TWO CARDINAL COMPACTNESS

BY
SAHARON SHELAH

ABSTRACT

We prove that if $\lambda \geqq \mu^{\aleph_0} = \mu \geqq |T|$ and if every finite subtheory of T has a (λ, μ)-model (i.e. a model with a domain of power λ, in which a distinguished predicate is interpreted as a set of μ elements) then T has such a model. There are generalizations for μ-like models (or, equivalently, to languages with generalized quantifiers).

Consider a first-order language L with a designated one-place predicate Q. A (λ, μ)-model for L is a model $\mathfrak{M} = \langle M, Q_{\mathfrak{M}}, \cdots \rangle$, such that $|M| = \lambda$ and $|Q_{\mathfrak{M}}| = \mu$, where $|X|$ is the cardinality of X and $Q_{\mathfrak{M}}$ is the interpretation of Q in $\mathfrak{M}$.

Our result is (This and more are in the notice by Shelah [4]):

THEOREM. *If T is a theory in the language L, and $\lambda \geqq \mu = \mu^{\aleph_0} \geqq |T|$ and if every finite subset of T has a (λ, μ)-model, then T has a (λ, μ)-model.*

This result is stronger than a previous result of Fuhrken (cf. [2]) that if $\lambda \geqq \mu = \mu^{\kappa}$ and $\kappa \geqq |T|$ and if every finite subset of T has a (λ, μ)-model then T has a (λ, μ)-model. Morley, in [3], p. 125, proves a similar result by means of partitions. Our proof generalizes his proof.

If $K(\lambda, \mu, L)$ is the class of (λ, μ)-models for L, then our result is that $K(\lambda, \mu, L)$ is μ-compact whenever $\lambda \geqq \mu = \mu^{\aleph_0} \geqq |L|$.

We shall say that a theory has names for Skolem-functions if for every formula $\phi(u_1, \cdots, u_n, u)$ of the language L there is a term $\tau_\phi(u_1, \cdots, u_n)$ such that the following sentences are all in T:

$$(1) \qquad \forall u_1, \cdots, u_n[\exists u \phi(u_1, \cdots, u_n, u) \to \phi(u_1, \cdots, u_n, \tau_\phi(u_1, \cdots, u_n))]$$

If T does not have this property, we can extend the language L by adding suitable function symbols. For every model of L one can interpret the new function

Received May 24, 1969 an in revised form June 23, 1970

symbols, without adding new individuals to the domain, so that all the sentences of (1) will hold. (See p. 112 in [3].) Hence it is enough to prove the result for theories which have names for Skolem-functions. For such a theory, T, if $\mathfrak{M} = \langle M, \cdots \rangle$ is a model of T and $A \subset M$, then, if we add to A all the members of the form $\tau_{\mathfrak{M}}(a_1, \cdots, a_n)$, where τ is an n-place term of L, $\tau_{\mathfrak{M}}$ is its interpretation in $\mathfrak{M}$ and $a_1, \cdots, a_n \in A$, we get a set N which together with all the relations and functions of $\mathfrak{M}$, restricted to it, is an elementary submodel of $\mathfrak{M}$.

Let $A = \{a_i : i < \lambda\}$ be a set of λ new individual constants. Consider the following condition on T:

(*) There exists an equivalence relation, E, on $\bigcup_{n<\omega} A^n$, with μ equivalence classes, such that equivalent sequences are of the same length and the set of all the following formulas is consistent with T:

(i) $a_i \neq a_j$, where $i \neq j$;

(ii) $Q(a_i)$, where $i < \mu$;

(iii) $\tau(\bar{b}) = \tau(\bar{c}) \vee [\neg Q(\tau(\bar{b})) \wedge \neg Q(\tau(\bar{c}))]$, where, for some n τ is an n-place term of L, $\bar{b}, \bar{c} \in A^n$ and $\bar{b} E \bar{c}$.

LEMMA 1. *If* $|T| \leqq \mu$, *T has names for Skolem-functions and T satisfies* (*) *then it has a* (λ, μ)*-model.*

PROOF. With no loss of generality we can assume that the language, L, of T is of power $\leqq \mu$. Let $\mathfrak{M}$ be a model satisfying T and (i), (ii), and (iii). To simplify notation we shall identify each a_i with the member which is denoted by it and, thus, we have $A \subset M$.

