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MORE ON COUNTABLY COMPACT, LOCALLY COUNTABLE SPACES

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ABSTRACT

Following [5], a T_3 space X is called good (splendid) if it is countably compact, locally countable (and ω-fair). $G(\kappa)$ (resp. $S(\kappa)$) denotes the statement that a good (resp. splendid) space X with $|X| = \kappa$ exists. We prove here that (i) Con(ZF) $\rightarrow$ Con(ZFC + MA + 2^ω is big + $S(\kappa)$ holds unless $\omega = \mathrm{cf}(\kappa) < \kappa$); (ii) a supercompact cardinal implies Con(ZFC + MA + $2^\omega > \omega_{\omega+1} + \neg G(\omega_{\omega+1})$); (iii) the "Chang conjecture" $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ implies $\neg S(\kappa)$ for all $\kappa \geqq \omega_\omega$; (iv) if $\mathcal{P}$ adds ω_1 dominating reals to V iteratively then, in $V^{\mathcal{P}}$, we have $G(\lambda^\omega)$ for all λ.

§0. Introduction

In this paper we countinue the investigations started in [5] concerning the following problem first raised by E. van Douwen [3]: What can be the cardinality of a countably compact, locally countable T_3 space?

Let us recall some notation and terminology from [5]: A T_3 space X is called *good* if it is both countably compact and locally countable, and it is called *splendid* if in addition it is also ω-fair. (A space X is called *κ-fair* if for every $Y \in [X]^\kappa$ we have $|\bar{Y}| = \kappa$ as well.) $G(\kappa)$ (resp. $S(\kappa)$) denotes the statement that a good (resp. splendid) space of cardinality κ exists.

The main results of [5] may now be summarized as follows:

0.1. For $\kappa > \omega$, $G(\kappa)$ implies $\mathrm{cf}(\kappa) \neq \omega$, moreover if $\kappa > 2^\omega$ then even $\kappa^\omega = \kappa$.

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0.2. For all $n \in \omega$ we have $S(\omega_n)$.

0.3. Martin's axiom implies $G((2^\omega)^{+n})$ for each $n \in \omega$.

0.4. If $V = L$ then $S(\kappa)$ is valid unless $\mathrm{cf}(\kappa) = \omega < \kappa$.

Recently, P. Nyikos has observed that the proof of 0.4 in [5], with practically no alternations, actually yields the same conclusion if one only uses the following consequence of $V = L$: if $\mathrm{cf}(\kappa) = \omega < \kappa$ then

(a) the cofinality of $[\kappa]^\omega$ under inclusion is κ^+, i.e. there is $\mathscr{A} \subset [\kappa]^\omega$ with $|\mathscr{A}| = \kappa^+$ such that every member of $[\kappa]^\omega$ is contained in some member of $\mathscr{A}$ (of course, if $\kappa > 2^\omega$ this implies $\kappa^\omega = \kappa^+$);

(b) $\square_\kappa$ holds.

Since (a) and (b) are also valid if one only assumes that the covering lemma holds over the core model, cf. [1] or [2], it is clear that large (in particular, many measurable) cardinals are needed if one intends to build a model in which the conclusion of 0.4 fails.

§1. Good spaces of size less than 2^ω

The results in [5] left the following problem open: Can $G(\kappa)$ be valid for some κ with $\omega_\omega < \kappa < 2^\omega$? In this section we are going to give a complete answer to this question.

The first half of this answer is based on the following simple lemma.

1.1. Lemma. *For any κ, $S(\kappa)$ is preserved under CCC forcing.*

Proof. Let X be a splendid space (of cardinality κ) in V and Q be any CCC notion of forcing; we claim that X remains splendid in V^Q. Since local countability and the T_3 property are obviously preserved in any extension of V, it remains only to show that X will remain countably compact and ω-fair in V^Q.

To see this, let A be any countable subset of X in V^Q. Since Q is CCC, there is a countable $B \subset X$ in V with $A \subset B$. Now $\bar{B}$ is a countable compact T_3 space in V, hence it is homoemorphic to a countable successor ordinal with its order topology. Consequently, $\bar{B}$ will trivially remain compact hence closed in V^Q, showing both that A has a limit point and that $\bar{A} \subset \bar{B}$ is countable.

Now, from 0.4 and 1.1 we immediately obtain the following corollary that gives an affirmative answer to the problem explicitly formulated on p. 206 of [5].

1.2. Corollary. *If ZF is consistent, so is "ZFC + MA + 2^ω is as large as you wish + $S(\kappa)$ is valid unless κ is singular of cofinality ω". In particular, we see that MA plus $\omega_{\omega+1} < 2^\omega$ is consistent with $S(\omega_{\omega+1})$.*

Now, in order to get consistency results going in the opposite direction we introduce the following definition.

