Sh:144

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COUNTABLY DECOMPOSABLE ADMISSIBLE SETS*

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The known results about Σ_1-completeness, Σ_1-compactness, ordinal omitting etc. are given a unified treatment, which yields many new examples. It is shown that the unifying theorem is best possible in several ways, assuming $V = L$.

Introduction

The generalizations of results from the theory of countable admissible sets and from definability theory for countable structures to certain uncountable sets or structures has been the subject of several works by Barwise, Chang-Moschsvakis, Green, Karp, Makkai, Nyberg, Grant and perhaps others. (Recently, while this paper was being written we have received a work in the same direction by S. Friedman.)

The common feature of all these generalizations, which are of concrete nature, was that the relevant set or structures, while not countable, can be still represented as a countable union of 'small' sets. Making this observation explicit we provide in this paper a unified treatment that contains the concrete results of all these authors on Σ_1-completeness, Σ_1-compactness, non-definability of well-orderings, inductive definability of Π^1_1-relations, ordinal pinning by statements of $L_{\infty\omega}$ etc. By 'concrete results' we mean results providing set-theoretic conditions on a set or a structure, that imply some definability property, as opposed to abstract results, asserting that one definability property implies another, e.g., that Π^1_1-reflection implies Σ_1-compactness, or that if M is a uniform Kleene structure, then any admissible set above M, projectible into M is Σ_1-compact.

In addition to all these results, we establish ordinal omitting theorems or in general type omitting theorems (for $L_{\aleph_{\omega+1}\omega}$ for instance) and via such theorems generalize H. Friedman's theorem [8] on the existence of models (e.g., of set theory) whose standard part has an ordinal α, where α is any given countable

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admissible ordinal. As an application we find (assuming $V=L$) a necessary and sufficient condition for ordinals $\kappa<\alpha<\kappa^+$ where $\mathrm{cf}(\kappa)=\omega$, to be of the form $\alpha(X)$ – the minimal X-admissible ordinal where $X\subseteq\kappa$. The problem of doing it was raised by Nyberg [22], who obtained some special cases. (S. Friedman in [9, 10] solved this problem independently of us. [22] influenced much of our earlier work on the subject of this paper.) Most of these results are included in the first part of this paper. In the second and third parts we verify that our results are in some sense the best possible. For instance, if one defines $h(A)$ for an admissible set A to be the minimal ordinal not pinned down by a sentence of $L_{\infty\omega}\cap A$, then if one assumes $V=L$, then $h(L_\alpha)=\alpha$ iff this is guaranteed by the decomposability conditions in the first part of the paper, which imply $h(A)=o(A)$. Similarly L_α is validity admissible iff it is guaranteed to be such by our decomposability condition.[1] Also we provide characterization of the limit ordinals α such that L_α has the Kleene property, i.e., every Π^1_1-relation on L_α is inductively definable. These results use the fine structure of $V=L$.

Trying to get a similar result for Σ_1-compactness stumbles on the fact that if κ is a regular cardinal $\{\alpha\mid\alpha<\kappa, L_\alpha \text{ is } \Sigma_1\text{-compact}\}$ contains a closed unbounded subset of κ. (See [1, VIII, 8.3] and [25]), hence one can find many L_α's which do not satisfy the decomposability conditions but which are Σ_1-compact. The fact that one has such α's seems to be completely accidental and if one strengthens the notion of Σ_1-compactness to stable Σ_1-compactness, then L_α is stably Σ_1-compact if and only if it satisfies the right decomposability conditions. The proof of this fact is the content of the third part of the paper.[2]

PART I

1. Preliminaries

We work in ZFC with a class UR of elements. For applications it is often best to assume that every set has a one to one mapping into UR, hence UR is a proper class. Capital letters range only over sets (occasionally classes), not over urelements. $\mathcal{P}(X)$ is the power set of X,

$$X^Y=\{f\mid f: Y\to X\},$$

[1] It is not the case, however, that L_α is validity admissible iff $h(L_\alpha)=\alpha$, for all admissible ordinals α. This is because $h(A)=o(A)$ is guaranteed by having some admissible structure $\langle A, \in, \boldsymbol{R}\rangle$ on A satisfying our conditions, whereas being validity-admissible is not inherited from $\langle A, \in, \boldsymbol{R}\rangle$ to $\langle A, \in\rangle$. See Part II for the details.

[2] The results of this paper were announced in the three abstracts [19]. As mentioned there the essential results of Parts I and II were already presented to an Ω-group meeting in Be'er Sheva in March 1978. The abstracts [19] are marred by two technical errors, corrected below in footnote 3. These were generated in later attempts to shorten the definitions and statements.

Sh:144

$\mathrm{Tc}(X)$ is the transitive closure of X, and for $B \subseteq \mathrm{UR}$,

$$H(\kappa)_B = \{x \mid |\mathrm{Tc}(\{x\})| < \kappa \text{ and } \mathrm{Tc}(\{x\}) \cap \mathrm{UR} \subseteq B\},$$

$H(\kappa) = H(\kappa)_{\Phi}$, κ, λ, μ denote infinite cardinals.

In admissible set theory we mainly follow the terminology and notation of Barwise's book [1]. However by a transitive structure we mean a single-sorted structure

$$\mathscr{A} = \langle A, \in \cap A^2, \mathrm{UR} \cap A, S_1, \ldots, S_k\rangle$$

($\langle A, \in, \boldsymbol{S}\rangle$ for short), where A is a transitive set and each S_i is a relation or operation on A. $\mathscr{A}$ is amenable if $S_i \cap a \in A$ for every $a \in A$. An admissible structure is a transitive structure $\mathscr{A} = \langle A, \in, \boldsymbol{S}\rangle$ satisfying the axioms of KPU for its language. An admissible set is a set A such that $\langle A, \in\rangle$ (i.e., $\langle A, \in \cap A^2, \mathrm{UR} \cap A\rangle$) is an admissible structure. $\Sigma_n(\mathscr{A})$ is the set of relations on A definable over $\mathscr{A}$ by a $\Sigma_n(\mathscr{A})$-formula (possibly including parameters from A). Sometimes consider more general $\Sigma_n(A)$-formulas which may contain, besides parameters from A and individual free variables, also free relation 'variables' (in addition to the relation constants S_i, denoting the relations S_i of $\mathscr{A}$). Such a formula in which the free relation variables $T_1, \ldots, T_l$ occur only positively is typically denoted by $\Phi = \Phi(x_1, \ldots, x_n, T_1^+, \ldots, T_l^+) = \Phi(\boldsymbol{x}, \boldsymbol{T}^+)$. Similar conventions apply to Π_n-formulas, Π_1^1- or strict Π_1^1-formulas etc.

By an $\mathscr{A}$-r.e. relation, where $\mathscr{A}$ is an admissible structure, we mean a $\Sigma_1(\mathscr{A})$-relation. An $\mathscr{A}$-partial-recursive function is a partial function from A^n into A (for some $n < \omega$) whose graph is an $\mathscr{A}$-r.e. relation. The identification of r.e. with Σ_1 stipulated here is not meant as a claim that Σ_1 is the 'correct generalization' of r.e.; it is made simply because Σ_1-relations on $\mathscr{A}$ are in the center of our interest in this paper.

Denote by LV the class of all logically valid $\mathscr{L}_{\infty\omega}$-sentences. An admissible structure $\mathscr{A}$ is *validity admissible* if $A \cap \mathrm{LV}$ is $\mathscr{A}$-r.e. [1, VII 1.7]. $\mathscr{A}$ is *Σ_1-complete* when for every $\mathscr{A}$-r.e. theory T, $\mathrm{Cn}(T) \cap A$ is $\mathscr{A}$-r.e. (A theory is a set of $\mathscr{L}_{\infty\omega}$-sentences and $\mathrm{Cn}(T)$ is the class of logical consequences of T in $L_{\infty\omega}$). $\mathscr{A}$ is *Σ_1-compact* when for every $\mathscr{A}$-r.e. theory T, if T has no model, then some $T_0 \subseteq T$, $T_0 \in A$ has no model. $\mathscr{A}$ is *uniformly Σ_1-complete* when there exists a $\Sigma_1(\mathscr{A})$-formula $\Phi(x, T^+)$ such that for every $\mathscr{A}$-r.e. theory T, $\mathrm{Cn}(T) \cap A = \{a \in A \mid \mathscr{A} \models \Phi[a, T]\}$. (This definition differs from definition 1.1(6) of [22], but is equivalent to it.) As Nyberg proves [22, 1.2] the following three conditions are equivalent for an admissible structure $\mathscr{A}$:

(1) $\mathscr{A}$ is validity-admissible and Σ_1-compact.
(2) $\mathscr{A}$ is uniformly Σ_1-complete.
(3) $\mathscr{A}$ is Σ_1-complete and Σ_1-compact.

Denote by $h(A)$ the least ordinal that cannot be pinned down by an $\mathscr{L}_{\infty\omega} \cap A$ sentence and by $h_\Sigma(\mathscr{A})$ the least ordinal that cannot be pinned down by an $\mathscr{A}$-r.e.

theory [1, VII, 3 and VII, 6]. h_β is the least ordinal that cannot be pinned down by a sentence of β-logic. By β-logic we understand first-order logic, with distinguished constants c_γ ($\gamma \leq \beta$) and a distinguished relation symbol $<$, but attention is restricted to 'β-models', i.e., models in which $<$ linearly orders its field and the sentence

$$\forall x \left(x < c_\gamma \leftrightarrow \bigvee_{\alpha<\gamma} x = c_\alpha \right)$$

is satisfied for each $\gamma \leq \beta$. (For $\beta = \omega$ this is essentially the familiar ω-logic and h_ω is the first non-recursive ordinal.)

For any object a we denote by HYP(a) the least admissible set containing a as an element. In particular, if $\mathcal{M} = \langle M, \in, S_1, \ldots, S_k \rangle$ is a transitive structure, then HYP($\mathcal{M}$) = HYP($\{M, S_1, \ldots, S_k\}$). Every structure $\langle M, S_1, \ldots, S_k \rangle$ where $M \subseteq \mathrm{UR}$ can be identified with the transitive structure

$$\mathcal{M} = \langle M, \in, \boldsymbol{S} \rangle = \langle M, \in \cap M^2, \mathrm{UR} \cap M, \boldsymbol{S} \rangle = \langle M, \emptyset, M, \boldsymbol{S} \rangle$$

and so the operation HYP applies to arbitrary structures (of finite signature) based on urelements. Note that for us HYP($\mathcal{M}$) (or generally HYP(a)) is simply an admissible set, not a structure. Let $\alpha(a) = o(\mathrm{HYP}(a)) = \min\{o(A) \mid A \text{ is admissible and } a \in A\}$. In particular, if $a = \beta$ (an ordinal), then by these definitions $\alpha(\beta)$ is the next admissible ordinal and $\mathrm{HYP}(\beta) = L_{\alpha(\beta)}$. An admissible set of the form HYP($\mathcal{M}$) is called a successor admissible set.

Recall (from [1, VI, §3–4]) that if $\mathcal{M}$ is a transitive structure having an inductive pairing function (this is true in particular if M is closed under pairs), then HYP($\mathcal{M}$) is projectible into M and a relation R on M is inductive on $\mathcal{M}$ iff R is Σ_1 over HYP($\mathcal{M}$). Still assuming that $\mathcal{M}$ has an inductive pairing function, it is implicit in [22] and not hard to prove directly that (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3); (4)$\Leftrightarrow$(5)$\Leftrightarrow$(6) and (3)$\Rightarrow$(4) where

(1) Every Π^1_1-relation on $\mathcal{M}$ is inductive;
(2) $A = \mathrm{HYP}(\mathcal{M})$ is validity-admissible;
(3) $A = \mathrm{HYP}(\mathcal{M})$ is Σ_1-complete;
(4) $A = \mathrm{HYP}(\mathcal{M})$ is Σ_1-compact;
(5) $h(A) = o(A)$ where $A = \mathrm{HYP}(\mathcal{M})$;
(6) $h_\Sigma(A) = o(A)$ where $A = \mathrm{HYP}(\mathcal{M})$.

In fact (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5)$\Leftrightarrow$(6) for every successor admissible set A. [*Hint*: (3)$\Rightarrow$(4) because A is resolvable [22, 1, 4], (4)$\Leftrightarrow$(6) by [1, VIII, 6.5], (2)$\Leftrightarrow$(3) and (5)$\Leftrightarrow$(6) because the class of models of an A-r.e. theory can be represented by a class of relativized reducts of models of a single $\mathcal{L}_{\infty\omega} \cap A$ sentence. (2)$\Rightarrow$(1) is easy and (1)$\Rightarrow$(3) occurs in the proof of [22, 2.9].]

Following Nyberg, we call a transitive structure $\mathcal{M}$ with an inductive pairing function satisfying (1) a *Kleene structure.*

2. Decomposability conditions and the main completeness and compactness results

In the sequel, $\mathscr{A}$ is always an admissible structure with domain A. Following the observations made in the introduction, we formulate a 'decomposability condition' for admissible structures meaning intuitively that every member of the set is a countable union of 'small sets' and that this decomposition is in some sense recognizable in the admissible structure.

Our main theorem which will be stated in this section claims that an admissible structure satisfying the decomposability condition has all the nice properties one associates with countable admissible sets, i.e., Σ_1-completeness, Σ_1-compactness, theorems about ordinal pinning and ordinal omitting, etc.

We first have to define what we mean by small sets.

Definition 2.1. Let $S \subseteq A$-UR, i.e., S is a set of sets and $S \subseteq A$. S is a smallness predicate for $\mathscr{A}$ if

(a) S is $\mathscr{A}$-r.e.
(b) $x \in S \rightarrow \mathscr{P}(x) \in \mathscr{A}$.
(c) The relation $\{(x, \mathscr{P}(x)) \mid x \in S\}$ is $\mathscr{A}$-r.e.

Definition 2.2. $\mathscr{A}$ has the first decomposition property (DP1) if for some smallness predicate S for $\mathscr{A}$ every member of $\mathscr{A}$ is a countable union of members of S.

Note that we are not assuming that the decomposition of $x \in A$ into a countable union of members of S is in $\mathscr{A}$ or even definable there.

In Section 4 we shall show that if $\mathscr{A}$ has (DP1), then one can assume that the smallness predicate is of the form (for some $\kappa \leqslant o(\mathscr{A})$)

$$S(X) \leftrightarrow (\exists f \in A)\,(\exists \alpha < \kappa)\,(f \text{ maps } X \text{ into } \alpha \wedge f \text{ is one to one}).$$

(Hence the smallness predicate can be assumed to mean "X has small $\mathscr{A}$-cardinality".)

The next condition claims that in some sense we know at least a trace of the decomposition of a general $X \in A$ into a countable union of sets whose power set is in A.

Definition 2.3. A relation $R \subseteq (A\text{-UR})^2$ is a decomposition relation for $\mathscr{A}$ if

(a) R is $\mathscr{A}$-r.e.
(b) $\forall X\, \exists Y\, R(X, Y)$.
(c) For every X and Y such that $R(X, Y)$, X is a countable union of members of Y whose (real) power set is a subset of Y. In symbols: $X = \bigcup_{n<\omega} X_n$ for some sequence of sets X_n satisfying $\mathscr{P}(X_n) \subseteq Y$ for all n.

(Note again that we are not assuming that the representation of X as countable union of members of Y is in A.)

Definition 2.4. (a) The admissible structure $\mathscr{A}$ is said to have the second decomposition property (DP2) if it has a decomposition relation.

(b) A is said to have the decomposition property (DP) if it has (DP1) and (DP2).

Note. It will follow from the results in Section 3 that if A contains an element with maximal (real) cardinality, then (DP2) $\Rightarrow$ (DP1) but one can find an example of an admissible structure satisfying (DP2) but not (DP1).

Definition 2.5. $\mathscr{A}$ is said to be *countably decomposable* if it has the decomposition property and A is a countable union of members of A.

Our main results can be stated for $\mathscr{A}$ having the decomposition property[3], provided we restrict our attention to theories T such that T is a countable union of members of A. (Call such theories σ-small.) Of course if $\mathscr{A}$ is countably decomposable, then every $T \subseteq A$ is σ-small over A.

Theorem 2.1. *Let $\mathscr{A}$ have the decomposition property and let $T \subseteq \mathscr{L}_{\infty\omega} \cap A$ be σ-small and $\mathscr{A}$-r.e. Then*

(a) $\mathrm{Cn}(T) \cap A$ *is $\mathscr{A}$-r.e.*

(b) *If T has no model, then some $T_0 \subseteq T$, $T_0 \in A$ has no model.*

(c) *The least ordinal not pinned down by T is a member of A.*

Corollary 2.2. *If $\mathscr{A}$ has the* DP, *then $\mathscr{A}$ is validity admissible and $h(A) = o(A)$. If $\mathscr{A}$ is countably decomposable, then $\mathscr{A}$ is Σ_1-complete and Σ_1-compact and $h_\Sigma(\mathscr{A}) = o(A)$.*

(The implication $2.1 \Rightarrow 2.2$ is easy.)

We shall prove 2.1 and related results in Section 5. The rest of this section contains examples of admissible structures which have the DP. These examples show that the known concrete completeness and compactness theorems are special cases of 2.1 or 2.2. Note that if $\mathscr{A}$ has DP, then every admissible expansion of $\mathscr{A}$ does. In particular if an admissible set A (i.e., the structure $\langle A, \in \rangle$) satisfies our condition, then so does every admissible structure with universe A.

Example 2.3. If $A \subseteq (\aleph_1)_{A \cap UR}$ (i.e., every member of A is countable), then A has the decomposition property. Define S by

$$S(X) \Leftrightarrow X \text{ is a singleton.}$$

(Hence DP1). Define R by

$$(X, Y) \in R \Leftrightarrow Y = \{\{a\} \mid a \in X\} \cup \{\emptyset\}.$$

[3] Our present definition of the decomposition property is a correction of the version announced in [19], where (DP1) was omitted because we had an (erroneous) proof that (DP2)$\Rightarrow$(DP1) always.

It can be easily verified that R is a decomposition relation for A. Thus $\mathscr{A}$ has the DP. If, moreover, $\mathscr{A}$ is countable, then $\mathscr{A}$ is countably decomposable. Thus Theorems 2.1 and 2.2 yield the well-known Barwise compactness and completeness theorems.

Example 2.4. Suppose that $\mathscr{A}$ is closed under $\mathscr{P}$ – the power set operation (i.e., $x \in A - \mathrm{UR} \Rightarrow \mathscr{P}(x) \in A$), and let $\mathscr{A} = \langle A, \in, \ldots \rangle$ be an admissible structure such that $\mathscr{P} \restriction A$ is $\mathscr{A}$-partial-recursive. Then $\mathscr{A}$ has the decomposition property, since one can take S to be all the sets which are member of A ($S = A - \mathrm{UR}$) and define R by $(X, Y) \in R \Leftrightarrow Y = \mathscr{P}(X)$. Thus we get from 2.1 and 2.2 the Barwise–Karp cofinality-ω compactness theorem [1, VII, 7.4] and an associated completeness theorem.

To simplify the statement of the following examples we define a *cardinality-maximal* (c-max) element of a set A to be an element D of A such that every set in A is an A-image of D. (For $X, Y \in A$ we say that Y is an A-*image* of X when there exists a function $f \in A$ from a subset of X onto Y.)

Lemma 2.5. *Let D be a c-max element of the admissible set A. Suppose that $(\forall X \in D)\ (\mathscr{P}(X) \subseteq D)$ and that $D = \bigcup_{n<\omega} D_n$ where $D_n \in D$ for each n. Then A has the DP.*

Proof. To verify (DP1) define

$$S = \{X \in A \mid X \text{ is an } A\text{-image of some member of } D\}.$$

The assumption that $X \in D \Rightarrow \mathscr{P}(X) \in D$ guarantee that this is a smallness predicate for A (i.e., for $\langle A, \in \rangle$). Since every $X \in A$ is an A-image of $D = \bigcup_{n<\omega} D_n$, X is a countable union of sets in S.

To verify (DP2) define $R \subseteq A^2$ by

$$(X, Y) \in R \Leftrightarrow (\exists f \in A)[(f : \mathrm{dom}(f) \xrightarrow{\text{onto}} X)$$

$$\wedge\, \mathrm{dom}(f) \subseteq D \wedge Y = \{f''W \mid W \in D \wedge W \subseteq D\}].$$

R is clearly A-r.e. For every $X \in A$ there exists $Y \in A$ such that $R(X, Y)$. (D is c-max hence $X = f''B$ for some $f \in A, B \subseteq D$, hence $Y = \{f''W \mid W \in D, W \subseteq D\}$ satisfies $R(X, Y)$.)

Since $D = \bigcup_n D_n$ and $D_n \in D, \mathscr{P}(D_n) \subseteq D$ we get that if $(X, Y) \in R$, then X is a countable union of members of Y, whose power set is included in Y. □

Lemma 2.6. *Let $D \in A$ satisfy $(\forall X \in D)\ (P(X) \subseteq D)$ and let $D = \bigcup_{n<\omega} D_n$ where $D_n \in D$. If A is projectible into D, then A is countably decomposable.* (See [1, V, 5.1] for definition of projectible.)

Proof. Let π be a notation system for A such that $D_\pi \subseteq D$ where $D_\pi = \bigcup_{a\in A} \pi(a)$. If $X \in A$ and $Y = \bigcup_{a\in X} \pi(a)$, then $Y \subseteq D$ and $f: Y \xrightarrow{\text{onto}} X$, given by $f(y) = |y|_\pi$, is a member of A, since π is A-recursive. Thus D is a c-max element of A, so by Lemma 2.5 A has DP.

A is a countable union of its members since if D_n are the sets described in Lemma 2.5 one can define $E_n = D_n \cap D_\pi$. For each n $E_n \in A$ (since $\mathcal{P}(D_n) \subseteq A$). Hence $A_n = \{|y|_\pi \mid y \in E_n\} \in A$, by Σ_1-replacement. But $A = \bigcup_{n<\omega} A_n$ because each $a \in A$ has a notation in E_n for some n. □

We now continue with our examples of structures having DP and countably decomposable structures.

Example 2.7. Let κ be a strong limit cardinal of cofinality ω and $D = H(\kappa)$. Let A be an admissible set such that $D \in A$ and D is a c-max element of A. Clearly D satisfies Lemma 2.5 for A. Hence A has DP. In particular if $A = H(\kappa^+)_B$ for any set $B \subseteq \mathrm{UR}$, we obtain that $H(\kappa^+)_B$ is validity-admissible. This is an abstract version of Karp's completeness theorem for $L_{\kappa^+\omega}$-[16]) and we obtain also that the well-ordering number for single sentences of $L_{\kappa^+\omega}$ is κ^+. The conditions $D \in A$ and D is c-max in A also hold whenever $\langle A, \in\rangle$ is a transitive elementary submodel of $\langle H(\kappa^+)_B, \in\rangle$ as well as when $A = \mathrm{HYP}(D)$, or more generally, when $D \in A$ and A is projectible into D. (In the latter case A is countably decomposable, hence it is Σ_1-compact, by Lemma 2.6.)

Example 2.8. Let D be a set representable in the form $D = \bigcup_{n<\omega} D_n$, where for each n D_n is transitive and $\mathcal{P}(D_n) \subseteq D_{n+1}$. Clearly Lemmas 2.5 and 2.6 are applicable to D. Typical examples are $D = H(\kappa)$ where κ is a strong limit cardinal of cofinality ω, or $D = V_\alpha$ where $\mathrm{cf}(\alpha) = \omega$. By Lemma 2.5 if D is a c-max element of A (and A is admissible), then A has the DP. By Lemma 2.6 and our main theorems (2.1 and 2.2) if $D \in A$ and A is admissible and projectible into D, then A is Σ_1-complete, Σ_1-compact and $h_\Sigma(A) = o(A)$. This example covers all the major concrete applications of the main result (Corollary 2.10) of [22].

Example 2.9. Let A be an admissible set containing a c-max element B. Assume that $B = \bigcup_{n<\omega} B_n$ for some sequence $\langle B_n \mid n<\omega\rangle \in A$ (hence $\omega \in A$), and that $\bigcup_{n<\omega} \mathcal{P}(B_n) \subseteq C$ for some $C \in A$ (hence $\bigcup_{n<\omega} \mathcal{P}(B_n) \in A$). Then A has the DP. Indeed (DP1) holds for

$$S = \{X \in A \mid X \text{ is an image of } B_n \text{ for some } n<\omega\},$$

and (DP2) holds for

$$R = \{(Y, Z) \in A^2) (\exists f \in A)\ [f \text{ is a function from a subset of } B \text{ onto } Y \text{ and } Z = \{f''W \mid W \in C\}\},$$

as the reader can verify. This example gives Makkai's compactness theorem [17, 5.2], which generalizes that of Green [11]. Green's theorem is also a special case of Example 2.8, as shown in [22]. Note that our result here slightly improves Makkai's in requiring only $(\exists C\in A)\ (\bigcup_{n<\omega}\mathcal{P}(B_n)\subseteq C)$ rather than $(\exists C\in A)$ $(\bigcup_{n<\omega}{}^{B_n}\omega\subseteq C)$. A more useful improvement would be to discard the hypothesis that $\langle B_n \mid n<\omega\rangle\in A$, and this is done in the next example.

Example 2.10. Assume that A is admissible, B is a c-max element of A, $B=\bigcup_{n<\omega}B_n$ where $\bigcup_{n<\omega}\mathcal{P}(B_n)\subseteq C$ for some $C\in A$. Then A has the DP.

(DP2) is verified precisely as in Example 2.9. The proof of (DP1) is considerably more tricky, so we refer to Section 4 where we prove that (DP2)$\Rightarrow$(DP1) whenever A has an element of maximum cardinality, i.e., $(\exists X\in A)\,(\forall \gamma\in A)$ $(|X|\geqslant|Y|)$. This is of course the case whenever A has a c-max element, as here.

Note that this example contains both the preceding ones as special cases.

Example 2.11. ('Iteration of Example 2.10'). Let $k\geqslant 1$ and let $(E_0,\dots,E_k)$ be a sequence of sets satisfying (with the notation $B=E_0$, $A=E_k$) the following:

(1) A is admissible.

(2) For $1\leqslant i\leqslant k-1$, E_i is a c-max element of E_{i+1}.

(3) For $1\leqslant i\leqslant k-1$, E_i has cofinality ω, i.e., it is a union of countably many of its members.

(4) There exist $C\in A$ and a sequence $\langle B_n \mid n<\omega\rangle$ such that $B=\bigcup_{n<\omega}B_n$ and $\bigcup_{n<\omega}\mathcal{P}(B_n)\subseteq C$.

Then A has the DP.

To see this, claim first that if $0\leqslant i\leqslant k-1$, then there exist $C\in A$ and a sequence $\langle X_n \mid n<\omega\rangle$ such that $E_i=\bigcup_{n<\omega}X_n$ and $\bigcup_{n<\omega}\mathcal{P}(X_n)\subseteq C$. For $i=0$ this is (4) above. Let $1\leqslant i\leqslant k-1$ and suppose the claim true for $i-1$. Thus $E_{i-1}=\bigcup_{m<\omega}Y_m$ where $\bigcup_{m<\omega}\mathcal{P}(Y_m)\subseteq D$ for some $D\in A$. By (2) and (3), E_i is a countable union of sets X_n each of which is an E_i-image of some Y_m. Therefore

$$\bigcup_{n<\omega}\mathcal{P}(X_n)\subseteq\{g''W \mid g \text{ is a function, } g\in E_i \text{ and } W\subseteq Y_m \text{ for some } m\}.$$

Letting $C=\{g''W \mid g$ is a function, $g\in E_i$ and $W\in D\}$ it follows that $\bigcup_{n<\omega}\mathcal{P}(X_n)\subseteq C$ and clearly $C\in A$. This completes the proof of the claim.

Taking $i=k-1$ in the claim we see that $E_{k-1}=\bigcup_{n<\omega}X_n$ where $\bigcup_{n<\omega}\mathcal{P}(X_n)\subseteq C$ for some $C\in A$. But E_{k-1} is a c-max element of A, so A has the DP by Example 2.10.

We shall now illustrate the application of the last few examples to more specific cases. For the rest of this section we assume that $V=L$ and κ is a limit cardinal of cofinality ω, say $\kappa=\sup\{\kappa_n \mid n<\omega\}$ where $\langle\kappa_n \mid n<\omega\rangle$ is a strictly increasing sequence of infinite cardinals. Since we are assuming $V=L$ κ is a strong limit cardinal and the set $D=L_\kappa=H(\kappa)$ is as required for Example 2.8.

Example 2.12. L_{κ^+} has the DP, for it contains L_κ as a c-max element. Similarly, if $\kappa<\delta<\kappa^+$ is an admissible ordinal and

$$L_\delta \vDash \text{“}\kappa \text{ is the last cardinal”},$$

then L_δ has the DP, as again it has L_κ as a c-max element.

Example 2.13. Let $\kappa<\delta<\kappa^+$ be admissible and denote $S=\{\gamma \mid \kappa\leqslant\gamma<\delta$ and γ is a cardinal of $L_\delta\}$ (in the previous example we had $S=\{\kappa\}$). We claim that if each $\gamma\in S$ has cofinality ω (in the 'real universe', not in L_δ) and S has a last element, then L_δ has the DP. In case S is finite this follows from Example 2.11 (with $B=L_\kappa$, $A=L_\delta$, $B_n=L_{\kappa_n}$, $C=L_\kappa$, $E_1=L_{\gamma_2}$, where $\gamma_1<\gamma_2<\cdots$ are the members of S). For the general case we argue by induction on $\gamma\in S$ that γ can be represented as $\bigcup_{n<\omega} X_n$ for some sequence $\langle X_n \mid n<\omega\rangle$ such that $\bigcup_{n<\omega}\mathcal{P}(X_n)\subseteq L_\gamma$ (and, in fact, $|X_n|<\kappa$ and $\sup(X_n)<\gamma$ for each n). For $\gamma=\kappa$ take $X_n=L_{\kappa_n}$, and the induction step is left to the reader (use the fact that if $\gamma\in S$, then any bounded subset of γ which is a member of L_δ is already a member of L_γ, by a well-known theorem of Gödel relativized to L_δ). Applying the result to the last element β of S we see that Example 2.10 applies to $A=L_\delta$ with $B=\beta$, $C=L_\beta$ (or with $B=C=L_\beta$), so L_δ has the DP.

A special case of this example is the case where δ is the least ordinal such that $\delta>\kappa$, $L_\delta\vDash \mathrm{ZF}^-$ (=ZF without the power-set axiom) and $L_\delta\vDash$“κ is not the last cardinal”. By the minimality of δ, L_δ has just one cardinal (call it γ) greater than κ, and each $a\in L_\delta$ is definable in $\langle L_\delta, \in\rangle$ by a first-order formula using ordinals $<\kappa$ as parameters. For each $n<\omega$ let

$$B_n=\{a\in L_\delta \mid a \text{ is definable in } \langle L_\delta, \in\rangle \text{ by a } \Sigma_n\text{-formula with ordinals } <\kappa_n \text{ (only) as parameters}\}.$$

As $L_\delta\vDash\mathrm{ZF}^-$ and there is (in ZF^-, or even in KP) a truth definition for Σ_n-formulas, we have $B_n\in L_\delta$ for each n, so L_δ $(=\bigcup_{n<\omega} B_n)$ has cofinality ω. Similarly, $B_n\cap\gamma$ is bounded below γ for each n (for γ is a regular cardinal in L_δ) hence $\mathrm{cf}(\gamma)=\omega$. Thus Example 2.13 applies and L_δ has the DP and is even countably decomposable (hence Σ_1-complete etc.).

The following example is of a similar character.

Example 2.14. Let δ be the least ordinal $>\kappa$ such that $L_\delta\vDash\mathrm{ZF}$ (assuming such an ordinal exists), and let $C_\delta=\{\gamma<\delta \mid L_\delta\vDash$“$\gamma$ is a cardinal”$\}$. The structure $\langle L_\delta, \in, C_\delta\rangle$ is admissible, as $L_\delta\vDash\mathrm{ZF}$. We claim that $\langle L_\delta, \in, C_\delta\rangle$ is countably decomposable. To see this let, for each $n<\omega$, $A_n=\{a\in L_\delta \mid a$ is definable in $\langle L_\delta, \in\rangle$ by a Σ_n-formula involving only κ and ordinals $<\kappa_n$ as parameters$\}$. Since $L_\delta\vDash\mathrm{ZF}$ and satisfaction for Σ_n-formulas is definable in ZF it is clear that $A_n\in L_\delta$ for each n and $L_\delta=\bigcup_{n<\omega} A_n$ (since by minimality of δ every member of L_δ is definable from κ and members of κ). We define R by $(X, Y) \in R$ iff

$$Y = \{Z \mid Z \subseteq X,\ Z \text{ has } L_\delta\text{-cardinality} < \kappa\}.$$

R is $\langle L_\delta, \in, C_\delta\rangle$-r.e. (not $\langle L_\delta, \in\rangle$-r.e.) and clearly $\forall X \exists Y ((X, Y) \in R)$. $X = \bigcup_{n<\omega} (X \cap A_n)$, and clearly if $(X, Y) \in R$, $X \cap A_n \in Y$, and $\mathcal{P}(X \cap A_n) \subseteq Y$ since the L_δ cardinality of $X \cap A_n$ is $\leqslant \kappa_n$.

Thus we have verified that A has (DP2). (DP1) can be inferred from (DP2) by the above-quoted result of Section 4, or verified directly by defining

$$S(X) \Leftrightarrow (X \text{ has } L_\delta\text{-cardinality} < \kappa).$$

S is clearly L_δ-r.e. (in the parameter κ). $\{(x, \mathcal{P}(x)) \mid x \in S\}$ is L_δ-r.e. since L_κ is closed under power set, hence for $x \in S$

$$y = \mathcal{P}(x) \Leftrightarrow \exists f \exists \alpha < \kappa\ (f \text{ maps } x \text{ into } \alpha \wedge f \text{ is one to one}$$
$$\wedge \forall z \in y\ (z \subseteq x) \wedge \forall z \in L_\kappa\ (z \subseteq \alpha \to f^{-1}(z) \in y)).$$

Thus L is countably decomposable, hence Σ_1-complete and compact.

The kind of considerations we applied in the last examples suggest a systematic approach to classification of admissible ordinals in L, with respect to completeness and compactness. We shall pursue such a classification in the second and the third part of this paper.