Let $\mathfrak{N} = \langle N, \cdots \rangle$ be the elementary submodel of $\mathfrak{M}$, such that N is the union of A and the set of all elements of the form $\tau_{\mathfrak{M}}(b_1, \cdots, b_n)$, where $b_1, \cdots, b_n \in A$. Since there are at most μ terms, we have $|N| = |A| = \lambda$. Now, $Q_{\mathfrak{N}}$ consists of all a_i, $i < \mu$, as well as all the members $\tau_{\mathfrak{M}}(b_1, \cdots, b_n)$ which are in $Q_{\mathfrak{M}}$, where $b_1, \cdots, b_n \in A$. The sentences of (iii) imply that, for every n, every X which is an equivalence class, such that $X \subset A^n$, contributes at most μ elements to $Q_{\mathfrak{N}}$. Hence $|Q_{\mathfrak{N}}| = \mu$ and $\mathfrak{N}$ is a (λ, μ)-model.

LEMMA 2. *If T has a (λ, μ)-model,* $\mu = \mu^{\aleph_0}$ and $|T| \leqq \aleph_0$ *then T satisfies* (*).

PROOF. Let $\mathfrak{M} = \langle M, \cdots \rangle$ be a (λ, μ)-model of T. Interpret the a_i so that they are the names of all the members of M and so that the a_i, for $i < \mu$, are the names of all the members of $Q_{\mathfrak{M}}$. Again, identify each a_i with the element denoted by it, so that we have $M = A$ and $Q_{\mathfrak{M}} = \{a_i : i < \mu\}$.

Let L_0 be the countable sublanguage of L such that all the sentences of T_0 are sentences in L_0. Change the model $\mathfrak{M}$ to a model $\mathfrak{M}'$ in the following way: For every n-place term τ which is not in L_0 put: $\tau_{\mathfrak{M}'}(b_1, \cdots, b_n) = a_0$ for all $b_1, \cdots, b_n \in A$ (as being the first member of A). The interpretation of all the predicates and all the other terms is the same as in $\mathfrak{M}$. It is obvious that $\mathfrak{M}'$ is a model of T and that (i) and (ii) hold in it.

Now, define an equivalence relation E by: If, for some n, $\bar{b}, \bar{c} \in A^n$, then $\bar{b} E \bar{c}$ if, for every n-place τ either $\tau_{\mathfrak{M}'}(\bar{b})$ and $\tau_{\mathfrak{M}'}(\bar{c})$ are equal or both are not in $Q_{\mathfrak{M}'}$.

It is obvious that (iii) holds and it remains to show that there are μ equivalence classes.

Now, $\bar{b} E \bar{c}$ if in the above definition the requirement concerning $\tau_{\mathfrak{M}}(\bar{b})$ and $\tau_{\mathfrak{M}}(\bar{c})$ is made only for τ in L_0, because for τ not in L_0 the requirement holds in any case.

Let Te_n be the set of all n-place terms of L_0. For every $\bar{b} \in A^n$ let f_b be the function from Te_n to $\{a_i : i < \mu\} \cup \{e\}$, (where e is a new individual) defined as follows:

$f_{\bar{b}}(\tau) = a_i$ if $\tau_{\mathfrak{M}'}(\bar{b}) = a_i$ and $i < \mu$

$f_{\bar{b}}(\tau) = e$ otherwise.

It is clear that $\bar{b} E \bar{c}$ iff $f_{\bar{b}} = f_{\bar{c}}$. Hence there are equivalence classes at most as there are such functions, implying that their number is $\leqq \mu^{\aleph_0}$. On the other hand every $\{a_i\}$, $0 < i < \mu$ forms an equivalence class, for the case $n = 1$, because the term v, where v is an individual variable, maps a_i to itself. Hence, there are at least μ equivalence classes. Since $\mu = \mu^{\aleph_0}$ the lemma follows.