1.3. Definition. For $\kappa \geqq \omega_\omega$ let $P(\kappa)$ denote the following statement: for any collection $\mathscr{A} \subset [\omega_\omega]^\omega$ with $|\mathscr{A}| = \kappa$ there is some $B \in [\omega_\omega]^\omega$ such that $|A \cap B| < \omega$ for all $A \subset \mathscr{A}$.

The reason for giving this definition is the following trivial observation: if $P(\kappa)$ holds then $G(\kappa)$ fails. Indeed, assume X is a locally countable T_1 space with $|X| = \kappa$. Since, by the definition, $P(\kappa)$ implies $\kappa \geqq \omega_\omega$, we may assume that $\omega_\omega \subset X$. For every point $p \in X$ let us pick a countable neighbourhood U_p and apply $P(\kappa)$ to the collection $\{U_p \cap \omega_\omega : p \in X\}$. This gives us a set $B \in [\omega_\omega]^\omega \subset [X]^\omega$ for which $B \cap U_p$ is finite for every $p \in X$, hence B has no limit point in X, i.e. X is not countably compact.

Comparing this observation with our remark made at the end of §0, it is clear that if we want to show the consistency of $P(\kappa)$ for some $\kappa > \omega_\omega$ then large cardinals have to be used. Fortunately, this has been done for us by Magidor in [8], where, for $\kappa = \omega_{\omega+1}$, the assumption of our next implication was shown to be consistent from a supercompact cardinal.

1.4. Lemma. *Assume that*

$$2^\omega < \omega_\omega < \kappa < (\omega_\omega)^\omega.$$

Then $P(\kappa)$ is valid.

Proof. Let $\mathscr{A} \subset [\omega_\omega]^\omega$ with $|\mathscr{A}| = \kappa$. By an old result of Sierpinski [9] there is an almost disjoint collection $\mathscr{B} \subset [\omega_\omega]^\omega$ with

$$|\mathscr{B}| = \omega_\omega^\omega > |\mathscr{A}| = \kappa.$$

For each $A \in \mathscr{A}$ let us put

$$\mathscr{B}_A = \{B \in \mathscr{B} : |B \cap A| = \omega\}.$$

Since $\mathscr{B}$ is almost disjoint, we clearly have

$$|\mathscr{B}_A| \leqq 2^\omega < \omega_\omega,$$

consequently

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$$|\bigcup\{\mathscr{B}_A : A \in \mathscr{A}\}| \leqq \kappa.$$

But then for any $B \in \mathscr{B} \setminus \bigcup\{\mathscr{B}_A : A \in \mathscr{A}\}$ we have $|B \cap A| < \omega$ for all $A \subset \mathscr{A}$, hence the proof is completed.

Of course, 0.1 immediately implies that $G(\kappa)$ is false if $2^\omega < \omega_\omega < \kappa < \omega_\omega^\omega$. In order to put 1.4 to use we still need the following lemma.

1.5. LEMMA. *$P(\kappa)$ is preserved under any CCC forcing.*

PROOF. Let Q be any CCC notion of forcing and let, in V^Q, $f: \kappa \to [\omega_\omega]^\omega$ be a function enumerating an $\mathscr{A} \subset [\omega_\omega]$ with $|\mathscr{A}| = \kappa$. Now by a theorem of [6, p. 206] there is a function $F: \kappa \to [\omega_\omega]^\omega$ *in* V such that $f(\alpha) \subset F(\alpha)$ for all $\alpha \in \kappa$. Then we have a $B \in [\omega_\omega]^\omega$ such that $|B \cap F(\alpha)| < \omega$ for all $\alpha < \kappa$. Then we have $|A \cap B| < \omega$ for all $A \in \mathscr{A}$.

As an immediate corollary of this, Magidor's above-mentioned result, and 1.4 we get the following result.

1.6. COROLLARY. *If there is a supercompact cardinal then it is consistent to have Martin's axiom plus $\omega_{\omega+1} < 2^\omega$ plus the failure of $G(\omega_{\omega+1})$.*

§2. When all splendid spaces are small

In view of our remark made at the end of §0, large cardinals are needed if one wants to establish e.g. the consistency of the statement that the cardinalities of splendid spaces are bounded. Of course, by 0.2, the least possible such bound is ω_ω.