Sy Friedman [10] independently obtained compactness and completeness results for structures of the form $\langle L_\alpha, \in\rangle$ where $|\alpha| = \kappa$ and $\mathrm{cf}(\kappa) = \omega$. He proved that if α is as above and $L_\alpha = \bigcup_{n<\omega} x_n$ where each $x_n \in L_\alpha$ and $L_\alpha \models |x_n| < \kappa$, and if δ is the largest α cardinal, then $\mathrm{cf}(\delta) = \omega$, then α is Σ_1-compact with ordinal omitting, and $\langle L_\alpha, \in, g\rangle$ is Σ_1-complete for some g such that $\langle L_\alpha, \in, g\rangle$ is admissible. Friedman's proof gives an alternative proof of Theorem 2.1 for structures of the form $\langle L_\alpha, \in, \ldots\rangle$

The ordinals studied by Friedman were exactly those studied by us, and in the second part of the paper, we show that $\langle L_\alpha, \in, g\rangle$ is countably decomposable for appropriate choice of g. Hence Theorem 2.2 applies and yields the Σ_1-compactness, completeness and ordinal omitting results (see Section 10). Friedman proved a converse theorem; see Theorem 13.2 for more clarification on the relation between the present paper and Friedman's [10].

3. Sufficient conditions for Kleene structures

Let $\mathcal{M} = \langle M, \in, \mathbf{S}\rangle$ be a transitive structure having an inductive pairing function. Recall from Section 1 that $\mathcal{M}$ is a Kleene structure iff $\mathrm{HYP}(\mathcal{M})$ is validity admissible. By 2.2 it suffices that $\mathrm{HYP}(\mathcal{M})$ will have the decomposition property. This is always the case when M is countable. Keeping in mind that $\mathrm{HYP}(\mathcal{M})$ is projectible into M, Example 2.8 shows that if $M = \bigcup_{n<\omega} M_n$ where M_n is transitive and $\mathcal{P}(M_n) \subseteq M_{n+1}$ for each n, then $\mathcal{M}$ is a Kleene structure. This result is essentially due to Chang and Moschovakis [5] and is explicitly proved in [23]. Example 2.10 leads to a large class of Kleene structures, described in the following proposition.

Proposition 3.1. *Let $\mathcal{M} = \langle M, \in, \mathbf{S} \rangle$ be a transitive structure having an inductive pairing function. Suppose that there exist sets M_n $(n < \omega)$ such that $M = \bigcup_{n<\omega} M_n$ and $\bigcup_{n<\omega} \mathcal{P}(M_n) \subseteq C$ for some $C \in \mathrm{HYP}(\mathcal{M})$. Then $\mathcal{M}$ is a Kleene structure.*

Proof. Use Example 2.10 with $A = \mathrm{HYP}(\mathcal{M})$, $B = M$. $\square$

A special case of Proposition 3.1 is when $M = \bigcup_{n<\omega} M_n$ and $M_n \subseteq M_{n+1}$, $\mathcal{P}(M_n) \in M$ for each $n < \omega$. This case is stated in [22, p. 113] but the proof seems to require something like the methods of this paper, unless one is willing to assume that the sets M_n are transitive.

Corollary 3.2. *Let M be a transitive set closed under (unordered) pairs for which there exists a sequence $\langle M_n \mid n < \omega \rangle$ such that $M = \bigcup_{n<\omega} M_n$ and $\bigcup_{n<\omega} \mathcal{P}(M_n) \subseteq M$. Then every structure of the form $\langle M, \in, \mathbf{S} \rangle$ is a Kleene structure.*

It seems likely that in this situation $\langle M, \in \rangle$ is a uniform Kleen structure in Nyberg's sense, but this involves checking the proofs of all the results involved for the required uniformity, and we have not done it.

We can get many more examples of Kleene structures by using Example 2.11 with $A = \mathrm{HYP}(\mathcal{M})$ and $E_{k-1} = M$. The most useful case is when $k = 2$ so that $B \in M \in A = \mathrm{HYP}(\mathcal{M})$. For this case Example 2.11 shows the following

Proposition 3.3. *Let $\mathcal{M} = \langle M, \in, \mathbf{S} \rangle$ be a transitive structure having an inductive pairing function. Assume that the set M has cofinality ω and has a* c-max *member B of the form $B = \bigcup_{n<\omega} B_n$ where $\bigcup_{n<\omega} \mathcal{P}(B_n) \subseteq C$ for some $C \in \mathrm{HYP}(\mathcal{M})$. Then $\mathcal{M}$ is a Kleene structure.* $\square$

Corollary 3.4. *Let κ be a strong limit cardinal of cofinality ω and let M be a transitive set closed under pairs such that $H(\kappa)$ is a c-max element of M. Then every structure $\langle M, \in, \mathbf{S} \rangle$ is a Kleene structure.* $\square$

Note that if $\langle M, \in \rangle$ is a transitive elementary substructure of $\langle H(\kappa^+), \in \rangle$ (or of $\langle H(\kappa^+)_W, \in \rangle$ for some set W of urelements) and M has cofinality ω, then Corollary 3.4 applies to M. This, together with [25, 4.1] proves [25, 4.2(3)] i.e., the Π^1_1-compactness of almost all levels of cofinality ω in any cumulative hierarchy of length κ^+ whose union is $H(\kappa^+)_W$ for some set W of urelements. Actually, the work reported in this paper started from the proof of Corollary 3.4 (for M admissible) for the purpose of getting this result on Π^1_1-compactness.

Corollary 3.4 was first proved directly, along the lines of [5] or [23] and later derived from the completeness Theorem 2.1.

In Part II of this paper we shall get a characterization of the limit ordinals α for which $\langle L_\alpha, \in\rangle$ is a Kleene structure, assuming $V = L$.

4. Overture to the Proof of Theorem 2.1

Before we prove Theorem 2.1 (in the next two sections) we shall elaborate on the DP and draw an equivalent form of it which will be easier to apply.

Both (DP1) and (DP2) state that every set is a countable union of sets z_i whose power set is still in the admissible set A, and the power set of z_i can in some sense be effectively found. The apparent strengthenings of (DP1) and (DP2) which we shall consider (but which are actually equivalent to (DP1) and (DP2) respectively) will state that every set in A is a countable union of sets z_i, such that $\mathcal{P}(z_i \times z_i) \in A$, and $\mathcal{P}(z_i \times z_i)$ can be found in some effective sense.

Definition 4.1. $\mathcal{A}$ has the strong first decomposition property (SDP1) if there is an $\mathcal{A}$-r.e. predicate S such that $\{(X, \mathcal{P}(X \times X)) \mid S(x)\}$ is $\mathcal{A}$-r.e. and such that every member of $\mathcal{A}$ is a countable union of elements satisfying S. Such S is called a strong smallness predicate for $\mathcal{A}$.

Definition 4.2. A strong decomposition relation for $\mathcal{A}$ is $R \subseteq A^2$ such that

(a) R is $\mathcal{A}$ r.e.

(b) $\forall X \exists Y R(X, Y)$.

(c) For every X, Y such that $R(X, Y)$, X can be represented as a countable union $X = \bigcup_{n<\omega} X_n$, where $X_n \in Y$ and $\mathcal{P}(X_n \times X_n) \subseteq Y$. $\mathcal{A}$ has the strong second decomposition property (SDP2) if it has a strong decomposition predicate.

The main result of this section is that the seemingly stronger (SDP1) and (SDP2) are respectively equivalent to (DP1) and (DP2). The reader who is satisfied with replacing (DP1) by (SDP1) in the statement of Theorem 2.1 can skip this section and go directly to the Proof of Theorem 2.1 in the next section. The only fact from this section we shall need is Lemma 4.7 which is independent from any thing else proved in this section. To motivate the proofs of these two main facts we give first a non effective version.

Theorem 4.1. *Let $\langle A, \in\rangle$ be a transitive set closed under primitive recursive set functions ('prim closed'). Assume that every $x \in A$ can be written as $\bigcup_{i\in\omega} z_i$, such that $\mathcal{P}(z_i) \in A$, then every $x \in A$ can be expressed as $\bigcup_{i\in\omega} z_i$ where $\mathcal{P}(z_i \times z_i) \in A$.*

Proof. If every member of A is countable, then the theorem is trivial (because every member of A is union of singletons and for singleton z_i, clearly $\mathcal{P}(z_i \times z_i) \in A$).

It is clearly enough to prove the theorem for x's such that $\mathcal{P}(x)\in A$. So let x be an infinite set such that $\mathcal{P}(x)\in A$. We first show that we can find a large enough member of A, whose power set is still in A.

Lemma 4.2. *Let A be as in Theorem* 4.1. *If $\mathcal{P}(x)\in A$ and x infinite, then for some $z\in A$*

$$|z|\geqslant 2^{2^{2^{|x|}}} \quad \textit{and} \quad \mathcal{P}(z)\in A.$$

Proof. Since $\mathcal{P}(x)\in A$, we can find $z_i\in A$ such that $\mathcal{P}(x)=\bigcup_{i<\omega} z_i$ and $\mathcal{P}(z_i)\in A$. The cofinality of $|\mathcal{P}(X)|$ is bigger than ω. Hence for some z_i we have $|z_i|=|\mathcal{P}(x)|=2^{|x|}$. Since $\mathcal{P}(z_i)\in A$ we can repeat the argument for x replaced by z_i. Repeating it twice we get $z\in A$

$$|z|=2^{2^{2^{|x|}}} \quad \text{and} \quad \mathcal{P}(z)\in A. \qquad \square \text{ Lemma 4.2}$$

The next fact states that we can find z of large enough cardinality, so that $\mathcal{P}(z\times z)\in A$.

Lemma 4.3. *Let x and A be as in Lemma* 4.2. *Then there exists $t\in A$, such that $|t|=(2^{|x|})^+$ and $\mathcal{P}(t\times t)\in A$.*

Proof. Let z be an element satisfying Lemma 4.2. Since A is prim closed $z\times z\in A$. By assumption $z\times z=\bigcup_{n<\omega} y_n$ where $y_n\in A$, $\mathcal{P}(y_n)\in A$. Enumerate z in a sequence $\langle z_\gamma \mid \gamma<|z|\rangle$ let $[z]^2$ be the set of unordered pairs of elements of z. Then

$$[z]^2=\bigcup_{m,n<\omega} W_{m,n}$$

where

$$W_{m,n}=\{\{z_\gamma, z_\delta\}\mid \gamma<\delta,\ (z_\gamma, z_\delta)\in y_m \text{ and } (z_\delta, z_\gamma)\in y_m\}.$$

By the Erdös–Rado partition Theorem [7] and the fact that $|z|\geqslant(2^{2^{|x|}})^+$, there exists a subset t of z such that

$$|t|=(2^{|x|})^+ \quad \text{and} \quad [t]^2\subseteq W_{m,n} \quad \text{for some fixed } m, n<\omega.$$

It follows that $t\times t\subseteq y_m\cup y_n\cup t'$ where $t'=\{(y,y)\mid y\in t\}$. (Note $t'\in A$ since A is Prim closed.) Since $\mathcal{P}(z)\in A$, we have $t\in A$. Recall that $\mathcal{P}(y_m), \mathcal{P}(y_n), \mathcal{P}(t')$ all belongs to A. (The later is because we have $\mathcal{P}(t)\in A$, and t' is an A image of t. Again we use the fact that A is prim closed.) Clearly we get $\mathcal{P}(t\times t)\in A$. $\square$ Lemma 4.3

We now conclude the proof of Theorem 4.1. So given infinite x such that $\mathcal{P}(x)\in A$. Let t satisfy Lemma 4.3 with respect to x. $x\times t\in A$, hence $x\times t=\bigcup_{n<\omega} z_n$ where $\mathcal{P}(z_n)\in A$.

Every member of t, l determines a partition of x into ω many parts. Since $|t| = (2^{|x|})^+$ and the cardinality of such partitions is $2^{|x|}$. We get $t' \subseteq t$, $|t'| = (2^{|x|})^+$ and every $l \in t'$ determines the same partition on x, namely $x =$ can be represented as a union $\bigcup_{n<\omega} x_n$ where $x_n \times t' \subseteq z_n$.

Since $|t'| \geq |x_n|$, there exists a function f_n from t' onto x_n. $f_n \subseteq x_n \times t'$, hence $f_n \in \mathcal{P}(z_n) \in A$, so we get $f_n \in A$. Recall that $(t' \times t') \in A$, x_n is an A image of t'. It follows that $\mathcal{P}(x_n \times x_n) \in A$. ($\mathcal{P}(x_n \times x_n)$ is the set $\{R \mid \text{for some } T \in \mathcal{P}(t' \times t')$: $R = \{(f_n(l), f_n(m)) \mid (l, m) \in T\}\}$.) Hence X is a countable union of sets the powerset of whose square is in A. □ Theorem 4.1.

We now handle (DP1).

Theorem 4.4. (DP1) *implies* (SDP1).

Proof. We begin the proof by showing that the smallness predicate S may be assumed to be closed under A images.

Lemma 4.5. *Let S be a smallness predicate for $\mathcal{A}$ and let*

$$\bar{S} = \{X \in A \mid (\exists f, Y \in A)[S(Y) \text{ and } f \text{ is a function from a subset of } Y \text{ onto } X]\},$$

then $\bar{S}$ is a smallness predicate for $\mathcal{A}$, $\bar{S} \supseteq S$ and $\bar{S}$ is closed under A images.

Proof. $\bar{S}$ is clearly $\Sigma_1(\mathcal{A})$ since S is. Next note that if f is a function from a subset of Y onto X, then $\mathcal{P}(X) = \{f''W \mid W \in \mathcal{P}(Y)\}$. Therefore $\bar{S}(X) \Rightarrow P(X) \in A$ as A is prim closed. Also $\{(X, \mathcal{P}(X) \mid X \in \bar{S}\}$ is $\Sigma_1(\mathcal{A})$ since

$$\begin{aligned}\bar{S}(X) \wedge z = P(X) \Leftrightarrow (\exists f, Y, t \in A)[&S(Y) \wedge t = P(Y)\\ &\wedge (f \text{ is a function from a subset of } Y \text{ onto } X)\\ &\wedge z = \{f''W \mid W \in t\}].\end{aligned}$$

This shows that $\bar{S}$ is a smallness predicate for A and the rest of Lemma 4.5 is obvious. □ Lemma 4.5

It follows from Lemma 4.1 that if $\mathcal{A}$ satisfies (DP1), then (DP1) is witnessed for $\mathcal{A}$ by some smallness predicate S which is closed under A images. Fix such S and note that every subset of a member of this S is also a member of S and that every $X \in A$ such that $|X| \leq 1$, is a member of S. Let κ be the least ordinal such that $\neg S(\kappa)$. Then for $\alpha \in A$, $S(\alpha) \Leftrightarrow \alpha < \kappa$. Clearly $1 < \kappa \leq o(A)$, and it is possible that $\kappa = o(A)$. The next lemma shows that every $x \in A$ is a countable union of sets which are A images of some $\alpha < \kappa$.

Lemma 4.6. *Every $X \in A$ is a countable union of $\bigcup_{n<\omega} X_n$ where for some $\alpha_n < \kappa$ and $f_n \in A$, $X_n = f_n''\alpha_n$ and f_n is one-to-one.*

Proof. By Theorem 4.1 we can express X as $\bigcup_{n<\omega} X_n$ where $\mathcal{P}(X_n \times X_n) \in A$. Without loss of generality we can assume that $S(X_n)$. (Otherwise replace each X_n by a decomposition of it into countable union of members of S.) X_n can be well-ordered by some relation R_n. Since $R_n \subseteq X_n \times X_n$, we have $R_n \in A$. But $\langle A, \in\rangle$ is an admissible set. Hence for some $f_n \in A$ and some ordinal α_n, we have $A \models$"f_n is an order preserving function from $\langle \alpha_n, \in\rangle$ onto $\langle X_n, R_n\rangle$".

By our assumption about S being closed under A images we have $S(\alpha_n)$, hence $\alpha_n < \kappa$. $\square$ Lemma 4.6.

Conclusion of the Proof of Theorem 4.4. We have to produce a strong smallness predicate for A. We define

$$\bar{S} = \{X \in A \mid X \text{ is an } A \text{ image of some } \alpha < \kappa\}$$

Note that $\bar{S} = \{X \in A \mid (\exists \beta < \kappa)(\exists f \in A)(f : \beta \to X \text{ is 1–1 and onto})\}$. Hence $\bar{S}$ is A-r.e. By Lemma 4.6 every member A is a countable union of members of $\bar{S}$. The only claim left to be verified is that $\{(X, \mathcal{P}(X \times X)) \mid \bar{S}(X)\}$ is $\mathcal{A}$-r.e. (This includes the claim that for $\bar{S}(X)$, $\mathcal{P}(X \times X) \in A$.)

We show first that $\alpha < \kappa \Rightarrow \mathcal{P}(\alpha \times \alpha) \in A$ and that $\{(\alpha, \mathcal{P}(\alpha \times \alpha) \mid \alpha < \kappa\}$ is $\mathcal{A}$-r.e. Note that if α is infinite, since A is admissible, we have a bijection $g : \alpha \to \alpha \times \alpha$, hence, using $\mathcal{P}(\alpha) \in A$ we get $\mathcal{P}(\alpha \times \alpha) \in A$. The case α finite is obvious. Moreover

$$\alpha \text{ infinite } \wedge \alpha < \kappa \wedge Y = P(\alpha \times \alpha) \Leftrightarrow \exists Z\, \exists g\, (g : \alpha \to \alpha \times \alpha$$
$$\text{is a bijection} \wedge Z = \mathcal{P}(\alpha) \wedge Y = \{f''z \mid z \in Z\}).$$

Hence we get that $\{(\alpha, \mathcal{P}(\alpha \times \alpha) \mid \alpha < \kappa\}$ is $\mathcal{A}$-r.e. (Recall that $\alpha < \kappa \to S(\alpha)$. Now if $S(X)$ we have some $\alpha < \kappa$ and $g : \alpha \xrightarrow{\text{onto}} X$. Define $\tilde{g} : \alpha \to X \times X$ by $\tilde{g}(\xi, \eta) = (g(\xi), g(\eta))$. Then $\mathcal{P}(X \times X) = \{\tilde{g}''t \mid t \in \mathcal{P}(\alpha \times \alpha)\}$. Hence

$$\bar{S}(X) \wedge Y = \mathcal{P}(X \times X) \Leftrightarrow \bar{S}(X) \wedge (\exists \alpha, Z, g \in A)$$
$$[\alpha < \kappa \wedge Z = \mathcal{P}(\alpha \times \alpha) \wedge g : \alpha \xrightarrow{\text{onto}} X \wedge Y = \{\tilde{g}''t \mid t \in Z\}].$$

This proves that $\{(X, \mathcal{P}(X, X) \mid \bar{S}(X)\}$ is $\mathcal{A}$-r.e. and $\bar{S}$ was shown to be a strong smallness predicate $\square$ Theorem 4.4.

Our subsequent use of strong smallness predicates depends on the following simple observation.

Lemma 4.7. *Let S be a strong smallness predicate on A. If $S(X)$ holds and X is an infinite set, then $\omega \in A$ and $\omega^X \in A$ where $\omega^X = \{f \mid f : X \to \omega\}$. Moreover, the relation $\{(X, \omega^X) \mid S(X)$ and X is infinite$\}$ is $\mathcal{A}$-r.e.*

Proof. If $S(X)$ holds, then all subsets of X and all binary relations on X are in A. If X is infinite, it has a subset X_0 and a well-ordering r of X_0, the order type of which is ω. It follows that $\omega \in A$ and that the unique isomorphism $g : \langle X_0, r\rangle$ onto

$\langle \omega, < \rangle$ is in A. Now $\omega^X = \{g \circ f \mid f : X \to X_0\}$. Since $P(X \times X) \in A$ it is clear that $X_0^X \in A$, hence $\omega^X \in A$.

For the moreover part note that

$$S(X) \wedge Y = \omega^X \Leftrightarrow [\exists X_0, g, Z, W \in A][S(X) \wedge$$
$$\wedge Z = \mathcal{P}(X \times X) \wedge X_0 \subseteq X \wedge g : X_0 \text{ is one-to-one and onto } \omega \wedge$$
$$\wedge W = \{f \mid f \in Z, f : X \to X_0\} \wedge Y = \{g \circ f \mid f \in W\}].$$

This relation is $\mathscr{A}$-r.e. because S is a strong smallness predicate on $\mathscr{A}$. $\square$ Lemma 14.7

We now wish to prove that if $\mathscr{A}$ has (DP2), then $\mathscr{A}$ satisfies (SDP2). This is not needed for the proof of Theorem 2.1, but will be used to show that (DP2)$\Rightarrow$ (DP1) whenever A contains an element of maximal cardinality. Several examples in Section 2 depend on this fact.

Theorem 4.8. *If $\mathscr{A}$ satisfies* (DP2), *then it satisfies* (SDP2).

Proof. The idea of the proof is to have an effective version of the proof of Theorem 4.1. We fix a decomposition predicate for $\mathscr{A}$, R. We shall assume without loss of generality that

$$R(X, Y) \Rightarrow \mathcal{P}(\{a\} \times \{a\}) \in Y \quad \text{for all } a \in X.$$

The first lemma is a variant of Lemma 4.2 except that we handle not just one x, but all collection of them, provided we are given a set y containing the power set of all of them.

Lemma 4.9. *Let $\mathscr{A}$ satisfy* (DP2). *There exists an $\mathscr{A}$-r.e. predicate T such that*

(a) $\forall Y \exists Z\, T(Y, Z)$.

(b) *If $T(Y, Z)$, then for every infinite $M \in Y$ such that $\mathcal{P}(M) \in Y$, Z contains a member F whose cardinality is $2^{2^{2^M}}$, and $\mathcal{P}(T) \subseteq Y$.*

Proof. Define $\mathcal{P}(Z, T)$ – the power set of Z relative to T – by

$$\mathcal{P}(Z, T) = \{Y \mid Y \subseteq Z, Y \in T\}.$$

$\mathcal{P}(Z, T)$ is clearly $\mathscr{A}$-recursive.

Define $Q(Y) = \{\mathcal{P}(Z, Y) \mid Z \in Y\}$.

R is a decomposition relation for $\mathscr{A}$.

Define $T_0(Y, Z)$ by

$$T_0(Y, Z) \leftrightarrow \forall T \in Z\, \exists S \in Q(Y)\, \exists U$$
$$[R(S, U) \wedge T \in U] \wedge \forall S \in Q(Y)\, \exists U [R(S, U) \wedge U \subseteq Z],$$

i.e. $T_0(Y, Z)$ means that Z is the union of decomposition candidate for members of $Q(Y)$.

Note that by properties of R and the admissibility of A, $\forall Y\,\exists Z\, T_0(Y, Z)$. Note also that if $M \in Y$, M infinite and $\mathcal{P}(M) \subseteq Y$, then if $T(Y, Z)$ holds, then for some U, $R(\mathcal{P}(M), U)$ holds, and $U \subseteq Z$. Hence for some X_n's, $X_n \in Z\, \mathcal{P}(M) = \bigcup_{n<\omega} X_n$. Therefore some X_n has cardinality $2^{|M|}$ and $P(X_n) \subseteq U \subseteq Z$.

We can now iterate the definition of T_0 to get

$$T(Y, Z) \leftrightarrow \exists Z_1, Z_2\, (T_0(Y, Z_1) \wedge T_0(Z_1, Z_2) \wedge T_0(Z_2, Z)).$$

$T(Y, Z)$ is clearly $\mathcal{A}$-r.e. $\forall Y\,\exists Z\, T(Y, Z)$. If $M \in Y$, M infinite and $\mathcal{P}(M) \subseteq Y$, then if $T(Y, Z)$ holds (witnessed by Z_1 and Z_2), then in Z_1 one can find M_1 such that $|M_1| = 2^{|M|}$, $\mathcal{P}(M_1) \subseteq Z_1$, hence in Z_2 one can find M_2, $|M_2| = 2^{|M_1|}$, $\mathcal{P}(M_2) \subseteq Z_2$. Similarly in Z one can find M_3, $|M_3| = 2^{|M_2|}$, $\mathcal{P}(M_3) \subseteq Z$. We can take $F = M_3$ and the lemma is verified. $\square$ Lemma 4.9

The next lemma corresponds to Lemma 4.3.

Lemma 4.10. *Let $\mathcal{A}$ satisfy* (DP2). *There exists an $\mathcal{A}$-r.e. predicate $M(X, Y)$ such that*

(a) $\forall X\,\exists Y\, M(X, Y)$.

(b) *If $M(X, Y)$ holds, then for every infinite $S \in X$, such that $\mathcal{P}(S) \subseteq X$, Y contains a set U, $|U| = (2^{|S|})^+$, $\mathcal{P}(U \times U) \subseteq Y$, $\mathcal{P}(U) \subseteq Y$.*

Proof. Let $T(X, Y)$ be the predicate satisfying Lemma 4.9. Let $M^*(X, Y)$ be the predicate expressing the fact that, for some Z, $T(X, Z)$ holds and Y contains (as subset) U satisfying $R(S \times S, U)$ for all $S \in Z$. Namely

$$\begin{aligned} M^*(X, Y) \leftrightarrow \exists Z\, [(T(X, Z) \wedge Z \subseteq Y \wedge \forall S \in Z \\ \exists U\, (R(S \times S, U) \wedge U \subseteq Y) \wedge \forall S \in Z \{(s, s) \mid s \in S\} \in Y]. \end{aligned}$$

$M^*(X, Y)$ is $\Sigma_1(\mathcal{A})$ and $\forall X\,\exists Y\, M^*(X, Y)$. (Since $\forall X\,\exists Z\, T(X, Y)$ and $\forall S \in T\,\exists U\, R(S \times S, U)$). Define $M(X, Y)$ as expressing the fact that Y contains the unions of all triples of elements of some Z satisfying $M^*(X, Z)$ i.e.

$$M(X, Y) \leftrightarrow \exists Z\, (M^*(X, Z) \wedge \forall Z_1, Z_2, Z_3 \in Z\, (Z_1 \cup Z_2 \cup Z_3 \in Y).$$

M is the required predicate. Clearly M is $\Sigma_1(\mathcal{A})$ and $\forall X\,\exists Y\, M(X, Y)$. Now assume that $M(X, Y)$ holds and $S \in X$, S infinite and $\mathcal{P}(S) \subseteq X$. For some Z we get that $M^*(X, Z)$ holds and that the union of every triple of elements of Z is in Y. By definition of M^*, Z contains an element $V, \mathcal{P}(V) \subseteq Z$ such that $|V| = 2^{2^{2^{|S|}}}$ (recall the properties of T) and $V \times V = \bigcup_{n<\omega} V_n$ where $\mathcal{P}(V_n) \subseteq Z$. By the arguments of the proof of Lemma 4.3 we get that for some $U \subseteq V$, $|U| = (2^{|S|})^+$ and for some m, n

$$U \times U \subseteq V_n \cup V_m \cup \{(u, u) \mid u \in U\}.$$

It follows that every subset of $U \times U$ is the union of three elements of Z. Hence it is in Y.

It follows that Y contains an element U, $|U| = (2^{|S|})^+$ and $\mathcal{P}(U \times U) \subseteq Y$, $\mathcal{P}(U) \subseteq X$. $\square$ Lemma 4.10

Lemma 4.11. *Let $\mathcal{A}$ satisfy* (DP2). *There exists an $\mathcal{A}$-r.e. predicate $Q(X, Y)$ such that*

(a) $\forall X \exists Y Q(X, Y)$.

(b) *If $Q(X, Y)$ holds, then for every infinite $S \in X$ such that $\mathcal{P}(S) \subseteq X$, there are* 1–1 *functions in Y, f_n, $(n < \omega)$, such that $S = \bigcup_{n<\omega}$* range(f_n), $\mathcal{P}(\mathrm{Dom}(f_n) \times \mathrm{Dom}(f_n)) \subseteq Y$.

Proof. Let $M(X, Y)$ be the predicate satisfying Lemma 4.10. Define $Q(X, Y)$ by

$$Q(X, Y) \leftrightarrow \exists Z\, [M(X, Z) \wedge Z \subseteq Y \wedge \forall U \in Z$$
$$\forall S \in X\, \exists L\, (R(U \times S, L) \wedge L \subseteq Y],$$

i.e. Q expresses the fact that for some Z satisfying $M(X, Z)$, Y includes as subsets a decomposition witness for every $U \times S$ where $U \in Z, S \in X$ (i.e. L such that $R(U \times S, L)$). That Q is $\Sigma_1(\mathcal{A})$ and that $\forall X \exists Y Q(X, Y)$ is easy to verify. Assume $Q(X, Y)$ and let $S \in X$ be infinite with $\mathcal{P}(S) \subseteq X$. Let Z witness $Q(X, Y)$, in particular $M(X, Z)$ holds. By properties of M, Z contains U, such that $\mathcal{P}(U \times U) \subseteq Z \subseteq Y$ and $|U| = (2^{|S|})^+$. Since R is a decomposition relation, and Y contains a decomposition witness for $U \times S$ we get $U \times S = \bigcup_{n<\omega} V_n$ where $\mathcal{P}(V_n) \subseteq Y$. Now the argument is as in the conclusion of Theorem 4.1, i.e. we get that for some $U' \subseteq U$, $|U'| = (2^S)^+$ and some sequence S_n $(n < \omega)$, $S = \bigcup S_n$ and $U' \times S_n \subseteq V_n$. Since $|U'| \geq |S_n|$ we get the existence of a 1–1 function $f: U_n \xrightarrow{\text{onto}} S_n$ where $U_n \subseteq U'$. Clearly $f_n \subseteq U' \times S_n \subseteq U_n$. Hence $f_n \in Y$. The facts $S = \bigcup_{n\in\omega} \mathrm{range}(f_n)$, $\mathcal{P}(\mathrm{Dom}(f_n) \times \mathrm{Dom}(f_n)) \subseteq Y$ are obvious. $\square$ Lemma 4.11

The conclusion of the proof of Theorem 4.8. We shall produce a strong decomposition predicate using the predicate $Q(X, Y)$ introduced by Lemma 4.11.

$\bar{R}(X, Y)$ will express that for some T such that $R(X, T)$, and some Z such that $Q(T, Z)$, for every function f in Z the preimage of f of a set of pairs is in Y, namely:

$$\bar{R}(X, Y) \leftrightarrow \exists T \exists Z\, (R(X, T) \wedge T \subseteq Y \wedge Q(T, Z) \wedge \forall f \in Z\, \forall L \in Z$$
$$[f \text{ is a one-to-one function from some } K \text{ to } N, \wedge L \subseteq K \times K$$
$$\to \{\langle f(r), f(s)\rangle \mid \langle r, s\rangle \in L\} \in Y]).$$

Again checking that $\bar{R}$ is $\Sigma_1(\mathcal{A})$ and that $\forall X \exists Y R(X, Y)$ is easy to verify. Assume that $\bar{R}(X, Y)$ holds and let T, Z witness it. Since $R(X, T)$ holds we get $X = \bigcup_{n<\omega} X_n$ where $\mathcal{P}(X_n) \subseteq T$. If X_n is finite, then we can assume $|X_n| = 1$. By the properties of $Q(T, Z)$, each infinite X_n can be represented as $X_n = \bigcup_{m<\omega}$ range$(f_{n,m})$ where $f_{n,m} \in Z$ and $\mathcal{P}(\mathrm{Dom}(f_{n,m}) \times \mathrm{Dom}(f_{n,m})) \subseteq Z$ for some one-to-one functions $f_{n,m}$. Let range $f_{n,m} = X_{n,m}$. By definition of $\bar{R}$, since

$\mathcal{P}(\mathrm{Dom}(f_{n,m})\times \mathrm{Dom}(f_{n,m}))\subseteq Z$ we get $\mathcal{P}(X_{n,m}\times X_{n,m})\subseteq Y$. Hence $X=\bigcup_{n<\omega} X_{n,m}$ where $X_{n,m}\in Y$ and $\mathcal{P}(X_{n,m}\times X_{n,m})\subseteq Y$ which proves that $\bar{R}(X, Y)$ is a strong decomposition relation. □ Theorem 4.8

The main reason for having Theorem 4.8 is

Proposition 4.12. *If A contains an element of maximal (real) cardinality, then if $\mathcal{A}$ has* (DP2) *it has* (DP1).

(*Note*: We have an example where the proposition fails if A does not contain an element of maximal cardinality, see Section 16.)

Proof. Assume that $\mathcal{A}$ satisfies (DP2) and let R be a strong decomposition relation for $\mathcal{A}$ (we use Theorem 4.8) and let D be an element of A with maximal cardinality.

Let B satisfy $R(D, B)$ and pick a representation $D=\bigcup_{n<\omega} D_n$ where $\mathcal{P}(D_n\times D_n)\subseteq B$. Let $|D_n|=\kappa_n$ and $\kappa=|D|$. Clearly $\kappa=\sup_{n<\omega}\kappa_n$. And since D is of maximal cardinality $2^{\kappa_n}=|\mathcal{P}(D_n)|\leqslant|D|$. Hence κ is a strong limit cardinal having cofinality ω. Since $P(D_n\times D_n)\in A$ it is clear that $P(\kappa_n)\in A$. (By picking a well-ordering of D_n of order type κ_n.)

Define the smallness predicate S by

$$S(X)\leftrightarrow\exists\alpha<\kappa\ \exists f\ (f:X\xrightarrow{1-1}\alpha).$$

S is clearly $\Sigma_1(\mathcal{A})$. If $S(X)$ holds, then $\mathcal{P}(X)\in A$, since $P(\alpha)\subseteq A$ for $\alpha<\kappa$. In fact even $\mathcal{P}(X\times X)\subseteq A$. We have to verify that $(X,\mathcal{P}(X))$ for X in S is $\mathcal{A}$-r.e. But if $W\in A$, $X\in S$, then

$$\begin{aligned}W=\mathcal{P}(X)\leftrightarrow&\forall w\in W\ (\omega\subseteq x)\wedge\exists\alpha<\kappa\ \exists f\ (f:\xrightarrow{1-1}\alpha\ \wedge\\ &\forall Y\in B\ (Y \text{ a well-ordering of order type } \alpha\\ &\rightarrow\exists g\ [g:\mathrm{Dom}(Y)\xrightarrow{\text{order preserving}}\alpha)\wedge\\ &\forall Z\subseteq Y\ Z\in B\wedge f^{-1}(g''\,\mathrm{Dom}\ Z)\in W])).\end{aligned}$$

We used the fact that relation "Y is a well-ordering of order type α" is $\mathcal{A}$-recursive since it is both $\Pi_1(\mathcal{A})$ and $\Sigma_1(\mathcal{A})$.

