Math. Log. Quart. **54**, No. 4, 340 – 344 (2008) / **DOI** 10.1002/malq.200710032

Groupwise density cannot be much bigger than the unbounded number

Saharon Shelah[*]

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
and Department of Mathematics, Hill Center – Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

Received 1 June 2007, revised 17 September 2007, accepted 18 September 2007
Published online 1 July 2008

Key words Cardinal characteristics of the continuum, unbounded number, groupwise density.
MSC (2000) 03E17, 03E05, 03E20

We prove that $\mathfrak{g}$ (the groupwise density number) is smaller or equal to $\mathfrak{b}^+$, the successor of the minimal cardinality of an unbounded subset of ${}^\omega\omega$. This is true even for the version of $\mathfrak{g}$ for groupwise dense ideals.

1 Introduction

In the present note we are interested in two cardinal characteristics of the continuum, the unbounded number $\mathfrak{b}$, and the groupwise density number $\mathfrak{g}$. The former cardinal belongs to the oldest and most studied cardinal invariants of the continuum (see, e. g., van Douwen [9] and Bartoszyński and Judah [2]) and it is defined as follows.

Definition 1.1

(a) The partial order $\leq_{J^{\rm bd}_\omega}$ on ${}^\omega\omega$ is defined by

$$f \leq_{J^{\rm bd}_\omega} g \quad \text{if and only if} \quad (\exists N < \omega)(\forall n > N)(f(n) \leq g(n)).$$

(b) *The unbounded number* $\mathfrak{b}$ is defined by

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \text{ has no } \leq_{J^{\rm bd}_\omega}\text{-upper bound in } {}^\omega\omega\}.$$

The groupwise density number $\mathfrak{g}$, introduced by Blass and Laflamme in [4], is perhaps less popular but it has gained substantial importance in the realm of cardinal invariants. For instance, it has been studied in connection with the cofinality $\mathrm{cf}(\mathrm{Sym}(\omega))$ of the symmetric group on the set ω of all integers, see Thomas [8] or Brendle and Losada [5]. The cardinal $\mathfrak{g}$ is defined as follows.

Definition 1.2

(a) We say that a family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* whenever

(a1) $B \subseteq A \in \mathcal{A}$, $B \in [\omega]^{\aleph_0}$ implies $B \in \mathcal{A}$;

(a2) for every increasing sequence $\langle m_i : i < \omega\rangle \in {}^\omega\omega$ there is an infinite set $\mathcal{U} \subseteq \omega$ such that

$$\bigcup\{[m_i, m_{i+1}) : i \in \mathcal{U}\} \in \mathcal{A}.$$

(b) *The groupwise density number* $\mathfrak{g}$ is the minimal cardinal θ for which there is a sequence $\langle \mathcal{A}_\alpha : \alpha < \theta\rangle$ of groupwise dense subsets of $[\omega]^{\aleph_0}$ such that

$$(\forall B \in [\omega]^{\aleph_0})(\exists \alpha < \theta)(\forall A \in \mathcal{A}_\alpha)(B \not\subseteq^* A).$$

(Recall that for infinite sets A and B, $A \subseteq^* B$ means $A \setminus B$ is finite.)

[*] e-mail: shelah@math.huji.ac.il

The unbounded number $\mathfrak{b}$ and the groupwise density number $\mathfrak{g}$ can be in either order, see Blass [3] and more Mildenberger and Shelah [7, 6], the latter article gives a bound on $\mathfrak{g}$. However, as we show in Theorem 2.3, $\mathfrak{g}$ cannot be bigger than $\mathfrak{b}^+$.

Notation 1.3 Our notation is rather standard and compatible with that of classical textbooks on set theory (like Bartoszyński and Judah [2]). We will keep the following rules concerning the use of symbols.

1. $A, B, \mathcal{U}$ (with possible sub- and superscripts) denote subsets of ω, infinite if not said otherwise.
2. m, n, ℓ, k, i, j are natural numbers; $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \zeta$ are ordinals, θ is a cardinal.