In this section our aim is to show that there is a reasonable assumption, first considered in [7] by Levinsky, Magidor and Shelah, which indeed implies that this is the case. This assumption is actually a model-theoretic statement, a case of Chang's conjecture, usually denoted by the symbol

$$(\omega_{\omega+1}, \omega_\omega) \to (\omega_1, \omega).$$

The meaning of this is as follows: if $\mathscr{A} = \langle A, U, R_n : n \in \omega \rangle$ is any structure such that $|A| = \omega_{\omega+1}$, $U \subset A$ is unary with $|U| = \omega_\omega$ then $\mathscr{A}$ has an elementary substructure $\mathscr{A}' = \langle A', U', R'_n : n \in \omega \rangle$ for which $|A'| = \omega_1$ and $|U'| = \omega_0$ (of course, here $U' = U \cap A'$ and $R'_n = R_n \cap (A')^{i_n}$ for $n \in \omega$, where i_n is the arity of R_n).

In [7] Levinsky, Magidor and Shelah proved that the existence of a 2-huge cardinal implies the consistency of GCH plus $(\omega_{\omega+1}, \omega_\omega) \to (\omega_1, \omega)$. For our purposes it will be convenient to first give the following topological consequence of this proposition. (Note that, as is easily seen by a simple induction, any first countable ω-fair space is also ω_n-fair for each $n \in \omega$.)

2.1. THEOREM. *If $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ holds then any first countable space that is ω-fair is also ω_ω-fair.*

PROOF. Assume, on the contrary, that there is an ω-fair first countable space X with $|X| = \omega_{\omega+1}$ and a dense subset $S \subset X$ with $|S| = \omega_\omega$. Let us fix for each $p \in X$ a neighbourhood base $\{V_n(p) : n \in \omega\}$ in X and then for each $n \in \omega$ we define a binary relation R_n on X as follows:

$$R_n(x, y) \leftrightarrow y \in V_n(x).$$

Now, applying $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ to the structure $\mathscr{X} = \langle X, S, R_n : n \in \omega \rangle$ we get a set $Y \in [X]^{\omega_1}$ such that $|S \cap Y| = \omega$ and $\mathscr{Y} = \langle Y, S \cap Y, R_n \cap Y^2 : n \in \omega \rangle$ is an elementary substructure of $\mathscr{X}$.

We claim that $S \cap Y$ is dense in Y, which contradicts the assumption that X is ω-fair. Indeed, since S is dense in X, for each $n \in \omega$ we have that the sentence

$$\forall x \; \exists y [R_n(x, y) \wedge y \in S]$$

is satisfied in $\mathscr{X}$, consequently the same sentence is also satisfied in $\mathscr{Y}$. Now, it is obvious that this actually means that $Y \cap S$ is dense in Y.

Since, by 0.1, no good space of size $\geqq \omega_\omega$ is ω_ω-fair we immediately get the following corollary.

2.2. COROLLARY. *If $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ holds then $S(\kappa)$ implies $\kappa < \omega_\omega$.*

P. Nyikos, after having heard of this result, gave the following strengthening of it: $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ implies that every first countable, ω-bounded and locally hereditarily Lindelöf space has Lindelöf degree $< \omega_\omega$. Below we show that already the assumption "every ω-fair first countable space is ω_ω-fair" yields the same conclusion. Moreover, our proof is completely different from and much simpler than his.

2.3. THEOREM. *Assume that every ω-fair first countable space is ω_ω-fair. Then for every first countable, ω-bounded and locally hereditarily Lindelöf space X we have $L(X) < \omega_\omega$.*

PROOF. The countable compactness of X clearly implies that $L(X) \neq \omega_\omega$ (as well as $L(F) \neq \omega_\omega$ for every closed subspace F of X). Moreover, since X is locally hereditarily Lindelöf we have $L(Y) = hL(Y)$ for every $Y \subset X$.

Now, if we had $L(X) > \omega_\omega$ then X would contain a right-separated subspace S with $|S| = \omega_\omega$. Then $L(\bar{S}) = hL(\bar{S}) \geqq \omega_\omega$ and $L(\bar{S}) \neq \omega_\omega$ imply that in fact $L(\bar{S}) > \omega_\omega$, hence we may choose a right-separated set $Z \subset \bar{S}$ with $|Z| = \omega_{\omega+1}$.

Let us now consider the subspace $Y = S \cup Z$ of X. We claim that Y is ω-fair, which will yield a contradiction since, of course, Y is not ω_ω-fair.

Indeed, since X is ω-bounded, for every countable set $A \subset X$ we have that $\bar{A}$ is compact, hence being covered by finitely many hereditarily Lindelöf sets it is also hereditarily Lindelöf. Consequently, for every $A \in [Y]^\omega$ we have both $|\bar{A} \cap S| \leqq \omega$ and $|\bar{A} \cap Z| \leqq \omega$, hence $|\bar{A} \cap Y| \leqq \omega$, which was to be shown.