Every $x\in A$ is countable union of sets in S since by R being a strong decomposition relation $X=\bigcup X_n$ where $\mathcal{P}(X_n\times X_n)\in A$. Therefore $|X_n|<\kappa$, hence in A we can find a well-ordering of X_n of order type α for some $\alpha<\kappa$. Therefore $S(X_n)$ holds. □ Lemma 4.9

5. Proof of the completeness theorem using games

In this section we prove Theorem 2.1(a), that an admissible structure having DP is Σ_1-complete for σ-small theories. Upon examination of the proof, one can check that it yields uniform Σ_1-completeness, hence Σ_1-compactness and one can deduce 2.1(b).

The use of games or game formulas to establish Σ_1-completeness, which originated with Vaught is well known today (see e.g. Makkai [18], also [17] and Grant [12]).

The general idea of the proof of Theorem 2.1(a) is to characterize the relation $\varphi \in \mathrm{Cn}(T)$ for $\varphi \in A$ and T a σ-small A-r.e. theory by the existence of a winning strategy for the second player in a certain open game $G_{T,\varphi}$. In this game the first player (call him White) attempts to produce an increasing sequence $\langle \Phi_n \mid n < \omega \rangle$ of sets of sentences whose union is a Hintikka set (see [18] and below), including $T \cup \{\neg \varphi\}$, while the second player (Black) provides him with pieces of T and with various challenges, especially concerning the choice of disjuncts from various disjunctions. If $T \cup \{\neg \varphi\}$ has a model, then White has a strategy for facing Black's challenges and hence win after ω steps. If on the other hand $\varphi \in \mathrm{Cn}(T)$, then White cannot have such a strategy and hence (since the game is open and therefore determined) Black has a winning strategy.

In order to be able to express the existence of a winning strategy for Black by a Σ_1-formula some assumptions must be made on the definability over $\mathscr{A}$ of the set of options available for each player. The following Corollary 4.2 takes care of the matter.

First we describe the open game $\mathrm{OG}(A, P)$ associated with any set A and any set P of finite sequences from A containing the empty sequence. (i.e. $\emptyset \in P \subseteq A^{<\omega}$). The two players White and Black alternately choose elements $a_0, a_1, a_2, \ldots$ of A. As soon as a position $(a_0, \ldots, a_b) \notin P$ is reached, the player who moved last (White if he is even, Black if he is odd) is declared loser and the game stops. If the game continues for ω steps so that $(a_0, \ldots, a_k) \in P$ for all $k < \omega$, then White is the winner. We denote by P_W (P_B) the set of all $x \in P$ of even (respectively odd) length. P_W (P_B) is thus the set of positions at which it is White's (Black's) turn to play. Define a function $O_W : P_W \to A$ by

$$O_W(x) = \{y \in A \mid x^\frown\langle y \rangle \in P\}.$$

$O_W(x)$ is the set of options available to White at position x and we call O_W the options function of White. The options function of Black is defined similarly and denoted by O_B. A position $x \in P$ is called a winning position for some player when that player has a winning strategy in the game obtained from $\mathrm{OG}(A, P)$ by taking x, instead of $\emptyset$ as the initial position (where White moves first if $x \in P_W$ and Black otherwise).

Lemma 5.1. *Let $\mathscr{A}$ be an admissible structure. Consider an open game $\mathrm{OG}(A, P)$ where the set P of positions is $\mathscr{A}$-r.e. and the options function O_W of White is $\mathscr{A}$-partial-recursive. Then the set of winning positions for Black in this game is $\mathscr{A}$-r.e.*

Proof. The set in question is the least fixed point of the following inductive definition: $x \in S$ if

$$[x \in P_W \wedge (\forall y \in O_W(x))(x^\frown\langle y\rangle \in S)] \cup [x \in P_B \wedge \exists y\ (x^\frown\langle y\rangle \in S)].$$

Thus the set is $\mathscr{A}$-r.e. by Gandy's theorem [1]. $\square$ Lemma 5.1

For later use we remark that one may assign to each winning position x for Black an ordinal $D(x)$ measuring 'how far' Black is from winning. $D(x)$ is simply the stage at which x enters into the least fixed point of the above inductive definition. Thus $D(x) \leqslant \alpha$ if

$$x \in P_W \wedge (\forall y \in O_W(x)) D(x^\frown\langle y\rangle) < \alpha] \vee [x \in P_B \wedge \exists y\ (D(x^\frown\langle y\rangle) < \alpha)].$$

We stipulate that $D(x) = \infty$ for any position x which is not a winning position for Black. ∞ is considered greater than any ordinal.

Noting the uniformity of the above inductive definition in which P and the graph of O_W occur only positively, and using the obvious uniform positive version of Gandy's theorem we conclude

Corollary 5.2. *There is a fixed Σ_1-formula $\rho(x, X^+, Y^+)$ such that whenever $\mathscr{A}$ and P are as in Lemma* 5.1 *and $a \in A$, then a is a winning position for Black in* OG(A, P) *if $\mathscr{A} \models \rho[a, P, O_W]$. (We identify O_W with its graph.) In particular (letting $a = \emptyset$)* there is a *Σ_1-formula $\rho_0(X^+, Y^+)$ such that for A and P as above, Black has a winning strategy in* OG(A, P) *if $A \models \rho_0[P, O_W]$.*

We need a few more preliminaries on sets of $L_{\infty\omega}$ sentences. As far as Theorem 2.1 is concerned there is no loss of generality in considering $\mathscr{L}_{\infty\omega}$ sentences built up from atomic formulas and their negations by means of $\wedge$, $\vee$, $\forall$, $\exists$ (i.e. sentences in negation normal form) and in which the only nonlogical symbols occurring are relation symbols and individual constants. ('$=$' is considered a logical symbol.) Let Φ be a set of such sentences. Put

$$C_0(\Phi) = \{c \mid c \text{ is a constant occurring in } \Phi \text{ or is the specific constant } c_0\},$$

$$C_1(\Phi) = \{\psi \mid \psi \text{ is a conjunct of a member of } \Phi\},$$

$$C_2(\Phi) = \{\psi(c) \mid \forall x\, \psi(x) \in \Phi \text{ and } c \in C_0(\Phi)\},$$

$$C_3(\Phi) = \{c = c \mid c \in C_0(\Phi)\},$$

$$C_4(\Phi) = \{c = d \mid \text{`}d = c\text{'} \in \Phi\},$$

$$C_5(\Phi) = \{c = e \mid \text{for some } d \text{ `}c = d\text{'} \in \Phi \text{ and `}d = e\text{'} \in \Phi\},$$

$$C(\Phi) = \bigcup_{i=1}^{5} C_i(\Phi) = \text{the set of `immediate logical consequences' of } \Phi\}.$$

Note that the operation C is set-primitive-recursive, hence its restriction to any admissible set A is A-recursive.

By a witnessing assignment for Φ we mean a one-to-one function h such that

$$\mathrm{dom}(h) = \{\exists x\, \psi(x) \mid \exists x\, \psi(x) \in \Phi \text{ and there is no constant } c \text{ such that } \psi(c) \in \Phi\},$$

the values of h are constants outside $C_0(\Phi)$. We assume that some fixed primitive-recursive operation H has been chosen, which assigns to any set Φ (of sentences of the form considered above). a witnessing assignment $h = H(\Phi)$ for Φ. Let

$$C'(\Phi) = \Phi \cup \{\psi(c) \mid \exists x\, \psi(x) \in \mathrm{Dom}(h) \text{ and } c = h(\exists x\, \psi(x))\}$$

where $h = H(\Phi)$. Thus $C''(\Phi)$ is the result of adding to Φ witnesses to existential statements in some fixed primitive-recursive way. Note that if Φ already contains the required witnesses, then $H(\Phi) = \emptyset$ and $C'(\Phi) = \Phi$.

Let $C''(\Phi) = C(\Phi) \cup C'(\Phi)$. Call Φ a *Hintikka* set when

(1) Φ does not contain an atomic sentence and its negation.

(2) $C''(\Phi) = \Phi$.

(3) Whenever $\bigvee \Psi \in \Phi$, some disjunct $\psi \in \Psi$ belongs to Φ.

We remind the reader that every Hintikka set has a model (cf [18]) and that every model of Φ in which only constants from $C_0(\Phi)$ are interpreted can be expanded to a model of $C'(\Phi)$, hence of $C''(\Phi)$ (for $\Phi \models C(\Phi)$).

The main result of this section is

Proposition 5.3. *Let A be an admissible structure having the* DP. *Then there is a $\Sigma_1(\mathscr{A})$-formula $\sigma(T^+)$ such that whenever T is σ-small $\mathscr{A}$-r.e. (consisting of sentences in negation normal form without function symbols but possibly with constants), then*

$$A \models \sigma(T) \quad \textit{if } T \textit{ has no model.}$$

This clearly implies Theorem 2.1(a) because $\varphi \in \mathrm{Cn}(T)$ if $T \cup \{\neg\varphi\}$ has no model and the translation to negation normal form and replacing function symbols by predicates are primitive recursive syntactical operations.

Proof of Proposition 5.3. Fix S and R witnessing the fact that $\mathscr{A}$ has (DP1) and (DP2) respectively. By Lemma 4.4 we can assume that S is a strong smallness predicate for $\mathscr{A}$. Let T be a σ-small $\mathscr{A}$-r.e. theory as assumed. We are going to describe an open Game $\mathrm{OG}(A, P)$ associated with T such that T has a model iff White has a winning strategy in this game. Then we shall use Corollary 5.2 to get the desired $\Sigma_1(\mathscr{A})$-formula which is independent of T (getting a uniform version of Proposition 5.3).

The rules of the game are as follows: At step n White chooses an element $a_n \in A$ and Black chooses some $b_n \in A$, each player being required to fulfill the condition below. The set P of positions in the game is the set of sequences

Sh:144

$\langle a_0, b_0, a_1, b_1, a_n, b_n\rangle$ or $\langle a_0, b_0, \ldots, a_n\rangle$ with the following conditions satisfied:

a_n is a quadruple $a_n = (F_n, \Phi_n, D_n, G_n)$,

b_n is a triple (T_n, X_n, Y_n).

[Informally White tries to construct $\Phi = \Phi_0 \subseteq \Phi_1 \subseteq \cdots$ such that $\bigcup_{n<\omega} \Phi_n$ is a Hintikka set Φ_ω and Black forces him to include in Φ_{n+1} a piece T_n of T, so that eventually $\Phi_\omega \supseteq T$. X_n is a challenge for White to choose disjuncts for a certain small set of disjunctions in Φ_n (this set is D_{n+1}). G_{n+1} is Whites reaction to this challenge. Usually G_{n+1} does not give the desired disjuncts but consists of a promise to choose each of them after a specified number of steps. F_{n+1} is the fulfillment of promises of this kind given earlier in the game. The role of Y_n is technical and has to do with restricting the options of White to choosing disjuncts in the future, so that the set of available options is a member of A, so we can apply Lemma 5.1. It will be helpful to note that any function choosing disjuncts from some members of X_n is a subset of $X_n \times \mathrm{Tc}(X_n)$.]

Now we state more formally the type of moves White and Black can play in the game.

$a_0 = (\emptyset, \emptyset, \emptyset, \emptyset)$.
$b_n = (T_n, X_n, Y_n)$

where $T_n \subseteq T$ ($T_n \in A$ as $b_n \in A$), $S(X_n)$ and $R(X_n \times \mathrm{Tc}(X_n), Y_n)$.

$$a_{n+1} = (F_{n+1}, \Phi_{n+1}, D_{n+1}, G_{n+1})$$

such that

(1) $D_{n+1} = \{\varphi \mid \varphi \in X_n \cap \Phi_n, \varphi$ is a disjunction, such that no disjunct of φ belongs to $\Phi_n\}$.

(2) If D_{n+1} is finite, then G_{n+1} is a function that chooses a disjunct from each $\varphi \in D_{n+1}$.

(3) If D_{n+1} is infinite, then $G_{n+1}: D_{n+1} \to \omega - (n+2)$. [Informally if $G_{n+1}(\varphi) = j$, it means that White promises to choose a disjunct from φ at step j.]

(4) $F_{n+1} = \langle f_i^{n+1} \mid 1 \leqslant i \leqslant n\rangle$. If $1 \leqslant i \leqslant n$ and D_i is an infinite set of disjunctions and $\sigma_i : D_i \to \omega - (i+1)$, then f_i^{n+1} is a function that chooses a disjunct from each $\varphi \in D_i$ satisfying $G_i(\varphi) = n+1$ *and* $f_i^{n+1} \in Y_i$. For other values of i between 1 and n, $f_i^{n+1} = \emptyset$.

(5) $\Phi_{n+1} = C''(\Phi_n) \cup T_n \cup \bigcup_{i=1}^{n}$ range f_i^{n+1} (if D_{n+1} is finite, we also add range G_{n+1} to the union) such that Φ_{n+1} contains no atomic sentence and its negation. Clearly the set of positions in this game is $\mathcal{A}$-r.e. (The distinction between D_n finite and infinite comes from the case $\omega \notin A$.)

Lemma 5.4. *The options function O_W is $\mathcal{A}$-partial-recursive.*

Proof. The main point to notice is that if $p \in P_W$ (i.e. p is a position in which White is to move), then $O_W(p)$ is a member of A. This is clear if $p = \emptyset$. So assume

$p=(a_0, b_0, \ldots, a_n, b_n)$. Note that there is no freedom in the choice of D_{n+1} which is a primitive recursive function of b_n. Given D_{n+1}, if it is finite, then A contains as a member the set of all functions from D_{n+1} into $\mathrm{Tc}(D_{n+1})$. (In fact there is an $\mathscr{A}$-r.e. function mapping D_{n+1} into this set.) If D_{n+1} is infinite note that $D_{n+1}\subseteq X_n$, and $S(X_n)$. Hence $P(X_n\times X_n)\in A$. (We assumed that S is a strong smallness predicate.) In particular, if X_n is infinite, $\omega^{X_n}\in A$ (see Lemma 4.7) and the function mapping $X\in S$ to ω^X is $\mathscr{A}$-r.e. Hence we have an $\mathscr{A}$-r.e. function computing the set of possible G_{n+1}'s from b_n.

Remember that f_i^{n+1} was required to be in Y_i. Hence we can find an $\mathscr{A}$-partial-recursive function computing from $\langle b_0, \ldots, b_n\rangle$ the set of possible F_{n+1}. $\square$ Lemma 5.4

It is clear from the preceding arguments and the proof of Lemma 5.4 that one can write $\Sigma_1(\mathscr{A})$-formulas $\rho_1(x, T^+)$ and $\rho_2(x, y, T^+)$ which define the set of positions P and the graph of O_W, in the game associated with T, for any choice of the $\mathscr{A}$-r.e theory T. (The only place T occurs is in the requirement $T_n\subseteq T$ on Black's move.)

Let $\sigma(T^+)$ be the formula

$$\rho_0(\{x\mid \rho_1(x, T^+)\}, \{(x, y)\mid \rho_2(x, y, T^+)\})$$

where ρ_0 comes from Corollary 5.2. Then if T is $\mathscr{A}$-r.e. we get

$$A\models\sigma[T] \text{ if Black has a winning strategy in the game associates with } T.$$

Now comes the model-theoretic part of the proof.

Lemma 5.5. *If T is σ-small, then T has a model if and only if White has a winning strategy in the associated game.*
(We are not using the $\mathscr{A}$-r.e.ness of T).

Proof. First suppose that T has a model and let $\mathscr{M}$ be a model for T. We describe the winning strategy for White. At the time of choosing his move $a_n=(F_n, \Phi_n, D_n, G_n)$ he also defines an interpretation in $\mathscr{M}$ of all the new witnessing constants appearing in Φ_n, interpretation that extends the interpretation he already picked for new constants which appeared in Φ_{n-1}.

More precisely at step n White picks a 7-triple

$$c_n=(F_n, \Phi_n, D_n, G_n, \mathscr{M}_n, H_n, \langle h_k^n\mid n<k<\omega\rangle)$$

of which the first four members form his move a_n, $\mathscr{M}_n$ is the model $\mathscr{M}$, expanded by the interpretation he picked for the new constants appearing in Φ_n. As induction assumption he makes sure that $\mathscr{M}_n\models\Phi_n$. H_n and h_k^n will be described below. (If $n=0$ the 7-triple is $\langle\emptyset, \emptyset, \emptyset, \emptyset, \mathscr{M}, \emptyset, \emptyset\rangle$). (We are not assuming $c_n\in A$, just that its first 4-members are.) Assume c_n has been chosen and Black's legal move is $b_n=(T_n, X_n, Y_n)$. Now White has to choose c_{n+1}.

D_{n+1} is of course determined by Φ_n and X_n. By assumption $\mathcal{M}_n \models \Phi_n$, hence $\mathcal{M}_n \models D_{n+1}$. Hence for every disjunction of D_{n+1}, some disjunct holds in $\mathcal{M}_n$. Let H_{n+1} be a function defined on D_{n+1} which picks such a disjunct for every member of D_{n+1}.

Note that $H_{n+1} \subseteq X_n \times \mathrm{Tc}(X_n)$, hence that by definition of Y_n (note that $R(X_n \times \mathrm{Tc}(X_n), Y_n)$ holds). H_{n+1} is a countable union of sets whose power set belongs to Y_n.

H_{n+1} can be represented as a countable disjoint union of members of Y_n, h_k^{n+1} $(i<k<\omega)$. (h_k^{n+1} is a partial function from D_{n+1} to $\mathrm{Tc}(D_{n+1})$.)

If D_{n+1} is finite, then White picks $G_{n+1} = H_{n+1}$. If D_{n+1} is infinite, then White defines $G_{n+1}: D_{n+1} \to \omega$ by $G_{n+1}(\rho) =$ the unique k such that $\rho \in h_k^{n+1}$. ($G_{n+1} \in A$ since $\omega^{X_n} \in A$.)

$$F_n = \langle f_j^i \mid i \leqslant j \leqslant n \rangle$$

where if D_j was infinite, then $f_j^i = h_i^j$ (h_i^j comes from his previous choice of c_j) and $f_j^i = \emptyset$ otherwise.

$$\Phi_{n+1} = C''(\Phi_n) \cup T_n \cup \bigcup_{i=1}^{n} \mathrm{range}(f_j^i).$$

(We add range (G_{n+1}) to Φ_{n+1} if D_{n+1} is finite.)

White has now to interpret the new constants appearing in Φ_{n+1}, but that can be easily done by expanding $\mathcal{M}_n$, noting that by induction assumption $\mathcal{M}_n \models \Phi_{n+1}$. The resulting expanded model is $\mathcal{M}_{n+1}$. Note that $\mathcal{M}_{n+1} \models \Phi_{n+1}$, (one has to use induction, noting the particular way in which h_i^j were defined for $j \leqslant n$, we of course use $\mathcal{M}_n \models T_n$ since $T_n \subseteq T$).

Thus we see that as long as Black plays legally, White can continue to choose the 7-triple c_n, but then he can continue his play. So White has a winning strategy.

Conversely assume that White has a winning strategy and we show that T has a model. We shall construct a sequence $\langle b_n \mid n < \omega \rangle$, $b_n = (T_n, X_n, Y_n)$ of legal moves for Black such that $\bigcup_{n<\omega} T_n = T$ and for all $k \in \omega$, $\bigcup_{k<n<\omega} X_n = \Phi_\omega = \bigcup_{n<\omega} \Phi_n$, where the sets Φ_n are generated by White using his winning strategy in response to the play $b_0, b_1, \ldots$ of Black. Then Φ_ω is a Hintikka set containing T. Indeed $\Phi_{n+1} \supseteq T_{n+1}$, $\Phi_{n+1} \supseteq C''(\Phi_n)$, no atomic sentence and its negation belongs to Φ_n. Moreover every disjunction $\delta \in \Phi_n$ either has a disjunct in Φ_n or (since it belongs to X_m for some $m > n$) it falls in some D_{m+1} and then has disjunct in Φ_j where $j = m+1$ if D_{m+1} is finite or in Φ_k where $k = G_{m+1}(\delta)$. As every Hinttika set has a model, T has a model.

It remains only to show that Black can play as desired. Since T is σ-small he can obviously pick $T_n \subseteq T$, $T_n \in A$ whose union is T. Also once X_n is chosen there always exists a suitable Y_n since R is a decomposition relation for $\mathcal{A}$. So the only problem is to choose $X_n \in S$ such that $\bigcup_{k<n<\omega} X_n = \Phi_\omega$ holds for every k.

Each Φ_n is a member of A, hence by S being a smallness predicate for $\mathcal{A}$ witnessing that $\mathcal{A}$ has (DP1), $\Phi_n = \bigcup_{l<\omega} Z_l^n$, where $S(Z_l^n)$. Once White picked Φ_n, Black picks a suitable sequence $\langle Z_l^n \mid l<\omega\rangle$ and plays $X_k = Z_l^n$ at step k where $f(n, l) = k$. (f is any 1–1 function from ω^2 onto ω such that $f(n, l) \geq n$ for each $n, l<\omega$.) Clearly $\bigcup_{k<n<\omega} X_n = \Phi_\omega$ for every $k>\omega$. □ Lemma 5.5

Since our game is open, it is determined therefore, for a σ-small theory, T, $\mathscr{A} \vDash \sigma(T)$ iff T has no model. □ Proposition 5.3 □ Theorem 2.1(a)

Proposition 5.3 also implies 2.1(b) (compactness), since if T is σ-small and T has no model, then $\mathscr{A} \vDash \sigma(T)$. Therefore by ordinary Σ_1-reflection (note that T appears positively in σ, hence $\sigma(T)$ is a Σ_1-formula) it follows that $\mathscr{A} \vDash \sigma(T')$ for some $T' \subseteq T$, $T' \in A$. Since $T' \in A$, it is trivially σ-small, hence by Proposition 5.3, T' has no model. □ Theorem 2.1(b)

(This argument is essentially a repetition of Nyberg's proof that uniform Σ_1-completeness implies Σ_1-compactness. See [22], Proof 1.2].)

Theorem 2.1(c) will be proved in Section 7. Actually the proof of Theorem 2.1 gives a uniform version of this theorem, (uniform across different admissible structures, the only parameter changing will be the specific smallness predicate and the specific decomposition relation).

Proposition 5.6. *There exists a (fixed) Σ_1-formula $\sigma_1(S^+, R^+, Q^+, T^+)$ such that if $\mathscr{A}$ is an admissible structure, S is a smallness predicate for $\mathscr{A}$ (witnessing* (DP1)), *R is a decomposition predicate for $\mathscr{A}$, Q is the predicate $\{\langle x, \omega^X\rangle \mid X \in S, X \text{ infinite}\}$ (hence S, Q, R, are $\mathscr{A}$-r.e. relations) and T is σ-small $\mathscr{A}$-r.e. theory, then*

$$T \text{ has no model} \quad \text{if } A \vDash \sigma_1[S, R, Q, T].$$

Proof. By examining the construction of the formula σ in the proof of Proposition 5.3 and noting that if T is $\mathscr{A}$-r.e. we can write (uniformally) a Σ_1-formula $\varphi(x, T^+)$ defining the theory obtained from T by replacing each function symbol by appropriate predicate and transforming each sentence to its negation normal form. We also use the fact that the transformation from a smallness predicate to a strong smallness predicate is uniform, as well as the proof of Lemma 4.7 □ Proposition 5.6

Corollary 5.7. *There exists a Σ_1-formula $\sigma_2(S^+, R^+, Q^+, T^+, X)$ such that if $\mathscr{A}$, S, R, and Q are as in Proposition 5.6 and T is a σ-small $\mathscr{A}$-r.e. theory, then*

$$\mathrm{Cn}(T) \cap A = \{a \in A \mid \mathscr{A} \vDash \sigma_2[S, R, Q, T, a]\}.$$

Proof. σ_2 is obtained from σ, by replacing T with $T \cup \{$the negation of $a\}$.
□ Corollary 5.7

From Corollary 4.7 we immediately obtain uniform versions of the completeness theorem for any example of a class of admissible structures for which we have a uniform definition of a smallness predicate witnessing (DP1) and a decomposition relation witnessing (DP2). Thus for instance Examples 2.3 and 2.4 yield uniform versions of the Barwise and Barwise Karp completeness theorems etc.

6. Axiom system and completeness theorem

From the proof of Proposition 5.3 one can extract an axiom system for $L_{\infty\omega} \cap A$ from which every validity can be proved in A. The first step is to analyze in model-theoretic terms what is the meaning of the fact that a certain position in the game associated with a σ-small theory T is a winning position for Black.

So let $p = \langle a_0, b_0, \ldots, a_n, b_n\rangle$ or $p = \langle a_0, b_0, \ldots, a_n\rangle$ be a position in the game where $a_i = \langle F_i, \Phi_i, D_i, G_i\rangle$, $b_i = \langle T_i, X_i, Y_i\rangle$. We shall define the formula associated with p, Ψ_p. (Ψ_p intuitively means that if it holds in a given model of T White can win by playing 'inside' this model.)

Define first $Y_{i,m}$ for $i<n$, $m>n$ by

$$\{f \mid f \in Y_i, \mathrm{Dom}(f) = \{\delta \mid \delta \in D_{i+1}, G_{i+1}(\delta) = m\},$$
$$\forall x \in \mathrm{Dom}(f): f(x) \text{ is one of the disjuncts appearing in } x\},$$

$$\Psi_p = \bigwedge \Phi_n \wedge \bigwedge_{i=1}^{n} \bigwedge_{m>n} \bigvee_{f \in Y_{i-1,m}} \bigwedge_{x \in \mathrm{Dom}(f)} f(x).$$

(For $n=0$ we take Ψ_p to be $c_0 \approx c_0$.) Note that $\Psi_{\langle a_0, b_0, \ldots, a_n\rangle} = \Psi_{\langle a_0, b_0, \ldots, a_n, b_n\rangle}$.

Lemma 6.1. *p is winning position for White if $T \cup \{\Psi_p\}$ has a model.*

Proof. Like the proof of Lemma 5.5. □ Lemma 6.1

Looking at the proof of Lemma 5.5 one can actually see that the proof yields

Lemma 6.2. *Let q be a position in the game at which it is Whites turn to play. Let $\mathcal{M}$ be a model of $T \cup \{\Psi_q\}$. (Note that $\mathcal{M}$ contains interpretation for all the new constants in Ψ_q.) Then there exists $p \in O_W(q)$ and an expansion of $\mathcal{M}$,$\mathcal{M}'$, by interpreting the constants in Ψ_p such that $\mathcal{M}' \models \Psi_p$.*

It follows from Lemma 6.1 that Black wins the game associated with T, he wins the game starting from any legal position p. (Otherwise $T \cup \{\Psi_p\}$ would have a model, hence T would have a model, contradicting Lemma 5.5.)

Our axiom system for the admissible structure $\mathcal{A}$ having DP (where S, R witness it, and S is a strong smallness property) is made up of the usual axioms for $L_{\infty\omega}$, for instance (A1)–(A7) of Barwise [1, III-4] together with the following scheme (which is a kind of distributive law). For each $\Phi \subseteq L_{\infty\omega} \cap A$, such that

Sh:144

$\Phi \in A$, $S(\Phi)$ and Φ is a set of disjunctions, Φ infinite and for each $Y \in A$ such that $R(\Phi \times \mathrm{Tc}(\Phi), Y)$ put an axiom of the form

$$\text{(A8)} \qquad \bigwedge \Phi \to \bigvee_{G \in \omega^{\Phi}} \bigwedge_{m<\omega} \bigvee_{f \in Y_{G,m}} \bigwedge_{\alpha \in \mathrm{Dom}(f)} f(x)$$

where $Y_{G,m} = \{f \mid f$ is a function defined on the set $\{\varphi \mid \varphi \in \Phi, G(\varphi) = m\}$, and such that $\forall \varphi \in \mathrm{Dom}(f)$ $f(\varphi)$ is one of the disjuncts appearing in $\varphi\}$. Note that since $S(\Phi)$ holds, $\omega^{\Phi} \in A$, hence any axiom of the form (A8) is in A and the set of all axioms is an $\mathscr{A}$-r.e. subset of $L_{\infty\omega} \cap A$.

The inference rules are (R1)–(R3) as in [1, III-4]. All axioms of the form (A8) are valid because assume that Φ is infinite set of disjunctions, $S(\Phi)$ holds as well as $R(\Phi \times \mathrm{Tc}(\Phi), Y)$. Consider a model $\mathscr{M}$ of $\bigwedge \Phi$, we have to verify that $\mathscr{M}$ satisfies

$$\bigvee_{G \in \omega^{\Phi}} \bigwedge_{m<\omega} \bigvee_{f \in Y_{G,m}} \bigwedge_{x \in \mathrm{Dom}(f)} f(x)$$

where $Y_{G,m}$ was defined above. By properties of R $\Phi \times \mathrm{Tc}(\Phi) = \bigcup X_n$ for some X_n $(n<\omega)$ such that $X_n \in Y$, $\mathscr{P}(X_n) \subseteq Y$. Since $\bigwedge \Phi$ holds in M, for every $\varphi \in \Phi$ we have some $f(\varphi) \in \mathrm{Tc}(\varphi)$ which is one of the disjuncts in φ which holds in $\mathscr{M}$. The pair $\langle \varphi, f(\varphi) \rangle \in \Phi \times \mathrm{Tc}(\Phi)$, so for some $G(\varphi)$, $\langle \varphi, f(\varphi) \rangle \in X_{G(\varphi)}$. G will be the witness for the outermost disjunction above. Let $m<\omega$. $f_m = f \restriction \{\omega \mid G(\varphi) = m\}$ is clearly in $Y_{G,m}$. Hence by definition of f, $\bigwedge_{x \in \mathrm{Dom}(f_m)} f_m(x)$ holds in $\mathscr{M}$, and we verified that there is $G \in \omega^{\Phi}$ such that for all $m<\omega$ there is $f_m \in Y_{G,m}$, $\mathscr{M} \models \bigwedge_{x \in \mathrm{Dom}(f)} f(x)$. This proves

$$\mathscr{M} \models \bigvee_{G \in \omega^{\Phi}} \bigwedge_{m<\omega} \bigvee_{f \in Y_{G,m}} \bigwedge_{x \in \mathrm{Dom}(f)} f(x).$$

What we have proved (together with the known facts) shows that our axiom system is sound. Now we prove that it is complete:

Theorem 6.3. *For a σ-small theory T, and $\varphi \in L_{\infty\omega} \cap A, \varphi \in \mathrm{Cn}(T)$ iff there is a proof of φ from T in our axiom system.*

Sketch of Proof. Define a theory to be $\mathscr{A}$-consistent if one can not prove a contradiction from it in our axiom system. Assume $T \cup \{\neg\varphi\}$ is consistent, then White can win the game associated with $T \cup \{\neg\varphi\}$. White simply makes sure that every position p arrived at the game $T \cup \{\neg\varphi\} \cup \{\Psi_p\}$ is $\mathscr{A}$-consistent. He can always do it because if the present position is $p = \langle a_0, b_0, \ldots, a_n, b_n \rangle$ and he is faced with $b_n = \langle T_n, X_n, Y_n \}$, then by induction assumption Ψ_p is consistent with $T \cup \{\neg\varphi\}$. Note that D_{n+1} is a set of disjunctions such that $S(D_{n+1})$ holds. We claim that White can pick a function $G_{n+1} : D_{n+1} \to \omega$ such that

$$T \cup \{\neg\varphi\} \cup \{\Psi_p\} \cup \{\bigwedge_{m<\omega} \bigvee_{f \in Y_{G,m}} \bigwedge_{x \in \mathrm{Dom}(f)} f(x)\}$$

is $\mathscr{A}$-consistent. (Otherwise using (A8) one gets a contradiction from $T \cup \{\neg\varphi, \Psi_p\}$) Using G_{n+1} and the consistency of Φ_p White can complete his move. We leave the details to the reader. Hence if $T\cup\{\neg\varphi\}$ is consistent, White wins the associated game, hence $T\cup\{\neg\varphi\}$ has a model. Contradicting $\varphi\in \mathrm{Cn}(T)$. □ Theorem 6.2.

7. Pinning down ordinals

Recall [1, III-7] that a theory $T \subseteq L_{\infty\omega} \cap A$ is said to pin down the ordinal α if it contains a symbol $<$ such that

(a) $\mathcal{M} \models T$ implies that $<$ interpreted in $\mathcal{M}$ is a well-ordering of its field.

(b) In some model of T, $<$ *has order type* α.

In this section we prove Theorem 2.1(c), i.e. if T is an $\mathscr{A}$-r.e. σ-small theory which pins down ordinals, then the ordinals pinned by it are bounded below $o(A)$. Let

$$T' = T\cup\{c_n \text{ is in the field of } <\}\cup \bigcup_{n<\omega} \{c_n < c_{n-1}\}$$

where $\{c_n \mid n<\omega\}$ is a set of new constants not appearing in T. By assumption T' has no models. (Otherwise T has a non-well-ordered model.) Hence Black wins the game associated with T'. We shall make a small change in the game by restricting the moves of Black, such that on his nth move he must play T_n which does not mention c_k for $k>n$ and $T_n \supseteq \{c_i$ in the field of $<\}$ and '$c_i < c'_{i-1} \in T_n$ for $0<i\leq n$. (It follows that c_k for $k>n$ does not appear in Φ_n.) It is easy to see that Black still wins the modified game. Recall that at the proof of Lemma 5.1 we assigned by $\mathscr{A}$-r.e. function an ordinal $D(p)\in A$ to any legal position in the game in which Black wins. (Hence to every position.)

Lemma 7.1. *Let $p=\langle a_0, b_0, \ldots, b_{n-1}, a_n\rangle$ or $p=\langle a_0,\ldots, b_{n-1}, a_n, b_n\rangle$ be a position in the modified game associated with T'. (Note that Ψ_p does not mention c_k, $k\geq n$.) Consider all models of $T\cup\{\Psi_p\}$, then the order type of the initial segment of $<$ determined in the model by the interpretation of c_{n-1} is $<D(p)$. (In case $n=0$ we take "the order type determined by c_{n-1}" to mean "the order type of $<$").*

Theorem 2.1(c) follows from Lemma 7.1, for $n=0$. Since $D(\emptyset)\in A$. The order type of $<$ in any model of T is less than $D(\emptyset)$.