2 The result

Lemma 2.1 *For some cardinal $\theta \le \mathfrak{b}$ there is a sequence $\langle B_{\zeta,t} : \zeta < \theta, t \in I_\zeta\rangle$ such that the following hold:*

(a) $B_{\zeta,t} \in [\omega]^{\aleph_0}$.

(b) *If $\zeta < \theta$ and $s \ne t$ are from I_ζ, then $B_{\zeta,s} \cap B_{\zeta,t}$ is finite* (*so* $|I_\zeta| \le 2^{\aleph_0}$).

(c) *For every $B \in [\omega]^{\aleph_0}$ the set $\{(\zeta, t) : \zeta < \theta \,\&\, t \in I_\zeta \,\&\, B_{\zeta,t} \cap B$ is infinite$\}$ is of cardinality $2^{\aleph_0}$.*

Proof. This lemma is a weak version of the celebrated base-tree theorem of Bohuslav Balcar and Petr Simon with $\theta = \mathfrak{h}$ which is known to be $\le \mathfrak{b}$, see Balcar and Simon [1, Theorem 3.4, p. 350]. However, for the sake of completeness of our exposition, let us present a proof.

Let $\langle f_\zeta : \zeta < \mathfrak{b}\rangle$ be a $\le_{J^{\rm bd}_\omega}$-increasing sequence of members of ${}^\omega\omega$ with no $\le_{J^{\rm bd}_\omega}$-upper bound in ${}^\omega\omega$. Moreover we demand that each f_ζ is increasing (clearly, this does not change $\mathfrak{b}$). By induction on $\zeta < \mathfrak{b}$ choose sets $\mathcal{T}_\zeta$ and systems $\langle B_{\zeta,\eta} : \eta \in \mathcal{T}_{\zeta+1}\rangle$ such that the following hold:

(i) $\mathcal{T}_\zeta \subseteq {}^\zeta(2^{\aleph_0})$, and if $\eta \in \mathcal{T}_{\zeta+1}$, then $B_{\zeta,\eta} \in [\omega]^{\aleph_0}$.

(ii) If $\eta \in \mathcal{T}_\zeta$ and $\varepsilon < \zeta$, then $\eta \restriction \varepsilon \in \mathcal{T}_\varepsilon$.

(iii) If ζ is a limit ordinal, then

$$\mathcal{T}_\zeta = \{\eta \in {}^\zeta(2^{\aleph_0}) : (\forall \varepsilon < \zeta)(\eta \restriction \varepsilon \in \mathcal{T}_\varepsilon) \,\&\, (\exists A \in [\omega]^{\aleph_0})(\forall \varepsilon < \zeta)(A \subseteq^* B_{\varepsilon,\eta\restriction(\varepsilon+1)})\}.$$

(iv) If $\varepsilon < \zeta$ and $\eta \in \mathcal{T}_{\zeta+1}$, then $B_{\zeta,\eta} \subseteq^* B_{\varepsilon,\eta\restriction(\varepsilon+1)}$.

(v) For $\eta \in \mathcal{T}_{\zeta+1}$ and $m_1 < m_2$ from $B_{\zeta,\eta}$ we have $f_\zeta(m_1) < m_2$.

(vi) If $\eta \in \mathcal{T}_\varepsilon$, then the set $\{B_{\varepsilon,\nu} : \eta \lhd \nu \in \mathcal{T}_{\varepsilon+1}\}$ is an infinite maximal subfamily of

$$\{A \in [\omega]^{\aleph_0} : (\forall \xi < \varepsilon)(A \subseteq^* B_{\xi,\eta\restriction(\xi+1)})\}$$

consisting of pairwise almost disjoint sets.

It should be clear that the choice is possible. Note that for some limit $\zeta < \mathfrak{b}$ we may have $\mathcal{T}_\zeta = \emptyset$ (and then also $\mathcal{T}_\xi = \emptyset$ for $\xi > \zeta$). Also, if we define $\mathcal{T}_\mathfrak{b}$ as in (iii), then it will be empty (remember clause (v) and the choice of $\langle f_\zeta : \zeta < \mathfrak{b}\rangle$).