We mention here, without proof, that the relation $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ is preserved under CCC forcing. Consequently, from a model satisfying it we may get one in which it remains true and MA $+ 2^\omega > \omega_{\omega+1}$ are also satisfied. This yields us a model in which $S(\kappa)$, hence by [5] also $G(\kappa)$, fails non-trivially whenever $\omega_\omega \leqq \kappa < 2^\omega$. But the existence of a 2-huge cardinal is a much stronger requirement than that of a supercompact cardinal, hence 1.6 is a better result.

§3. Another model with arbitrarily large good spaces

Our aim here is to show that a very simple forcing yields a model as described in the title.

3.1. THEOREM. *If $\mathcal{P}$ is the partial order that adds iteratively ω_1 dominating reals to V then in $V^{\mathcal{P}}$, $G(\lambda^\omega)$ holds for each cardinal λ.*

In this section we shall use D to denote the standard notion of forcing that adds a dominating real to V, i.e. a function $r : \omega \rightarrow \omega$ such that $r(n) > f(n)$ for all but finitely many $n \in \omega$ whenever $f \in {}^\omega\omega \cap V$, cf [4].

A space X is called nice iff it is a locally countable, locally compact T_2 space. Let us remark that each nice space is both first countable and regular.

The proof of 3.1 is based on the following lemma:

3.2. LEMMA. *Let $\langle X, \tau \rangle$ be a nice space. Then, in V^D, $\langle X, \tau \rangle$ can be embedded as a dense, open subspace into a nice space $\langle Y, \sigma \rangle$ satisfying property* (∗) *below*:

(∗) *Each* $Z \in [X]^\omega \cap V$ *has an accumulation point in* $\langle Y, \sigma \rangle$.

PROOF. First we fix, in V, a function $F : X \times \omega \rightarrow [X]^{\leq \omega}$ satisfying for each $x \in X$ 3.1.1 and 3.1.2 below:

3.1.1. $F(x,0) \supseteq F(x,1) \supseteq \cdots \supseteq F(x,n) \supseteq \cdots$.

3.1.2. $\{F(x,n) : n \in \omega\}$ forms a local base of x in $\langle X, \tau \rangle$ consisting of compact open neighbourhoods.

Next we choose a maximal almost disjoint family $\mathscr{A} \subset [X]^\omega$ of countable, closed discrete subsets of X. For each $A \in \mathscr{A}$ we use $\vec{A}$ to denote a one-to-one enumeration of A in V in type ω.

Now we will extend X in such a way that for each $A \in \mathscr{A}$ the sequence $\vec{A}$ will become convergent.

The underlying set of $\langle Y, \sigma \rangle$ will be

$$Y = X \cup \{y_A : A \in \mathscr{A}\}$$

where the y_A's are new and different points.

From now on we work in V^D. Let us consider the function $F^* : Y \times \omega \rightarrow [Y]^{\leq \omega}$ given by 3.1.3 and 3.1.4:

3.1.3. F^* extends F.

3.1.4. $F^*(y_A, n) = \{y_A\} \cup \bigcup \{F(\vec{A}(k), r(k)) : k > n\}$.

We define the topology σ on Y as follows: for each $y \in Y$ we choose $\{F^*(y,n) : n \in \omega\}$ as a local base of y in Y.

Obviously, $\langle X, \tau \rangle$ is an open, dense subspace of Y. It is also clear that $\langle Y, \sigma \rangle$ is locally countable.

In order to prove that $\langle Y, \sigma \rangle$ is locally compact let us first remark that each $F(x,n)$ remains compact in V_D, because it is a countable, compact T_2 space, i.e., homeomorphic to a countable successer ordinal. Consequently Y is locally compact at every $x \in X$. Next we prove that every $F^*(y_A, n)$ is also compact.

Indeed, if S is any infinite subset of $F^*(y_A, n)$ then either $S \cap F(\vec{A}(k), r(k))$ is infinite for some fixed $k > n$ or S intersects $F(\vec{A}(k), r(k))$ for infinitely many $k \in \omega$, hence, in either case, S has a limit point.

Let us now check property (∗). Consider a $B \in [X]^\omega \cap V$ having no accumulation point in X. By the maximality of $\mathscr{A}$ we can find $A \in \mathscr{A}$ having infinite intersection with B. But then y_A is a limit point of $A \cap B$, for $\vec{A}$ converges to y_A.