Proof. By induction on $D(p)$. We distinguish two cases:

(a) $p\in P_B$. In this case $D(p)=D(q)+1$ for some $q\in O_B(p)$. But $\Psi_q=\Psi_p$, $D(q)<D(p)$, hence by induction assumption in any model of $T\cup\{\Psi_q=\Psi_p\}$ the initial segment determined by c_{n-1} has order type $<D(q)<D(p)$.

(b) $p\in P_W$, $p=\langle a_0, b_0,\ldots, a_n, b_n\rangle$. Let us distinguish two cases:

(I) $T\cup\{\Psi_p\}$ has no model. In this case the Lemma is true trivially.

(II) $T \cup \{\Psi_p\}$ does have models. In this case, by Lemma 6.2, any model of $T \cup \{\Psi_p\}$ can be expanded to a model of $T \cup \{\Psi_q\}$ for some $q \in O_W(p)$.

Note that since c_n did not appear in Φ_n, the only mention of it in Φ_q comes from T_n. Hence all that Φ_q implies about c_n is that c_n is in the field of $<$ and $c_n < c_{n-1}$. Assume that in some model $\mathcal{M}$ of $T \cup \{\Psi_p\}$ the order type of the initial segment of $<$ determined by c_{n-1} is $\geqslant D(p)$. $\mathcal{M}$ can be expanded to a model of Φ_q for some $q \in O_W(p)$.

By definition of D, $D(p) > D(q)$. Hence we can find in $\mathcal{M}$ an element x which is $<$ the interpretation of c_{n-1}, and such that the order type of the initial segment of $<$ determined by x is $D(q)$. If we re-interpret c_n as x we still get a model of $T \cup \{\Psi_q\}$. Hence we got a contradiction to our induction assumption for q, and the Lemma is verified. $\square$ Lemma 7.1 $\square$ Theorem 2.1(c).

8. Ordinal omitting

If $<$ is a linear ordering, denote by $\mathrm{Wf}(<)$ the largest well-ordered initial segment of $<$. Without loss of generality we can identify $\mathrm{Wf}(<)$ with the ordinal which is its order type. The following theorem generalizes Theorem 7.5 in [1, III] (which is a generalization of H. Friedman's theorem on the existence of models of Set Theory with a given well-founded part [8].)

Theorem 8.1. *Let $\mathcal{A}$ be countably decomposable. Let T be a Σ_1-theory in $L_{\infty\omega} \cap A$ such that $T \vDash$ "$<$ is a linear ordering" and for each $\beta < o(A)$, T has a model $\mathcal{M}$ such that $\mathrm{Wf}(<) \geqslant \beta$ holds in the model. Then T has a model with $\mathrm{Wf}(<) = o(A)$.*

Proof. Let $\tilde{T}$ be the theory T expanded with the addition of the constants c_β for $\beta < o(A)$ and the sentences

$$\text{“}c_\beta \text{ is the field of } <\text{”} \wedge \forall x\, (x < c_\beta \rightarrow \bigvee_{\alpha<\beta} x = c_\alpha).$$

$\tilde{T}$ is clearly $\Sigma_1(\mathcal{A})$ and it has a model (using Theorem 2.2-compactness). We want to get a model of T such that the ordinal $\alpha = o(A)$ is not in $\mathrm{Wf}(<)$. (For $\beta < \alpha, \beta$ is in $\mathrm{Wf}(<)$ by construction of $\tilde{T}$.) We expand $\tilde{T}$ further to $\tilde{\tilde{T}}$ by adding constants $c_0, c_1, \ldots$ with the sentences "$c_{i+1} < c_i$". $\tilde{\tilde{T}}$ is consistent since otherwise T pins down ordinals, hence for some $\beta < o(A)$ it can not have a model whose well-founded part (hence itself) has order type $\geqslant \beta$.

We shall define a modified version of our basic game associated with T. The intuitive idea is that in addition to the steps in the original game, White is given at the nth step a small set of constants. (We assume that no constant c_k, $k > n$, appeared so far.) He has to divide them into three subsets C_1, C_2, C_3. C_1 is the set of those which he decides to put in the well-founded part of the constructed model. He has to witness it by picking an ordinal $\beta < o(A)$ and putting the

Sh:144

resulting set of sentences $c<c_\beta$ for every $c\in C_1$. C_2 is the set of those which White decides to put in the non-well-founded part, so he makes $c_{n+1}<c$ for every $c\in C_2\cup\{c_0, c_1, \ldots, c_n\}$. (Note that the game does not satisfy the fact that $O_W(p)\in A$ for every position p but that does not interfere with the following arguments.) C_3 is the set of constants which White decides to put outside the field of $<$.

So the formal definition of the game is that Black plays $b_n=(T_n, X_n, Y_n)$ where T_n, X_n, Y_n are the same as in our original game except that T_n does not mention any constant c_k $k>n$. White's move is

$$a_n=(F_{n+1}, \Phi_{n+1}, D_{n+1}, G_{n+1}, C_1^{n+1}, C_2^{n+1}, C_3^{n+1}, \beta)$$

where F_{n+1}, D_{n+1}, G_{n+1} are like in our original game. Define

$$C^{n+1}=\{\text{the constants appearing in } \Phi_n\}\cap X_n.$$

(By induction assumption $C^{n+1}\cap\{c_0, c_1, \ldots\}\subseteq\{c_0, \ldots, c_n\}$.) Then

$$C_1^{n+1}\cup C_2^{n+1}\cup C_3^{n+1}=C^{n+1}, \qquad C_i^{n+1}\cap C_j^{n+1}=\emptyset, \quad i\neq j.$$

β is an ordinal, $\beta<o(A)$.

$$\begin{aligned}\Phi_{n+1}=C''(\Phi_n)\cup T_n\cup\bigcup_{i=1}^{n}\operatorname{range}(f_i^{n+1})\cup\{c<c_\beta\mid c\in C_1^{n+1}\}\\ \cup\{c_{n+1}<c\mid c\in C_2^{n+1}\}\cup\{c_{n+1}<c_n\}\cup\{\text{“}c\text{ is not in the}\\ \text{field of }c\text{”}\mid c\in C_3^{n+1}\}.\end{aligned}$$

We put range(G_{n+1}) in Φ_{n+1} if D_{n+1} is finite. Again Φ_{n+1} should not contain an atomic sentence and its negation. This finishes the definition of the games which we denote by G.

For a position p in our present game, Ψ_p is defined as in Section 6. The proof of Theorem 2.1 is divided into two parts:

Lemma 8.2. *If White has a winning strategy in the game G, then $\tilde{T}$ has a model in which* $\mathrm{Wf}(<)=o(A)$.

Proof. Since $\mathscr{A}$ is countably decomposable we pick X_n such that $S(X_n)$. (S is a strong smallness predicate witnessing (DP1) for $\mathscr{A}$) and $\bigcup_{n<\omega}X_n=A$. We pick also $T_n\in A$ such that $\bigcup_{n<\omega}T_n=\tilde{T}$ but T_n does not mention any c_k for $k>n$. We define Y_n to be any member of A satisfying $R(X_n\times\mathrm{Tc}(X_n), Y_n)$. ($R$ is a decomposition predicate witnessing (DP2).)

At his nth move we let Black play (T_n, X_n, Y_n). Clearly White's strategy gives a sequence Φ_n such that $\Phi_\omega=\bigcup_{n<\omega}\Phi_n$ is a Hinttika set, containing T. For Φ_ω we can define a model in which every element is the interpretation of some constant in the language of Φ_ω. Each such constant, c, appears in some X_n, hence it was put in C_1^{n+1}, C_2^{n+1} or C_3^{n+1}. In the first case for some β, Φ_{n+1} contains “$c<c_\beta$”, hence by definition of $\tilde{T}$, C determines an initial segment of $<$ of ordertype $<\beta<o(A)$.

Sh:144

If c was put in C_2^{n+1}, then c is not in the well-founded part of $<$ since $\cdots < c_{n+2} < c_{n+1} < c$. If $c \in C_3^{n+1}$, c is not in the field of $<$. So in the resulting model every element in $\mathrm{Wf}(<)$ is less than $o(A)$, but since it is a model of $\tilde{T}$, $\alpha \subseteq \mathrm{Wf}(<)$, hence $\mathrm{Wf}(<) = \alpha$. □ Lemma 8.2

Lemma 8.3. *In the game G White has a winning strategy.*

Proof. White plays such that for each position p arrived in the game $\tilde{\tilde{T}} \cup \{\Psi_p\}$ is consistent. Since Ψ_p does not change by Black's move we just have to show that given a position p in which White is to move, White can move to a position q such that $\tilde{\tilde{T}} \cup \{\Psi_q\}$ is consistent. So assume $p = \langle a_0, b_0, \ldots, a_n, b_n \rangle$, where $b_n = (T_n, X_n, Y_n)$, let C^n be defined as above. Note that $\mathcal{P}(C^n) \in A$, $\mathcal{P}(C^n \times C^n) \in A$ since $C^n \subseteq X_n$ and $S(X_n)$ holds. Let $\mathcal{M}$ be a model of $T \cup \{\Psi_p\}$. We shall not distinguish between constants and their interpretation in $\mathcal{M}$. Let C_3^{n+1} be the set of those constants in C^{n+1} which are not in the field of $<$. Let $E = \{(c, c') \mid c \in C^{n+1}, \mathcal{M} \models c < c'\}$. Note that C_3^{n+1} and $E \in A$. Put

$$\Psi = \Psi_p \wedge \bigwedge \{\text{'}c \text{ is not in the field of } <\text{'} \mid c \in C_3^{n+1}\}$$
$$\wedge \{c < c' \mid (c, c') \in E\}.$$

Clearly $\tilde{\tilde{T}} \cup \{\Psi\}$ is consistent. ($\mathcal{M}$ is a model for it.) We already determined C_3^{n+1}. We next let White determine C_2^{n+1}, C_2^{n+1} and q.

E is an ordering of $C^{n+1} - C_3^{n+1}$. Let c be the minimal element of $\mathrm{Wf}(E)$ such that $\tilde{\tilde{T}} \cup \{\Psi\} \cup \{c_\beta < c \mid \beta \in o(A)\}$ is consistent. (c does not necessarily exist.)

Let D be $\{d \mid (d, c) \in E\}$ if c exists and $D = \mathrm{Wf}(E)$ if c as above does not exist. Note $D \in A$. ($\mathcal{P}(C^{n+1}) \in A$.) We claim that $\exists \beta < o(A)$ such that $\tilde{\tilde{T}} \cup \{\Psi\} \vdash d < c_\beta$ for all $d \in D$. The reason is that

$$\forall d \in D \ \exists \beta \ \ulcorner d < c_\beta \urcorner \in \mathrm{Cn}(\tilde{\tilde{T}} \cup \{\Psi\}).$$

By our completeness Theorem 2.2(a) and Σ_1-reflection

$$\exists \beta \ \forall d \in D \ \ulcorner d < c_\beta \urcorner \in \mathrm{Cn}(T \cup \{\Psi\}).$$

So we let White play $C_1^{n+1} = D$, $C_2^{n+1} = C^{n+1} - (C_1^{n+1} \cup C_3^{n+1})$, and the ordinal β above.

Let Ψ be $\Psi \wedge \bigwedge_{d \in D} d < c_\beta$. Clearly $\tilde{\tilde{T}} \cup \{\Psi\}$ is consistent. We claim that

$$\tilde{\tilde{T}} \cup \{\Psi\} \cup \{c_{n+1} < d \mid d \in C_2^{n+1}\}$$

is consistent.

Note that c_k $(k > n)$ is not mentioned in Ψ, and the only way it is mentioned in $\tilde{\tilde{T}}$ is in the sentences $c_k < c_{k-1}$ and $c_{k+1} < c_k$. We first show that it is consistent with $\tilde{\tilde{T}} \cup \{\Psi\}$ that some $e < d$ for all $d \in C_2^{n+1}$ such that $e \notin \mathrm{Wf}(<)$. Otherwise, if it is inconsistent, it means that

$$\tilde{\tilde{T}} \cup \{\Psi\} \vdash < \text{restricted to } \left\{ x \,\middle|\, \bigwedge_{d \in C^{n+1}} \ulcorner x < d \urcorner \right\} \text{ is well ordered.}$$

Sh:144

By Theorem 2.1(c), $\exists\gamma<o(A)$ such that

$$\tilde{\tilde{T}}\cup\{\Psi\}\vdash \text{the order type of } < \text{ restricted to } \{x \mid \bigwedge_{d\in C_2^{n+1}} d\} \text{ is less than } \gamma.$$

Since

$$\tilde{\tilde{T}}\vdash \text{the order type of the initial segment of } c_\gamma \text{ is } \gamma,$$

We get

$$T\cup\{\Psi\}\vdash \bigvee_{d\in C^{n+1}} d<c_\gamma.$$

Since $\tilde{\tilde{T}}$ contains the statement $\forall x\,(x<c_0\rightarrow\bigvee_{\beta<\gamma} x=c_\beta)$, it is clear that C_2^{n+1} contains E minimal element, which we denoted before by c. Hence

$$\tilde{\tilde{T}}\cup\{\Psi\}\vdash c<c_\gamma,$$

contradicting the definition of D and C_2^{n+1}.

Therefore we conclude that $\tilde{\tilde{T}}\cup\{\Psi\}$ has a model in which some $e<d$ for all $d\in C_2^{n+1}$, where $e\notin \mathrm{Wf}(<)$. Since Ψ does not mention c_{n+1}, we can interpret c_{n+1} as e and c_k for $k>n+1$ as some decreasing sequence below e, which exists since $e\notin\mathrm{Wf}(<)$.

We conclude that $\tilde{\tilde{T}}\cup\{\Psi\}\cup\{c_{n+1}<d \mid d\in C_2^{n+1}\}$ is consistent. It has a model $\mathcal{M}$. Since $\Psi\rightarrow\Psi_p$, by the arguments of Lemma 6.2 White can now complete his move by picking G_{n+1} and F_{n+1}, and interpret the new constants of $C''(\Phi_n)$ in $\mathcal{M}$, such that if q is the resulting position $\mathcal{M}\models\Psi_q$. □ Lemma 8.3

Lemmas 8.2 and 8.3 obviously lead to □ Theorem 8.1.

A typical application of Theorem 8.3 is the following generalization of Friedman's Theorem.

Corollary 8.4. *Assume that there exists a standard model of Set Theory whose ordinals has order type $>o(A)$ where $\mathcal{A}$ is countably decomposable. Then $o(A)$ is the order type of the well-founded part of the ordinals of some non-standard model of Set Theory.*

This gives many examples of standard parts of models of Set Theory. Note that if $\mathcal{A}$ is countably decomposable, $o(A)$ has cofinality ω. The problem of characterizing the well-founded part of non-standard models of Set Theory in uncountable cofinality has a different flavour. See [20].

Another application is the problem of trying to characterize the ordinals of $\kappa<\alpha<\kappa^+$ such that $\alpha=\alpha(X)$ where $X\subseteq\kappa$. ($\alpha(X)$ is the minimal ordinal admissible in X.) If $\mathrm{Cf}(\kappa)=\omega$, and α is $o(A)$ for some countably decomposable $\mathcal{A}$, then $\alpha=\alpha(X)$ for some $X\subseteq\kappa$. The proof is by looking at a theory T of the form (c is a new constant):

(a) KP+ $<$ is the ordering of the ordinals.

(b) $\forall x\,(x<c_\gamma \rightarrow \bigvee_{\beta<\gamma} x=c_\beta)$ for every $\gamma<\alpha$.

(c) $\forall x\,(x\in c \rightarrow x\in c_\kappa)$.

(d) There is no ordinal admissible in c.

T has well-founded models where $<$ has order type arbitrary large in κ^+. By Theorem 8.1 there is a model of T whose well-founded part is α. This model gives an element c which is really a subset of κ, X. It is easily verified that $\alpha(X)=\alpha$. In the next part we shall see that in the constructible universe $\alpha=\alpha(X)$ (where cofinality $(|\alpha|=\omega)$. If $\alpha=o(A)$ for some countably decomposable $\mathscr{A}$. This was our independent proof to some of the results of Sy Friedman [9,10]. (Friedman dealt also with the cases $|\alpha|$ regular, and cofinality $(|\alpha|)>\omega$.)

9. Interpolation

It is well known that the Craig Interpolation Theorem, while holding for $L_{\omega_1\omega}$ fails in general for $L_{\infty\omega}$. However there are some cases in which one gets an interpolant for $L_{\kappa\omega}$, provided one allows the interpolant to lie in a stronger language. Thus Chang's Theorem yields [4] that if κ is strong limit of cofinality ω, that an implication in $L_{\kappa\omega}$ has an interpolant in $L_{\kappa\kappa}$. Under the same assumption an implication in $L_{\kappa^+\omega}$ has an interpolant in $L_{\kappa^+\kappa}$.

Since the Craig interpolation holds in any admissible countable fragment of $L_{\omega_1\omega}$, naturally there arises the question whether the Chang interpolant can be found in an admissible structure containing φ and ψ. We shall get a positive answer for admissible structures satisfying (DP). (By the proof of Theorem 4.4 we may assume in this section the smallness predicate witnessing (DP1) is always "having $\mathscr{A}$ cardinality $<\kappa$" where κ is the first non-small ordinal.)

Note that if $\mathscr{A}$ is an admissible structure satisfying (DP1), then either $\kappa=o(A)$ or κ is the maximal cardinality of a member of A, because every member of A is a countable union of elements having cardinality $<\kappa$.

Theorem 9.1. *Let $\mathscr{A}$ be an admissible structure having* (DP). *Let κ be the first non-small cardinal of $\mathscr{A}$. Let φ and $\psi\in L_{\infty\omega}\cap A$ such that it is logically valid that $\varphi\rightarrow\psi$. Then there exists a sentence χ in $L_{\infty\kappa}$, $\chi\in A$ such that every non-logical symbol in χ appears in both ψ and φ, and it is logically valid that $\varphi\rightarrow\chi$ and $\chi\rightarrow\psi$.* (*We assume that equality is a logical symbol.*)

Proof. We shall define a variant of our basic game (defined in Section 5). The game will be in principle like the basic game except that now White is constructing two sets formulas Φ_n, Ψ_n such that $\bigcup_{n<\omega}\Phi_n\cup\bigcup_{n<\omega}\Psi_n$ is a Hinttika set containing $\{\varphi, \neg\psi\}$. (Of course White must fail.) In White's construction all the non-logical symbols of Φ_n will be the non-logical symbols of φ together with the new witnessing constants introduced so far. Similarly all the non-logical symbols of Ψ_n will be the non-logical symbols of ψ together with the new witnessing constants. (To simplify terminology we shall assume that the type of Φ_n, denoted by $\tau(\Phi_n)$ contains all non-logical symbols introduced thus far. Similarly for $\tau(\Psi_n)$.) At each stage n we shall have less than κ new witnessing constants.

Without loss of generality we shall assume that φ and ψ contain no function symbols or constants. (We replace n-ary functions by $n+1$ predicates, we replace constants by unary predicates and φ by

$$\varphi \wedge \text{"all these predicates are functions . . . ."}$$

and ψ by

$$\text{"all these predicates are functions} \to \psi\text{"}.$$

Upon finding the interpolant we substitute back the function symbols and the constants.) We also assume that whenever we mention φ and ψ, we actually mean their negation normal form.

As in the basic game Black supplies a small set of disjunctions to take care of. White will handle disjunctions and existential statements separately for the Φ_n part and the Ψ_n part. Say for odd n's taking care of the Φ_n part and of the Ψ_n part at even steps. Since we want to have few witnessing constants we shall use Black's move also to determine which existential statements should be taken care of at a particular stage.

We would like to limit Black's moves to a set in A, at each stage. For guaranteeing it we pick before the game starts Y such that $R(X, Y)$ where $X = \mathrm{Tc}(\{\varphi, \psi\})$. We actually would like to limit Black to playing small sets from Y, however there may be new witnessing constants, hence relevant formulas may not be in $\mathrm{Tc}(Y)$. Therefore we define, for a set C of constants, $h(Z, C) =$ all formulas obtained from formulas in Z by substituting constants from C for their free variables.

$$\tilde{h}(Y, C) = \{h(Z, C) \mid Z \in Y\}.$$

A move by Black is $b_n = (X_n, Y_n)$ where $Y_n = h(Y, C)$ where C is the set of all witnessing constants introduced so far. $X_n \in Y$ such that $S(X_n)$ holds. White's move is now a 6-tuple.

$$a_n = \langle F_n, \Phi_n, \Psi_n, D_n, E_n, G_n \rangle,$$

$$a_0 = \langle \emptyset, \{\varphi\}, \{\neg \psi\}, \emptyset, \emptyset, \emptyset \rangle.$$

In general we require that $\Phi_n \cup \Psi_n$ would contain no-atomic sentences and its negations. The specific requirements are $(n > 0)$

(1) $D_n = \{\lambda \mid \lambda \in X_{n-1} \cap \Phi_n, \lambda$ is a disjunction no disjunct of λ belongs to $\Phi_n\}$,

Sh:144

if n is odd, and

$$D_n = \{\lambda \mid \lambda \in X_{n-1} \cap \Psi_n, \lambda \text{ is a disjunction}$$
$$\text{no disjunct of } \lambda \text{ belongs to } \Psi_n\},$$

if n is even.

Similarly

$$E_n = \{\lambda \mid \lambda \in X_{n-1} \cap \Phi_n, \lambda \text{ has the form } \exists x\, \mu(x)\},$$

if n is odd, and

$$E_n = \{\lambda \mid \lambda \in X_{n-1} \cap \Psi_n, \lambda \text{ has the form } \exists x\, \mu(x)\},$$

if n is even.

(2) If D_n is finite, then G_n is a function that chooses a disjunct of λ for each $\lambda \in D_n$.

(3) If D_n is infinite, then $G_n : D_n \to \mathrm{Odd} - (n+1)$ if n is odd (where Odd is the set of odd integers).

If n is even, then $G_n : D_n \to \mathrm{Even} - (n+2)$ where Even is the set of even integers.

(4) F_n is like in the basic game. (Note that f_i^n is empty if $i = n (\mathrm{mod}\, 2)$.)

(5) If n is odd, then

$$\Phi_n = \Phi_{n-1} \cup C(\Phi_{n-1}) \cup C'(E_n) \cup \bigcup_{i=1}^{n-1} \mathrm{range}(f_i^n),$$

$$\Psi_n = \Psi_{n-1}.$$

If n is even, then

$$\Phi_n = \Phi_{n-1},$$

$$\Psi_n = \Psi_{n-1} \cup C(\Psi_{n-1}) \cup C'(E_n) \cup \bigcup_{i=1}^{n-1} \mathrm{range}(f_i^n).$$

(6) If n is odd (even) and D_n is finite, we add range G_n to $\Phi_n(\Psi_n)$. (The operations C, C' were defined in Section 5, but note that we assume that every new witnessing constant 'appears' in both Φ_{n-1} and Ψ_{n-1}. Hence all the witnessing constants appearing so far are in $C_0(\Psi_n)$ and $C_0(\Phi_n)$.)

Recall that we assumed that the predicate $S(X)$ simply expresses "X has A cardinality $< \kappa$". Hence one can easily verify by induction on n that at every stage in the game the set of new constants introduced so far, has cardinality $< \kappa$. Note also that in the present version of the basic game $O_{\mathrm{B}}(x)$ is also $\mathscr{A}$-r.e. ($O_{\mathrm{B}}(x)$ is Black's options at position X.)

Following Section 6 we define for each position p in our game two sentences in $L_{\infty\omega}$, Γ_p and Δ_p. Assume $p = \langle a_0, b_0, \ldots, a_n, b_n \rangle$ or $p = \langle a_0, b_0, \ldots, a_n \rangle$ (Γ_p and Δ_p do not depend on b_n), where $a_i = \langle F_i, \Phi_i, \Psi_i, D_i, E_i, G_i \rangle$, $b_i = \langle X_i, Y_i \rangle$. Recall from

Section 6 the notation

$$Y_{i,m}=\{f \mid f\in Y_i, \mathrm{Dom}(f)=\{\delta \mid \delta\in D_{i+1}, G_{i+1}(\delta)=m\},$$
$$\forall x\in \mathrm{Dom}(f), f(x) \text{ is one of the disjuncts of } x\}.$$

Then

$$\Gamma_p=\bigwedge \Phi_n \bigwedge_{\substack{i=1\\ i \text{ odd}}}^{n} \bigwedge_{\substack{m \text{ odd}\\ m>n}} \bigvee_{f\in Y_{i-1,m}} \bigwedge_{x\in \mathrm{Dom}(f)} f(x),$$

$$\Delta_p=\bigwedge \Psi_n \wedge \bigwedge_{\substack{i=1\\ i \text{ even}}}^{n} \bigwedge_{\substack{m>n\\ m \text{ even}}} \bigvee_{f\in Y_{i-1,m}} \bigwedge_{x\in \mathrm{Dom}(f)} f(x).$$

Note that the type of Γ_p is $\tau(\Phi_n)$ and $\tau(\Delta_p)=\tau(\Psi_n)$. For $p=\emptyset$ we put $\Gamma_\emptyset=\varphi$, $\Delta_\emptyset=\neg\psi$.

Lemma 9.2. *If White has a winning strategy in our game, starting from position p, then $\{\Gamma_p, \Delta_p\}$ has a model.*

Proof. Like the proof of Lemma 5.5 by letting Black pick $\langle X_n \mid n<\omega\rangle$ such that

$$\bigcup_{\substack{n<\omega\\ n \text{ even}}} X_n \supseteq \bigcup_{n<\omega} \mathrm{Tc}(\Phi_n) \bigcup_{\substack{n<\omega\\ n \text{ odd}}} X_n \supseteq \bigcup_{n<\omega} \mathrm{Tc}(\Psi_n),$$

Black can do it since $R(\mathrm{Tc}(\{\varphi\}\cup\{\psi\}), Y)$ holds, and by the definition of the Y_n's. It follows that $\bigcup_{n<\omega}\Phi_n \cup \bigcup_{n<\omega}\psi_n$ is a Hinttika set containing $\{\varphi, \neg\psi\}$. □ Lemma 9.2

Since it is logically valid that $\Gamma_p\to\varphi$ and $\Delta_p\to\neg\psi$ we get that $\{\Gamma_p, \Delta_p\}$ has no model. Hence White has no winning strategy from any position in the game. Recall from Section 5, the function D on the positions in the game in which Blacks wins, into ordinals. Since every position is a winning position for Black, D is defined on all positions in the game.

By recursion on $D(p)$ we define a formula in $L_{\infty\omega}$, χ_p such that $\tau(\chi_p)\subseteq \tau(\Gamma_p)\cap\tau(\Delta_p)$ and such that

$$\vdash \Gamma_p\to\chi_p \quad \text{and} \quad \vdash \chi_p\to\neg\Delta_p.$$

As one can see from the definition below if p is a position in $P_W(P_B)$, then χ_p is a primitive recursive function of

$$\{\chi_q \mid q\in O_W(p), D(q)<D(p)\} \qquad (\{\chi_q \mid q\in O_B(p), D(q)<D(p)\}).$$

Hence since O_W and O_B are $\mathscr{A}$-r.e. the function $p\to\chi_p$ is $\mathscr{A}$-r.e. and we get $\chi_p\in A$.

Assume first that $p\in P_W$. Hence $D(p)>D(q)$ for every $q\in O_W(p)$. Hence χ_q is defined for every $q\in O_W(p)$. Without loss of generality assume that $p=$

Sh:144

$\langle a_0, b_0, a_1, b_1, \ldots, a_n, b_n\rangle$ where n is even. The treatment of the other case is completely analogous. For each $q \in O_W(p)$ white introduced some new witnessing constant for existential formulas in E_{n+1}. Let $\boldsymbol{c}$ be a sequence enumerating those constants. (Note that the cardinality of the new witnessing constants is less than κ.) Note that $\boldsymbol{c}$ does not appear in Δ_q. Hence by induction assumption

$$\vdash \Gamma_q(\boldsymbol{c}) \to \chi_q(\boldsymbol{c}), \qquad \vdash \chi_q(\boldsymbol{c}) \to \neg\Delta_q$$

Hence we get

$$\text{(I)} \qquad \vdash \forall \boldsymbol{x}\, [\Gamma_q(\boldsymbol{x}) \to \chi_q(\boldsymbol{x})], \qquad \vdash \exists \boldsymbol{x}\, \chi(\boldsymbol{x}) \to \neg\Delta_q$$

where $\boldsymbol{x}$ is a vector of new variables replacing one by one the elements of $\boldsymbol{c}$. Note that since n is even, we have $\Delta_p = \Delta_q$ for every $q \in O_W(p)$.

By an argument similar to the proof of Lemma 6.2

$$\text{(II)} \qquad \vdash \Gamma_p \to \bigvee_{q \in O_W(p)} \exists \boldsymbol{x}\, \Gamma_q(\boldsymbol{x})$$

We define χ_p as $\bigvee_{q \in O_W(p)} \exists \boldsymbol{x}\, \chi_q(\boldsymbol{x})$. χ_p is as required since by (I) and (II), $\vdash \Gamma_p \to \chi_p$ and by (I), $\vdash \chi_p \to \neg\Delta_p$. (Note that this case includes the case $D(p) = 0$. In that case $p \in P_W$ and $O_W(p) = \emptyset$. It means that any answer by White to Black's challenge will yield an atomic sentence and its negation in $\Phi_{n+1} \cup \Psi_{n+1}$. It follows that if Ξ is the set of all atomic sentences or their negation appearing in Ψ_n, and which are in $\tau(\Phi_n) \cap \tau(\Psi_n)$ we have $\Gamma_p \vdash \bigvee_{\xi \in \Xi} \neg\xi$, hence we can take $\chi_p = \bigvee_{\xi \in \Xi} \neg\xi$. χ_p clearly satisfies the requirements.

If $p \in P_B$, then χ_q is defined for all $q \in O_B(p)$, $D(q) < D(p)$. Note the set of all q such that $q \in O_B(p)$, $D(q) < D(p)$ is in A, Since D is $\Sigma_1(A)$, and for every $q \in O_B(p)$, D is defined, hence by Σ_1-reflection we get $X \in A$ and $\forall q \in O_B(p)\, \exists \alpha \in X\, (X \models D(q, \alpha))$. Now we can use Δ_0-separation to get that $\{q \mid q \in O_B(p),\ D(p) < D(q)\} \in A$. Denote this set by $E(p)$. In the case $p \in P_B$, $\Gamma_p = \Gamma_q$, $\Delta_p = \Delta_q$ for every $q \in O_B(p)$, hence if we define $\chi_p = \bigvee_{q \in E(p)} \chi_q$, then $\vdash \Gamma_p \to \chi_p$ and $\vdash \chi_p \to \neg\Delta_p$. (Actually any χ_q for $q \in E(p)$ would have done, but we do not want any choice involved so that $p \to \chi_p$ will be $\mathscr{A}$-r.e.)

We can now conclude the proof of Theorem 9.1. $\chi_\emptyset$ where $\emptyset$ is the first position in the game is of type $\tau(\varphi) \cap \tau(\psi)$ and $\vdash \varphi \to X_q \vdash X_\emptyset \to \psi$. $\chi_\emptyset$ is clearly the required interpolant. $\square$ Theorem 9.1

PART II

In this part we study the problem whether Theorem 2.2 is the best possible. In this part we shall concentrate on the problem of validity admissibility and ordinals pinning. (Σ_1-compactness will be handled in the third part). We shall assume $V = L$ and treat just admissible structures of the form $\langle L_\alpha, \in\rangle, \ldots$. We feel that a theorem yielding validity admissibility results for all structures of the form $\langle L_\alpha, \in \rangle$ for which such a result can be obtained is rather comprehensive.

10. A fine structure detour

In this section we prove a technical fact emerging from the fine structure theory of the constructible universe [14] which will be our main technical tool. This fact is probably implicit in some of the proofs in the fine structure theory (e.g., the existence of morasses, see [6]) but we were not able to find a direct reference to it.

Lemma 10.1 ($V = L$). *Let κ be a singular cardinal. Let $\kappa \leqslant \alpha < \kappa^+$. Then there exists a subset of L_α, D_α, $|D_\alpha| < \kappa$ such that every element of L_α is first order definable in $\langle L_\alpha, \in \rangle$ from $\kappa \cup D_\alpha$.*

Proof. We prove it by induction on α. The case $\alpha = \kappa$ is obvious by taking $D_\alpha = \emptyset$. So assume that the lemma holds for all $\gamma < \alpha$, i.e., for all $\gamma < \alpha$ there exists D_γ, $|D_\gamma| < \kappa$ such that all members of $\langle L_\gamma, \in \rangle$ are first-order definable in $\langle L_\gamma, \in \rangle$ from $D_\gamma \cup \kappa$.

If $\alpha = \delta + 1$, then one can take $D_\alpha = D_\delta \cup \{\alpha\}$. (Remember that every member of L_α is a subset of L_δ, definable with parameters from L_δ.) So we treat the case α limit. α is not a cardinal so let $\alpha \leqslant \beta$ be the first ordinal such that there is a $\Sigma_n(L_\beta)$-map of a subset of some $\gamma < \alpha$ onto α. Let n be the minimal such n. Fix γ, and a function f which is $\Sigma_n(L_\beta)$, such that f maps a subset of γ onto α. Since κ is singular fix $B \subseteq \alpha$, $|B| < \kappa$, B cofinal in α.

Let ρ be the Σ_{n-1}-projectum of β. (See [14] for definition though all the relevant facts about ρ will be stated below.) It follows from [14] that

(I) $\alpha \leqslant \rho \leqslant \beta$ ($\alpha \leqslant \rho$ by definition of n). ρ is a limit ordinal, unless $n = 1$ and $\rho = \beta$ is a successor ordinal.

(II) $\exists A \subseteq L_\rho$ (A is usually known as a Σ_{n-1} master code) such that $\langle L_\rho, \in, A \rangle$ is amenable (i.e., $A \cap L_\delta \in L_\rho$ for all $\delta < \rho$), A is $\Sigma_{n-1}(L_\beta)$ and a subset of L_ρ is Σ_1 over $\langle L_\rho, \in, A \rangle$ iff it is $\Sigma_n(L_\beta)$.