The lemma will readily follow from the following fact.

Fact 2.2 *For every $A \in [\omega]^{\aleph_0}$ there is $\xi < \mathfrak{b}$ such that $|\{\eta \in \mathcal{T}_{\xi+1} : B_{\xi,\eta} \cap A$ is infinite$\}| = 2^{\aleph_0}$.*

To show Fact 2.2 let $A \in [\omega]^{\aleph_0}$ and define

$$S = \bigcup_{\zeta<\mathfrak{b}} \{\eta \in \mathcal{T}_\zeta : (\forall \varepsilon < \zeta)(A \cap B_{\varepsilon,\eta\restriction(\varepsilon+1)} \text{ is infinite})\}.$$

Clearly S is closed under taking the initial segments and $\langle\rangle \in S$. By the "maximal" in clause (vi), we have that

($\circledast$) if $\eta \in S \cap \mathcal{T}_\zeta$, where $\zeta < \mathfrak{b}$ is non-limit or $\mathrm{cf}(\zeta) = \aleph_0$, then $(\exists \nu)(\eta \lhd \nu \in \mathcal{T}_{\zeta+1} \cap S)$.

Now if $\eta \in S$ and $\lg(\eta)$ is non-limit or $\mathrm{cf}(\lg(\eta)) = \aleph_0$, then there are $\lhd$-incomparable $\nu_0, \nu_1 \in S$ extending η, i. e., $\eta \lhd \nu_0$ and $\eta \lhd \nu_1$. [Why? As otherwise $S_\eta = \{\nu \in S : \eta \trianglelefteq \nu\}$ is linearly ordered by $\lhd$, so let $\varrho = \bigcup S_\eta$. It follows from ($\circledast$) that $\lg(\varrho) > \lg(\eta)$ is a limit ordinal (of uncountable cofinality). Moreover, by (iv) + (vi),

$$\lg(\eta) \le \varepsilon < \lg(\varrho) \Rightarrow A \cap B_{\lg(\eta),\varrho\restriction(\lg(\eta)+1)} =^* A \cap B_{\varepsilon,\varrho\restriction(\varepsilon+1)}.$$

Hence, by (iii) + (ii), $\varrho \in \mathcal{T}_{\lg(\varrho)}$, so necessarily $\lg(\varrho) < \mathfrak{b}$. Using (vi) again we may conclude that there is $\varrho' \in S$ properly extending ϱ, getting a contradiction.]

Consequently, we may find a system $\langle \eta_\varrho : \varrho \in {}^{\omega>}2 \rangle \subseteq S$ such that for every $\varrho \in {}^{\omega>}2$

1. $k < \lg(\varrho) \Rightarrow \eta_{\varrho \restriction k} \lhd \eta_\varrho$,
2. $\eta_{\varrho \frown \langle 0 \rangle}, \eta_{\varrho \frown \langle 1 \rangle}$ are $\lhd$-incomparable.

For $\varrho \in {}^{\omega>}2$ let

$$\zeta(\varrho) = \sup\{\lg(\eta_\nu) : \varrho \trianglelefteq \nu \in {}^{\omega>}2\}.$$

Pick ϱ such that $\zeta(\varrho)$ is the smallest possible (note that $\mathrm{cf}(\zeta(\varrho)) = \aleph_0$). Now it is possible to choose a perfect subtree T^* of ${}^{\omega>}2$ such that

$$\nu \in \lim(T^*) \Rightarrow \sup\{\lg(\eta_{\nu \restriction n}) : n < \omega\} = \zeta(\varrho).$$

We finish by noting that for every $\nu \in \lim(T^*)$ we have that

$$\bigcup\{\eta_{\nu \restriction n} : n < \omega\} \in \mathcal{T}_{\zeta(\varrho)} \cap S$$

and there is $\eta^* \in \mathcal{T}_{\zeta(\varrho)+1} \cap S$ extending $\bigcup\{\eta_{\nu \restriction n} : n < \omega\}$. □

Theorem 2.3 $\mathfrak{g} \leq \mathfrak{b}^+$.

