \in \boldsymbol{N}$, of words s.t. $f_i = W_i(x_1, \dots, x_m)$. $(f_i | i \in \boldsymbol{N})$ is *effective* if $f_i = f_j f_k$ is a recursive relation in i, j, k.

(1.4) LEMMA. *Let $(f_i | i \in \boldsymbol{N})$ be an effective enumeration of the group F. Then F can be embedded in M s.t. $(f_i \mid i \in \boldsymbol{N})$ is compatible.*

PROOF. In [4] a recursive sequence of words $W_i(v_1, v_2)$ is given, s.t. the system of equations $\{W_i(v_1, v_2) \doteq \underline{f}_i \mid i \in \boldsymbol{N}\}$ is solvable in an extension G of F. Take an element $x \in M \setminus \{e\}$ ($\{e\}$ denotes the trivial group). The r.e. system of implications

$$\{W_i W_j \doteq W_k \,|\, f_i f_j = f_k\} \cup \{W_i W_j \doteq W_k \to \underline{x} \doteq e \mid f_i f_j \neq f_k\}$$

is solvable in $M \times G$ and has therefore a solution x_1, x_2 in M. $f_i \mapsto W_i(x_1, x_2)$ is the desired embedding of F in M.

A sequence $(F_i \mid i \in \boldsymbol{N})$ of subgroups of M is *effective* if there are elements $x_1, \dots, x_m \in M$ and a recursive sequence Ω_i, $i \in \boldsymbol{N}$ of recursively enumerable sets of words in $v_1, \dots, v_m$ s.t.

$$F_i = \langle W(x_1, \dots, x_m) \,\big|\, W \in \Omega_i \rangle, \quad i \in \boldsymbol{N}.$$

We remark

(1.5). Let the enumeration $(f_i | i \in \boldsymbol{N})$ of $F \subset M$ be compatible, and $(F_i \mid i \in \boldsymbol{N})$ a sequence of subgroups of F s.t. $f_j \in F_i$ is recursive in i, j. Then $(F_i \mid i \in \boldsymbol{N})$ is effective.

Let K_i, $i \in \boldsymbol{N}$ be a family of subgroups of K, and φ_i a family of isomorphic embeddings $\varphi_i\colon K_i \to F_i$. Define

$$C = \langle K^* F_0^* F_1^* \cdots ;\ \varphi_i(h) = h,\ h \in K_i,\ i \in \boldsymbol{N} \rangle,$$

the free product of $K, F_0, F_1, \dots$ where the K_i are amalgamated with $\varphi_i[K_i]$ via φ_i. We identify $K, F_0, \dots$ with subgroups of C. Then $F_i \cap K = K_i$ and $F_i \cap F_j = K_i \cap K_j$. We will use the notation

$$C = \prod_{i \in \boldsymbol{N}}{}^{*}(K, F_i; K_i).$$

A product $c_0 c_1 \cdots c_n$ of elements of $K \cup \bigcup_{i \in \boldsymbol{N}} F_i$ is in *normal* form, if $c_k \in K \to c_{k+1} \notin K$, $c_k \in F_i \to c_{k+1} \notin F_i$, $k = 0, 1, \dots, n-1$.

(1.6) Lemma. (1) *Every element of $\prod^*_{i \in \boldsymbol{N}}(K, F_i; K_i)$ can be written as a product in normal form.*

(2) *A product $c_0 c_1 \cdots c_n$ in normal form equals e iff $n = 0$, $c_0 = e$.*

PROOF. (1) If $d_0 d_1 \cdots d_n$ is not in normal form because $d_k, d_{k+1} \in K$ or $d_k, d_{k+1} \in F_i$, we proceed to the product $d_0 d_1 \cdots d_{k-1}(d_k d_{k+1}) d_{k+2} \cdots d_n$. Finally we arrive at a product in normal form.

(2) This can easily be derived from the normal form theorem for free products with amalgamation. Note that $\prod^*_{i \in \boldsymbol{N}}(K, F_i; K_i)$ is the union of the sequence $\cdots ((K^*_{K_0} F_0)^*_{K_1} F_1)^*_{K_2} F_2 \cdots$. ($\prod^*_{i \in \boldsymbol{N}}(K, F_i; K_i)$ is a tree product in the sense of [11], in the tree K is connected with each F_i.)

§2. Models with operation.

(2.1) DEFINITION. Let $\underline{M}$ be a model, I a set. An *operation $\mathscr{M}$ of I on $\underline{M}$* is a map-

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ping, which assigns to every finite $s \subset I$ a submodel $\mathcal{M}\, s \subset \underline{M}$ and to every bijection $f\colon s \to t$ an isomorphism $\mathcal{M} f\colon \mathcal{M}s \to \mathcal{M}t$ s.t.

(1) $\underline{M} = \bigcup\{\mathcal{M}s \mid s \subset I\}$,

(2) $\mathcal{M}\mathrm{id}_s = \mathrm{id}_{\mathcal{M}s}$,

(3) $\mathrm{dom}\, f = \mathrm{rng}\, g \Rightarrow \mathcal{M}f \circ \mathcal{M}g = \mathcal{M}(f \circ g)$,

(4) $f \subset g \Rightarrow \mathcal{M} f \subset \mathcal{M}g$.

Note that (2), (3), (4) imply

$s \subset t \Rightarrow \mathcal{M}s \subset \mathcal{M}t$,

$\mathrm{dom}\, f \subset \mathrm{rng}\, g \Rightarrow \mathcal{M} f \circ \mathcal{M}g = \mathcal{M}(f \circ g)$.