Lemma 3. *If $\mu = \mu^{\aleph_0}$, $|T| \leqq \aleph_0$ and every finite subtheory of T satisfies* (*) *then so does T.*

Proof. Let $T = \{\phi_0, \phi_1, \cdots\}$ and let E_i be the equivalence relation such that all the sentences of (iii) are consistent with $\phi_0 \wedge \cdots \wedge \phi_i$. Let E be defined as:

$$\bar{b} E \bar{c} \text{ if, for every } i < \omega, \ \bar{b} E_i \bar{c}.$$

Obviously (iii) is consistent with T for E thus defined. Since each E_i has μ equivalence classes, E has at most $\mu^{\aleph_0}$ equivalence classes. It has at least μ equivalence classes. Hence the lemma follows from $\mu = \mu^{\aleph_0}$.

Lemma 4. *T satisfies* (*) *iff every countable subtheory of T satisfies* (*).

PROOF. Given T, it suffices to construct a countable subtheory T_0 such that if T_0 satisfies (*), so does T.

Say that a sentence ϕ of L is of the same type as a sentence ψ if there exists a permutation of the function symbols, mapping each n-place symbol to an n-place symbol, and a similar permutation of the predicates, which keeps Q fixed, such that by applying these permutations and by a legitimate change of bound variables one can pass from ϕ to ψ (Also equality is fixed).

Obviously, being of the same type is an equivalence relation. Its equivalence classes will be called types. It can be seen that there are $\aleph_0$ types (all of them can be represented in a countable language containing Q and containing, for each n, $\aleph_0$ n-place function symbols and predicates).

Let T_0 be the countable theory obtained in the following way:

For each type t such that there exists $\phi \in t$ which can be written as: $\phi = \phi_0 \wedge \cdots \wedge \phi_i$, with $\phi_0, \cdots, \phi_i \in T$, choose one such ϕ and one representation $\phi = \phi_0 \wedge \cdots \wedge \phi_i$, with $\phi_0, \cdots, \phi_i \in T$ and put in T_0 the sentences $\phi_0, \cdots, \phi_i$.

Assume that E is an equivalence relation such that the set consisting of the sentences (i), (ii), and (iii) is consistent with T_0. We shall show, by contradiction, that it is consistent also with T. If it is not consistent with T then there is a finite conjunction ϕ of members of T with whom it is inconsistent. There is a finite conjunction ψ of members of T_0 which is of the same type as ϕ. We can pass from ϕ to ψ by applying certain permutations of the function symbols and the predicates and by changing, legitimately, the bound variables. It can be easily seen that such permuations keep each of the sentences of (i) and (ii) fixed and map (iii) onto itself. It follows that ψ is inconsistent with (i), (ii), and (iii). Contradiction.

If every finite subtheory of T has a (λ, μ)-model and $\mu = \mu^{\aleph_0}$, then, by Lemma 2, every finite subtheory satisfies (*). By Lemma 3, every countable subtheory of T satisfies (*) and, by Lemma 4, T satisfies (*). By Lemma 1, T has a (λ, μ)-model. This proves the theorem.

A similar theorem is true for the class of λ-like ordered models where $\aleph_0$ is small for λ (μ is small for λ, if for every $\lambda_i < \lambda$ $i < \mu$, $\prod_{i<\mu} \lambda_i < \lambda$. In other words $\lambda_0 < \lambda$ implies $\lambda_0^{\mu} < \lambda$ and the cofinality of λ is greater than μ).

A λ-like ordered model is a model $\mathfrak{M} = \langle M, \cdots \rangle$ for a language with a distinguished binary relation, such that this relation is interpreted as a linear λ-like ordering of M, that is, $|M| = \lambda$ and $\{x : x < b\}$ has power $< \lambda$, for all $b \in M$. In this case the following condition which is analogous to (*) should be used:

(**) There exists a family $\{E_l : l < \lambda\}$ of equivalence relation on $\bigcup_{n<\omega} A^n$, and a function $h : \bigcup_{n<\omega} \lambda^n \to \lambda$ such that each E_l has less than λ equivalence classes and the set of all the following sentences is consistent with T

(α) $a_i < a_j$ for $i < j < \lambda$

(β) $\tau(a_{i_1}, \cdots, a_{i_n}) < a_{h(i_1 \ldots i_n)}$

where τ is an n-place term

(γ) $\tau(\bar{b}) = \tau(\bar{c}) \vee [a_l < \tau(\bar{b}) \wedge a_l < \tau(\bar{c})]$

whenever $\bar{b} E_l \bar{c}$ and τ is a term with the appropriate number of places.