Proof. Assume towards contradiction that $\mathfrak{g} > \mathfrak{b}^+$.

Let $\langle f_\alpha : \alpha < \mathfrak{b} \rangle \subseteq {}^\omega\omega$ be a $\leq_{J^{\mathrm{bd}}_\omega}$-increasing sequence with no $\leq_{J^{\mathrm{bd}}_\omega}$-upper bound. We also demand that all functions f_α are increasing and $f_\alpha(n) > n$ for $n < \omega$. Fix a list $\langle \bar{m}_\xi : \xi < 2^{\aleph_0} \rangle$ of all sequences

$$\bar{m} = \langle m_i : i < \omega \rangle$$

such that $0 = m_0$ and $m_i + 1 < m_{i+1}$.

For $\alpha < \mathfrak{b}$ we define:

$$n_{\alpha,0} = 0, \qquad n_{\alpha,i+1} = f_\alpha(n_{\alpha,i}) \quad (\text{for } i < \omega), \qquad \bar{n}_\alpha = \langle n_{\alpha,i} : i < \omega \rangle;$$
$$\bar{n}^0_\alpha = \langle 0, n_{\alpha,2}, n_{\alpha,4}, \ldots \rangle = \langle n^0_{\alpha,i} : i < \omega \rangle, \qquad \bar{n}^1_\alpha = \langle 0, n_{\alpha,3}, n_{\alpha,5}, n_{\alpha,7}, \ldots \rangle = \langle n^1_{\alpha,i} : i < \omega \rangle.$$

Observe that if $\bar{m} \in {}^\omega\omega$ is increasing, then for every large enough $\alpha < \mathfrak{b}$ we have:

(a) $(\exists^\infty i < \omega)(m_{i+1} < f_\alpha(m_i))$, and hence

(b) for at least one $\ell \in \{0, 1\}$ we have

$$(\exists^\infty i < \omega)(\exists j < \omega)([m_i, m_{i+1}) \subseteq [n^\ell_{\alpha,j}, n^\ell_{\alpha,j+1})).$$

Now for $\xi < 2^{\aleph_0}$ we put:

$$\gamma(\xi) = \min\{\alpha < \mathfrak{b} : (\exists^\infty i < \omega)(f_\alpha(m_{\xi,i}) > m_{\xi,i+1})\};$$
$$\ell(\xi) = \min\{\ell \leq 1 : (\exists^\infty i < \omega)(\exists j < \omega)([m_{\xi,i}, m_{\xi,i+1}) \subseteq [n^\ell_{\gamma(\xi),j}, n^\ell_{\gamma(\xi),j+1}))\};$$
$$\mathcal{U}^1_\xi = \{i < \omega : (\exists j < \omega)([m_{\xi,i}, m_{\xi,i+1}) \subseteq [n^{\ell(\xi)}_{\gamma(\xi),j}, n^{\ell(\xi)}_{\gamma(\xi),j+1}))\}.$$

Note that $\gamma(\xi)$ is well defined by (a), and so also $\ell(\xi)$ is well defined (by (b)). Plainly, $\mathcal{U}^1_\xi$ is an infinite subset of ω. Now for each $\xi < 2^{\aleph_0}$, we may choose $\mathcal{U}^2_\xi$ so that $\mathcal{U}^2_\xi \subseteq \mathcal{U}^1_\xi$ is infinite and for any $i_1 < i_2$ from $\mathcal{U}^2_\xi$ we have

$$(\exists j < \omega)(m_{\xi,i_1+1} < n^{\ell(\xi)}_{\gamma(\xi),j} \,\&\, n^{\ell(\xi)}_{\gamma(\xi),j+1} < m_{\xi,i_2}).$$

Let a function $g_\xi : \mathcal{U}^2_\xi \longrightarrow \omega$ be such that

$(*)_1$ $i \in \mathcal{U}^2_\xi \,\&\, g_\xi(i) = j \Rightarrow [m_{\xi,i}, m_{\xi,i+1}) \subseteq [n^{\ell(\xi)}_{\gamma(\xi),j}, n^{\ell(\xi)}_{\gamma(\xi),j+1})$.