(2.2) Definition. Let $\underline{M}$ be a submodel of $\underline{N}$ and $\mathcal{N}$ an operation of $I \subset J$ on $\underline{N}$. $\mathcal{N}$ is an *extension* of $\mathcal{M}$, if for all $s \subset I$, $\mathcal{N}s = \mathcal{M}s$ and for all bijections $f\colon s \to t \subset I$, $\mathcal{N} f = \mathcal{M}f$.

(2.3) Lemma. *Let $\mathcal{M}$ be an operation of the infinite set I on the model $\underline{M}$, and J a superset of I. Then there is a unique extension of $\mathcal{M}$ to an operation $\mathcal{N}$ of J on a suitable model $\underline{N} \supset \underline{M}$.*

Proof. (Uniqueness) Let $\mathcal{N}'$ be another extension of $\mathcal{M}$ to an operation of J on $\underline{N}' \supset \underline{M}$. Choose for every $x \in \underline{N}$ a finite $s \subset J$ s.t. $x \in \mathcal{N}s$ and a bijection $f\colon t \to s$, $t \subset I$. Define

$$\varphi(x) = \mathcal{N}'f(\mathcal{N} f^{-1}(x)).$$

φ is a well-defined isomorphism $\varphi\colon \underline{N} \to \underline{N}'$, which satisfies $\varphi_{|\underline{M}} = \mathrm{id}_{\underline{M}}$, $\varphi[\mathcal{N}s] = \mathcal{N}'s$, $\varphi \circ \mathcal{N} f = \mathcal{N}'f \circ \varphi$.

(Existence) Choose for every finite $s \subset J$ a bijection $f_s\colon s' \to s$, $s' \subset I$. (In case $s \subset I$, we take $f_s = \mathrm{id}_s$.) Set $\underline{N}_s = \mathcal{M}s'$. For every $s \subset t \subset J$, there is a unique injection $f_{st}\colon s' \to t'$ s.t. $f_t f_{st} = f_s$. Define an isomorphic embedding $\underline{N}_{st}\colon \underline{N}_s \to \underline{N}_t$ by $\underline{N}_{st} = \mathcal{M} f_{st}$. (Note that $\underline{N}_{st}$ is the inclusion map of $\mathcal{M}s \subset \mathcal{M}t$, if $s \subset t \subset I$.)

The $\underline{N}_{st}$ form a commutative system: $\underline{N}_{ss} = \mathrm{id}_{\underline{N}_s}$, $\underline{N}_{st}\underline{N}_{rs} = \underline{N}_{rt}$, $r \subset s \subset t \subset J$. Define $\underline{N}$ as the direct limit of the system $\underline{N}_s$, $s \subset J$; $\underline{N}_{st}$, $s \subset t \subset J$. (The index set $P_\omega(J)$, the set of all finite subsets of J, is partially ordered by inclusion.)

We have embeddings $\underline{N}_{s\infty}\colon \underline{N}_s \to \underline{N}$ s.t. $s \subset t \Rightarrow \underline{N}_{t\infty}\, \underline{N}_{st} = \underline{N}_{s\infty}$ and $\underline{N} = \bigcup\{\mathrm{rng}\ \underline{N}_{s\infty} \mid s \subset J\}$. We can assume that for $s \subset I$, $\underline{N}_{s\infty} = \mathrm{id}_{\mathcal{M}s}$. As a consequence $\underline{M} = \bigcup\{\mathcal{M}s \mid s \subset I\} \subset \underline{N}$.

We define the operation of J on $\underline{N}$ by $\mathcal{N}s = \mathrm{rng}\ \underline{N}_{s\infty}$ and for $f\colon s \to t$, $\mathcal{N} f = \underline{N}_{t\infty}\, \mathcal{M}(f_s^{-1} f f_s)\underline{N}_{s\infty}^{-1}$.

The proof is now completed by chasing commutative diagrams.

Our construction admits the following generalization (which we use only in the proof of Theorem 3, §5).

Let $\underline{I}$ be a relational structure (a model, where no constants and functions are defined). An *operation* $\mathcal{M}$ of $\underline{I}$ on $\underline{M}$ assigns to every finite substructure $s \subset \underline{I}$ a submodel $\mathcal{M}s \subset \underline{M}$ and to every isomorphism $f\colon s \to t$ an isomorphism $\mathcal{M}f\colon \mathcal{M}s \to \mathcal{M}t$ s.t. (1), (2), (3), (4) are satisfied.

Lemma (2.3) generalizes to

(2.4) Lemma. *Let $\mathcal{M}$ be an operation of the relational structure $\underline{I}$ on the model $\underline{M}$. Let $\underline{I}$ be a substructure of $\underline{J}$, which contains—up to isomorphy—the same finite substructures as $\underline{I}$. Then there is a unique extension of $\mathcal{M}$ to an operation $\mathcal{N}$ of $\underline{J}$ on a suitable model $\underline{N} \supset \underline{M}$.*

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PROOF. Same as the proof of (2.3).

We describe now the way to use (2.4) for the construction of algebraically closed groups (no other applications of (2.4) are known to us).

Let Π be a group of automorphisms of the relational structure $\underline{I}$, which contains an extension of every isomorphism of two finite substructures of $\underline{I}$. We define for every finite $s \subset \underline{I}$, $\Pi_s = \{\pi \in \Pi \mid \pi|_s = \mathrm{id}_s\}$.