Lastly, we prove that $\langle Y, \sigma \rangle$ is T_2. Till now we have not used that r is a dominating real. Let us fix two different points of Y, say u and v. Since X is an open subspace of Y we can assume that $u \in Y \setminus X$, $u = y_A$.

We distinguish two cases:

Case 1. $v \in X$

First we fix a $k \in \omega$ with $v \notin \{\vec{A}(i) : i \geqq k\}$. Since A does not have an accumulation point we can choose a neighbourhood $F(v, n)$ of v, having empty intersection with $\{\vec{A}(i) : i \geqq k\}$.

Now let us consider the function $f \colon \omega \setminus k \to \omega$ defined in V as follows:

$$f(l) = \min\{i : F(\vec{A}(l), i) \subset X \setminus F(v, n)\}.$$

We know that r dominates f, i.e. we have $m \geqq k$ with $f(i) < r(i)$ for each $i > m$. Thus $F^*(y_A, m)$ and $F(v, n)$ are disjoint neighbourhoods of y_A and v.

Case 2. $v = y_B \in Y \setminus X$

First we fix $k \in \omega$ with $\{\vec{A}(l) : l > k\} \cap \{\vec{B}(l) : l > k\} = \emptyset$. Since $\{\vec{A}(l) : l > k\}$ and $\{\vec{B}(l) : l > k\}$ are disjoint, countable closed subsets of the regular space X, they can be separated by open sets, i.e. there are functions $f, g \in {}^{\omega}\omega \cap V$ with

$$\bigcup\{F(\vec{A}(l), f(l)) : l > k\} \cap \bigcup\{F(\vec{B}(l), g(l)) : l > k\} = \emptyset.$$

But r dominates both f and g, i.e. we have $n > k$ with $r(i) > f(i), g(i)$ for each $i > n$. This means that $F^*(y_A, n) \cap F^*(y_B, n) = \emptyset$.

This completes the proof of Lemma 3.2.

Proof of Theorem 3.1. The poset $\mathscr{P} = \mathscr{P}_{\omega_1}$ is given by the finite support iteration $\langle P_\alpha : \alpha \leqq \omega_1, \dot{Q}_\alpha : \alpha < \omega_1 \rangle$ where

$$V^{P_\alpha} \vDash \dot{Q}_\alpha = D$$

for each $\alpha < \omega_1$.

Given a cardinal λ with $\lambda^\omega = \lambda$ we define nice spaces X_α with $X_\alpha \in V^{P_\alpha}$ so that X_α is an open subspace of X_β for each $\alpha < \beta$, by induction on $\alpha \leqq \omega_1$ as follows. We denote by τ_α the topology of X_α.

We set a discrete space of cardinality λ as X_0 in V. For every limit α we put $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$ with the topology τ_α that is generated by $\bigcup\{\tau_\beta : \beta < \alpha\}$. Standard tricks (cf. e.g. [6] p. 281) will insure that $X_\alpha, \tau_\alpha \in V^{P_\alpha}$. Obviously X_α will be nice and every X_β is open in X_α.

If $\alpha = \beta + 1$ and X_β is defined then we apply Lemma 3.2 for X_β in $V^{\mathscr{P}_\beta}$. We get a nice space Y in $V^{\mathscr{P}_\beta * D} = V^{\mathscr{P}_{\beta+1}}$ and we put Y as $X_{\beta+1}$. This completes the construction.

We claim that X_{ω_1} is as required. It is easy to see by induction that for $\alpha \leqq \omega$, $|X_\alpha| = \lambda$. Let $A \in [X_{\omega_1}]^\omega$. Since we iterated by finite support there is $\alpha < \omega_1$ with $A \in [X_\alpha]^\omega \cap V^{P_\alpha}$. Then, by Lemma 3.2, A has an accumulation point in $X_{\alpha+1}$, hence in X_{ω_1} as well.

Thus X_{ω_1} is a countably compact nice space with cardinality λ, i.e., $G(\lambda)$ holds. The proof is completed.

Let us note finally that $\mathscr{P}$, being CCC and of cardinality continuum, is a very "mild" notion of forcing. Thus e.g. forcing with $\mathscr{P}$ does not change cardinal exponentiation and preserves large cardinals. In particular, as was mentioned at the end of §2, $\mathscr{P}$ preserves the relation $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$, consequently this enables us to get a model in which $S(\kappa)$ implies $\kappa < \omega_\omega$ but $G(\lambda^\omega)$ is valid for all cardinals λ.

Moreover, this leads to the following intriguing problem: Is it true in ZFC that $G(\lambda^\omega)$ is valid for all λ? Note that by 0.1 this would be equivalent to the statement that $G(\kappa)$ is valid for arbitrarily large cardinals κ.

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