Clearly, g_ξ is well defined and one-to-one. (This is very important, since it makes sure that the set $g_\xi[\mathcal{U}^2_\xi]$ is infinite.)

Fix a sequence $\bar{B} = \langle B_{\zeta,t} : \zeta < \theta, t \in I_\zeta \rangle$ given by Lemma 2.1 (so $\theta \le \mathfrak{b}$ and $\bar{B}$ satisfies the demands in Lemma 2.1(a) – (c)). By Lemma 2.1(c), for every $\xi < 2^{\aleph_0}$, the set

$$\{(\zeta, t) : \zeta < \theta \text{ and } t \in I_\zeta \text{ and } B_{\zeta,t} \cap g_\xi[\mathcal{U}^2_\xi] \text{ is infinite}\}$$

has cardinality continuum.

Now for each $\beta < \mathfrak{b}^+$ and $\xi < 2^{\aleph_0}$ we choose a pair $(\zeta_{\beta,\xi}, t_{\beta,\xi})$ such that

$(*)_2$ $\zeta_{\beta,\xi} < \theta$ and $t_{\beta,\xi} \in I_{\zeta_{\beta,\xi}}$,

$(*)_3$ $B_{\zeta_{\beta,\xi},t_{\beta,\xi}} \cap g_\xi[\mathcal{U}^2_\xi]$ is infinite, and

$(*)_4$ $t_{\beta,\xi} \notin \{t_{\alpha,\varepsilon} : \varepsilon < \xi \text{ or } \varepsilon = \xi \,\&\, \alpha < \beta\}$.

To carry out the choice we proceed by induction *first* on $\xi < 2^{\aleph_0}$, then on $\beta < \mathfrak{b}^+$. As there are $2^{\aleph_0}$ pairs (ζ, t) satisfying clauses $(*)_2 + (*)_3$, whereas clause $(*)_4$ excludes $\le (\mathfrak{b}^+ + |\xi|) \times \theta < 2^{\aleph_0}$ pairs (recalling that towards contradiction we are assuming $\mathfrak{b}^+ < \mathfrak{g} \le 2^{\aleph_0}$), there is such a pair at each stage $(\beta, \xi) \in \mathfrak{b}^+ \times 2^{\aleph_0}$.

Lastly, for $\beta < \mathfrak{b}^+$ and $\xi < 2^{\aleph_0}$ we let

$(*)_5$ $\mathcal{U}_{\beta,\xi} = g_\xi^{-1}[B_{\zeta_{\beta,\xi},t_{\beta,\xi}}] \cap \mathcal{U}^2_\xi$

(it is an infinite subset of ω) and we put

$(*)_6$ $A^+_{\beta,\xi} = \bigcup\{[m_{\xi,i}, m_{\xi,i+1}) : i \in \mathcal{U}_{\beta,\xi}\}$, and $\mathcal{A}_\beta = \{A \in [\omega]^{\aleph_0} : \text{for some } \xi < 2^{\aleph_0} \text{ we have } A \subseteq A^+_{\beta,\xi}\}$.

By the choice of $\langle \bar{m}_\xi : \xi < 2^{\aleph_0}\rangle$, $A^+_{\beta,\xi}$, and $\mathcal{A}_\beta$ one easily verifies that for each $\beta < \mathfrak{b}^+$, $\mathcal{A}_\beta$ is a groupwise dense subset of $[\omega]^{\aleph_0}$. Since we are assuming towards contradiction that $\mathfrak{g} > \mathfrak{b}^+$, there is an infinite $B \subseteq \omega$ such that

$$(\forall \beta < \mathfrak{b}^+)(\exists A \in \mathcal{A}_\beta)(B \subseteq^* A).$$