(2.5) LEMMA. *Suppose Π is a subgroup of M and let for every finite $s \subset \underline{I}$ a subgroup $M_s \subset M$ be given s.t.*

(5) $s \subset t \Rightarrow M_s \subset M_t$,

(6) $M_s \subset C_M(\Pi_s)$,

(7) $\pi M_s \pi^{-1} = M_{\pi[s]}$,

(8) $M_s \cong M$.

Then

(1) $\mathscr{M}s = M_s$, $\mathscr{M}f(x) = \pi x \pi^{-1}$ *(if $\pi \in \Pi$, $f \subset \pi$) defines an operation $\mathscr{M}$ of $\underline{I}$ on $\bigcup\{M_s \mid s \subset \underline{I}\}$.*

(2) *If $\mathscr{N}$ is an extension of $\mathscr{M}$ to an operation of $\underline{J}$ on N, N is an algebraically closed group with $N \equiv_{\infty\omega} M$, $|N| \leq |\underline{J}| + \aleph_0$.*

PROOF. (1) We prove that $\mathscr{M}f$ is well defined. Let $f: s \to t$ be a finite bijection, $x \in \mathscr{M}s$ and $f \subset \pi$, $\rho \in \Pi$. Then $\pi|_s = \rho|_s \Rightarrow \pi^{-1}\rho \in \Pi_s \Rightarrow \pi^{-1}\rho x = x\pi^{-1}\rho \Rightarrow \rho x \rho^{-1} = \pi x \pi^{-1}$.

(2) N is the direct union of the $\mathscr{N}s$, $s \in P_w(\underline{J})$, which are isomorphic to M. This fact alone entails that N is algebraically closed, possesses the same finitely generated subgroups as M and $|N| \leq |J| + \aleph_0$.

§3.

PROOF OF THEOREM 1. Set

$I = \boldsymbol{N}$, the set of natural numbers;

$\Pi =$ the group of finite permutations of I (i.e. the permutations π with finite support $\{i \in I \mid \pi(i) \neq i\}$).

We fix an effective enumeration $(\mu_i \mid i \in \boldsymbol{N})$ s.t. $\mu_i(j) = k$ is a recursive relation between i, j, k.

By (1.4) we find Π as a subgroup of M with compatible enumeration. Set $M_s = C_M(\Pi_s)$, $s \in P_\omega(I)$.

(3.1). Π, $(M_s \mid s \subset I)$ satisfy the conditions (5), (6), (7), (8) of (2.5).

PROOF. (5) and (6) are clear. (7) follows from

$$\pi M_s \pi^{-1} = \pi C_M(\Pi_s)\pi^{-1} = C_M(\pi \Pi_s \pi^{-1}) = C_M(\Pi_{\pi[s]}) = M_{\pi[s]}.$$

For (8) we use (1.2). Let H be a finitely generated subgroup of M_s and $h \in M$. Π_s has trivial center. Thus the group $\langle H, \Pi_s\rangle$ is the direct product $H \times \Pi_s$. We choose a group $M' \supset H$, isomorphic to M by an isomorphism which leaves the elements of H fixed. Let M'' be the free product of $M' \times \Pi_s$ and M with amalgamated subgroup $H \times \Pi_s$. Suppose $h_1, \dots, h_n$ are the generators of H and $h' \in M'$ the element corresponding to h. The theorem of Higman-Neumann-Neumann gives an element y in a supergroup of M'' s.t. $yhy^{-1} = h'$ and $yh_iy^{-1} = h_i$, $i = 1, \dots, n$. By (1.5), $(\Pi_s \mid s \in P_\omega(I))$ is effective (here we identify $\boldsymbol{N}$ and $P_\omega(I)$). Thus there are $x_1, \dots, x_m \in M$ and a r.e. set Ω of words s.t.

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$$\Pi_s = \langle W(x_1, \ldots, x_m) \mid W(v_1, \ldots, v_m) \in \Omega \rangle.$$

y solves the following r.e. system of equations.

$$\{[v, \underline{h}_i] \doteq e \mid i = 1, \ldots, n\} \cup \{[v\underline{h}v^{-1}, W(\underline{x}_1, \ldots, \underline{x}_m)] \doteq e \mid W(v_1, \ldots, v_m) \in \Omega\}$$

$([x, y] = x^{-1}y^{-1}xy)$.

If x is a solution in M, we have $x \in C_M(H)$ and $xhx^{-1} \in C_M(\Pi_s)$. This proves (3.1).

Let now $\mathscr{M}$ be the operation of Lemma 2.5, and $J \supset I$ a set of power κ. If we extend $\mathscr{M}$ to an operation $\mathscr{N}$ of J on N using (2.3), we obtain an algebraically closed N, $\infty - \omega$-equivalent to M, $|N| \leq \kappa$.

On order to show $|N| = \kappa$, we partition J in two-element subsets $\{i_\alpha, j_\alpha\}$, $\alpha < \kappa$. Let $a \in \Pi$ be the 2-cycle (01) and $b = (23)$. Clearly $a \in \mathscr{N}\{0, 1\}$, $b \in \mathscr{N}\{2, 3\}$. We fix the following *notation* for later use: $f^{j_1 \cdots j_m}_{k_1 \cdots k_m}$ is the finite bijection which maps k_i to j_i. Put $a_\alpha = \mathscr{N} f^{i_\alpha j_\alpha}_{01}(a)$. If $\alpha \neq \beta$, the isomorphism $\mathscr{N} f^{0123}_{i_\alpha j_\alpha i_\beta j_\beta}$ maps a_α to a and a_β to $\mathscr{M} f^{23}_{01}(a) = b$. Whence all a_α, $\alpha < \kappa$, are different.

§4.