We distinguish two cases:

Case I: *ρ is a limit ordinal.* Again using the fact that κ is singular and $\rho < \kappa^+$, we get $E \subseteq \rho$, $|E| < \kappa$, E cofinal in ρ.

f is $\Sigma_n(L_\beta)$. Hence it is $\Sigma_1(\langle L_\rho, \in, A \rangle)$. Let $\boldsymbol{p}$ be a finite set of parameters in L_ρ such that f is Σ_1 definable in $\langle L_\rho, \in, A \rangle$ from $\boldsymbol{p}$, by the Σ_1-definition $\exists x\, \Phi(x, y, z, A, \boldsymbol{p})$ where Φ is Σ_0. For $\delta \in E$, $\eta \in B$ define

$$f_{\delta,\eta} = \{(y, z) \mid y < \gamma, z < \eta, \exists x \in L_\delta\ \Phi(x, y, z, A, \boldsymbol{p})\}$$

(Recall that $\delta \in E$ implies $\delta < \rho$, $\eta \in B$ implies $\eta < \alpha$.) Since $A \cap L_\delta \in L_\rho$ and ρ is limit, $f_{\delta,\eta} \in L_\rho$. Note that $f_{\delta,\eta} \subseteq \gamma \times \eta$. By standard arguments $f_{\delta,\eta} \in L_\alpha$. (Otherwise the minimal ordinal μ such that $f_{\delta,\eta} \in L_\mu$, satisfies $L_{\mu+1} \models |\mu| \leq \max(\gamma, \eta)$ and $\alpha < \mu < \rho$. Hence $L_{\mu+1} \models |\alpha| \leq \max(\gamma, \eta)$. Contradicting $\mu < \rho \leq \beta$ and the definition of β.)

Note that

$$f=\bigcup_{\substack{\delta\in E\\ \eta\in B}} f_{\delta,\eta}.$$

We now define D_α by

$$D_\alpha=D_\gamma\cup\{\gamma\}\cup\{f_{\delta,\eta}\mid \delta\in E,\ \eta\in B\}.$$

Every member x of L_α is first-order definable in $\langle L_\alpha, \in\rangle$ from some $\xi<\alpha$. ξ is in the range of f, hence for some δ, η, $(\delta\in E, \eta\in B)$ and $\mu<\gamma$, $\xi=f_{\delta,\eta}(\mu)$. μ is first-order definable in $\langle L_\gamma, \in\rangle$ from $\kappa\cup D_\gamma$. Hence it is definable in $\langle L_\alpha, \in\rangle$ from $\{\gamma\}\cup\kappa\cup D_\gamma$. We conclude that x is first-order definable in $\langle L_\alpha, \in\rangle$. We verified that D_α satisfies the requirements.

Case II: $m=1, \rho=\beta$ *and* β *is a successor ordinal* $\beta=\delta+1$. Remember that every member of L_β is a subset of L_δ, definable from parameters in $\langle L_\delta, \in\rangle$. f is $\Sigma_1(L_\beta)$ in this case, so let $\exists x\, \Phi(x, y, z, \boldsymbol{p})$ be the Σ_1-definition of f ($\boldsymbol{p}$ is a finite set of parameters), where Φ is Σ_0. Define for f_η $(n<\omega, \eta\in B)$, $f_{n,\eta}$ by

$$f_{n,\eta}=\{(y, z)\mid y\in\gamma,\ z\in\eta, \exists x\ (x \text{ is a } \Sigma_n\text{-definable subset of } \langle L_\delta, \in\rangle \text{ such that } \Phi(x, y, x, \boldsymbol{p})\}.$$

It can be easily verified that $f_{n,\eta}\in L_\beta$ since it is a subset of L_δ, definable in $\langle L_\delta, \in\rangle$. (One should replace the parameters $\boldsymbol{p}$ by their definitions over $\langle L_\delta, \in\rangle$.) By arguments as above in Case I, we can show that $f_{n,\eta}\in L_\alpha$. Hence if we note again that $f=\bigcup_{n<\omega,\, \eta\in B} f_{n,\eta}$ and we define

$$D_\alpha=D_\gamma\cup\{\gamma\}\cup\{f_{n,\eta}\mid n<\omega,\ \eta\in B\},$$

we get a D_α satisfying the requirements of the lemma. □ Lemma 10.1

Recall from Section 4 that if $\mathscr{A}$ is an admissible structure, having (DP1) then the smallness predicate can be assumed to be $S(x)\leftrightarrow \mathscr{A}\models x$ has cardinality $<\kappa$, for some strong limit cardinal κ. Note that if $\kappa\in\mathscr{A}$, then $\mathrm{cf}(\kappa)=\omega$. So for $\langle L_\alpha, \in\rangle$ to have (DP1) we must have either α is a limit cardinal, or for some κ, $\mathrm{cf}(\kappa)=\omega$ $|\alpha|=\kappa$. In the latter case we must have that for each β, β is a countable union of members of L_α, having cardinality $<\kappa$. This observation motivates the following definition:

Definition 10.1 $(V=L)$. Let κ be a cardinal in L such that $\mathrm{cf}(\kappa)=\omega$. Let $\kappa\leq\beta<\kappa^+$. $d(\beta)$ – the decomposition ordinal of β – is the minimal ordinal γ, such that in $\langle L_\gamma, \in\rangle$ one can define sets $A_n\subseteq\beta$, $|A_n|<\kappa$, $\beta=\bigcup_{n<\omega} A_n$. (Note that we are not assuming that $\langle A_n \mid n<\omega\rangle$ is definable in $\langle L_\gamma, \in\rangle$, just that each A_n is definable.) $d(\beta)$ exists since once we get γ such $L_\gamma\models|\beta|=\kappa$. Then in L_γ we can find such A_n's. It follows that a necessary condition for $\langle L_\alpha, \in\rangle$ to have (DP1) is that $d(\beta)\leq\alpha$ for all $\beta<\alpha$. Note that if $\langle L_\alpha, \in\rangle$ has (DP2), then we must have

$d(\beta)<\alpha$ for all $\beta<\alpha$ since if Y satisfies $R(\beta, Y)$, $Y\in L_\alpha$. Then $Y\in L_\gamma$ for some $\gamma<\alpha$ and clearly $d(\beta)\leq\gamma$.

The connection to Lemma 10.1 is

Lemma 10.2. *$d(\beta)$ is the minimal ordinal γ such that $\beta\leq\gamma$ and the minimal cardinality of D_γ satisfying Lemma 10.1 is $\leq\omega$.*

Proof. Let γ be the minimal such that $\beta\leq\gamma$ and $|D_\gamma|\leq\omega$. We first show $d(\beta)\leq\gamma$. Let $\langle\kappa_n \mid \omega\rangle$ be a cofinal sequence in κ, and $D_\gamma=\langle a_n \mid n<\omega\rangle$. Define

$$A_n=\{\alpha \mid \alpha<\beta,\ \alpha \text{ is } \Sigma_n\text{-definable from } \kappa_n\cup\{a_0,\ldots,a_n\}\}.$$

Since every member of L_γ, hence of β, is first order-definable from $\kappa\cup\{a_0, a_1,\ldots\}$ we get $\beta=\bigcup_{n<\omega}A_n$. Note that A_n is definable in L_γ and its cardinality is κ_n. Hence by definition of $d(\beta)$, $d(\beta)\leq\gamma$.

We now prove $\gamma\leq d(\beta)$. By definition of $d(\beta)$ one can define in $L_{d(\beta)}$ subsets of β, A_n for $n<\omega$, $|A_n|<\kappa$ and $\beta=\bigcup_{n<\omega}A_n$. Each A_n is defined using a finite sequence of parameters p_n. Let ρ be the $d(\beta)$ cardinality of β, i.e., the minimal ordinal such that some $L_{d(\beta)}$ definable map maps ρ onto β. (Note that $d(\beta)=\beta$ and $\rho=\beta$ are possible.) Let p be a finite set of parameters from which one can define such a map. Let $D=p\cup\{\rho\}\cup\bigcup_{n<\omega}p_n$. (If $\rho=d(\beta)=\beta$ we omit it.)

Let M be the Skolem hull in $L_{d(\beta)}$ of $D\cup\kappa$ (i.e., all elements definable from $D\cup\kappa$). Since $M\prec L_{d(\beta)}$, M is isomorphic to a structure of the form L_δ for some $\delta\leq d(\beta)$ by the well known collapsing map h.

Claim. $\beta\subseteq M$.

Assume the claim for a while. Hence h is the identity on β. Hence if we apply in L_δ the definition of A_n, using $h(p_n)$ as the sequence of parameters rather than p_n, we get A_n again. Hence each A_n is L_δ-definable, and since $\beta=\bigcup_{n<\omega}A_n$, we get $d(\beta)\leq\delta$. Therefore $\delta=d(\beta)$. Let $\tilde{D}=h''D$. Since every member of M is definable in M from $D\cup\kappa$, every member of $h''M=L_{d(\beta)}$ is definable from $D\cup\kappa$, and we get that $\tilde{D}$ can be taken as a set $D_{d(\beta)}$ satisfying the requirements of Lemma 10.1. Since $|\tilde{D}|\leq\omega$, we proved $\gamma\leq d(\beta)$. So we just have to verify the Claim.

Since $\rho\leq\beta$ $\rho=\bigcup_{n<\omega}(A_n\cap\rho)$. Note that ρ, being an $L_{d(\beta)}$-cardinal, is admissible. Also by standard facts about the constructible hierarchy, $A_n\cap\delta\in L_\rho$ for $\delta<\rho$.

It follows that the unique function mapping $A_n\cap\rho$ order preservingly onto some ordinal $<\kappa$ is definable in $L_{d(\beta)}$, from p_n. (For $\delta<\rho$ the function mapping $A_n\cap\delta$ onto its order type is in L_ρ by admissibility. Since this function is unique for each $\delta<\rho$, and the function for different δ's are mutually coherent, one can define their union which is the required function for A_n.) We conclude that every ordinal less than ρ is definable from some $p_n \cup \kappa \cup \{\rho\}$. (This ordinal is a member of some A_n. The function mapping A_n onto some $\delta < \kappa$ is definable from p_n. Hence this ordinal is definable from some p_n and some ordinal $<\kappa$.)

Using the fact that from p one can define a map from ρ onto β, we get that every member of β is definable in $L_{d(\beta)}$ from $D \cup \kappa$. Hence $\beta \subseteq M$. $\square$ Lemma 10.2

Corollary 10.3. *For all $\beta < \kappa^+$, $d(d(\beta)) = d(\beta)$.*

Proof. By Lemma 10.2, $d(d(\beta))$ is the minimal ordinal $\gamma \geq d(\beta)$ such that there exists a set D_γ satisfying Lemma 10.1 for γ such that $|D_\gamma| \leq \omega$. But $\gamma = d(\beta)$ clearly has all the required properties. $\square$ Corollary 10.3

Corollary 10.3 is a special case of the next lemma.

Lemma 10.4. *Let γ be the minimal admissible ordinal $>d(\beta)$. Then for every $d(\beta) \leq \delta < \gamma$ $d(\delta) = \delta$.*

Proof. We have to show that for every $d(\beta) \leq \delta < \gamma$ we can find a set $D_\delta \subseteq L_\delta$ such that all members of L_δ are first-order definable in $\langle L_\delta, \in \rangle$ from $D_\delta \cup \kappa$, and $|D_\delta| \leq \omega$. The proof is by induction on δ. The case $\delta = d(\beta)$ is covered by Corollary 10.3. If $\delta = \rho + 1$ use the induction hypothesis by defining $D_\delta = D_\rho \cup \{\rho\}$. For limit ρ, since $\rho < \gamma$, ρ is not admissible. Since it is limit Σ-collection fails for it. Hence for some $\beta < \rho$, the minimal Σ_1-substructure of L_ρ containing L_β is isomorphic to L_ρ.

It follows that every member of L_ρ is Σ_1-definable from members of L_β, hence we can take $D_\pi = D_\beta \cup \{\beta\}$. ($D_\beta$ exists by induction assumption.) $\square$ Lemma 10.4

It was remarked by the referee that there is a close connection between our decomposition function $d(\beta)$ and the critical projecta of β, defined in [9]. There it is defined for each ordinal β a finite sequence $\{(\beta_i, n_i) \mid i < l\}$, $\{\rho_i \mid i \leq l\}$ $\{\rho_i' \mid i \leq l\}$ where ρ_i is the Σ_{n_i}-projectum of β_i, ρ_i' is the Σ_{n_i-1}-projectum of β_i, ρ_i decreasing, $\beta_0 = \beta$, $n_0 = 0$, and (β_{i+1}, n_{i+1}) is picked to be minimal in the lexicographic order so as to make $\rho_{i+1} < \rho_i$, and the sequence is as long as possible. Note that $\rho_l = \kappa$. One can show that the proof of Lemma 10.1 yields that the minimal cardinality of the set D_β is exactly the maximal cofinality of $\{\rho_{j_0-1}, \rho_{j_0}, \ldots, \rho_1\} \cup \{\rho_j', \ldots, \rho_1'\}$, where j_0 is the minimal such that $\beta_{j_0} > \beta$. Hence one can verify that $d(\beta)$ is exactly β_j where j is the minimal such that for $j \leq i$, $\mathrm{cf}(\rho_i) = \mathrm{cf}(\rho_i') = \omega$.

We have noted before that being closed under the function d is a necessary condition for $\langle L_\alpha, \in \rangle$ to have (DP2).

Is it sufficient? It is almost sufficient as claimed by the next two lemmas:

Lemma 10.5. *Let $\mathscr{A} = \langle L_\alpha, \in, \ldots \rangle$ be an admissible structure such that $\kappa \leq \alpha \leq \kappa^+$,*

$\mathrm{cf}(\kappa)=\omega$, α is closed under d, and let $L_\alpha \models$. There exists a last cardinal. Then $\mathscr{A}$ has (DP2).

Proof. Let δ be the last cardinal in the sense of L_α. Note that $d(\delta)<\alpha$. Note that for a set of ordinals in L_α of cardinality less than κ, the function mapping it to its powerset is $\mathscr{A}$-r.e. for

$$x=\mathscr{P}(z) \leftrightarrow \exists g \text{ (} g \text{ is one-to-one, } g \text{ maps } z \text{ onto some ordinal less than } \kappa \wedge x=\{g^{-1}(t) \mid t \in L_\kappa\}).$$

Note also that since α is admissible, there exists $d(\delta)<\rho<\alpha$ such that for every set of ordinals in $L_{d(\delta)+1)}$ there exists a one-to-one function in L_ρ mapping it onto some ordinal. hence the set B of all subsets of δ belonging to $L_{d(\delta)+1}$, having cardinality less than κ is in $L_{\rho+1}$, hence in L_α. Since every member of B has cardinality less than κ, and the function mapping it to its powerset is $\mathscr{A}$-r.e. we get $B' \in L_\alpha$, $B \subseteq B'$ and for $z \in B$, $\mathscr{P}(z) \subseteq B'$.

Define $R(X, Y)$ by

$$R(X, Y) \leftrightarrow \exists f \text{ (} f \text{ is one-to-one, } f \text{ maps } X \text{ onto a subset of } \delta \wedge Y=\{f^{-1}(z) \mid z \in B'\}).$$

$R(X, Y)$ is $\Sigma_1(\mathscr{A})$ and for every X, $\exists Y R(X, Y)$ since δ is a maximal cardinal in L_α. Note that by definition of $d(\delta)$, δ is a countable union of members of B, hence if $R(X, Y)$ holds, X is a countable union of members of Y whose powerset is a subset of y. Hence we verified (DP2). □ Lemma 10.5

We shall see later that Lemma 10.5 gives a necessary and sufficient condition for a structure of the form $\langle L_\alpha, \in\rangle$ to have (DP2), i.e., we shall see later that if $\langle L_\alpha, \in\rangle$ has (DP2), then $L_\alpha \models$ "there exists a maximal cardinal". In case L_α has no maximal element, then $\langle L_\alpha, \in\rangle$ does not have (DP2), but it can always be expanded to an admissible structure satisfying (DP2).

Lemma 10.6. *Let α be admissible $\kappa<\alpha<\kappa^+$, $\mathrm{cf}(\kappa)=\omega$, such $L_\alpha \models$. There is no largest cardinal, and α is closed under d. Then $\langle L_\alpha, \in\rangle$ can be expanded by an additional predicate to an admissible structure satisfying* (DP2).

(We shall later see that Lemma 10.6 cannot be improved to an admissible structure of the form $\langle L_\alpha, \in, \ldots\rangle$.)

Lemma 10.6 was proved independently by S. Friedman. See [10].

Proof. The additional predicate we introduce is the function $x \to P_\kappa^\alpha(x)$ where $P_\kappa^\alpha(x)$ is the set of all subsets of x, whose L_α cardinality is less than κ. Note that since L_α has no largest cardinal $P_\kappa^\alpha(x) \in L_\alpha$ for every $x \in L_\alpha$. (Since if $x \in L_\beta$ let γ be the next L_α-cardinal after β, then $P_\kappa^\alpha(x) \subseteq L_\gamma$ and as can be easily verified $P_\kappa^\alpha(x) \in L_{\gamma+1}$.)

We shall verify that $\langle L_\alpha, \in, P_\kappa^\alpha\rangle$ is an admissible structure. The first fact to note is that if γ is a successor of a successor cardinal in L_α, then L_γ is closed under P_κ^α. We claim that for such γ, $\langle L_\gamma, \in, P_\kappa^\alpha \restriction L_\gamma\rangle$ is a Σ_1-elementary substructure of $\langle L_\alpha, \in, P_\kappa^\alpha\rangle$. Let $x \in L_\gamma$, (say $x \in L_\beta$ where $\beta < \gamma$) and assume that some $y \in L_\alpha$ satisfies $\Phi(x, y, P_\kappa^\alpha)$ where Φ is Σ_0. Let δ be an L_α successor, of a successor cardinal such that $y \in L_\delta$. In L_α we can define an increasing sequence of elementary submodels of $\langle L_\delta, \in, P_\kappa \restriction L_\gamma\rangle$, $\langle M_\alpha \mid \alpha < \bar{\kappa}\rangle$, where $\bar{\kappa}$ is the successor L_α cardinal of κ, $P_\kappa^\alpha(M_\alpha) \subseteq M_{\alpha+1}$, $L_\alpha \models |M_\alpha| < \gamma$, $L_\beta \cup \{\gamma\} \subseteq M_0$. Let $M = \bigcup_{\alpha<\bar{\kappa}} M_\alpha$. Clearly $L_\alpha \models |M| < \gamma$, hence M is isomorphic in L_α to some structure of the form $\langle L_\rho, \in, Q\rangle$ where $\rho < \gamma$. (Note that γ as a successor cardinal is supposed to be regular in L_α.) Note also that $P_\kappa^\alpha(M) \subseteq M$. (Remember that the length of the sequence $\langle M_\alpha \mid \cdots\rangle$ is $\bar{\kappa}$.) Hence Q is really $P_\kappa^\alpha \restriction L_\rho$. Since $L_\beta \subseteq M_0$, $x \in L_\rho$, hence if $\bar{y}$ is the isomorphic image of γ, $\langle L_\rho, \in, P_\kappa^\alpha \restriction L_\rho\rangle \models \Phi(x, \bar{y}, P_\kappa^\alpha \restriction L_\rho)$. Since $\rho < \gamma$, we get $\langle L_\gamma, \in, P_\kappa^\alpha \restriction L_\gamma\rangle \models \Phi(x, \bar{y}, P_\kappa^\alpha \restriction L_\gamma)$ and we proved that $\langle L_\gamma, \in, P_\kappa^\alpha \restriction L_\gamma\rangle <_{\Sigma_1} \langle L_\alpha, \in, P_\kappa^\alpha\rangle$ and thus the admissibility of $\langle L_\alpha, \in, P_\kappa^\alpha\rangle$.

The fact that $\langle L_\alpha, \in, P_\kappa^\alpha\rangle$ has (DP2) is now very easily verified. Take $R(x, y)$ simply to be $y = P_\kappa^\alpha(x)$. Since α is closed under d, every $x \in L_\alpha$ is a countable union of sets of cardinality less than κ, hence sets in $P_\kappa^\alpha(x)$. Note that if z has cardinality less than κ in L_α, its power set is in L_α. Hence if $z \in P_\kappa^\alpha(x)$, $P(z) \subseteq P_\kappa^\alpha(x)$. □ Lemma 10.6

11. Pinning down ordinals – revisited

In this section we show (assuming $V = L$) that L_α is closed under ordinals pinned down by sentences in $L_{\infty\omega} \cap L_\alpha$ iff either α is a limit cardinal or $\mathrm{cf}(|\alpha|) = \omega$ and α is closed under d. (Hence in the latter case, Lemmas 10.5 and 10.6 guarantee that an appropriate expansion of $\langle L_\alpha, \in\rangle$ satisfies (DP), hence Theorem 2.1(c) is the best possible for this class of structures.)

Theorem 11.1 ($V = L$). *Let $\mathcal{A}$ be an admissible structure of the form $\langle L_\alpha, \in, \ldots\rangle$. Then*

(a) *$h(\mathcal{A}) = \alpha$ iff either α is a limit cardinal or $\kappa \leq \alpha \leq \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$ and α is closed under d.*

(b) *$h_\Sigma(\mathcal{A}) = \alpha$ iff $\mathcal{A}$ is Σ_1-compact and either α is a limit cardinal or $|\alpha| = \kappa$ where $\mathrm{cf}(\kappa) = \omega$ and α is closed under d.*

Proof. The first fact to note is that (a) implies (b). Since if we assume that (a) is given, then a necessary condition for $h_\Sigma(\mathcal{A}) = \alpha$ is $h(\mathcal{A}) = \alpha$. Moreover, $\mathcal{A}$ must be Σ_1-compact, otherwise one can easily pindown α itself by a Σ_1-theory. (Using a particular counterexample to compactness.) On the other hand by Proposition 3.3(iv) of Barwise [1, V II] if $\mathcal{A}$ is Σ_1-compact, then $h_\Sigma(\mathcal{A}) = h(\mathcal{A})$, hence by (a) if either α is a limit cardinal or α is closed under d, then $h(\mathcal{A}) = \alpha$. (Note that if $\alpha = \kappa^+$, then $\mathcal{A}$ is not Σ_1-compact.)

In proving (a) we prove first the 'if' part. If α is a limit cardinal, then by Barwise-Kunen [2], $h(L_\alpha) \leq \sup_{\beta<\alpha}(2^\beta)^+$, but since GCH holds in L and α is a limit cardinal we get $h(L_\alpha) \leq \alpha$.

Now assume $\kappa < \alpha \leq \kappa^+$, $\mathrm{cf}(\kappa) = \omega$ and α is closed under d. Let φ be a sentence in $L_{\infty\omega} \cap L_\alpha$ pinning down ordinals (i.e., all the models of φ are well-ordered). $\varphi \in L_\beta$ for some $\beta < \alpha$. Since α is closed under d, $d(\beta) < \alpha$. Let γ be the next admissible after $d(\beta)$. Since α is admissible, $\gamma \leq \alpha$. By Lemma 10.4, γ is closed under d. (Note that if $\delta \leq \beta$, $d(\delta) \leq d(\beta)$). In $\langle L_\gamma, \in\rangle$ there exists a maximal cardinal, since every ordinal in L_γ can be mapped into $d(\beta)$. Hence $|d(\beta)|^{L_\gamma}$ is the largest cardinal. By Lemma 10.5, $\langle L_\gamma, \in\rangle$ satisfies (DP2) (hence (DP)). By Theorem 2.1(c) the ordinals which are models of φ are bounded below γ, therefore they are bounded below α, and the 'if' part of (a) is established.

Now we prove the 'only if' part of (a). So assume $h(\langle L_\alpha, \in\rangle) = \alpha$. If α is a non-limit cardinal, say $\alpha = \kappa^+$, then if $\mathrm{cf}(\kappa) > \omega$, α can be pinned down by a sentence in $L_{\infty\omega} \cap L_\alpha$. The construction of this sentence is by Chang's trick, which is writing a sentence φ expressing

(a) $<$ is a linear order of the universe of the model.

(b) U is a unary predicate such that $< \restriction U$ orders it in order type κ.

(c) f is a binary function, such that for fixed x, $f(x, y)$ is a one-to-one function $\{y \mid y < x\}$ onto U.

(d) g is a ternary function such that for fixed x, $z \in U$ $g(x, y, z)$ maps $\{y \mid y < x, f(x, y) < z\}$ order preservingly into a proper initial segment of U.

$\langle\kappa^+, <\rangle$ can be easily expanded to a model of φ (noting that κ is a cardinal), and any model of φ is well-ordered by $<$ since $\mathrm{cf}(\kappa) > \omega$, hence a decreasing sequence $\cdots < y_1 < y_0$ must satisfy $f(y_0, y_i) < z$ for some $z \in U$ and all $0 < i$ (note that U has order type κ). Hence $g(x, y_i, z)$ gives a decreasing sequence of members of κ. Hence if $\alpha = \kappa^+$, then $\mathrm{cf}(\kappa) = \omega$, but then α is clearly closed under d. Hence we handled the case "α is a cardinal".

If α is not a cardinal, say $\kappa < \alpha < \kappa^+$, then Chang's trick again yields that $\mathrm{cf}(\kappa) = \omega$. We have to verify that α is closed under d. We shall establish this by showing that for all $\kappa < \beta < \kappa^+$ one can construct a sentence in $L_{\infty\omega}$, φ_β, such that φ_β is a primitive recursive function of β and κ, (hence if $\beta < \alpha$, $\varphi_\beta \in L_\alpha$ if α is admissible) and φ_β pins down $d(\beta)$. In fact φ_β will characterize the order type of $d(\beta)$ up to isomorphism. For the construction we need:

Lemma 11.2. *For every $\beta \leq \gamma < d(\beta)$, $\langle L_\gamma, \in\rangle$ can be represented as the direct limit of structures of the form $\langle L_\delta, \in\rangle$ where $\delta < \beta$, where the directed system is indexed by $P_{\omega_1}(\lambda)$ where $\lambda < \kappa$, ($P_{\omega_1}(\lambda)$ is the set of all countable subsets of λ partially ordered by inclusion).*

Proof. Let $\beta \leq \gamma \leq d(\beta)$, and let $D_\gamma \subseteq L_\gamma$ be a set of minimal cardinality satisfying Lemma 10.1. Let $\lambda = |D_\gamma|$. Note that $\lambda < \kappa$. By Lemma 10.2, since $\gamma < d(\beta)$ we have $|D_\gamma| > \omega$. Let G be a one-to-one function mapping λ onto D_γ. For $P \in P_{\omega_1}(\lambda)$

let M_p be the Skolem hull (in $\langle L_\gamma, \in\rangle$) of $G''P \cup \kappa$. $\langle M_p, \in\rangle$ is isomorphic to a structure of the form $\langle L_\delta, \in\rangle$ where $\delta \leq \gamma$. We claim that $\delta < \beta$. Let h be the collapsing isomorphism of M_p onto L_δ. Note that every member of L_δ is first-order definable in $\langle L_\delta, \in\rangle$ from $h''G''P \cup \kappa$. Hence D_δ can be taken to be countable. If $\beta \leq \delta$ we get $d(\beta) \leq \delta$, which is clearly a contradiction. Hence $\delta < \beta$.

$\langle L_\gamma, \in\rangle$ is the union of structures M_p where $P \in P_{\omega_1}(\lambda)$. Hence the directed system induced by $P_{\omega_1}(\lambda)$, where for P we take the transitive isomorphic of M_p, has $\langle L_\gamma, \in\rangle$ for its limit. □ Lemma 11.2

Lemma 11.3. *There exists an $L_{\infty\omega}$ sentence φ_β (primitive recursive in β and κ) such that all models of φ_β have the form $\langle L_\delta, \in, \ldots\rangle$ for some δ and for every $\beta \leq \gamma < d(\beta)$, $< L_\gamma, \in\rangle$ can be expanded to a model of φ_β.*

Proof. φ_β describes the directed system given by Lemma 11.2. More specifically it expresses the following statements.

(a) E is an extensional binary relation which satisfies the first-order sentence Φ, where Φ is the first-order sentence, whose existence was proved by Boulos [3], which guarantees that our model, if it is well-founded is isomorphic to a structure of the form L_δ.

(b) c_β, c_κ, c are individual constants which are ordinals (in the sense of E) such that c_β is isomorphic to β, c_κ to κ and cEc_κ. (c plays the role of λ if Lemma 11.2.)

(c) R is a binary relation such that if $R(x, y)$ holds, then "y is a countable subset of c" such that $\forall yy'$ "$R(x, y) \wedge y \subseteq y' \wedge y'$ is a countable subset of $c \rightarrow R(x, y')$" and such that $\forall x \, \exists y$ (y is a countable subset of $c \wedge R(x, y)$).

(d) F is a binary function such that if y is a fixed countable subset of c, $F(x, y)$ maps the x satisfying $R(x, y)$ into L_{c_β}, preserving E.

By Lemma 11.2 for every $\beta \leq \gamma < d(\beta)$, $\langle L_\gamma, \in\rangle$ can be expanded to a model of φ_β. On the other hand we claim that every model of φ_β is well-founded. This claim is proved as follows. Given a model of φ_β, M, $\langle L^M_{c_\beta}, E^M\rangle$ is clearly isomorphic to $\langle L_\beta, \in\rangle$. Hence we assume without loss of generality that $\langle L_\beta, \in\rangle \subseteq \langle M, E^M\rangle$. Hence c_κ is realized as κ and c as some ordinal less than κ, λ. Since κ is a cardinal every subset of λ is in L_κ, hence in M, and it is countable in M iff it is really countable. If $\cdots x_2E^Mx_1E^Mx_0$, then we can pick $y_i \subseteq \lambda$, y_i countable such that $R(x_i, y_i)$ holds, but then $y = \bigcup_{i<\omega} y_i \in M$ and it is a countable subset of λ. By (b) we must have $R(x_i, y)$ for $i < \omega$. But F maps $\{x \mid R(x, y)\}$ into L_β, preserving E. We get a contradiction to L_β being well-founded. Hence every model of φ_β is well founded. By (a) it is isomorphic to $\langle L_\gamma, \in\rangle$ for some γ. □ Lemma 11.3

Lemma 11.3 is sufficient for the proof of Theorem 11.1 but for future application we need.

Lemma 11.4. *For all $\kappa < \beta < \kappa^+$ there exists a sentence of $L_{\infty\omega}\psi_\beta$ primitive recursive in β and κ such that every model of ψ_β has the form $\langle L_{d(\beta)}, \in, \ldots\rangle$.*

Proof. ψ_β expresses the following statements:

(a) E is an extensional binary relation satisfying Φ, (Φ is as in Lemma 11.3).

(b) R is a ternary relation, F is a ternary function c_β, c_κ constants and c a unary function such that if y is an 'ordinal'

$$\langle L_y, E, R_y, F_y, c_\beta, c_\kappa, c(y)\rangle \models \varphi_\beta$$

where $R_y = \{(z, w) \mid R(y, z, w)\}$ and $F_y(z, w) = F(y, z, w)$ and φ_β is the sentence constructed in Lemma 11.3.

(c) $c_1, c_2, c_3, \ldots, c_n, \ldots$ are constants such that every element is first-order definable from finitely many of the c_n and finitely many 'members' of c_κ. (This is a sentence in $L_{\omega_1\omega}$.)

Obviously by definition of $d(\beta)$ a model of ψ_β must have the form $\langle L_{d(\beta)}, \in, \ldots\rangle$. □ Lemma 11.4

The proof of Theorem 11.1 is now concluded. Since for all $\beta<\alpha$, $\langle L_\alpha, \in\rangle$ contains a sentence pinning down $d(\beta)$ (as the set of ordinals in a model of ψ_β, constructed by Lemma 11.4), we must have by h $(\langle L_\alpha, \in\rangle) = \alpha$, $d(\beta)<\alpha$ and α is closed under d. □ Theorem 11.1

Note that the proof of Theorem 11.1 was yielding some bounds on the ordinals which can be pinned down by sentences in β-logic, where β-logic is first-order logic such that the similarity type contains distinguished constants $\langle c_\gamma \mid \gamma<\beta\rangle$ and a binary relation $<$ with the semantics $<$ well-orders its domain in order type β, and c_γ is the member of the domain of $<$ determining an initial segment of order type γ.

Let $h(\beta)$ be the minimal ordinal not pinned down in β-logic. Then

Corollary 11.5 ($V=L$). *If $\kappa<\beta<\kappa^+$, $\mathrm{cf}(\kappa)=\omega$, then $d(\beta)<h(\beta)\leq d(\beta)^+$ (where $d(\beta)^+$ is the minimal admissible ordinal above $d(\beta)$).*

Proof. Note that ψ_β constructed in the proof of Lemma 11.5 is really a sentence in β-logic, hence $d(\beta)<h(\beta)$. Since $d(\beta)^+$ is closed under d we get $h(\beta)\leq d(\beta)^+$. □ Corollary 11.5

A natural guess is that $h(\beta)=d(\beta)^+$ however we have examples in which $h(\beta)=d(\beta)^+$ as well as examples in which $h(\beta)<d(\beta)^+$. In fact:

Lemma 11.6. *For β as above $h(\beta)<d(\beta)^+$ iff every subset of β which has a Σ_1-definition in $L_{d(\beta)^+}$ from $\{\kappa, \beta, d(\beta)\}$ is a member of $L_{d(\beta)^+}$.*

Proof. Note that $L_{d(\beta)^+}$ is closed under d and it has a last cardinal. Hence it has (DP). Therefore by Theorem 2.1 the set of validities in $L_{\infty\omega}\cap L_{d(\beta)^+}$ is $\Sigma_1(L_{d(\beta)^+})$, where the parameters appearing in the Σ_1-definition are those appearing in the relations S and R witnessing (DP1) and (DP2) respectively. A simple observation (consider the proof of Lemma 10.5) will show that as parameters for a Σ_1-definition of S one can take κ, and for R κ, β, $d(\beta)$. Hence the set of validities of $L_{\infty\omega} \cap L_{d(\beta)^+}$ is Σ_1 in $\{\kappa, \beta, d(\beta)\}$.