Hence for every $\beta < \mathfrak{b}^+$ we may choose $\xi(\beta) < 2^{\aleph_0}$ such that $B \subseteq^* A^+_{\beta,\xi(\beta)}$. Plainly,

$$\gamma(\xi(\beta)) < \mathfrak{b} \quad \text{and} \quad \zeta_{\beta,\xi(\beta)} < \theta \le \mathfrak{b} \quad \text{and} \quad \ell(\xi(\beta)) \in \{0, 1\},$$

and therefore for some triple $(\gamma^*, \zeta^*, \ell^*)$ the set

$$W := \{\beta < \mathfrak{b}^+ : (\gamma(\xi(\beta)), \zeta_{\beta,\xi(\beta)}, \ell(\xi(\beta))) = (\gamma^*, \zeta^*, \ell^*)\}$$

is unbounded in $\mathfrak{b}^+$. Note that if $\beta \in W$, then

(1)
$$\begin{aligned} B &\subseteq^* A^+_{\beta,\xi(\beta)} \\ &= \bigcup\{[m_{\xi(\beta),i}, m_{\xi(\beta),i+1}) : i \in \mathcal{U}_{\beta,\xi(\beta)}\} \\ &\subseteq \bigcup\{[n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j}, n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j+1}) : j = g_{\xi(\beta)}(i) \text{ for some } i \in \mathcal{U}_{\beta,\xi(\beta)}\} \\ &\subseteq \bigcup\{[n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j}, n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j+1}) : j \in B_{\zeta_{\beta,\xi(\beta)},t_{\beta,\xi(\beta)}}\}. \end{aligned}$$

[Why? By the choice of $(\beta, \xi(\beta))$, by $(*)_6$, and by $(*)_1$ as $\mathrm{Dom}(g_{\xi(\beta)}) \subseteq \mathcal{U}_{\beta,\xi(\beta)} \subseteq \mathcal{U}^2_{\beta,\xi(\beta)}$; by $(*)_5$.]

Also, for $\beta \in W$ we have $\ell(\xi(\beta)) = \ell^*$, $\gamma(\xi(\beta)) = \gamma^*$, and $\zeta(\beta, \xi(\beta)) = \zeta^*$, so it follows from (1) that

$$B \subseteq^* \bigcup\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\beta,\xi(\beta)}}\}$$

for every $\beta \in W$.

Consequently, if $\beta \ne \alpha$ are from W, then the sets

$$\bigcup\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\beta,\xi(\beta)}}\} \quad \text{and} \quad \bigcup\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\alpha,\xi(\alpha)}}\}$$

are *not* almost disjoint. Hence, as $\langle n^{\ell^*}_{\gamma^*,j} : j < \omega\rangle$ is increasing, necessarily the sets $B_{\zeta^*,t_{\beta,\xi(\beta)}}$ and $B_{\zeta^*,t_{\alpha,\xi(\alpha)}}$ are not almost disjoint. So applying Lemma 2.1(b) we conclude that $t_{\beta,\xi(\beta)} = t_{\alpha,\xi(\alpha)}$. But this contradicts $\beta \ne \alpha$ by $(*)_4$, and we are done. □

Definition 2.4 We define a *cardinal characteristic* $\mathfrak{g}_\mathfrak{f}$ as the minimal cardinal θ for which there exists a sequence $\langle \mathcal{I}_\alpha : \alpha < \theta \rangle$ of groupwise dense *ideals* of $\mathcal{P}(\omega)$ (i. e., $\mathcal{I}_\alpha \subseteq [\omega]^{\aleph_0}$ is groupwise dense and $\mathcal{I}_\alpha \cup [\omega]^{<\aleph_0}$ is an ideal of subsets of ω) such that

$$(\forall B \in [\omega]^{\aleph_0})(\exists \alpha < \theta)(\forall A \in \mathcal{A}_\alpha)(B \not\subseteq^* A).$$

Observation 2.5 $2^{\aleph_0} \geq \mathfrak{g}_\mathfrak{f} \geq \mathfrak{g}$.

Theorem 2.6 $\mathfrak{g}_\mathfrak{f} \leq \mathfrak{b}^+$.