PROOF OF THEOREM 2. We use I and Π of §3. Let A be the free abelian group with generators a_i, $i \in I$. Every $\pi \in \Pi$ gives rise to an automorphism $\bar{\pi}$ of A by $\bar{\pi}(a_i) = a_{\pi(i)}$. Let B be the *split extension* of A w.r.t. the homomorhpism $\pi \mapsto \bar{\pi}$. (B is generated by A and Π, and is determined by the conditions $A \cap \Pi = E$, $\pi a \pi^{-1} = \bar{\pi}(a)$. B can explicitly given by the set $A \times \Pi$ together with the multiplication $(a, \pi)(a', \pi') = (a\bar{\pi}(a'), \pi\pi')$.) We find an effective enumeration $(b_i \mid i \in \mathbf{N})$ of B s.t. $b_i = \mu_j$ is a recursive relation. By (1.4) we get an embedding of B in M, where $(b_i \mid i \in \mathbf{N})$ is compatible. Then $(\mu_i \mid i \in \mathbf{N})$ is a compatible enumeration of $\Pi \subset B \subset M$, and we can use the results of §3:

The family $M_s = C_M(\Pi_s)$ gives rise to an operation $\mathscr{M}$ of I on $\bigcup\{M_s \mid s \in P_\omega(I)\}$ according to (2.5). The extension $\mathscr{N}$ of $\mathscr{M}$ to an operation of $J \supset I$ on N yields an algebraically closed group of cardinality κ, which is $\infty - \omega$-equivalent to M.

a_0 is an element of $\mathscr{N}\{0\}$. Since $a_i = \mathscr{N} f^i_0(a_0)$ for all $i \in I$, a_j is well defined by $a_j = \mathscr{N} f^j_0(a_0)$, $j \in J$. If $j_0, \ldots, j_n$ are different elements of J, the isomorphism

$$\mathscr{N} f^{01\cdots n}_{j_0 j_1 \cdots j_n} : \mathscr{N}\{j_0, \ldots, j_n\} \to \mathscr{N}\{0, 1, \ldots, n\}$$

maps $a_{j_0}, \ldots, a_{j_n}$ to $a_0, \ldots, a_n$. Whence the a_j, $j \in J$ freely generates an abelian group of rank κ.

§5.

PROOF OF THEOREM 3. Set

$\underline{I} = (\mathbf{Q}, <)$, the ordering of the rationals;

$\Pi =$ the group of all piecewise linear automorphisms of $\underline{I}$ (i.e. all order-preserving permutations of $\mathbf{Q}$ of the form

$$\pi(x) = r_i x + p_i, \quad x \in [q_i, q_{i+1})$$

where $-\infty = q_0 < q_1 < \cdots < q_n < q_{n+1} = \infty$, $r_i, p_i, q_i \in \mathbf{Q}$).

We will use only the following properties of Π:

Π satisfies the assumptions preceding (2.5).

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There is an effective enumeration $(\mu_i \mid i \in \boldsymbol{N})$ of Π s.t. $\mu_i(p) = q$ is recursive in i, p, q.

The Π_s, $s \in P_\omega(\underline{I})$, have trivial center.

Let A be presented by the generators a_p, c_p, $p \in \boldsymbol{Q}$ and the relations $[a_p, c_q] = e$, $p < q$. It is easy to see that A has solvable word problem and $q \leq p \Rightarrow [a_p, c_q] \neq e$. Every $\pi \in \Pi$ permutes the defining relations of A by $a_p \mapsto a_{\pi(p)}$, $c_p \mapsto c_{\pi(p)}$. We call the resulting automorphism of A $\bar{\pi}$.

Let B be the split extension of A w.r.t. the homomorphism $\pi \mapsto \bar{\pi}$. We find an effective enumeration $(b_i \mid i \in \boldsymbol{N})$ of B s.t. $b_i = \mu_j$ is recursive. By (1.4) we get an embedding of B in M, where $(b_i \mid i \in \boldsymbol{N})$ is compatible. Then $(\Pi_s \mid s \in P_\omega(\underline{I}))$ is an effective family.

We set $M_s = C_M(\Pi_s)$, $s \in P_\omega(\underline{I})$. The proof of Theorem 1 shows that we can apply (2.5) to obtain an operation $\mathscr{M}$ of $\underline{I}$ on $\bigcup \{M_s \mid s \subset \underline{I}\}$. We choose a linear ordering $\underline{J}$ which contains $\underline{I}$ and embeds $(\kappa, <)$. Let $\mathscr{N}$ extend $\mathscr{M}$ to an operation of $\underline{J}$ on N (2.4). N is algebraically closed and $\infty - \omega$-equivalent to M.

a_0, c_0 are elements of $\mathscr{N}\{0\}$. Since $a_i = \mathscr{N} f_0^i(a_0)$ and $c_i = \mathscr{N} f_0^i(c_0)$ for all $i \in \boldsymbol{Q}$, we can define $a_j = \mathscr{N} f_0^j(a_0)$ and $c_j = \mathscr{N} f_0^j(c_0)$ for all $j \in \underline{J}$.

If $j_0 < j_1$ are elements of $\underline{J}$, the isomorphism $\mathscr{N} f_{j_0 j_1}^{01}\colon \mathscr{N}\{j_0, j_1\} \to \mathscr{N}\{0, 1\}$ maps a_{j_0}, a_{j_1} to a_0, a_1 and c_{j_0}, c_{j_1} to c_0, c_1.

This implies that $[a_{j_0}, c_{j_1}] = e \Leftrightarrow j_0 < j_1$, for all $j_0, j_1 \in \underline{J}$.