Note that if φ is a sentence in β-logic which pins down ordinals, then $\varphi^* = \varphi \to \sim[\bigwedge_{i<\omega} c_{i+1} < c_i]$ is a logical validity where $\{c_i \mid i<\omega\}$ is a sequence of new constants. (We are not distinguishing between φ and an equivalent statement in $L_{\infty\omega}$.) φ^* is clearly in any admissible set containing β and it is primitive recursive in φ, and β. Since the logical validities of $L_{\infty\omega} \cap L_{d(\beta)^+}$ are $\Sigma_1(\kappa, \beta, d(\beta))$, we get that the set of all sentence in β-logic which pin down ordinals is Σ_1-definable in $L_{d(\beta)^+}$ from $\{\kappa, \beta, d(\beta)\}$. This last set can easily be coded as a subset of β. Now assume that every $\Sigma_1(\beta, \kappa, d(\beta))$-subset of β is a member $L_{d(\beta)^+}$. In particular the set of sentences in β-logic which pin down ordinals. Denote this set by A. For φ in A one can define by a $\Sigma_1(\kappa, \beta, d(\beta))$-definition the supremum of all ordinals pinned down by φ. (Denote this sup by $h(\varphi)$.) An ordinal γ is $\geqslant$ the sup of the ordinals pinned down by φ iff the sentence expressing '$\varphi \wedge$ "c is in the domain of $<$" $\wedge \neg\Phi_\gamma(c)$ (where Φ_γ is the canonical sentence of $L_{\infty\omega}$ characterizing order type γ) is logically valid. Hence the minimal such γ is $\Sigma_1(\varphi, \kappa, \beta, d(\beta))$-definable.

By admissibility, since $A \in L_{d(\beta)^+}$

$$\sup\{h(\varphi) \mid \varphi \in A\} < d(\beta)^+,$$

but clearly $h(\varphi) = \sup\{h(\varphi) \mid \varphi \in A\}$, and the 'if' part of the lemma is verified.

For the 'only if' part assume that $h(\beta) < d(\beta)^+$ and let $\exists y\, \varphi(x, y, \kappa, \beta, d(\beta))$ be a Σ_1-formula defining in $L_{d(\beta)^+}$ a subset of β which is not in $L_{d(\beta)^+}$ (φ is Σ_0). Denote this subset of β by B. For each $\delta \in B$ consider the sentence Φ_δ of β-logic expressing the following statements:

(a) E is a binary relation for which KP holds. (KP is really an infinite theory, but using β-logic, in particular $\beta \geqslant \omega$, one can guarantee, possibly with additional predicates that E gives a model of KP.)

(b) c is a constant denoting ordinal for which ψ_β holds where ψ_β is the sentence of β-logic, guaranteed to exist by Lemma 11.4, pinning down exactly $d(\beta)$.

(c) d is a constant, the minimal ordinal such that in L_d, there exists y satisfying $\varphi(c_\delta, y, c_\beta, c_\kappa, c)$ where c_κ, c_β are the canonical constants of β-logic denoting κ and β respectively.

Clearly a model of Φ_δ must contain an ordinal isomorphic to $d(\beta)$, hence since it is a model of KP its well-founded part contains an initial segment isomorphic to $L_{d(\beta)^+}$. Since $\delta \in B$ in $L_{d(\beta)^+}$ one can find y satisfying $\varphi(\delta, y, \beta, \kappa, d(\beta))$, hence d in our model has order type which is exactly the order type of the minimal ordinal $\rho(\delta)$ such that in L_ρ there exists such a y. Hence Φ_δ can be used to pin down ordinals.

Recall that $B \notin L_{d(\beta)^+}$, hence $\sup_{\rho \in B} \rho(\delta) = d(\beta)^+$. (Otherwise we would easily have $B \in L_{d(\beta)^+}$.) Since $\rho(\delta)$ for $\delta \in B$ can be pinned down by a sentence in β-logic we get that $h(\beta)=d(\beta)^+$ and we get a contradiction, proving the 'only if' part of our lemma. $\square$ Lemma 11.6

Using Lemma 11.6 one can easily get examples both for $h(\beta)=d(\beta)^+$ and for $h(\beta)<d(\beta)^+$. For instance if $\beta=d(\beta)$ (recall that $d(d(\beta))=\beta$), then every member of $L_{d(\beta)^+}$ is Σ_1-definable from members of β, hence one can easily find a Σ_1-subset of β (no parameters are necessary) which is not in $L_{d(\beta)^+}$, hence $h(\beta)=\beta^+$.

For another example take the minimal model of ZFC (actually large enough finite part is sufficient) containg $\kappa+1$. Let this model have the form L_δ. $d(\delta)=\delta$ since every member of L_δ is first-order definable from κ. Let ρ be the ω_1-th cardinal after κ in L_δ and let μ be its successor in L_δ. We claim that $d(\rho)=\mu$. $d(\rho)\leqslant\mu$ since every subset of ρ definable in L_δ is already in L_μ (recall that L_δ is a model of ZFC), since $d(\delta)=\delta$ once can find ω-many members of L_μ of cardinality $<\kappa$ whose union is ρ. On the other hand we cannot have $d(\rho)<\mu$ because by applying Chang's tricks inside L_δ (using $\mathrm{cf}(\rho)>\omega$) we can pin down μ. Hence if $d(\rho)<\mu$, also $d(\rho)^+<\mu$, contradicting Theorem 11.1. Since, if μ^+ is the next admissible after μ, then $\mu^+<\delta$, hence every subset of ρ definable in L_{μ^+} is already in L_δ, hence in L_μ. Thus by Lemma 11.6 $h(\rho)<\mu^+$.

12. Completeness revisited

Corollary 2.1 asserted in particular that every admissible structure satisfying (DP) is validity admissible. In this section we show that this is the best possible for structures of the form $\langle L_\alpha, \in\rangle$. Namely if such a structure is validity admissible, then it has (DP). Of course we cannot extend this result for structures of the form $\langle L_\alpha, \in, \ldots\rangle$ because $\langle L_\alpha, \in\rangle$ can be admissible for the validity predicate, even if L_α does not have (DP). (For instance $\alpha=\kappa^+$ where $\mathrm{cf}(\kappa)>\omega$.) A similar question for Σ_1-completeness will be handled in the next part.

Our first observation connects pinning down of ordinals and validity admissibility.

Lemma 12.1. *If L_α contains a sentence φ such that φ pins down some ordinal $\geqslant\alpha$, then $\langle L_\alpha, \in\rangle$ is not validity admissible.*

Proof. We show that if $\langle L_\alpha, \in\rangle$ is validity admissible then every Π_1-predicate in L_α is Σ_1, which is clearly a contradiction. Let $x\in L_\alpha$ and φ a Σ_1-formula such that $L_\alpha\models\neg\varphi(x)$ but for some $\delta>\alpha$, $L_\delta\models\varphi(x)$ if such exists. Without loss of generality fix x and φ such that δ is minimal. Let Ψ be a sentence in $L_{\infty\omega}\cap L_\alpha$ which pins down some $\beta\geqslant\alpha$. Let χ be a sentence expressing the following statements:

(a) E is a binary relation for which the sentence Φ characterizing L_α's holds.

(b) c is an individual constant, "c is an ordinal", Ψ holds for c. (which means that φ holds for the order type of c).

Sh:144

(c) d is an individual constant, "d is an ordinal". "$d \leq c$".

(d) e is an individual constant for which Φ_x holds where Φ_x is the canonical sentence characterizing x. (Φ_x is defined by induction as

$$\Phi_x(z) \leftrightarrow \forall y \left(yEz \leftrightarrow \bigvee_{t \in x} \Phi_t(y) \right).$$

(e) $e \in L_d$, $L_d \models \neg\varphi(e)$. (In case no x an φ exists we drop (e) altogether.)

Let $\rho(z)$ be a Π_1-formula.

Claim. $L_\alpha = \rho(z)$ holds for z iff the following sentence is a logical validity:

$$\mu_z = \chi \rightarrow \forall t\ (\Phi_z(t) \wedge t \in L_d \rightarrow 'L_d \models \rho(t)').$$

Proof of the Claim. Let $\langle M, E\rangle$ be a model of χ. By (a), $\langle L_c^M, E\rangle$ is well-founded, hence isomorphic to some $\langle L_\beta, \in\rangle$ (remember that Ψ pins down ordinals). Pick $t \in M$, for which $\Phi_z(t)$ holds and $t \in L_\alpha$. Since $d \leq c$, $\langle L_d, E\rangle$ is isomorphic to some $\langle L_\gamma, \in\rangle$ and by our assumptions $x, z \in L_\gamma$.

Since $L_\gamma \models \neg\varphi(x)$ (remember $L_\alpha \models \neg\varphi(e)$) we have $\gamma < \delta$ where δ is the minimal for which $L_\delta \models \varphi(x)$. By choice of x and φ, there is no Σ_1-sentence λ such that $L_\alpha \models \neg\lambda(z)$ but $L_\gamma \models \lambda(z)$. Hence we cannot have $L_\gamma \models \neg\rho(z)$, but $L_\alpha \models \rho(z)$, hence if $L_\alpha \models \rho(z)$ we must have $L_\gamma \models \rho(z)$. Therefore in our original model M we had $L_d \models \rho(t)$.

For the other direction if $L_\alpha \models \neg\rho(z)$, then one can easily construct a model of χ which includes L_α such that $d = \alpha$, and (recall that Ψ pins down some ordinal $\geq \alpha$) hence if we take $t = z$ we get t such that $\Phi_z(t)$ holds, $t \in L_d$ but "$L_d \models \neg\rho(t)$". Hence the sentence in the claim is not a logical validity. □ Claim.

Now if $LV \cap L_\alpha$ was $\Sigma_1(L_\alpha)$ every Π_1-formula ρ would be equivalent over L_α to a Σ_1-formula, namely $\rho(z) \leftrightarrow \mu_z \in LV \cap L_\alpha$ (Note that μ_z is Σ_1-definable in L_α from z.) □ Lemma 12.1

Lemma 12.2 ($V = L$). *Let $\mathscr{A}$ be an admissible structure of the form $\langle L_\alpha, \in, \ldots\rangle$ such that $\kappa < \alpha \leq \kappa^+$. where $\mathrm{cf}(\kappa) = \omega$. Assume further that α is closed under the function d. Then A is validity admissible iff the function d is $\Sigma_1(\mathscr{A})$.*

Proof. Assume that d is $\Sigma_1(\mathscr{A})$, then clearly $\mathscr{A}$ has (DP). (DP1) because α is closed under d. (DP2) because one can take as $R(X, Y)$ the sentence

$$\exists\gamma\ \exists\delta\ (x \in L_\gamma \wedge \delta = d(\gamma) \wedge Y = L_{\delta+1}).$$

Hence by Theorem 2.1, $\mathscr{A}$ is validity admissible.

For the other direction assume that $\mathscr{A}$ is validity admissible. Recall the sentence Ψ_β defined in Lemma 11.4 which characterizes the order type $d(\beta)$. Also for $\gamma < \alpha$ let Φ_γ be the canonical sentence of $L_{\infty\omega}$ characterizing order type γ. (Note that if $\beta<\alpha$, then Ψ_β, $\Phi_\beta \in L_\alpha$.) Then

$$\gamma = d(\beta) \leftrightarrow \text{“}\Psi_\beta \rightarrow \text{the ordinals of the model satisfy } \Phi_\gamma\text{” is a logical validity.}$$

Since Ψ_β and Φ_γ are Σ_1-definable from Ψ_β and Φ_γ, and since the set of logical validities is $\Sigma_1(\mathscr{A})$ we get that d is $\Sigma_1(\mathscr{A})$. □ Lemma 12.1

Our goal is

Theorem 12.3 ($V=L$). *The following are equivalent* ($\alpha>\omega$, α *admissible*):

(a) $\langle L_\alpha, \in\rangle$ *is validity admissible.*

(b) $\langle L_\alpha, \in\rangle$ *satisfies* (DP).

(c) $\kappa<\alpha\leqslant\kappa^+$ *where* $\mathrm{cf}(\kappa)=\omega$, *and* α *is closed under the decomposition* d *and* d *is* $\Sigma_1(L_\alpha)$.

(d) $\kappa<\alpha\leqslant\kappa^+$ *where* $\mathrm{cf}(\kappa)=\omega$, α *is closed under* d, $L_\alpha \vDash$ *there exists a last cardinal or* $\kappa=\omega$.

Proof. By Lemma 10.5, if $\kappa<\alpha\leqslant\kappa^+$ where $\mathrm{cf}(\kappa)=\omega$, α is closed under d and L_α has a last cardinal, then $\langle L_\alpha, \in\rangle$ has (DP2) (hence DP). (Similarly if $\kappa=\omega$.) By Theorem 2.1, $\langle L_\alpha, \in\rangle$ is validity admissible. Hence by Lemma 12.1, d is $\Sigma_1(L_\alpha)$. So we proved (d)$\rightarrow$(c).

(c)$\rightarrow$(b) should be obvious by now. (Actually it was proved in the proof of Lemma 12.2.)

(b)$\rightarrow$(a) is a consequence of Theorem 2.1.

Hence we just have to verify (a)$\rightarrow$(d). So assume that $\langle L_\alpha, \in\rangle$ is validity admissible. Now use Lemma 12.1. It follows that no sentence in L_α pins down an ordinal $\geqslant\alpha$, i.e. $h(\langle L_\alpha, \in\rangle)=\alpha$. By Theorem 11.1 either α is a limit cardinal or $\kappa<\alpha\leqslant\kappa^+$ where $\mathrm{cf}(\kappa)=\omega$ and α is closed under d. Our first step in proving (a)$\rightarrow$(d) is to rule out the possibility “α is a limit cardinal”. So assume that heading for contradiction that α is a limit cardinal and $\langle L_\alpha, \in\rangle$ is validity admissible.

Lemma 12.4. *Let* α *be a limit cardinal, then the set of cardinals* $<\alpha$ *is not* $\Sigma_1(L_\alpha)$.

Proof. Suppose that the set of cardinals $<\alpha$ (denote it by Card) was $\Sigma_1(L_\alpha)$. Then every $\Pi_1(L_\alpha)$ would be $\Sigma_1(L_\alpha)$ because for φ, a Π_1-formula,

$$\varphi(x) \leftrightarrow \exists\varphi\ (\beta\in\mathrm{Card} \wedge x\in L_\beta \wedge L_\beta \vDash \varphi(x))$$

which is a clear contradiction. □ Lemma 12.4

Now use the fact that the α is validity admissible. We claim that Card is $\Sigma_1(L_\alpha)$, because let Ψ be the sentence expressing: “F is a one-to-one function of the domain of $<$, where $<$ is a linear ordering onto a proper initial segment of itself". Then:

$$\beta \in \mathrm{Card} \leftrightarrow \text{“}\Phi_\beta \to \neg\Psi\text{” is a logical validity.}$$

(Φ_β is the canonical sentence characterizing ordering of order type β.)

If $LV \cap L_\alpha$ is $\Sigma_1(L_\alpha)$ we get $\mathrm{Card} \in \Sigma_1(L_\alpha)$, which contradicts Lemma 12.4.

Hence we conclude that if $\langle L_\alpha, \in\rangle$ is validity admissible we must have $\kappa < \alpha \leqslant \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$ and α is closed under d. By Lemma 12.2 we get that d is $\Sigma_1(L_\alpha)$. The proof of (a)$\to$(d), hence of our theorem, will be concluded by showing

Lemma 12.5. *Let $\kappa < \alpha \leqslant \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$, and $\kappa > \omega$, α is closed under d, then if d is $\Sigma_1(L_\alpha)$, then $L_\alpha \vDash$"there exists a last cardinal".*

Proof. Assume that $\langle L_\alpha, \in\rangle \vDash$"there is no last cardinal". Let $\exists z\, \varphi(x, y, z, p)$ be a Σ_1-formula defining $x = d(y)$, where φ is Σ_0 and p is the parameter. Let $p \in L_\beta$ where $\beta > \kappa$ and let β^+ be the L_α-cardinal which is the successor of $|\beta|^{L_\alpha}$. By assumption

$$L_\alpha \vDash \exists z\, \varphi(\beta, d(\beta^+), z, p).$$

Assume z witnesses the above statement, $z \in L_\gamma$ where $\gamma > d(\beta^+)$. Consider the structure $M = \langle L_\gamma, \beta, d(\beta)^+, z, p\rangle$. In L_α one can find an elementary substructure of M, N, such that $|N|^{L_\alpha} = |\beta|$, $L_\beta \subseteq N$, and such that $N \cap \beta^+$ is an ordinal having cofinality ω_1. (Here we used $\kappa > \omega$.)

(Note that if β^+ is regular in L_α, then every subset of β^+ of cardinality $|\beta|^{L_\alpha}$, is bounded in L_{β^+}. N is constructed by defining in L_α an increasing sequence of elementary submodels of M, $\langle N_\rho \mid \rho < \omega_1\rangle$, where each of them has cardinality $|\beta|^{L_\alpha}$, and such that $\sup(N_\rho \cap \beta^+) \subseteq N_{\rho+1}$. N is $\bigcup_{\rho<\omega_1} N_\rho$.)

N is the isomorphic to the structure of the form $\tilde{N} = \langle L_\delta, \eta, \mu, \bar{z}, \bar{p}\rangle$ by the usual collapsing isomorphism. Note that $\eta = N \cap \beta^+$ and that $\bar{p} = p$ since $\bar{p} \in L_\beta \subseteq N$.

Since $\tilde{N}$ can be elementary embedded into M we have

$$\tilde{N} \vDash \varphi(\eta, \mu, \bar{z}, p),$$

hence, since φ is Σ_0,

$$L_\alpha \vDash \exists z\, \varphi(\eta, \mu, z, p).$$

By definition of φ, $\mu = d(\eta)$. In particular $d(\eta) < \delta$.

By definition of the decomposition function d, in L_δ there are subsets of η, A_n for $n < \omega$, such that $|A_n| < \kappa$ and $\bigcup_{n<\omega} A_n = \eta$.

Recall that $\mathrm{cf}(\eta) = \omega_1$, hence for some $n < \omega$, A_n is cofine in η, its order type $< \kappa < \eta$. Therefore

$$L_\delta \vDash \eta \text{ is singular.}$$

But since η is mapped to β^+ in the elementary embedding of $\tilde{N}$ into M, we have

$L_\gamma \models \beta^+$ is singular which is a clear contradiction. $\square$ Lemma 12.5 $\square$ Theorem 12.3

It follows from Theorem 12.3 combined with Lemma 10.6 that we have a structure of the form $\mathcal{A} = \langle L_\alpha, \in, \ldots \rangle$ which has (DP) but $\langle L_\alpha, \in \rangle$ does not have (DP). (We can find α closed under d such that in L_α there is no last cardinal. For instance the minimal $\alpha > \aleph_\omega$: satisfying that there are ω cardinals is such an α.) On the other hand the proof of Theorem 12.3 can show that if $\mathcal{A} = \langle A, \in, \ldots \rangle$ satisfies (DP), then (assuming $V = L$) $\alpha = o(A)$ is either a limit cardinal or $\kappa < \alpha < \kappa^+$ and α closed under d. Hence $\langle L_\alpha, \in \rangle = L^A$ can be expanded to an admissible structure having (DP).

The analysis we have done in this section allows us to give (in $V = L$) an equivalent definition of (DP) and countably decomposability for admissible structures of the form $\langle L_\alpha, \in, \ldots \rangle$.

Theorem 12.6 ($V = L$). (a) *An admissible structure of the form* $\mathcal{A} = \langle L_\alpha, \in, \ldots \rangle$ *has* (DP) *iff either* (I) α *is a limit cardinal and the function* $x \to \mathcal{P}(x)$ (*note that in this case* L_α *is closed under the function* $\mathcal{P}$) *is* $\Sigma_1(\mathcal{A})$, *or* (II) $\kappa < \alpha \leqslant \kappa^+$ *where* $\mathrm{cf}(\kappa) = \omega$, α *is closed under the function* d *and* d *is* $\Sigma_1(\mathcal{A})$.

(b) $\mathcal{A}$ *as above is countably decomposable iff one of* (I) *and* (II) *above holds and* $\mathrm{cf}(\alpha) = \omega$.

Proof. (b) follows easily from (a), because if L_α is a countable union of its members iff $\mathrm{cf}(\alpha) = \omega$. Clearly if (I) or (II) holds, then $\mathcal{A}$ has (DP) ((I) is handled in Example 2.4, (II) is handled by the proof of Lemma 12.2) and the 'if' direction of (a) is verified.

Assume that $\mathcal{A} = \langle L_\alpha, \in, \ldots \rangle$ has (DP). By Corollary 2.2, $h(A) = o(A)$. By Theorem 11.1 either α is a limit cardinal or $\kappa < \alpha \leqslant \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$ and α is closed under d. By Corollary 2.2 again $\mathcal{A}$ is validity admissible, hence by Lemma 12.2 if $\kappa < \alpha \leqslant \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$ we must have that d is $\Sigma_1(\mathcal{A})$. Hence if α is not a limit cardinal case (II) holds. If α is a limit cardinal we shall use

Lemma 12.7 ($V = L$). *Let* $\mathcal{A}$ *be a structure of the form* $\langle L_\alpha, \in, \ldots \rangle$ *such that* α *is a limit cardinal, then* $\mathcal{A}$ *is validity admissible iff* $\mathcal{P} \restriction L_\alpha$ *is* $\Sigma_1(\mathcal{A})$.

Proof. If $\mathcal{P} \restriction L_\alpha$ is $\Sigma_1(\mathcal{A})$, then $\mathcal{A}$ has (DP), hence by Theorem 2.2 it is validity admissible. If $\mathcal{A}$ is validity admissible, then given y, x, one can (in $\Sigma_1(L_\alpha)$ way) find a sentence $\varphi_{x,y}$ in $L_{\infty\omega} \cap L_\alpha$ such that $y = \mathcal{P}(x)$ iff $\varphi_{x,y}$ is valid. ($\varphi_{x,y}$ has a binary relation E, and expresses the fact that if E is extensional such that $\langle \mathrm{dom}(E), E \rangle$ is isomorphic to $\langle \mathrm{Tc}(x), \in \rangle$ and P is a unary predicate on $\mathrm{dom}(E)$, *then* P 'appears' in y.)

Hence $y = \mathcal{P}(x)$ iff $\varphi_{x,y} \in LV$, therefore $y = \mathcal{P}(x)$ is A-r.e. $\square$ Lemma 12.7

Our theorem is established since A is validity admissible (having DP) $\square$ Theorem 12.6

13. Characterizing $\alpha(X)$ where $X \subseteq \kappa$

In Section 8 we got as an application for Theorem 8.1 the fact that if α is $o(\mathscr{A})$ for some countably decomposable structure $\mathscr{A}$ where $\kappa < \alpha < \kappa^+$, then $\alpha = \alpha(X)$ for some $X \subseteq \kappa$. In this section we prove the opposite implication. (As we mentioned in Section 8 the results of this section were independently obtained by Sy Friedman.)

Lemma 13.1 $(V = L)$. *Let* $\mathrm{cf}(\kappa) = \omega$ *and let* $X \subseteq \kappa$. *Then* $\langle L_{\alpha(X)}(X), \in \rangle$ *is countably decomposable.*

Proof. $\mathscr{A} = \langle L_{\alpha(X)}(X), \in \rangle$ satisfies the requirements of Lemma 2.6 where for D one takes L_κ. Recall that since we assume $V = L$, $x \in L_\kappa \to \mathscr{P}(x) \in L_\kappa$. Also $L_\kappa = \bigcup_{n<\omega} L_{\kappa_n}$ if $\langle \kappa_n \mid n < \omega \rangle$ is a cofinal sequence in κ. Also L_κ is a c-max element of $\mathscr{A}$, and $\mathscr{A}$ is projectable onto L_κ. □ Lemma 13.1

It follows from Lemma 13.1 that $\mathrm{cf}(\alpha(X)) = \omega$. Remember that $L_{\alpha(X)} \subseteq L_{\alpha(X)}(X)$, hence by Theorem 2.1(c) no ordinal $\geqslant \alpha(X)$ is pinned down by a sentence in $L_{\alpha(X)}$. Therefore, by Theorem 11.3, $\alpha(X)$ is closed under d.

Theorem 13.2 $(V = L)$. *If* $\kappa < \alpha < \kappa^+$, *$\alpha$ admissible* $\mathrm{cf}(\kappa) = \omega$, *then the following are equivalent*:

(a) $\alpha = \alpha(X)$ *for some* $X \subseteq \kappa$.

(b) $\mathrm{cf}(\alpha) = \omega$ *and α is closed under d.*

(c) $\langle L_\alpha, \in, P \rangle$ *is countably decomposable for appropriate predicate P.*

(d) $\langle L_\alpha, \in \rangle$ *satisfies the Barwise compactness theorem with ordinal omitting.*

Proof. In Lemma 13.1 and the remarks following it we actually proved (a) → (b), (b) → (c) follows from Lemma 10.6, (c) → (d) follows from Corollary 2.2 and Theorem 8.1. (d) → (a) was essentially proved when we made the application of Theorem 8.1 in Section 8. □ Theorem 13.2

Sy Friedman in [10] gave additional equivalent definitions of the ordinals described by Theorem 13.2, in terms of the sequence of 'critical' projecta of α. Also in that paper he characterizes ordinals $\kappa < \alpha < \kappa^+$, $\alpha = \alpha(X)$, $X \subseteq \kappa$ where $\mathrm{cf}(\kappa) > \omega$. (The case κ regular was handled in the first part of his paper.) We did not attend this problem $(\mathrm{cf}(\kappa) \neq \omega)$ at all.

PART III

14. Stable Compactness

As we described in the introduction, there is no hope of getting that an admissible structure $\mathscr{A}$ is Σ_1-compact just if it is countably decomposable.

Because in view of Stavi's result [25] many L_α's are Σ_1-compact even though they definitely do not have (DP). Our feeling that this compactness is accidental and not an intrinsic property of the admissible structure studied, motivated the following definition.

Definition 14.1. An admissible structure $\mathcal{A} = \langle A, R_1, \ldots, R_n\rangle$ is *stably Σ_1-compact* if every admissible expansion of $\mathcal{A}$ (i.e., an admissible structure of the form $\langle A, R_1, \ldots, R_n, Q_1, \ldots, Q_n\rangle$) is Σ_1-compact. Note that we just consider expansions of $\mathcal{A}$ (not extensions!), i.e., adding more relations, which must be of course amenable, i.e., if R is a relation introduced, then $R \cap x \in \mathcal{A}$ for every $x \in \mathcal{A}$.

Clearly every countably decomposable admissible structure is stably Σ_1-compact, because every expansion of such a structure is countably decomposable and hence by Corollary 2.2 it is Σ_1-compact.

There is another noteworthy class of stably Σ_1-compact structures. Namely let κ be a weakly compact cardinal (see [15] for definition) and let $\mathcal{A}$ be an admissible structure, $\mathcal{A} = \langle A, \in, Q_1, \ldots, Q_n\rangle$, $A \subseteq H(\kappa)$, $|A| = \kappa$, and every subset of $\mathcal{A}$ of cardinality less than κ included in an A-finite set. Then $\mathcal{A}$ is stably Σ_1-compact. In fact the conditions imposed on $\mathcal{A}$ guarantee that any admissible expansion of $\mathcal{A}$ satisfies Σ_1-separation. (Let $\mathcal{B}$ be such an expansion, $x \in A$, $\varphi(y, z)$ a $\Delta_0(\mathcal{B})$-formula. Since $|x| < \kappa$ there exists a transitive set $t \subseteq A$ such that $|t| < \kappa$ and if $z \in x$ and $\exists y \in A\ \mathcal{B} \models \varphi(y, z)$, then $\exists y \in t\ \mathcal{B} \models \varphi(y, z)$. By our assumption we can assume without loss of generality that $t \in A$ (if necessary we pass to a super set). Hence $\{z \mid z \in x, \mathcal{B} \models \exists y\ \varphi(z, y)\} = \{z \mid \in x\ \mathcal{B} \models \exists y \in t\ \varphi(z, y)\}$ which by Δ_0-separation lies in A.) Let $T \subseteq A \cap L_{\infty\omega}$ be $\Sigma_1(\mathcal{B})$ where $\mathcal{B}$ is an admissible expansion of $\mathcal{A}$. By definition of weak compactness, if T does not have a model, some $b \subseteq T$, $|b| < \kappa$, does not have a model. Note $L_{\infty\omega} \cap A \subseteq L_{\kappa\omega}$), but for some $a \in A$, $b \subseteq a$. By Σ_1-separation for $\mathcal{B}$, $T \cap a \in A$. Hence some A-finite subset of T does not have a model and $\mathcal{B}$ was verified to be Σ_1-compact.

The main theorem of this part is that these two cases exhaust all possibilities of stably Σ_1-compact admissible structures of the form $\langle L_\alpha, \in, R_1, \ldots, R_n, \ldots\rangle$ if one assumes $V = L$.

Theorem 14.1 ($V = L$). *Let $\mathcal{A}$ be an admissible structure of the form $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$. Then $\mathcal{A}$ is stably Σ_1-compact if and only if either:*

(I) *α is a weakly compact cardinal, or*

(II) *α is a limit cardinal, $\mathrm{cf}(\alpha) = \omega$ and $\mathcal{P} \restriction L_\alpha$ is $\Sigma_1(\mathcal{A})$.*

(III) *$\kappa < \alpha < \kappa^+$ where $\mathrm{cf}(\kappa) = \mathrm{cf}(\alpha) = \omega$, α is closed under the function d and $d \restriction L_\alpha$ is $\Sigma_1(\mathcal{A})$.*

Using Theorem 12.6 we get

Corollary 14.2 ($V=L$). *Let $\mathscr{A}$ be an admissible structure of the form $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$. Then $\mathscr{A}$ is stably Σ_1-compact if and only if either α is a weakly compact cardinal or $\mathscr{A}$ is countably decomposible.*

Proof of Theorem 14.1. The 'if' part of Theorem 14.1 follows from the remarks before the theorem (for the weak compact case) and Theorem 12.4 and Corollary 2.2. Hence we just have to prove the 'only if' part. So assume that $\mathscr{A}$ is stably Σ_1-compact.

Case I: *α is regular.* If α is not weakly compact, then some $T \subseteq L_{\infty\omega} \cap L_\alpha$ witnesses that α is not weakly compact. Namely T does not have a model, but every subset of T of cardinality less than α has a model. But since α is regular, every L_α-finite subset of T is of cardinality $<\alpha$. Hence $\langle L_\alpha, \in, R_1, \ldots, T\rangle$ is not Σ_1-compact, but it is clearly admissible, since α is regular and by well known facts, (we assume $V=L$), $T \cap L_\beta \in L_\alpha$ for $\beta<\alpha$.

Case II: *α is singular.* In this case we show that either α is not stably Σ_1-compact or (I) or (II) holds.

We shall use a variant of the combinatorial principle $\square$ introduced by Jensen [14] and proved by him to hold in L. See also [24].

Theorem 14.3 (Jensen [14]) ($V=L$). *One can assign to each singular limit ordinal α a set $C_\alpha \subseteq \alpha$ such that*:

(a) *C_α is a closed unbounded subset of α.*

(b) *The order type of C_α is less then α.*

(c) *If β is a limit point of C_α then $L_\alpha \models \beta$ is singular and $C_\beta = C_\alpha \cap \beta$.*

(d) *C_α is uniformly definable from α in $L_{\beta(\alpha)+1}$ where $\beta(\alpha)$ is the first ordinal such that $L_{\beta(\alpha)} \models \alpha$ is singular.*

Note. The way Theorem 14.3 is phrased in [14] or [24] does not yield that if β is a limit point of C_α, then $L_\alpha \models \beta$ is singular, and that C_α is uniformly definable from α in $L_{\beta(\alpha)+1}$ but easy checking of Jensen's proof show that it holds for the C_α's defined by Jensen.

For the rest of this paper we fix an assignment $\langle C_\alpha \mid \alpha \text{ singular}\rangle$ satisfying Theorem 14.3, and for α singular we define $\gamma_\alpha =$ the order type of C_α. The following theorem is the main fact we need to conclude the proof. We shall delay its proof to the next section.

Theorem 14.4 ($V=L$). *Let $\mathscr{A}$ be an admissible structure of the form $\langle L_\alpha, \in, \ldots\rangle$. Assume that α is singular. (Hence C_α and γ_α are defined.) We can expand $\mathscr{A}$ to an admissible structure $\mathscr{B} = \langle L_\alpha, \in, \ldots, G\rangle$ such that $G \subseteq \alpha$ and*

(a) *G is a closed unbounded subset of α.*

(b) *If β is a limit point of G, then either $\mathscr{A} \restriction L_\beta = \langle L_\beta, \in, R_1 \restriction L_\beta, \ldots\rangle$ is not admissible or $L_\alpha \models \beta$ is singular and $\gamma_\beta \neq \gamma_\alpha$.*

Theorem 14.1 will be proved by using an appropriate forcing notion, and getting G from a sufficiently generic filter over $\mathcal{A}$. For our given $\mathcal{A}$ assume that we picked G satisfying the conditions in Theorem 14.4 and $\mathcal{B}$ is the admissible structure $\langle L_\alpha, \in, R_1, \ldots, G\rangle$. The treatment of Case II will be divided into four subcases which are not mutually exclusive.

Subcase IIa: $h(L_\alpha)>\alpha$. I.e., some ordinal bigger than α is pinned down by a sentence in L_α, Ψ. In this case we claim that $\mathcal{B}$ is not Σ_1-compact. Hence $\mathcal{A}$ is not stably Σ_1-compact. We shall construct a theory T witnessing the fact that $\mathcal{B}$ is not Σ_1-compact.

We first construct a theory $T_{\mathcal{B}}$ expressing the fact that a model of it is an end extension of $\mathcal{B}$. (This construction applies generally to any admissible structure and we shall use $T_{\mathcal{B}}$ for other admissible structures as well.) $T_{\mathcal{B}}$ has a binary relation symbol E (representing $\in$) as well as a relation symbol R for every relation appearing in $\mathcal{B}$. We have also a constant c_x for every $x\in L_\alpha$ and another constant c. $T_{\mathcal{B}}$ will express the following facts:

(I) E is an extensional relation such that KP holds for E with all other predicates.

(II) $V=L$.

(III) For each $x\in L_\alpha$ we have a sentence $\forall z\,(z\,E\,c_x \leftrightarrow \bigvee_{y\in x} z=c_y)$ and if $R(x)$ holds, then $R(c_x)\in T_{\mathcal{B}}$. If $\sim R(x)$ holds, then $\neg R(c_x)\in T_{\mathcal{B}}$.