Proof. We repeat the proof of Theorem 2.3. However, for $\beta < \mathfrak{b}^+$ the family $\mathcal{A}_\beta \subseteq [\omega]^{\leq\aleph_0}$ does not have to be an ideal. So let $\mathcal{I}_\beta$ be an ideal on $\mathcal{P}(\omega)$ generated by $\mathcal{A}_\beta$ – so also $\mathcal{I}_\beta$ is the ideal generated by

$$\{A^+_{\beta,\xi} : \xi < 2^{\aleph_0}\} \cup [\omega]^{<\aleph_0}.$$

Lastly, let $\mathcal{I}'_\beta = \mathcal{I}_\beta \setminus [\omega]^{<\aleph_0}$.

Assume towards contradiction that $B \in [\omega]^{\aleph_0}$ is such that

$$(\forall \alpha < \mathfrak{b}^+)(\exists A \in \mathcal{I}_\alpha)(B \subseteq^* A).$$

So for each $\beta < \mathfrak{b}^+$ we can find $k_\beta < \omega$ and $\xi(\beta, 0) < \xi(\beta, 1) < \cdots < \xi(\beta, k_\beta) < 2^{\aleph_0}$ such that

$$B \subseteq^* \bigcup\{A^+_{\beta,\xi(\beta,k)} : k \leq k_\beta\}.$$

Let D be a non-principal ultrafilter on ω to which B belongs. Then for every $\beta < \mathfrak{b}^+$ there exists $k(\beta) \leq k_\beta$ such that $A^+_{\beta,\xi(\beta,k(\beta))} \in D$. As in the proof there for some $(\gamma^*, \zeta^*, \ell^*, k^*, k(*))$ the following set is unbounded in $\mathfrak{b}^+$:

$$W := \{\beta < \mathfrak{b}^+ : k(\beta) = k(*), k_\beta = k^*, \gamma_{\xi(\beta,k(*))} = \gamma^*, \zeta_{\beta,\xi(\beta,k(*))} = \zeta^*, \text{ and } \ell(\xi(\beta, k(*))) = \ell^*\}.$$

As there it follows that if $\beta \in W$, then

$$\bigcup\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\beta,\xi(\beta,k(*))}}\}$$

belongs to D. But for $\beta \neq \alpha \in W$ those sets are not almost disjoint, whereas $(\zeta^*, t_{\beta,\xi(\beta,k(*))}) \neq (\zeta^*, t_{\alpha,\xi(\alpha,k(*))})$ are distinct, giving us a contradiction. □

Acknowledgements We would like to thank Shimoni Garti and the anonymous referee for corrections.

The author acknowledges support from the United States-Israel Binational Science Foundation (Grant no. 2002323). This is publication 887 of the author.

References

[1] B. Balcar and P. Simon, Disjoint refinement. In: Handbook of Boolean Algebras, volume 2 (D. Monk and R. Bonnet, eds.), pp. 333 – 388 (North-Holland, 1989).
[2] T. Bartoszyński and H. Judah, Set Theory: On the Structure of the Real Line (AK Peters, 1995).
[3] A. Blass, Applications of superperfect forcing and its relatives. In: Set Theory and Its Applications. Lecture Notes in Mathematics 1401, pp. 18 – 40 (Springer, 1989).
[4] A. Blass and C. Laflamme, Consistency results about filters and the number of inequivalent growth types. J. Symbolic Logic **54**, 50 – 56 (1989).
[5] J. Brendle and M. Losada, The cofinality of the infinite symmetric group and groupwise density. J. Symbolic Logic **68**, 1354 – 1361 (2003).
[6] H. Mildenberger and S. Shelah, Increasing the groupwise density number by c. c. c. forcing. To appear in Annals of Pure and Applied Logic. Preprint available at `http://www.logic.univie.ac.at/~heike/843rev3.pdf`.
[7] H. Mildenberger and S. Shelah, The relative consistency of $\mathfrak{g} < \mathrm{cf}(\mathrm{Sym}(\omega))$. J. Symbolic Logic **67**, 297 – 314 (2002).
[8] S. Thomas, Groupwise density and the confinality of the infinite symmetric group. Arch. Math. Logic **37**, 483 – 493 (1998).
[9] E. K. van Douwen, The integers and topology. In: Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), pp. 111 – 167 (Elsevier Science Publishers, 1984).