Theorem 3 follows now from the next lemma. (We set $k = 2$, $\varphi(v_1, v_2, w_1, w_2) = [v_1, w_2] \doteq e$, take for ψ a Scott-sentence of M.)

(5.1) Lemma ([5]). *Let ψ be an $L_{\omega_1\omega}$-sentence and $\varphi(v_1, v_2, \ldots, v_k, w_1, \ldots, w_k)$ an $L_{\infty\omega}$-formula. Suppose that for all κ, there is a model $\underline{N}$ of ψ which contains a sequence $\mathbf{a}_\alpha \in \underline{N}^k$, $\alpha < \kappa$ s.t. $\underline{N} \models \varphi(\mathbf{a}_\alpha, \mathbf{a}_\beta) \Leftrightarrow \alpha < \beta$. Then ψ has 2^κ nonisomorphic models of cardinality κ, for all uncountable κ.*

§6.

Proof of Theorem 4. We use I and Π of §3. Let A be the free group with generators a_i, $i \in I$. Every $\pi \in \Pi$ determines an automorphism $\bar{\pi}$ of A by $\bar{\pi}(a_i) = a_{\pi(i)}$. Let B be the split extension of A w.r.t. the homomorphism $\pi \mapsto \bar{\pi}$. There is an effective enumeration $(b_i \mid i \in \boldsymbol{N})$ of B s.t. $b_i = \mu_j$ is recursive. We can assume that $B \subset M$ and $(b_i \mid i \in \boldsymbol{N})$ compatible (1.4).

Define $A_s = \langle a_i \mid i \in s \rangle$ for every finite $s \subset I$. Then $(A_s \mid s \subset I)$ is *admissible* in the following sense.

(6.1) Definition. A family $(L_s \mid S \in P_\omega(I))$ of subgroups of M is admissible, if (5), (6), (7) of (2.5) hold (where M_s is replaced by L_s) and

(9) $L_s \cap \Pi = e$.

(10) $(L_s \mid s \in P_\omega(I))$ is effective.

(11) Suppose s, $t \in P_\omega(I)$; $\pi_0, \ldots, \pi_n \in \Pi_s$ s.t. $\pi_k[t] \cap \pi_{k+1}[t] = \varnothing$ for $k = 0, \ldots, n-1$; $d \in L_{s \cup t}$; $\varepsilon_0, \ldots, \varepsilon_n \in \{1, -1\}$.

Then $\prod_{k \leq n} (\pi_k d \pi_k^{-1})^{\varepsilon_k} \neq e$ or $d \in C_M(\Pi_s)$.

$(A_s \mid s \subset I)$ satisfies (10), since A_s is generated by $\{\mu_i a_0 \mu_i^{-1} \mid \mu_i(0) \in s\}$.

To prove (11) we write $d \in A_{s \cup t}$ in reduced form:

$$d = a_{k_1}^{z_1} a_{k_2}^{z_2} \cdots a_{k_m}^{z_m}, \qquad k_i \in s \cup t, \quad k_i \neq k_{i+1}, \quad z_i \in \boldsymbol{Z} \setminus \{0\}.$$

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If $k_i \in s$ for all i, $d \in C_M(\Pi_s)$. If $k_j \in t$, we have $\pi_k(k_j) \neq \pi_{k+1}(k_j)$. Thus the $a^{z_j}_{\pi_k}(k_j)$, $k = 0, \ldots, n$, are not cancelled in the product

$$\prod_{k \le n} (\pi_k d \pi_k^{-1})^{\varepsilon_k} = \prod_{k \le n} (a^{z_1}_{\pi_k(k_1)} \cdots a^{z_m}_{\pi_k(k_m)})^{\varepsilon_k}.$$

Thus $\prod_{k \le n}(\pi_k d \pi_k^{-1})^{\varepsilon_k} \neq e$.

(6.2) LEMMA. *Let $(K_s \mid s \in P_\omega(I))$ be an admissible family, $h \in M$ and $r \in P_\omega(I)$. Then there is an element $x \in M$ and an admissible family $(L_s \mid s \in P_\omega(I))$ s.t. $xhx^{-1} \in L_r$, $x \in C_M(K_r)$ and $K_s \subset L_s$ for all $s \subset I$.*

We need two facts in our proof of (6.2).

(6.3) REMARK. (1) $C_{M'}(\Pi_{r_1 \cap r_2}) = C_{M'}(\Pi_{r_1}) \cap C_{M'}(\Pi_{r_2})$, for all finite r_1, $r_2 \subset I$ and all $M' \supset \Pi$.

(2) Suppose s, $t \in P_\omega(I)$; $\pi_0, \ldots, \pi_n \in \Pi_s$ s.t. $\pi_k[t] \cap \pi_{k+1}[t] = \varnothing$ for $k = 0, \ldots, n-1$; $d \in L_{s \cup t}$; $\varepsilon_0, \ldots, \varepsilon_n \in \{1, -1\}$. Then $\prod_{k \le n}(\pi_k d \pi_k^{-1})^{\varepsilon_k} \notin C_M(\Pi_s)$ or $d \in C_M(\Pi_s)$.

PROOF. (1) Clearly $C_{M'}(\Pi_{r_1 \cap r_2}) \subset C_{M'}(\Pi_{r_1}) \cap C_{M'}(\Pi_{r_2})$.

Let $b \in C_{M'}(\Pi_{r_1}) \cap C_{M'}(\Pi_{r_2})$ and $\pi \in \Pi_{r_1 \cap r_2}$. Let s be the support of π. There is $\rho \in \Pi_{r_2}$ s.t. $\rho[s] \cap r_1 = \varnothing$. We have $\rho\pi\rho^{-1} \in \Pi_{r_1}$. Since $\rho\pi\rho^{-1}$ commutes with b, π does also.