(IV) $c_x\,E\,c$ for every $x\in L_\alpha$. $T_{\mathcal{B}}$ is clearly $\Sigma_1(\mathcal{B})$. Our theory T will be $T_{\mathcal{B}}$ together with

(V) Ψ (recall that Ψ pins some ordinals $\geqslant\alpha$). We assume that all the Ψ relation symbols of Ψ including '$<$' do not appear in $\mathcal{B}$.

(VI) f is an order preserving function from the E ordinals into the field of $<$.

(VII) G is a closed unbounded class of ordinals and if x is a limit point of G, then either $\langle L_x, R_1 \upharpoonright L_x, \ldots\rangle$ is not admissible or x is a singular ordinal such that if we define C_x (using the uniform definition of the C_α's) we get a closed unbounded subset of x having order type $\neq c_{\gamma_\alpha}$.

T is clearly $\Sigma_1(\mathcal{B})$. Every L_α-finite subset of T has a model because if $t\in L_\alpha$, $t\subseteq T$, we can expand $\mathfrak{B}$ to a model of t by interpreting c_x as x, c as L_δ where δ is large enough such that if c_x appears in t $x\in L_\delta$, and we interpret the relations appearing in Ψ such that we get a model of Ψ with the order type of $<$ bigger than α. We let f be any order preserving function from α into the domain of $<$.

The handling of this subcase is concluded by:

Lemma 14.5. *T has no model.*

Proof. A model of T must be well-founded because Ψ pins down ordinals, Hence it is isomorphic to a structure of the form $\langle L_\beta, \in, \bar{R}_1, \bar{R}_2, \ldots, \bar{G}\rangle$. Clearly, by (III–IV), $\bar{R}_i\cap L_\alpha = R_i$, $\bar{G}\cap\alpha = G$. Since G is unbounded in α, α is a limit point of $\bar{G}$. Since $\langle L_\alpha, \in, R_1\cap L, R_2\cap L_\alpha, \ldots\rangle$ is admissible, we must have by (VII) that

Sh:144

$L_\beta \models \alpha$ is singular. Hence if $\beta(\alpha)$ is the minimal ordinal such that $L_{\beta(\alpha)} \models \alpha$ is singular we have $\beta(\alpha) < \beta$. Hence C_α defined in L_β is the same as the real C_α. Therefore it has order type γ_α. We have found a limit point of $\bar{G}$, contradicting the conditions imposed on $\bar{G}$ by (VIII). □ Lemma 14.5 □ Subcase IIa.

Subcase IIb: $\mathrm{cf}(\alpha) > \omega$ (*α singular*). Also in this case we claim that $\mathscr{A}$ is never stably Σ_1-compact. We form $\mathfrak{B}$ as before by expanding $\mathscr{A}$ as before using G satisfying Theorem 14.4. We again claim that $\mathfrak{B}$ is not Σ_1-compact, hence $\mathscr{A}$ is not stably Σ_1-compact. The theory which will witness this is a small variant of T, defined above in Subcase IIa. Namely we take $T_{\mathfrak{B}}$ together with (VII) of T. Like the previous case, every L_α-finite subset of T has a model. Again we finish this case by showing

Lemma 14.6. *T has no models.*

Proof. Suppose T has a model of the form $\langle M, E, \bar{R}_1, \ldots, \bar{R}_n, \bar{G}\rangle$. This proof is different from the proof of Lemma 14.5 because $\langle M, E\rangle$ cannot be assumed now to be well-founded. But we know that the ordinals of M have a maximal initial segment which is well-founded. Without loss of generality we can assume that this initial segment is some ordinal β and that $L_\beta \subseteq M$. By definition of T (recall $T_{\mathfrak{B}} \subseteq T$), $\beta \geq \alpha$ hence $L_\alpha \subseteq M$ and $E \restriction L_\alpha = \in \restriction L_\alpha, \ldots$ We shall, in what follows, misuse the language by pretending that $\langle M, E, \ldots\rangle$ is a standard model of KP. For instance we shall systematically confuse $x \in M$ and $\{y \mid y E x\}$. We shall use superscript M for relativizing different set-theoretic notions to M. Thus L_z^M is the element of M considered by M to be 'L' of the M ordinal z.

We distinguish two cases:

Case A: $\alpha < \beta$. Hence $\alpha \in M$. Since $G = \bar{G} \cap L_\alpha$ and G is unbounded in α, we get as in Lemma 14.5 that $\alpha \in \bar{G}$. Since $\bar{R}_i \cap L_\alpha = R_i$ for $i = 1, \ldots, n$, $\langle L_\alpha, \in, \bar{R}_1 \cap L_\alpha, \ldots, \bar{R}_n \restriction L_\alpha\rangle$ is admissible. Hence $M \models \alpha$ is singular $\wedge$ the order type of $C_\alpha^M \neq \gamma_\alpha$.

We shall use the following fact which was called Lemma 8 in [21].

Fact. *Let $\langle M, E\rangle$ be a model of* KP$+V=L$. *Let α be an ordinal in the well-founded part of $\langle M, E\rangle$, with* $\mathrm{cf}(\alpha) > \omega$. *If $\langle M, E\rangle \models \alpha$ is singular, then $\beta^M(\alpha)$ is in the well-founded part of M. Hence $\beta^M(\alpha) = \beta(\alpha)$.* (Recall that $\beta(\alpha)$ is the minimal ordinal such that $L_{\beta(\alpha)} \models \alpha$ is singular.)

(The proof uses [14] to represent, in M, $L^M_{\beta(\alpha)}$ as a directed limit of structures of the form $\langle L_\gamma, \in\rangle$, $\gamma < \alpha$, where the direct limit is indexed by $\delta < \alpha$, ordered by their natural order. Since $\mathrm{cf}(\alpha) > \omega$ every countable set of elements of the directed limit is already in one of the members of the directed system. Hence $L^M_{\beta(\alpha)}$ is well-founded.)

Since the real $\beta(\alpha)$ is in M, we get by the uniform definability of C_α's that

$C_\alpha^M = C_\alpha$, and we immediately get a contradiction to our assumption that the order type of $C_\alpha^M \neq \gamma_\alpha$.

Case B: $\alpha = \beta$. In this case we follow an argument we have used in [20]. (See also Section 1 in [21]. Since $\bar{G}$ is unbounded in M, we let y be a non-standard limit point of $\bar{G}$. Note that since M is a model of KP, hence 'amenable', we get that $\bar{G} \cap (y+1)$ as well as $\bar{R}_i \cap y$ for $1 \leq i \leq n$ are 'in' M. (What we really mean is that for some $z \in M$, $\forall t \in M$, $t E x \leftrightarrow t \in \bar{G} \wedge t E y$. Similarly for $\bar{R}_i$.) Since every limit point of $\bar{G} \cap (y+1)$ is singular in M and $M \vDash \mathrm{KP}$, there is $z \in M$ such that $\bar{G} \cap (y+1)$, $\bar{R}_1 \cap y, \ldots, \bar{R}_2 \cap y$ belongs to L_z^M and $L_z^M \vDash$"every limit point of $\bar{G} \cap (y+1)$ is singular". Let H be the set of all elements of L_z^M first-order definable in $\langle L_z^M, E\rangle$ from y, $\bar{G} \cap y$, $\bar{R}_1 \cap y$, and from members of α. By Lemma 2 in [20] we get that $\langle H, E\rangle$ is well-founded. Hence it is isomorphic to a structure of the form $\langle L_\gamma, \in\rangle$. Let $h : \langle H, E\rangle \to \langle L_\gamma, \in\rangle$ be the collapsing homomorphism. Note that for $\delta < \alpha$, $h(\delta) = \delta$. Hence $\alpha \leq h(y)$ and that $h(\bar{G} \cap y) \cap \alpha = G$. Therefore,

$$\langle L_\gamma, \in\rangle \vDash \alpha \text{ is a limit point of } h(\bar{G} \cap (y+1)).$$

(Recall that G is unbounded in α.)

Similarly $h(\bar{R}_i \cap y) \cap L_\alpha = R_i$ for $1 \leq i \leq n$.

By construction of z and M we get $\langle L_\gamma, \in\rangle \vDash$"every limit point δ of $h(\bar{G} \cap (y+1))$ is either such that $\langle L_\delta, \in, h(\bar{R}_1 \cap y) \cap L_\delta, h(\bar{R}_2 \cap y) \cap \delta, \ldots\rangle$ is not admissible or δ is singular and the order type of $C_\delta \neq \gamma_\alpha$".

Hence we get

$$\langle L_\gamma, \in\rangle \vDash \alpha \text{ is singular and the order type of } C_\alpha \neq \gamma_\alpha.$$

But C_α in the sense of L_γ is the real C_α and we get a contradiction. □ Lemma 14.6 □ Subcase IIb.

Subcase IIc: Case II holds, i.e., α is singular, but Subcases IIa and IIb fail. Hence in this case $h(L_\alpha) = \alpha$ and $\mathrm{cf}(\alpha) = \omega$. By Theorem 11.1 we know that since $h(L_\alpha) = \alpha$ we have that either α is a limit cardinal or $\kappa < \alpha < \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$, and α is closed under the function d. For each of these two possibilities we shall need one of the following two lemmas.

Lemma 14.7 ($V = L$). *Let α be a limit cardinal. Let $\mathscr{A} = \langle L_\alpha, \in, R_1, \ldots, R_n\rangle$ be an admissible structure which is Σ_1-compact. (Note that $\mathscr{A}$ is closed under the power set operation $x \to \mathscr{P}(x)$). Then the expansion of $\mathscr{A}$ by the power set operation is admissible.*

Proof. Since α is a limit cardinal, the structure $\mathscr{A}$ expanded by the power set operation namely $\mathscr{B} = \langle L_\alpha, \in, R_1, \ldots, R_n, \mathscr{P}\rangle$ is amenable. So we just have to verify the Σ_1-bounding axiom. Let $\varphi(z, y)$ be a Δ_0-formula in the language of $\mathscr{L}$, and let $a \in L_\alpha$ be such that $\mathscr{L} \vDash \forall z \in a\, \exists y\, \varphi(z, y)$. We have to show that there exists a t such that

$$\mathscr{L} \models \forall z \in a\, \exists y \in t\, \varphi(z, y).$$

Consider the theory

$$T_{\mathscr{A}} \cup \{\text{“}P \text{ is the power set operation” (namely: } \forall x\, \forall y\, y = P(x) \leftrightarrow \forall z\, (z\, E\, y \leftrightarrow \forall t (t\, E\, z \to t\, E\, x))\}$$
$$\cup \{\text{“}\neg\exists t\, \forall z\, E\, c_a\, \exists y\, Et\, \varphi(z, y)\text{”}\}.$$

(Recall that c_z is the constant mean to denote a.) ($T_{\mathscr{A}}$ was defined for any admissible structure $\mathscr{A}$ above in Case IIa.)

T is clearly $\Sigma_1(\mathscr{A})$. Any $\mathscr{A}$-finite subset of it, T', has a model, namely $\mathscr{B}$ expanded by an interpretation of the constant c as some L_δ ($\delta < \alpha$) containing all x's such that c_x appears in T'. We assumed that $\mathscr{A}$ is Σ_1-compact, hence T has a model. $\mathscr{C} = \langle M, E, \bar{R}_1, \ldots, \bar{R}_n, \bar{P}, \ldots, \mathbf{c}\rangle$. Without loss of generality $L_\alpha \subseteq M$. Since α is a limit cardinal, the interpretation of $\bar{P}$, restricted to L_α, must be the real power set operation. Since L_α is closed under the real power set. Let $\mathbf{c}$ be the interpretation of the constant c. Note that $x\, E\, \mathbf{c}$ for $x \in L_\alpha$. Also $R_i = L_\alpha \cap \bar{R}_i$. Hence it follows by assumption that

$$\mathscr{C} \models \forall z\, E\, a\, \exists y\, E\mathbf{c}\, \varphi(x, z),$$

which clearly contradicts $\mathscr{C} \models T$. $\square$ Lemma 14.7

The next lemma is similar in character.

Lemma 14.8. *Let $\kappa < \alpha < \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$. Assume that α is closed under the function d. Let $\mathscr{A}$ be an admissible structure of the form $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$, which is Σ_1-compact. Then the expansion of $\mathscr{A}$ by the function d, namely $\mathscr{B} = \langle L_\alpha, \in, R_1, \ldots, R_n, d\rangle$ is admissible.*

Proof. We first show that $\mathscr{B}$ is amenable. We have to show that for $\gamma < \alpha$, $d \restriction \gamma \in L_\alpha$. If there exists a last cardinal in L_α, then by Lemma 10.5, $\langle L_\alpha, \in\rangle$ has (DP). If not then by Lemma 10.6, $\langle L_\alpha, \in\rangle$ can be expanded to a structure having (DP). Over such a possibly expanded structure d must be Σ_1 (by Lemma 12.2). Since d is a total function (α is closed under d!), it is Δ_1, hence it follows that for $\gamma < \alpha$, $d \restriction \gamma \in L_\alpha$. Now we have to show Σ-reflection for $\mathscr{B}$. Let $\varphi(z, y)$ be Δ_0-formula in the language of d such that for some $a \in L_\alpha$, $\mathscr{L} \models \forall z \in a\, \exists y\, \varphi(z, y)$ but (heading for contradiction)

$$\mathscr{L} \models \neg\exists t\, \forall z \in a\, \exists y \in t\, \varphi(z, y).$$

Recall that by Lemma 11.4 for all $\kappa < \beta < \kappa^+$ there exists a sentence of $L_{\infty\omega}$, ψ_β, which is primitive recursive in β, and ψ_β is consistent, all of its models have the form $\langle L_{d(\beta)}, \in, \ldots\rangle$ and it has a finite similarity type. Expand the language of T by introducing an extra symbol for every symbol appearing in the language of

ψ_β's, but with arity which is larger by one from the original arity. Thus if R is a ternary relation $\mathbf{R}$ introduced is quadruple relation etc. For $\beta<\alpha$ let $\Phi_\beta(x)$ be the formula expressing "ψ_β holds in $\langle L_x, E, \mathbf{R}_{c_\beta}, \ldots\rangle$ where $\mathbf{R}_{c_\beta}$ etc. is the c_β section of $\mathbf{R}$, namely if r is $n+1$-ary it is the set of n-tuples $\boldsymbol{x}$ such that $\langle c_\beta, \boldsymbol{x}\rangle \in \mathbf{R}$, where $\mathbf{R}$ is the relation corresponding to R.

Consider the following theory

$$T = T_{\mathscr{A}} \cup \{\forall x\ (x = \mathbf{d}(\beta) \rightarrow \Phi_\beta(x) \mid \beta < \alpha\}$$
$$\cup \{\text{“}\neg\exists t\ \forall z\ E\, c_a\ \exists y\ Et\, \varphi(z, y)\text{”}\}.$$

(Recall that c_x is the constant of $T_{\mathscr{A}}$ mean to denote a.)

T is $\mathscr{A}$-r.e. because the function $\beta \rightarrow \psi_\beta$ is $\mathscr{A}$-r.e. Hence $\beta \rightarrow \Phi_\beta$ is $\mathscr{A}$-r.e.

Like in the previous lemma every $\mathscr{A}$-finite subset of T has a model. By Σ_1-compactness of $\mathscr{A}$, T has a model. Again if $\mathscr{C} = \langle M, E, \ldots, \mathbf{c}, \mathbf{d}, \mathbf{R}, \ldots\rangle$ is such a model we can assume $L_\alpha \subseteq M$ and hence $x\, E\, \mathbf{c}$ for all $x \in L_\alpha$.

If $\mathscr{C} \models x = \mathbf{d}(\beta)$ for some $\beta < \alpha$, then $\langle L_x^{\mathscr{C}}, E\rangle$ can be easily expanded to a model of ψ_β. Hence by the construction of ψ_β in Lemma 11.4, $\langle L_x^{\mathscr{C}}, E\rangle$ must be isomorphic to $\langle L_{d(\beta)}, \in\rangle$. Since $L_{d(\beta)}$ is in $\mathscr{C}$, we must have $x = d(\beta)$. Thus $\mathbf{d} \restriction L_\alpha = d \restriction L_\alpha$. Therefore the interpretation of the constant c, $\mathbf{c}$, is easily verified to satisfy $\mathscr{C} \models \forall z\, E\, c_a\, \exists y\, E\mathbf{c}\, \varphi(z, y)$ which clearly contradicts $\mathscr{C} \models T$. □ Lemma 14.8

To conclude the proof we need to use a Theorem by L. Harrington [13] which very elegantly replaced the previous arguments we had for the case we are handling. In order to state Harrington's theorem we need

Definition 14.2. An ordinag α is called amenably admissible iff there is *no* function $f: \kappa \rightarrow \alpha$ $(\kappa < \alpha)$ whose range is cofinal in α, and such that for all $\rho < \kappa$, $h \restriction \rho \in L_\alpha$.

Note. If α is singular in L, then α is not amenably admissible, since a function enumerating C_α monotonically is a witness to α not being amenably admissible.

Theorem 14.9 (Harrington [13]). *Let $\mathscr{A}$ be an admissible structure of the form $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$ such that α is not amenably admissible. Let Q be a relation on L_α which is not $\mathscr{A}$-r.e. Then for some relation T, $\langle L_\alpha, \in, R_1, \ldots, R_n, T\rangle$ is admissible, but $\langle L_\alpha, \in, R_1, \ldots, R_n, T, Q\rangle$ is not admissible.* (In [13] Theorem 14.9 was formulated for the case $\mathscr{A} = \langle L_\alpha, \in\rangle$, but the proof applies as well for expanded structures.)

Lemma 14.10 $(V = L)$. *Let α be singular $\mathscr{A}$ an admissible structure of the form $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$*

(a) *If α is a limit cardinal, then if $\mathscr{P} \restriction L_\alpha$ is not $\mathscr{A}$-r.e., then $\mathscr{A}$ is not stably Σ_1-compact.*

(b) *If $\kappa<\alpha<\kappa^+$, $\mathrm{cf}(\kappa)=\omega$ and α closed under d, then if d is not $\mathcal{A}$-r.e., then $\mathcal{A}$ is not stably Σ_1-compact.*

Proof. (a) If $\mathcal{P}\restriction L_\alpha$ is not $\mathcal{A}$-r.e. we can use Theorem 14.9 (since α is singular in L, it is not amenably admissible) and get T such that $\mathcal{B}=\langle L_\alpha, \in, R_1, \ldots, R_n, T\rangle$ is admissible, but $\langle L_\alpha, \in, R_1, \ldots, R_n, T, \mathcal{P}\restriction L_\alpha\rangle$ is not admissible. By Lemma 14.7, $\mathcal{B}$ is not Σ_1-compact. Hence some admissible expansion of $\mathcal{A}$ yields a structure which is not Σ_1-compact. Hence $\mathcal{A}$ is not stably Σ_1-compact.

(b) The proof is like in (a), replace $\mathcal{P}\restriction L_\alpha$, by $d\restriction L_\alpha$ and Lemma 14.7 by Lemma 14.8. □ Lemma 14.10

So we conclude from Lemma 14.10 that the only way $\mathcal{A}=\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$ can be stably Σ_1-compact (α singular) is that $\mathrm{cf}(\alpha)=\omega$ and either α is a limit cardinal and $\mathcal{P}\restriction L_\alpha$ is $\mathcal{A}$-r.e. or $\kappa<\alpha<\kappa^+$ and α is closed under the function d, which is $\mathcal{A}$-r.e. We conclude that in our case either (II) or (III) holds. □ Case IIc □ Case II □ Theorem 14.1 (modulo Theorem 14.4)

15. Proof of 14.4 by forcing[4]

We start proving Theorem 14.4. So let the admissible structure $\mathcal{A}=\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$ be given, such that α is singular in L.

Recall (see [14]) that ρ_1, the Σ_1-projectum of $\mathcal{A}$ is the minimal ordinal $\leqslant\alpha$ such that there exists an $\Sigma_1(\mathcal{A})$ one-to-one map of L_α into ρ_1. ρ_1 can also be characterized as the minimal ordinal such that some $\Sigma_1(\mathcal{A})$-subset of ρ, A, is not in L_α. (Hence for $\beta<\alpha$, every $\Sigma_1(\mathcal{A})$-subset of β is in L_α.) Note also that $\mathrm{cf}(\rho_1)=\mathrm{cf}(\alpha)$ and that if $\rho_1<\alpha$, then ρ_1 is the last L_α-cardinal.

Let $C=C_\alpha$ be the canonical closed unbounded subset of α, having order type $<\alpha$, and such that if γ is a limit point of C_α, $C\cap\gamma\in L_\alpha$. Let $\gamma=\gamma_\alpha$ be the order type of C. We have to introduce a closed unbounded subset of α, G, such that $\langle L_\alpha, R_1, \ldots, R_n, G\rangle$ is admissible, and for every limit point of G, δ, either $\langle L_\delta, \in, R_1\cap L_\delta, \ldots, R_n\cap L_\delta\rangle$ is not admissible or δ is singular in L_α and the order type of the canonical C_δ is different from γ.

G will be introduced by forcing over $\mathcal{A}$. Our forcing notion will be the obvious: approximations to G which lie in L_α, namely:

Definition 15.1. Let $\langle\mathcal{P}, \leqslant\rangle$ be the set of all closed bounded subsets of α, g, such hat g has a last element which is not a limit point of g, μ, and such that for every limit point of g, δ, either $\langle L_\delta, \in, R_1\cap L_\delta, \ldots, R_n\cap L_\delta\rangle$ is not admissible or δ is singular in L_μ and the order type of $C_\delta\neq\gamma$, $\mathcal{P}$ is partially ordered by $g\leq h$ if g is an initial segment of h. ($\langle\mathcal{P}, \leq\rangle$ is clearly $\Delta_1(\mathcal{A})$.)

[4] We acknowledge with thanks simplificiations in the proof suggested by the referee.

Our objective is to find a generic enough filter $\mathcal{G}$ in $\langle \mathcal{P}, \leqslant \rangle$ such that if $G = \bigcup_{g \in \mathcal{G}} g$ we have that $\langle L_\alpha, \in, R_1, \ldots, R_n, G \rangle$ is admissible. The fact that G will be unbounded in α will follow from the genericity of $\mathcal{G}$ because for any condition in $\mathcal{P}$, g, and for any $\delta < \alpha$, g can be extended to a condition containing some ordinal $> \delta$.

Let us introduce some notations. If $\mathcal{G}$ is a filter in $\mathcal{P}$, we denote by $G(\mathcal{G})$ the corresponding closed subset of α, i.e., $\bigcup_{g \in \mathcal{G}} g$. We say that $\mathcal{G}$ is unbounded if $G(\mathcal{G})$ is unbounded. $\mathcal{A}(\mathcal{G})$ is the structure $\mathcal{A}$ expanded by $G(\mathcal{G})$, and $\mathcal{A}(\mathcal{G}) \restriction \delta$ is $\langle L_\delta, \in, R_1 \cap L_\delta, \ldots, R_n \cap L_\delta, G(\mathcal{G}) \cap L_\delta \rangle$. It also is clear that if $G(\mathcal{G})$ is unbounded, then for all δ there is $g \in \mathcal{G}$ such that $\delta \cap G(\mathcal{G}) = g \cap \delta$. The following lemma should be now obvious.

Lemma 15.1. *Let $\mathcal{G}$ be an unbounded filter over $\langle \mathcal{P}, \leqslant \rangle$. Let $x \in L_\alpha$ and $\varphi(x)$ a Σ_1-formula in the language of $\mathcal{A}(\mathcal{G})$. Then $\mathcal{A}(\mathcal{G}) \models \varphi(x)$ iff for some $\delta < \alpha$, $\mathcal{A}(\mathcal{G}) \restriction \delta \models \varphi(x)$, hence for some $g \in \mathcal{G}$,*

$$\langle L_\delta, \in, R_1 \restriction L_\delta, \ldots, R_n \restriction L_\delta, g \cap L_\delta \rangle \models \varphi(x).$$

Definition 15.2. An unbounded filter $\mathcal{G}$ over $\langle \mathcal{P}, \leqslant \rangle$ is called generic if for every Δ_0-formula $\varphi(y, z)$ (with parameters) in the language of $\mathcal{A}(\mathcal{G})$ and for every $x \in L_\alpha$ for some $g \in G$ either

(a) for some $y \in x$, for no extension h of g, no $\delta < \alpha$, and $z \in L_\delta$ we have

$$\langle L_\delta, \in, R_1 \restriction L_\delta, \ldots, h \rangle \models \exists z\, \varphi(y, z),$$

or (b) for some $\delta < \alpha$ we have

$$\langle L_\delta, \in, R_1 \restriction L_\delta, \ldots, g \rangle \models \forall y \in x\, \exists z\, \varphi(y, z).$$

Remark. In the definition above one could replace "for every $x \in L_\alpha$" by the seemingly weaker assumption "for every L_α-regular ordinal x". The reason is that every element of L_α is the image of some ordinal under a function that lies in L_α. Hence we can restrict the definition to "x's which are ordinals". The reason we can restrict it to "an L_α-regular ordinal" is because if $\mathcal{G}$ is not generic, we let ρ be the minimal ordinal for which Definition 15.2 fails. We let $\varphi(y, z)$ be the Δ_0-formula witnessing it. We claim that ρ must be regular in L_α. Otherwise let $h : \kappa \to \rho$ be a function, $h \in L_\alpha$, range(h) cofinal in ρ. Consider the Δ_0-formula (which uses h as parameter)

$$\psi(t, \gamma) \leftrightarrow \text{“}h \in L_\gamma \wedge \langle L_\gamma, R_1 \restriction L_\gamma, \ldots, R_n \restriction L_\gamma, G \cap \gamma \rangle \models \forall y < h(t)\, \exists z\, \varphi(y, z)\text{”}.$$

By minimality of ρ, $\forall t \in \kappa\ \exists \gamma\, \psi(t, \gamma)$. But since $\kappa < \rho$ we get that there exists δ such that

$$\langle L_\delta, \in, R_1 \restriction L_\delta, \ldots, G \cap \delta \rangle \models \forall t \in \kappa\ \exists \gamma\, \psi(t, \gamma),$$

Sh:144

but since h is cofinal in ρ we get

$$\langle L_\delta, \in, R_1 \cap L_\delta, \ldots, G \cap \delta\rangle \models \forall z \in \rho\, \exists y\, \varphi(y, z),$$

which is a contradiction. $\square$ Remark.

So when constructing our generic filter we have to fulfil several assignments.

Definition 15.3. (a) An assignment is a pair $\langle \mu, \varphi\rangle$ where $\mu < \alpha$ and $\varphi(y, z)$ is Δ_0 in the language containing symbols for $\in, R_1, \ldots, R_n, G$ (φ may contain parameters from L_α.)

(b) $g \in \mathcal{P}$ fulfills the assignment $\langle \mu, \varphi\rangle$ if (a) or (b) of Definition 15.2 holds for g (where in case (b) we can take $\delta = \sup g$). (Note that if one δ works for g, any larger δ works. Note also that fulfilling on a given assignment is a Π_1-statement about g.)

By the remark above it is enough to fulfil those assignments for which μ is L_α regular.

Our basic lemma is the following which uses arguments due to Harrington [13]. They replaced much more complicated arguments we had originally used.

Lemma 15.2. *Let κ be an L_α-regular cardinal or an L_α-cardinal which is smaller than ρ_1 (the Σ_1-projectum of $\mathcal{A}$). Let $H = \{\langle \mu_\rho, \varphi_\rho\rangle \mid \rho < \eta\}$ be a sequence of assignments, $H \in L_\alpha$ where $\eta < \kappa$ and each $\mu_\rho \leqslant \kappa$ for $\rho < \eta$. Let $g \in \mathcal{P}$. Then some extension of g, h, fulfills all the assignments in H.*

Proof. We define by recursion in L_α (hence by a Σ_1 way) a sequence $\langle g_\delta \mid \delta < \chi\rangle$ (where χ is some ordinal $\leqslant \alpha$. We shall later use that $\chi = \alpha$), $g_\delta \in \mathcal{P}$, for $\delta < \delta'$ $g_\delta \leqslant g_{\delta'}$ and a function $f_\delta : \eta \to \kappa + 1$ such that for $\delta < \delta'$ and for $\rho < \eta$, $f_\delta(\rho) \leqslant f_{\delta'}(\rho)$. To get started we let g_0 be any extension of g containing an element bigger than κ. We let $f_0(\rho) = 0$ for all $\rho < \eta$.

For δ limit we try to define a common extension of all g_β ($\beta < \delta$) namely we consider $h = \bigcup_{\beta<\delta} g_\beta$, and we try to set $g_\delta = h \cup \{\sup(h)\} \cup \{$some ordinal μ such that $\sup(h)$ is singular in $L_\mu\}$. The problem may be that $g_\delta \notin \mathcal{P}$. (The only reason for that is that either $\sup h$ is regular in L_α or that the order type of $C_{\sup(h)} = \gamma$ and $\langle L_{\sup(h)}, R_1 \restriction L_{\sup(h)}, \ldots\rangle$ is admissible.) If this happens we stop the construction, i.e., we let $\chi = \delta$. If $g_\delta \in \mathcal{P}$ we define $f_\delta(\rho) = \sup_{\beta<\delta} f_\beta(\rho)$, for $\rho < \eta$.

For $\delta = \beta + 1$, we consider the following two questions.

(a) Is $g_\beta \in L_\beta$ and $L_\beta \models$ every limit point of g_β is singular?

(b) Does there exist $\rho < \eta$ such that $f_\beta(\rho) < \mu_\rho$ and $z \in L_\beta$, $g_\beta \leqslant h \in L_\beta$ such that

$$\langle L_\beta, \in, R_1 \cap L_\beta, \ldots, R_n \cap L_\beta, h\rangle \models \varphi_\beta(f_\beta(\rho), z).$$

If the answer is "No" to either of these questions, then we let $g_{\beta+1} = g_\beta$, $f_{\beta+1} = f_\beta$. If the answer is "Yes" to both (a) and (b) we pick the minimal ρ_0 witnessing a "Yes" answer to (b). For ρ_0 we pick the minimal $h \in L_\beta$, witnessing a "Yes" answer for (b). We define $g_{\beta+1} = h \cup \{\beta\}$. (Note that $g_{\beta+1} \in \mathcal{P}$.) We define $f_{\beta+1}(\rho_0) = f_\beta(\rho_0) + 1$ and $f_{\beta+1}(\rho) = f_\beta(\rho)$ for $\rho \neq \rho_0$.

This definition clearly gives a $\Sigma_1(\mathcal{A})$-function on some ordinal $\leq \alpha$. Note that since $f_\delta(\beta)$ is increased by 1 at most at each step, we get that for fixed $\{f_\delta(\rho) \mid \delta < \chi\}$ is an initial segment of μ_ρ.

Note also that since we increased g_β just in case we already knew in L that every limit point of g_β is singular, hence if for some limit δ $\langle g_\beta \mid \beta < \delta \rangle$ is not eventually fixed, then the inductive definition of $\langle g_\beta \mid \beta < \delta \rangle$, $\langle f_\beta \mid \beta < \delta \rangle$, can be done inside $\langle L_\delta, \in, R_1 \cap L_\delta, \dots \rangle$ and $\sup(\bigcup_{\beta<\delta} g_\beta) = \delta$.

We claim that the $\langle g_\beta \mid \beta < \chi \rangle$ are defined for all $\beta < \alpha$ (namely $\chi = \alpha$) and that they are eventually constant. This follows from:

Lemma 15.3. *Let $\delta \leq \alpha$ be such that $\mathcal{A} \restriction \delta = \langle L_\delta, \in, R_1 \cap L_\delta, \dots, R_n \cap L_\delta \rangle$ is admissible $\delta > \kappa$. Then $\langle g_\beta \mid \beta < \delta \rangle$ (if they are defined) are eventually constant.*

Our claim follows from Lemma 15.3 because if $\chi < \alpha$, then $\langle g_\beta \mid \beta < \chi \rangle$ are not eventually constant but then $\chi = \sup(\bigcup_{\beta<\chi} g_\beta)$, hence by the lemma $\mathcal{A} \restriction \chi$ is not admissible, hence $\bigcup_{\beta<\alpha} g_\beta \cup \{\chi\} \cup \{\chi+1\} \in \mathcal{P}$ and our inductive definition goes behind χ.

Proof of Lemma 15.3. Assume that $\langle g_\beta \mid \beta < \delta \rangle$ are not eventually constant. (Note that this implies that $g_\beta \in L_\delta$ for $\beta < \delta$.) Let

$$A = \{\langle \rho, \mu \rangle \mid \text{for some } \beta < \delta, f_\beta(\rho) = \mu\}.$$

$A \subseteq \eta \times (\kappa + 1)$, A is clearly $\Sigma_1(\mathcal{A} \restriction \delta)$. (Recall that the sequence $\langle f_\beta \mid \beta < \delta \rangle$ is $\Sigma_1(\mathcal{A} \restriction \delta)$.) Note also that for fixed ρ, $\{\mu \mid \langle \rho, \mu \rangle \in A\}$ is an initial segment of $\kappa + 1$.

Claim. $A \in L_\delta$.