(2) Choose $\pi \in \Pi_s$ s.t. $\pi_n[t] \cap \pi\pi_n[t] = \varnothing$. Set $\varepsilon_{2n+1-k} = -\varepsilon_k$ and $\pi_{2n+1-k} = \pi\pi_k$ for $k \le n$. By (11), $\prod_{k \le 2n+1}(\pi_k d \pi_k^{-1})^{\varepsilon_k} \neq e$ or $d \in C_M(\Pi_s)$. The first term equals $[c^{-1}, \pi^{-1}]$, where $c = \prod_{k \le n}(\pi_k d \pi_k^{-1})^{\varepsilon_k}$. Whence $c \notin C_M(\Pi_s)$ or $d \in C_M(\Pi_s)$.

PROOF OF (6.2). We are looking for an element $x \in C_M(K_r)$ s.t.

(12) $L_s = \langle K_s \cup \{\pi(xhx^{-1})\pi^{-1} \mid \pi \in \Pi, \pi[r] \subset s\}\rangle$, $s \in P_\omega(I)$ is admissible.

(5), (7), (10) are true for any x. (6) is equivalent to

(13) $xhx^{-1} \in C_M(\Pi_r)$.

Thus $x \in C_M(K_r)$ has to satisfy (13), (9), (11), where $(L_s \mid s \subset I)$ is defined by (12). There are elements $x_1, \ldots, x_l \in M$, a recursive sequence W_i, $i \in \boldsymbol{N}$ of words s.t. $\mu_i = W_i(x_1, \ldots, x_l)$, and a recursive sequence Ω_s, $s \in P_\omega(I)$ of r.e. sets of words s.t. $K_s = \langle W(x_1, \ldots, x_l) \mid W \in \Omega_s\rangle$. We write $\underline{\mu}_i$ for $W_i(\underline{x}_1, \ldots, \underline{x}_l)$, and choose a recursive enumeration $W_{i,s}(v)$, $i \in \boldsymbol{N}$, of all terms of the form

$$W(V_1(\underline{x}_1, \ldots, \underline{x}_l), \ldots, V_k(\underline{x}_1, \ldots, \underline{x}_l), \underline{\mu}_{i_1} v\underline{h}v^{-1} \underline{\mu}_{i_1}^{-1}, \ldots, \underline{\mu}_{i_m} vhv^{-1} \underline{\mu}_{i_m}^{-1}),$$

where W, $V_1, \ldots, V_k$ are words, $\mu_{i_1}[r] \subset s, \ldots, \mu_{i_m}[r] \subset s$. Thus $W_{i,s}(x)$, $i \in \boldsymbol{N}$, denotes the elements of L_s.

An element $x \in M$ is in $C_M(K_r)$ and satisfies (9), (13), (11), where $(L_s \mid s \subset I)$ is defined by (12), iff x is a solution of the following r.e. system of implications with coefficients $x_1, \ldots, x_l$, h.

(14)
$$\begin{aligned}
&[v, W(\underline{x}_1, \ldots, \underline{x}_l)] \doteq e \; ; W \in \Omega_r. \\
&W_{i,s}(v) \doteq \underline{\mu}_j \rightarrow \underline{\mu}_j \doteq e \; ; i, j \in \boldsymbol{N}, s \in P_\omega(I). \\
&[v\underline{h}v^{-1}, \underline{\mu}_i] \doteq e \; ; i \in \boldsymbol{N}, \mu_i \in \Pi_r. \\
&\prod_{k \le n} (\underline{\mu}_{m_k} W_{i, s \cup t}(v)\, \underline{\mu}_{m_k}^{-1})^{\varepsilon_k} \doteq e \rightarrow [W_{i, s\cup t}(v), \underline{\mu}_j] \doteq e \; ; s, t \in P_\omega(I), \\
&m_0, \ldots, m_n, i, j \in \boldsymbol{N}, \varepsilon_0, \ldots, \varepsilon_n \in \{1, -1\}, \text{ s.t. } \mu_{m_k} \in \Pi_s, \mu_j \in \Pi_s \text{ and} \\
&\mu_{m_k}[t] \cap \mu_{m_{k+1}}[t] = \varnothing \text{ for } k = 0, 1\ldots .
\end{aligned}$$

We have to show that (14) has a solution in M. In view of (1.3) it is enough to find

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an element x in a supergroup M' of M which is contained in $C_{M'}(K_r)$ and satisfies (9), (13), (11) (if $(L_s \mid s \subset I)$ is defined by (12), and M replaced by M').

Set $K = \bigcup \{K_s \mid s \subset I\}$. $\langle K, \Pi\rangle$ is a split extension of K, every $\pi \in \Pi$ determines an automorphism $\bar{\pi}$ of K by $\bar{\pi}(y) = \pi y \pi^{-1}$. Choose an extension H of K_r which is isomorphic to $\langle K_r, h\rangle$ by an isomorphism φ which leaves the elements of K_r fixed and maps h to h'. For every finite bijection f: $r \to s \subset I$, we choose an isomorphism $f^*: H \to H_f$ onto an extension H_f of K_s which extends $\bar{\pi}_{\restriction K_r}: K_r \to K_s$, where $f \subset \pi \in \Pi$. We take $\mathrm{id}_r^* = \mathrm{id}_H$. Let C be the free product of the K, H_f, $\operatorname{dom} f = r$, with amalgamated subgroups $K_{\operatorname{rng} f}$:

$$C = \prod_{\operatorname{dom} f = r}^{*} (K, H_f; K_{\operatorname{rng} f}).$$

We extend $\bar{\pi}$ to an automorphism of C by $\bar{\pi}(f^*(y)) = (\pi f)^*(y)$, $y \in H$.

We can regard $\langle K, \Pi\rangle$ as a subgroup of D, the split extension of C w.r.t. the homomorphism $\pi \mapsto \bar{\pi}$.

Using the free product of M and D with amalgamated subgroup $\langle K, \Pi\rangle$, we find a common supergroup M' of D and M. By the Higmann-Neumann-Neumann theorem, we can assume that M' contains an element x which yields an inner automorphism extending φ, i.e. $x \in C_{M'}(K_r)$ and $xhx^{-1} = h'$.

If we define $(L_s \mid s \subset I)$ by (12), (13) and (9) are clearly satisfied. To prove (11), we define for all $r' \subset I$, $|r| = |r'|$,

$$F_{r'} = \langle H_f \mid f: r \to r'\rangle.$$

Obviously

(15) $C = \prod^*_{|r'|=|r|} (K, F_{r'}; K_{r'})$, $F_{r'} \cap K = K_{r'}$ and $L_s = \langle K_s \cup \bigcup \{\mathscr{F}_{r'} \mid r' \subset s, |r'| = |r|\}\rangle$.

Suppose now s, $t \in P_\omega(I)$; $\pi_0, \ldots, \pi_n \in \Pi_s$ s.t. $\pi_k[t] \cap \pi_{k+1}[t] = \varnothing$ for $k = 0, 1, \ldots, n-1$; $d \in L_{s \cup t}$; $\varepsilon_0, \ldots, \varepsilon_n \in \{1, -1\}$. We want to show that $c \neq e$ or $d \in C_{M'}(\Pi_s)$, where $c = \prod_{k \leq n} (\pi_k d \pi_k^{-1})^{\varepsilon_k}$. We write d in normal form w.r.t. (15)

$$d = c_0 c_1 \cdots c_m\ ; \quad c_i \in K \cup \bigcup \{F_{r'} \mid |r'| = |r|\};$$

and use the abbreviations $c_{i,k}$ for $\pi_k c_i \pi_k^{-1}$, d_k for $\pi_k d \pi_k^{-1}$. We distinguish five cases.

Case 1. All c_i, $i = 0, \ldots, m$, are in $C_{M'}(\Pi_s)$. Then $d \in C_{M'}(\Pi_s)$.

We assume in the sequel that there is a $c_i \notin C_{M'}(\Pi_s)$. In fact we can assume that $c_m \notin C_{M'}(\Pi_s)$. (Otherwise we replace d by a conjugate bdb^{-1}, $b \in C_{M'}(\Pi_s)$.)

Case 2. There is $c_l \in F_{r'} \setminus K$, $r' \subset s \cup t$, $r' \not\subset s$. Then, passing to the normal form of $\prod_{k \leq n} (c_{0,k} \cdots c_{m,k})^{\varepsilon_k}$, the $c_{l,k}^{\varepsilon_k}$ do not vanish. Whence

$$c = \prod_{k \leq n} d_k^{\varepsilon_k} \neq e.$$

We can now assume that $c_k \notin K \Rightarrow c_k \in F_{r'}$, $r' \subset s \Rightarrow c_k \in C_{M'}(\Pi_s)$. Let l be the smallest number s.t. $c_l \notin C_{M'}(\Pi_s)$.

Case 3. $l < m$. It follows $l + 1 < m$. If $c_0 \notin K$, $k \leq n$, $c_{0,k} \cdots c_{m,k} c_{0,k+1} \cdots c_{m,k+1}$ is a normal form of $d_k d_{k+1}$. If $c_0 \in K$, $k \leq n$, $c_{0,k} \cdots c_{m-1,k}(c_{m,k}\, c_{0,k+1}) c_{1,k+1} \cdots c_{m,k+1}$ is a normal form of $d_k d_{k+1}$. (For, $c_{m-1,k} \in F_{r'} \setminus K$, $c_{1,k+1} \in F_{r''} \setminus K$, r', $r'' \subset s$. But $(c_{m,k}\, c_{0,k+1}) \notin C_{M'}(\Pi_s)$, since otherwise we would have by (6.3.1), $c_{m,k} \in C_{M'}(\Pi_{s \cup \pi_k[t]}) \cap C_{M'}(\Pi_{s \cup \pi_{k+1}[t]}) = C_{M'}(\Pi_s)$.)

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Similarily one verifies that $c_{m,k}^{-1}c_{m-1,k}^{-1} \cdots (c_{l,k}^{-1}c_{l,k+1})c_{l+1,k+1} \cdots c_{m,k+1}$ is a normal form of $d_k^{-1}d_{k+1}$ and that $c_{0,k} \cdots (c_{m,k}c_{m,k+1}^{-1}) \cdots c_{0,k+1}^{-1}$ is a normal form of $d_k d_{k+1}^{-1}$.

Gluing all these normal forms together (and the related normal form of $d_k^{-1}d_{k+1}^{-1}$), one obtains a nontrivial normal form of $\prod_{k\le n}(c_{0,k} \cdots c_{m,k})^{\varepsilon_k}$, whence $c \ne e$.

Case 4. $l = m = 1$. Then $d \in K_{s\cup t}$ and $c \ne e$ or $d \in C_M(\Pi_s)$ follows from the hypothesis.