Assume the claim, and use the admissibility of $\mathcal{A} \restriction \delta$ to get $\beta_0 < \delta$ such that for $(\rho, \mu) \in A$, $\exists \beta < \beta_0$ such that $f_\beta(\rho) = \mu$. It follows that for all $\beta < \delta$, $\rho < \eta$, $f_\beta(\rho) \leq f_{\beta_0}(\rho)$. Since for fixed ρ, $f_\beta(\rho)$ are increasing as β increases, $f_\beta = f_{\beta_0}$ for $\beta > \beta_0$, but then by construction $g_\beta = g_{\beta_0}$ for $\beta > \beta_0$ which is clearly a contradiction. □ Lemma 15.3 modulo the Claim

Proof of the Claim. Since A is $\Sigma_1(\mathcal{A} \restriction L_\delta)$ the Claim is clear for the case $\kappa < \rho_1$, In case κ is L_α regular note that:

$$B = \{\rho \mid \rho < \eta, \{\rho\} \times \kappa \subseteq A\} \text{ is } \Sigma_1(\mathcal{A} \restriction L_\delta) \text{ as well as } C = \{\rho \mid (\rho, \kappa) \in A\}.$$

We claim that B and C are in L_δ. This follows from

Lemma 15.4. *Let κ be a regular L_δ-cardinal, D a $\Sigma_1(\mathcal{A} \restriction \delta)$-subset of $\eta \times \kappa$ where $\eta < \kappa$. Then either $D \in L_\delta$ or for some $\rho < \eta$, $\{\mu \mid \langle \rho, \mu \rangle \in D\}$ is unbounded in κ.*

Proof. In L_δ one can enumerate monotonically the β's such that in L_β one 'generates' a new member of D (i.e., in L_β one finds for the first time evidence for the Σ_1 fact that some element is a member of D). If the β's thus enumerated are bounded in δ, then D is $\Delta_0(A \restriction \delta)$, hence $D \in L_\delta$. Otherwise one can enumerate at least $\kappa+1$ such β's $\langle \beta_\mu \mid \mu < \kappa \rangle$. Hence D contains a subset, D', of cardinality κ which is $\Delta_0(\mathscr{A} \restriction \delta)$. Hence $D' \in L_\delta$. Since $\eta < \kappa$, and κ is regular in L_δ we must have some $\rho \in L_\delta$ such that $\{\mu \mid \langle \rho, \mu \rangle \in D'\}$ has order type κ. The same ρ works for D. $\square$ Lemma 15.4

It follows from Lemma 15.4 that any subset of η, (consider it as a subset of $\eta \times \{1\} \subseteq \eta \times \kappa$) which is $\Sigma_1(\mathscr{A} \restriction \delta)$ is in L_δ (we cannot have κ-many values with a given ρ). In particular C and B are in L_δ. Consider $D = A - C \times \{\kappa\} - B \times \kappa$. D is $\Sigma_1(\mathscr{A} \restriction \delta)$. For every $\rho < \eta$, $\{\mu \mid \langle \rho, \mu \rangle \in D\}$ is bounded in κ (it is an initial segment of $\kappa+1$ and it is neither κ nor $\kappa+1$ since otherwise we have $\rho \in C$ or $\rho \in B$ respectively).

Hence by Lemma 15.4, $D \in L_\delta$, but $A = D \cup C \times \{\kappa\} \cup B \times \kappa$. Hence $A \in L_\delta$. $\square$ Claim.

We resume the proof of Lemma 15.2. We have proved that the sequence $\langle g_\beta \mid \beta < \alpha \rangle$ must be eventually constant. Let h be this constant, say $h = g_\beta$. h is the required extension of g, because let $\langle \mu_\rho, \varphi_\rho \rangle$ be one of the assignments in H. We claim that h fulfills this assignment. If $f_\beta(\rho) = \mu_\rho$, then it means that for every $\xi < \mu_\rho$ we had some g_{β_ξ} and z such that

$$\langle L_{\beta_\xi}, \in, R_1 \cap L_{\beta_\xi}, \ldots, g_{\beta_\xi} \rangle \models \varphi_\rho(\xi, z).$$

Hence since $\delta = \sup(h) \geq \beta_\xi$ and $h \geq g_{\beta_\xi}$, we have

$$\langle L_\delta, \in, R_1 \cap L_{\beta_\xi}, \ldots, h \rangle \models \varphi_\rho(\xi, z)$$

and the assignment is fulfilled.

If on the other hand $f_\beta(\rho) < \mu_\rho$, then no extension of h, h', no $\delta < \alpha$ and no z satisfy

$$\langle L_\delta, \in, R_1 \mid L_\delta, \ldots, h' \rangle \models \varphi(f_\beta(\rho), z).$$

Because if such h', z, δ exist, then z belongs to L_δ. We can assume that $\delta > \beta$ and δ is large enough so that every limit point of $h = g_\beta$ is singular in L_δ. Hence when we arrived to $\delta+1$ in our definition of $\langle g_\beta \mid \beta < \alpha \rangle$, we get answer "Yes" to both our questions, hence we must have that $g_{\delta+1} \neq g_\beta$, (we added at least δ to g_β). Hence we got a contradiction to the assumption that h is the eventually constant value of $\langle g_\beta \mid \beta < \alpha \rangle$. $\square$ Lemma 15.2

For proving Theorem 14.4 we need one more fact, which follows from $\square$ (the version in Theorem 14.3). This fact, or actually much more than it, is essentially proved in [24] but we shall include a proof for completeness.

Lemma 15.5 ($V = L$). *Let η be a regular cardinal. To each α having cofinality η one can assign a closed unbounded subset D_α of order type η such that for all $\beta < \alpha$, $D_\alpha \cap \beta \in L_\alpha$. In particular in our proof we get also $D_\alpha \subseteq C_\alpha$.*

Proof. The proof is by induction on α.

For $\alpha = \eta$ (which is the minimal ordinal of cofinality η) we take $D_\eta = \eta$. For general α consider C_α. The order type of C_α is γ, where $\gamma < \alpha$. Clearly $\mathrm{cf}(\gamma) = \mathrm{cf}(\alpha) = \eta$. Hence D_γ is already defined. We let

$$D_\alpha = \{\beta \mid \beta \in C_\alpha, \quad \text{the order type of } C_\alpha \cap \beta \quad \text{is in } D_\gamma\}.$$

D_α is easily verified to satisfy the requirement. (Note that for $\beta \in C_\alpha$, $C_\alpha \cap \beta \in L_\alpha$, hence if $C_\alpha \cap \beta$ has order type $\mu < \gamma$, $D_\gamma \cap \mu \in L_\gamma \subseteq L_\alpha$. But then $D_\alpha \cap \beta = \{\delta \mid \delta \in C_\alpha \cap \beta$, the order type of $C_\alpha \cap \beta \cap \delta \in D_\gamma \cap \mu\}$. Hence $D_\alpha \cap \beta \in L_\alpha$.) □ Lemma 15.5

Another fact we shall need is definition by Σ_2-recursion, provided we want a sequence of length less than the cofinality of α.

Lemma 15.6. *Let $\mathscr{A}$ be as above, where $\mathrm{cf}(\alpha) = \eta$. Let H be a $\Sigma_2(\mathscr{A})$-function. Let $\beta < \eta$. Then there exists in L_α a sequence $\langle a_\delta \mid \delta < \beta\rangle$ such that for all δ, $a_\delta = H(\delta, \langle a_\mu \mid \mu < \delta\rangle)$.*

Proof. We prove the lemma by induction on β. Non-limit β is easy. For limit β, $\langle a_\delta \mid \delta < \mu\rangle$ exists for $\mu < \beta$. Hence the sequence $\langle a_\delta \mid \delta < \beta\rangle$ is defined. The only problem is showing that $\langle a_\delta \mid \delta < \beta\rangle \in L_\alpha$. Let H be defined by $\exists z\, \varphi(z, x, y, t)$ where φ is Π_1.

By induction we define χ_ρ for $\rho < \beta \times \omega$. χ_0 be large enough so that for all $\mu < \beta$, $\langle a_\delta \mid \delta < \mu\rangle \in L_{\chi_0}$, and in L_{χ_0} we can find z_μ such that

$$\mathscr{A} \models \varphi(z, a_\mu, \mu, \langle a_\delta \mid \delta < \mu\rangle).$$

χ_0 exists because $\mathrm{cf}(\alpha) > \beta$.

For ρ limit we let χ_ρ be $\sup_{\mu < \rho} \chi_\mu$. (Note $\chi_\rho < \alpha$ because $\mathrm{cf}(\alpha) > \beta \times \omega$.)

For $\rho = \beta \times n + \delta + 1$ (where $\delta < \beta$) we pick χ_ρ as an ordinal bigger than $\chi_{\rho-1}$ such that $\mathscr{A} \restriction \chi_\rho$ satisfies the following Π_2-sentence

$$\psi \models \forall x\, (x \neq a_\delta \rightarrow \neg \exists z\, \varphi(z, x, \delta, \langle a_\rho \mid \rho < \delta\rangle).$$

χ_ρ exists by Π_2-reflection since clearly $\mathscr{A} \models \psi$. Let $\chi = \sup_{\rho < \beta \times \omega} \chi_\rho$. Again $\chi < \alpha$.

If we try to use in $\mathscr{A} \restriction \chi$ the inductive definition like in $\mathscr{A}$ we get the same sequence $\langle a_\rho \mid \rho < \beta\rangle$. Hence the sequence $\langle a_\rho \mid \rho < \beta\rangle$ is in L_α □ Lemma 15.6

Proof of Theorem 14.4. We shall prove Theorem 14.4 by proving the existence of a generic filter in $\mathscr{P}$. Recall $\mathrm{cf}(\alpha) = \eta$.

Fix $f: L_\alpha \to \rho_1$ which is injective and $\Sigma_1(\mathscr{A})$. By Lemma 15.5, fix $D \subseteq \alpha$ such that D is closed unbounded in α, the order type of D is η and for $\beta < \alpha$, $D \cap \beta \in L_\alpha$. Similarly since $\mathrm{cf}(\rho_1) = \mathrm{cf}(\alpha) = \eta$ and $\rho_1 \leqslant \alpha$ fix $E \subseteq \rho_1$, E closed unbounded in ρ_1, the order type of E is η and for $\beta < \rho_1$, $E \cap \beta \in L_{\rho_1} \subseteq L_\alpha$. Let $\{\gamma_\beta \mid \beta < \eta\}$ be an increasing enumeration of the elements of E. We define by induction an increasing sequence $\langle g_\rho \mid \rho < \eta \rangle$ of elements of $\mathscr{P}$. $g_0 = \{\eta + 1\}$. If g_β is defined, we pick $\delta > \sup\{g_\beta\}$, $\delta \in D$. Let $h_\beta = g_\beta \cup \{\delta\}$. Let H_β be the set of all assignments of the form $\langle \mu, \varphi \rangle$ where, if a is the set of parameters appearing in φ, then $f(a) < \gamma_\beta$ and either $\mu < \gamma_\beta$ or in case ρ_1 is L_α-regular, $\mu = \rho_1$. Clearly, since $\{x \mid f(x) < \gamma_\beta\}$ is $\Sigma_1(\mathscr{A})$ and $\gamma_\beta < \rho_1$, $\{x \mid f(x) < \gamma_\beta\} \in L_\alpha$, hence it is easily verified that $H \in L_\alpha$. Note that the L_α-cardinality of H is at most γ_β. Hence by Lemma 15.2, there exists g which extends h_β and which fulfills all the assignments in H. We let $g_{\beta+1}$ be the minimal such g, (in the canonical well-ordering of L). At limit stage we take $g_\beta = \bigcup_{\delta<\beta} g_\delta \cup \{\mu, \chi\}$ where $\mu = \sup(\bigcup_{\delta<\beta} g_\delta)$ and χ is the first ordinal in which μ is singular. (Note that since μ is a limit point of D, μ is singular in L_α, because $\mu < \alpha$, since $\beta < \eta$ and $\mathrm{cf}(\mu) = \mathrm{cf}(\beta)$ while $\mathrm{cf}(\alpha) = \eta$.)

The sequence $\langle g_\delta \mid \delta < \beta \rangle$ is in L_α, because it can be defined by Σ_2-recursion from $D \cap \mu$ and $E \cap \gamma_\beta$. (We use lemma 15.6.) The following observations explain why the recursive definition of $\langle g_\delta \mid \delta < \beta \rangle$ is a *Σ_2-definition*: from $D \cap \mu$ and $E \cap \gamma_\beta$

(A) The condition h_δ described above is Δ_0 in g_β and $D \cap \mu$.

(B) The set H_δ, as a set, is Σ_1 in δ and $E \cap \gamma_\beta$.

(C) $g_{\delta+1}$ is the minimal condition which extends h and which satisfies all assignments in H_δ. The statement $g_{\delta+1}$ is an extension of h is Δ_1, and to say that it satisfies all assignment in H_δ, is Π_1 in δ and $E \cap \beta$ (since H_δ is Σ_1). To express the fact that for every element of L_α, smaller than g_δ, it is either not a condition or does not satisfy all assignments in H_δ is Σ_1 in the above parameters. Hence $g_{\delta+1}$ is Σ_2-definable from g_δ, δ, and $D \cap \mu$, $E \cap \gamma_\beta$.

(D) For limit δ, g_δ is clearly Σ_1-definable from $\langle g_\mu \mid \mu < \delta \rangle$.

Hence our recursive definition is Σ_2 and $\langle g_\delta \mid \delta < \beta \rangle \in L_\alpha$. Therefore $g_\beta \in L_\alpha$ and clearly $g_\beta \in \mathscr{P}$.

We let $\mathscr{G}$ be the filter in $\mathscr{P}$ generated by $\langle g_\delta \mid \delta < \eta \rangle$. $\mathscr{G}$ is unbounded ($G(\mathscr{G})$ contains η points of D) and every assignment of the form $\langle \mu, \varphi \rangle$ where μ is L_α-regular is fulfilled by $\mathscr{G}$, because $\mu \leqslant \rho_1$ and E is cofinal in ρ_1, hence for some $\beta < \eta$ if a is the set of parameters of φ, $f(a) < \gamma_\beta$ and either $\mu < \gamma_\beta$ or ρ_1 is regular and $\mu = \rho_1$. Hence $\langle \mu, \varphi \rangle \in H_\beta$ and therefore $\langle \mu, \varphi \rangle$ is fulfilled by $\mathscr{G}$. □ Theorem 14.4

16. Miscellaneous

The problem of characterizing structures having Σ_1-completeness is similar to characterizing Σ_1-compactness. They are of course connected.

Lemma 16.1. *If a structure of the form $\mathscr{A}=\langle L_\alpha, \in, R_1 \cdots R_n\rangle$ is Σ_1-complete then it is Σ_1-compact.*

(The converse does not hold because we have many α's, $\omega_1<\alpha<\omega_2$, such that $\langle L_\alpha, \in\rangle$ is Σ_1-compact, but by Theorem 12.3, $\langle L_\alpha, \in\rangle$ is not validity admissible, hence not Σ_1-complete.)

Proof. Assume that $\mathscr{A}$ is not Σ_1-compact. Let H be a $\Sigma_1(\mathscr{A})$-theory witnessing it. Let $\varphi(x)$ define it. Consider the following theory T. T has a constant for each $x\in L_\alpha$, c_x, and a relation symbols E (for $\in$), $\mathbf{R}_i$ for R_i $(1<i\leq n)$ also have enough extra relations so that one can define (uniformally in T) for every x ordinal a model of

$$H_x=\{\psi \mid \psi\in L_x, L_x\models\varphi(\psi)\}.$$

T also expresses KP+ $V=L$ and "$\forall z(z\,E\,c_x\leftrightarrow \bigvee_{y\in x} z=c_y)$" for $x\in L_\alpha$.

$$\mathbf{R}_i(c_x)\in T \quad \text{if } R_i(x) \quad \text{holds,}$$

$$\mathbf{R}_i(c_x)\in T \quad \text{if } {\sim}R_i(x) \quad \text{holds.}$$

Any model of T must have the form

$$\langle L_\alpha, \in, R_1, \ldots, R_n, \ldots\rangle.$$

(It can not be an end extension of $\langle L_\alpha, \in, R_1, \ldots, R_n, \ldots\rangle$ otherwise we get a model for H.)

T is clearly Σ_1. If the consequences of T are $\Sigma_1(\mathscr{A})$, we get that the truth of sentences in $\mathscr{A}$ are $\Sigma_1(\mathscr{A})$ which is clearly a contradiction. □ Lemma 16.1

Note that like the problem of compactness, we cannot expect Σ_1-completeness to imply any kind of decomposability, since if $\kappa>\omega$ is regular and $\langle L_\alpha, \in\rangle\prec\langle L_\kappa, \in\rangle$, then if P is the predicate of being logically valid, then $\langle L_\alpha, \in, P\cap L_\alpha\rangle\prec\langle L_\kappa, \in, P\rangle$, but $\langle L_\alpha, \in, P\cap L_\alpha\rangle$ is Σ_1-compact (as an elementary substructure of $\langle L_\kappa, \in, P\rangle$ and it is validity admissible, hence by [22] it is Σ_1-complete.) Definitely we can pick such α's such that $\langle L_\alpha, \in, P\cap L_\alpha\rangle$ satisfy no reasonable decomposability condition.

Similarly to Definition 14.1 we can *define* that $\mathscr{A}$ is stably Σ_1-complete if any expansion of $\mathscr{A}$ into admissible structure is Σ_1-complete.

Like Theorem 14.1 we have:

Theorem 16.2 ($V=L$). *An admissible structure of the form $\mathscr{A}=\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$ is stably Σ_1-complete iff either*

(a) *α is weakly compact and the power set function is $\Sigma_1(\mathscr{A})$, or*

(b) *$\mathscr{A}$ is countably decomposable.*

Proof. If α is weakly compact and $\mathscr{P}(x)$ is $\Sigma_1(\mathscr{A})$, then $\mathscr{A}$ has (DP), hence it is validity admissible, and it is stably Σ_1-compact (because α is weakly compact). By

Sh:144

Nyberg [22] every admissible expansion of $\mathscr{A}$ is Σ_1-complete. If $\mathscr{A}$ is countably decomposable, then every admissible expansion of it is countably decomposable. Hence by Corollary 2.2 it is Σ_1-complete.

For the other direction, by Lemma 16.1, if $\mathscr{A}$ is stably Σ_1-complete it is stably Σ_1-compact. Now the theorem follows from Theorem 14.1 and the proof of Theorem 12.4 which shows that for a limit cardinal α if $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$ is validity admissible we must have that $\mathscr{P} \restriction L_\alpha$ is $\Sigma_1(\mathscr{A})$. $\square$ Theorem 16.2

We conclude this section by providing the example (promised after Proposition 4.12) of a structure satisfying (DP2), without satisfying (DP1). We shall construct our example in L.

Let γ be a limit cardinal such that $\langle L_\gamma, \in\rangle$ is a model of large enough finite subset of ZFC. We shall force over $\langle L_\gamma, \in\rangle$ and expand it by introducing a function f, defined on γ, such that for every $\alpha<\gamma$, $f(\alpha)$ is either α^+ or $\kappa<\alpha<\kappa^+$ where κ is cardinal, $\mathrm{cf}(\kappa)=\omega$ and $d(\alpha)\leq f(\alpha)$. The set of forcing conditions will be the set of all functions defined on initial segment of γ satisfying the requirements on f.

Using the fact that $\langle L_\gamma, \in\rangle$ is a model of enough axioms of set theory, one can prove using arguments like in Section 15 (though simpler because one has more axioms available) that there exists a 'generic' F such that $\langle L_\gamma, \in, F\rangle$ is admissible. $\langle L_\gamma, \in, F\rangle$ clearly satisfies (DP2) because one can define

$$R(X, Y):\quad \exists\alpha\, \exists f\, (f \text{ maps } \alpha \text{ onto } X \wedge Y=\{f''Z \mid Z\in L_{F(\alpha)}\}).$$

$R(X, Y)$ is clearly $\Sigma_1(\langle L_\gamma, \in, F\rangle)$ and it is a decomposition predicate for this structure.

Claim. *The power set function $X\to P(X)$ is not $\Sigma_1(\langle L_\gamma, \in, F\rangle)$.*

Once the claim is verified it is clear that $\langle L_\gamma, \in, F\rangle$ does not satisfy (DP1) in view of Lemma 12.7, because if it satisfies (DP1), it is validity admissible.

Proof of the Claim. Assume that $\varphi(x, y, \boldsymbol{a}, F)$ gives a Σ_1-definition of the function. We shall prove that for every condition g there exists an extension h and some x, y in L_γ, such that $h\vDash\varphi(x, y, \boldsymbol{a}, F)$ but $y\neq P(x)$. By genericity of F, such an h can be assumed to be an initial segment of F, hence we get a contradiction to our assumption.

Given a condition g, a, and φ where $\varphi(x, y, a, F)=\exists Z\,\psi(x, y, a, F, Z)$. By assumption about φ, let η be a limit cardinal of cofinality ω such that $\eta<\delta$ and g, $a\in L_\eta$. By assumption there exists an extension of g_0, $\tilde{h}$ whose domain is $\rho>\eta$ such that

$$\langle L\rho, \in, \tilde{h}\rangle\vDash\exists Z\,\psi(\eta, P(\eta), a, h, Z).$$

Fix Z. Let μ be a limit cardinal less than 0 such that $\rho\leq\mu$ and $\tilde{h}\in L_\mu$. Let μ be the set of elements definable in $\langle L_\mu, \in\rangle$ from $(\eta+1)\cup\{a, g, \tilde{h}, \rho, z\}$ and let $\langle L_\chi, \in\rangle$

Sh:144

be the transitive isomorph of M where $\tilde{h}$ is collapsed to h. Let j be the inverse of the collapsing isomorphism. We shall show that h is a condition. Granted that, since g is collapsed to itself, h extends g (recall that $\tilde{h}$ was an extension of g) and if $\bar{\rho}$ is the collapse of ρ and y the collapse of $P(\eta)$ we get

$$\langle L_{\bar{\rho}}, \in, h\rangle \vDash \exists Z\, \psi(\eta, y, a, h, Z).$$

But $|y| = \eta$, and therefore $y \neq P(\eta)$, hence we got the required extension of g. So we just have to verify that h is a condition. h is clearly a function from $\bar{\rho}$ into $\bar{\rho}$. Since $h \restriction \eta = \tilde{h} \restriction \eta$, $h \restriction \eta$ satisfies the requirements.

We have to verify that for $\eta < \alpha < \bar{\rho}$, $d(\alpha) \leq h(\alpha)$. We first prove that for every α such that $L_\chi \vDash \alpha$ is a cardinal, $\alpha > \eta$, then $d(\alpha) \leq (\alpha^+)$ (where α^+ is taken in the sense of L_χ).

This follows from the fact that since μ is a limit cardinal every definable subset of $j(\alpha)$ definable over L_μ is an element of $j(\alpha^+)$. Every element of α is definable in L_χ from η and finitely many parameters, hence in L_{α^+} one can find a countable family A_n such that $|A_n| < \eta$ and $\bigcup A_n \supseteq \alpha$. (Recall that $\mathrm{cf}(\eta) = \omega$.) Now let $\alpha < \rho$, in L_χ we have either $h(\alpha) = \alpha^+$ and in that case $d(\alpha) \leq h(\alpha)$ or $L_\chi \vDash \kappa < \alpha < \kappa^+$ where κ is a cardinal of cofinality ω and $d(\alpha) \leq h(\alpha)$.

In the last case one can define sets B_n in $L_{h(\alpha)}$ such that $L_\chi \vDash |B_n| < \kappa$ and $\alpha = \bigcup_{n<\omega} B_n$. For each n, let $\beta_n = |B_n|$ in the sense of L_χ. Where $f_n : \beta_n \to B_n$ is onto, and definable in $L_{h(\alpha)}$. By previous arguments, since $d(\beta_n) = \beta_n^+ < \kappa$ we can define in L, A_n^m such that $L_\chi \vDash |A_n^m| < \eta$ and $\beta_n = \bigcup_m A_n^m$. One can now easily show that $f''\, A_n^m$ is definable in $L_{h(\alpha)}$ and $\alpha = \bigcup_{m,n} f''\, A_n^m$, hence $d(\alpha) \leq h(\alpha)$.

Appendix. Admissable ordinals are not immortal (by Leo Harrington)

For $X \subseteq \alpha$ let $M_\alpha[X]$ be the structure $\langle L_\alpha, \in, X\rangle$.

Call α amenably admissible if: for all $X \subseteq \alpha$, if X is L_α-amenable, then $M_\alpha[X]$ is admissible.

Notice: (α a limit) α is not amenably admissible iff $\exists \kappa < \alpha\, \exists h : \kappa \to \alpha$ s.t. h is unbounded in α to $\forall \rho < \kappa$ $(h \restriction \rho \in L_\alpha)$.

Notice: if $\mathrm{cof}\, \alpha = \omega$, or if α is a singular ordinal in L, then α is not amenably admissible.

Theorem. *Suppose α is not amenably admissible. Let $R \subseteq \alpha$ be s.t. $M_\alpha[R]$ is admissible and R is not $\mathbf{\Delta}_1$ over L_α. Then: There is $Q \subseteq \alpha$ s.t. $M_\alpha[Q]$ is admissible and $M_\alpha[R, Q]$ is not admissible.*

Proof. (1) Let $\kappa =$ least ord $< \alpha$ which demonstrates that α is not amenably admissible.

(2) Notice: $\kappa \leq (\mathbf{\Delta}_2$ in $M_\alpha[R]) -$cof of α.

Notation: by a set of ords x we mean that $x:\beta \to 2$ for some ord β ($\beta = \operatorname{dom} x$). We say that x is a subset of β.

Let $F = \{F \mid f \in L_\alpha, f$ is a func, $\operatorname{dom} f$ is an ord, and $\forall \xi < \operatorname{dom} f$ ($f(\xi)$ is a subset of ξ)$\}$.

(3) For $f, g \in F: f \leqslant g$ if $f = g \restriction \operatorname{dom} f$.

For a, a set of ords, $f \leqslant_a g$ if $f \leqslant g$ and

$$\forall \xi\, (\operatorname{dom} f \leqslant \xi < \operatorname{dom} g \Rightarrow g(\xi) \restriction \operatorname{dom} a = a \restriction \xi).$$

Let R be as in the statement of the Theorem.

(4) For $f \in F$ and for $\xi < \operatorname{dom} f$ let:

$$\sigma(f, \xi) = \max\{\delta < \xi \mid f(\xi) \restriction \delta = R \restriction \delta\}, \qquad \hat{\sigma}(f, \xi) = \limsup_{\nu < \xi} \sigma(f, \nu).$$

Let $A(f) = \{\xi < \operatorname{dom} f \mid \sigma(f, \xi) < \hat{\sigma}(f, \xi)\}$.

(5) Notice (for $f, g \in F$):

(i) If $g \geqslant_R f$ then $A(g) = A(f)$.

(ii) If $\operatorname{dom} a > \operatorname{dom} f$, $a \restriction \operatorname{dom} f = R \restriction \operatorname{dom} f$, and $a \neq R \restriction \operatorname{dom} a$ (a in L_α), and if $g \geqslant_a f$, then $A(g) = A(f)$.

Let $P = \{\langle c, f\rangle \mid f \in F, c \in L_\alpha, c$ is a set, $\operatorname{dom} c \leqslant \operatorname{dom} f\}$. For $p \in P$, let $p = \langle c_p, f_p\rangle$. Order P by: $p \leqslant q$ iff $c_p \leqslant c_q$ (i.e., $c_p = c_q \restriction \operatorname{dom} c_p$), and $f_p \leqslant_{c_p} f_q$. Let $\hat{F} = \{f \in F \mid A(f)$ has order type $< \kappa\}$. Let $\hat{P} = \{p \in P \mid c_p \leqslant R$ and $f_p \in \hat{F}\}$. For G a filter on P, let $Q_G = \bigcup\{f_p \mid p \in G\}$.

Claim. *There is a filter G on $\hat{P}$ s.t. Q_G has domain α, $M_\alpha[Q_G]$ is admissible, and $A(Q_G)$ has order type exactly κ.* [Notice, since $A(Q_G)$ is $\mathbf{\Delta}_1$ in $M_\alpha[R, Q_G]$, the claim yields the Theorem.]

Proof of the Claim. For d a set, μ an ord, and for $Y \subseteq d \times \mu$, Y is called initial in $d \times \mu$ if $\forall i \in d$ (Y_i is an initial segment of μ) [where $Y_i = \{j < \mu \mid (i, j) \in Y\}$].

(6) For $d \in L_\alpha$, $\mu < \alpha$, $d \times \mu$ is called small if every initial subset of $d \times \mu$, which is $\mathbf{\Sigma}_1$ in $M_\alpha[R]$, is actually in L_α.

(7) Notice $d \times \mu$ is small if either:

(i) μ is a regular card of L_α and (α-card of d) $< \mu$.

(ii) (α-card of d) $\leqslant \mu$ and every $\mathbf{\Sigma}_1$ in $M_\alpha[R]$ subset of μ is actually in L_α.

Let φ be the universal Σ_1-formula.

Fact 1. *Let $\langle c, f\rangle \in \hat{P}$; $d \times \mu$ small. Then there is $g \geqslant_c f$ s.t. $A(g) = A(f)$ and s.t. for all $i \in d$ either (letting $\gamma = \operatorname{dom} g$):*

(a) $M_\gamma[g] \models \forall j < \mu\, \varphi(i, j)$,

(b) $\exists j < \mu$ *s.t.* $M_\gamma[g] \models \forall k < j \varphi(i, k)$, *but for all* $g' \geqslant_{R \restriction \gamma} g$, $M_{(\operatorname{dom} g')}[g'] \models \neg\varphi(i, j)$.

Proof. In a $\mathbf{\Sigma}_1$ over $M_\alpha[R]$ way, build a sequence g_δ, Z^δ ($\delta < \theta$, some $\theta \leqslant \alpha$) as follows:

$$g_0 = f, \qquad z^0 = \emptyset.$$
$$g_\lambda = \bigcup \{g_\delta \mid \delta < \lambda\}, \qquad Z^\lambda = \bigcup_{\delta<\lambda} Z^\delta \quad \text{for } \lambda \text{ limit.}$$

Given g_δ and Z^δ; if:

(8) There are g^*, i^*, j^* s.t.:

$$g^* \geq_c g_\delta, \qquad A(g^*) = A(g_\delta), \qquad i^* \in d, \qquad j^* < \mu, \qquad (i^*, j^*) \notin Z^\delta,$$
$$\forall k < j^* \ (i^*, k) \in Z^\delta, \quad \text{and} \quad M_{(\operatorname{dom} g^*)}[g^*] \models \varphi(i^*, j^*).$$

Then: pick such g^*, i^*, j^* and let $g_{\delta+1} = g^*$, $Z^{\delta+1} = Z^\delta \cup \{(i^*, j^*)\}$.

Let $Z = \bigcup_{\delta<\theta} Z^\delta$. Clearly Z is an initial subset of $d \times \mu$, and Z is $\mathbf{\Sigma}_1$ in $M_\alpha[R]$; so Z is in L_α. Thus $\theta < \alpha$, and so $\theta = \hat{\delta} + 1$ for some $\hat{\delta}$. Let $\hat{g} = g_{\hat{\delta}}$. Let $\gamma = \operatorname{dom} \hat{g}$. We have:

$$\hat{g} \geq_c f; \qquad A(\hat{g}) = A(f).$$

Let $W = \{i \in d \mid \forall j < \mu \ (i, j) \in Z\}$. Let $V = d \setminus W$. For $i \in V$, let $j(i)$ be the first $j < \mu$ s.t. $(i, j) \notin Z$. We have:

(9) $\forall i \in W M_\gamma[\hat{g}] \models \forall j < \mu \ \varphi(i, j)$.

(10) $\forall i \in V M_\gamma[\hat{g}] \models \forall k < j(i) \ \varphi(i, k)$.

(11) For all $\hat{f} \geq_c \hat{g}$, if $A(\hat{f}) = A(\hat{g})$, then $\theta(\hat{f})$ holds [where $\theta(\hat{f}) \equiv \forall i \in V$ $M_{(\operatorname{dom} \hat{f})}[\hat{f}] \models \neg\varphi(i, j(i))$].

For a set in L_α s.t. $\operatorname{dom} a \geq \hat{g} \,\&\, a \restriction \gamma = R \restriction \gamma$, let f_a be the unique f' s.t. $f' \geq_a \hat{g} \,\&\, \operatorname{dom} f' = \operatorname{dom} a$. Notice: by (5), $A(f_a) = A(\hat{g})$; also notice: if $a \neq R \restriction \operatorname{dom} a$, then for all $\hat{f}$, $\hat{f} \geq_a f_a \Rightarrow A(\hat{f}) = A(\hat{g})$ [use 5(ii)]. Thus: if $a \neq R \restriction \operatorname{dom} a$, then $\psi(a)$ holds [where $\psi(a) \equiv \forall \hat{f} \ (\hat{f} \geq_a f_a \Rightarrow \theta(\hat{F})$]. But R is not $\mathbf{\Delta}_1$ over L_α, and ψ is Π_1 over L_α. Thus there is $\beta \geq \gamma$ s.t. $\psi(R \restriction \beta)$ holds.

Let $g = f_{(R \restriction \beta)}$. g is as desired by Fact 1 [i.e. for $i \in w$, (a) holds because of (9); for $i \in V$, (b) holds because of (10), and $\psi(R \restriction \beta)$]. $\square$ Fact 1

Fact 2. *There is a sequence $\langle b_\delta \rangle$, $\delta < \kappa$, s.t.*

(1) *$\langle b_\delta \rangle$, $\delta < \rho$, is in L_α (all $\rho < \kappa$).*

(2) *$b_\delta = d_\delta \times \mu_\delta$ is small (all δ).*

(3) *For all $i < \alpha$, all regular α-cards ν, there is δ s.t. $i \in d_\delta \,\&\, \nu \leq \mu_\delta$.*

Proof. Let $h : k \to \alpha$ demonstrate that α is not amenably admissible.

Case 1: There is no last α-card. Then let $b_\delta = h(\delta) \times h(\delta)$ (this is small by (2)(ii)).

Case 2: There is a last α-card, β.

Subcase 1: Every $\mathbf{\Sigma}_1$ over $M_\alpha[R]$ subset of β is in L_α. Then let $b_\delta = h(\delta) \times \beta$ (this is small again by (2)(ii)).

Subcase 2: There is a $\mathbf{\Delta}_1$ over $M_\alpha[R]$ 1–1 function $p : \alpha \to \beta$. Then let $d_\delta = \{i < \alpha \mid p(i) < p(h(i)\}$, and let $\mu_\delta = \beta$ (if β is regular in L_α); otherwise let μ_δ be the first regular α-card $> p(h(\delta))$ ($d_\delta \times \mu_\delta$ is small by (2)(i)).

These cases are exhaustive. $\square$ Fact 2

Now, build a sequence p_δ, $\delta<\kappa$, of conditions in $\hat{P}$ so that:

$$p_0=\theta, \qquad p_\lambda=\bigcup_{\delta<\lambda} p_\delta;$$

given $p_\delta=\langle c, f\rangle$, let g be as in Fact 1 for $d_\delta\times\mu_\delta$ (where $d_\delta\times\mu_\delta$ are as in Fact 2). Pick $p_{\delta+1}\geqslant\langle R\restriction \operatorname{dom} g, g\rangle$ s.t. $A(f(p_{\delta+1}))\supsetneqq A(g)$. The filter generated in $\hat{P}$ by $\langle p_\delta \mid \delta<\kappa\rangle$ is easily seen to be a witness to the proof of the claim.

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