Case 5. $1 < l = m$. We look at the cancellation in the product

$$d_{k+1}^{-1}d_{k+2}d_{k+3}^{-1} \cdots d_{k'-2}^{-1}d_{k'-1}, \qquad k \le n, \quad k' - k \text{ odd},$$

alternating exponents. This product equals $b = c_{m,k+1}^{-1}c_{m,k+2}c_{m,k+3}^{-1} \cdots c_{m,k'-2}^{-1}c_{m,k'-1}$. By (6.3.2), $b \notin C_{M'}(\Pi_s)$. This shows that $c_{0,k} \cdots c_{m-1,k}(c_{m,k}\, bc_{m,k'}^{-1})c_{m-1,k'}^{-1} \cdots c_{0,k'}^{-1}$ is a normal form of $d_k(d_{k+1}^{-1} \cdots d_{k'-1})d_{k'}^{-1}$; $c_{0,k} \cdots (c_{m,k}b)c_{0,k'} \cdots c_{m,k'}$ or $c_{0,k} \cdots (c_{m,k}bc_{0,k'}) \cdots c_{m,k'}$ is a normal form of $d_k(d_{k+1}^{-1} \cdots)d_{k'}$; and $\cdots (c_{0,k}^{-1}bc_{0,k'}) \cdots$ or $\cdots c_{0,k}^{-1}bc_{0,k'} \cdots$ is a normal form of $d_k^{-1}(d_{k+1}^{-1} \cdots)d_{k'}$.

We produce now a normal form of $\prod_{k\le n}(c_{0,k} \cdots c_{m,k})^{\varepsilon_k}$ by using the above bracketings. We handle the products $d_k^{-1}(d_{k+1}^{-1} \cdots)d_{k'}^{-1}$ similarily, and the products $d_k d_{k+1}$, $d_k d_{k+1}^{-1}$, $d_k^{-1}d_{k+1}^{-1}$ as in Case 3. One observes that the brackets do not interfere. Thus our normal form is nontrivial and $c \ne e$.

This proves Lemma 6.2.

(6.4) Corollary. *There is a family* $(M_s \mid s \in P_\omega(I))$ *of subgroups of* M *which satisfies* (5), (6), (7), (8) *of* (2.5) ; (11) *of* (6.1) *and* $A_s \subset M_s$ *for all* $s \in P_\omega(I)$.

Proof. Let (h_i, r_i), $i \in N$, be an enumeration of $M \times P(I)$, where every pair occurs infinitely often. We define a sequence of admissible families $(L_s^i \mid s \subset I)$ s.t. $L_s^i \subset L_s^{i+1}$. We start with $L_s^0 = A_s$ and define $(L_s^{i+1} \mid s \subset I)$ by an application of (6.2) to h_i, r_i and $(L_s^i \mid s \subset I)$. Finally we set $M_s = \bigcup \{L_s^i \mid i \in N\}$. (8) holds by (1.2).

Let $\mathscr{M}$ be the operation which belongs to $(M_s \mid s \subset I)$ by (2.5). We extend $\mathscr{M}$ to an operation $\mathscr{N}$ of $J \supset I$—J a set of power κ—on N. N is algebraically closed and $\infty - \omega$-equivalent to M. As in the proof of Theorem 2, one sees that N contains a free-group of rank κ. Thus $|N| = \kappa$.

Let now λ be an uncountable regular cardinal and P a subgroup of N of power λ. Every $b \in N$ is of the form $\mathscr{N}f_b(d_b)$, where $f_b\colon s_b \to r_b \subset J$, $s_b \subset I$, $d_b \in M_{s_b}$.

Since λ is uncountable and regular, I and M countable, we find a subset P' of P of power λ s.t.:

the r_b, $b \in p'$, are almost disjoint *i.e.* the r_b can be represented as a union $r \cup t_b$, where the t_b, $b \in P'$, are disjoint;

the $f_b^{-1}[r]$ are all equal, say to $s \subset I$;

the $f_b^{-1}[t_b]$ are all equal, say to $t \subset I$;

the d_b are all equal to $d \in M_{s\cup t}$;

the functions $f_b^{-1} \mid_r$ are all equal to f'; $b \in P'$.

(6.5). For all $n \ge 1$, $b_0, \ldots, b_n \in P'$, $b_k \ne b_{k+1}$ for $k = 0, \ldots, n-1$, $\varepsilon_0, \ldots, \varepsilon_n \in \{1, -1\}$ we have $\prod_{k\le n} b_k^{\varepsilon_k} \ne e$.

Proof. We take an injection $f\colon r \cup t_{b_0} \cup \cdots \cup t_{b_n} \to I$ s.t. $f' \subset f$. Then we choose $\pi_0, \ldots, \pi_n \in \Pi$ s.t. $ff_{b_k} \subset \pi_k$. All π_k are in Π_s and $\pi_k[t] \cap \pi_{k+1}[t] = \varnothing$. Since there are two different $\pi_k d\pi_k^{-1} = \mathscr{N}f(b_k)$, d does not belong to $C_M(\Pi_s)$. By (11)

$$e \ne \prod_{k\le n} (\pi_k d\pi_k^{-1})^{\varepsilon_k} = \mathscr{N}f(\prod_{k\le n} b_k^{\varepsilon_k}).$$

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Let $\{c_\alpha, b_\alpha\}$, $\alpha < \lambda$ be a partition of P' in two-element subsets. Then the products $c_\alpha b_\alpha$ generate a free subgroup of P of power λ.

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HEBREW UNIVERSITY
JERUSALEM, ISRAEL

TECHNISCHE UNIVERSITÄT
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