One can generalize the method further and obtain a similar result for λ-like ordered models, in which there are distinguished individuals, say $b_1, \cdots, b_k$, which are denoted by constants of the language, such that $\{x : x < b_i\}$ is ordered by $<$ in a λ_i-like order. One has to require that for each λ_i, $|T| < \lambda_i$ and $\aleph_0$ is small for λ_i. (Of course, $\lambda_i < \lambda$, $i = 1, \cdots k$.)

The result for λ-like ordered models implies the analogous result for the language with the additional quantifier: "there exist at least λ elements such that..." provided $|T| < \lambda$ and $\aleph_0$ is small for λ. Similarly, one gets the result for languages with several quantifiers of this kind, provided that for each λ_i in question $|T| < \lambda_i$ and $\aleph_0$ plus the number of quantifiers is small for λ_i.

Note that the lemmas imply that for T which has names for Skolem functions and for $\lambda \geqq \mu = \mu^{\aleph_0} \geqq |T|$, T has a (λ, μ)-model iff (*) holds. Note also that, if for a certain equivalence relation E, the sentences of (iii) are not consistent with T then there is a finite subset of A such that an inconsistency is implied by the sentences of (iii) involving elements of this subset only. This leads to the following result:

Define an identification as a pair $\langle I, n \rangle$ where I is an equivalence relation on $\bigcup_{k<n} \{i : i < n\}^k$ $(n < \omega)$. Say that $\langle I, \mathrm{n} \rangle$ is realized by the equivalence relation E on $\bigcup_{k<\omega} A^k$ if there is a one-to-one embedding of $\{i : i < k\}$ into A such that:

1) $i < j$ iff $F(i) < F(j)$ where F is the embedding.

2) if $\bar{i}, \bar{j} \in \bigcup_{k<n} \{i : i < n\}^k$, and $\bar{i} I \bar{j}$, then $\bar{i}, \bar{j}$ has the same length, and the embedding carries them to sequences equivalent to E.

We say that $\langle I, n \rangle$ is an identification of the pair (λ, μ) if for every set A of cardinality λ and equivalence relation E on $\bigcup_{k<\omega} A^k$ with μ equivalence classes, $\langle I, n \rangle$ is realized by E.

Now the result is:

If every finite subtheory of T has a (λ_1, μ_1)-model, every identification of (λ_2, μ_2) is an identification of (λ_1, μ_1), and $\lambda_2 \geqq \mu_2 = \mu_2^{\aleph_0} \geqq |T|$, then T has a (λ_2, μ_2)-model.

With little changes in the proof of the main theorem we can generalize a result of Ehrenfeucht [1] to (λ, μ)-models. The type an element $a \in M$ realizes in $\mathfrak{M}$ is the set of formulas $\phi(x_0)$ of L such that $\mathfrak{M}$ satisfies $\phi(a)$. The set of types realized in $\mathfrak{M}$ is the set of types P such that for some $a \in M$, P is the type a realizes in $\mathfrak{M}$. We can prove that:

If a theory T has a (λ, μ)-model, $\lambda \geqq \mu^{\aleph_0} = \mu \geqq |T|$, then T has a (λ, μ)-model such that the cardinality of the set of types realized in it is $\leqq 2^{\aleph_0} + |T|$.

A similar result is true for λ-like models.

ACKNOWLEDGEMENT

I wish to thank the referee for essentially rewriting this article.

REFERENCES

1. A. Ehrenfeucht, *Theories having at least continuum many non-isomorphic models in each infinite power*, Notices Amer. Math. Soc. **5** (1958), 680.

2. G. Fuhrken, *Languages with added quantifier "there exists at least $\aleph_\alpha$,"* Proc. 1963 Berkeley Symposium on Theory of Models, North-Holland Publ. Co., pp. 121–131, 1965.

3. M. Morley, *Partition and Models,* Proc. of the Summer School in Logic, Leeds, 1967. Lecture Notes in Mathematics 70, Springer-Verlag, Berlin-Heidelberg, New York, pp. 109–158.

4. S. Shelah, *Two-cardinal and power-like models: compactness and large group of authomorphisms*, Notices Amer. Math. Soc. **18** (1971).

THE HEBREW UNIVERSITY OF JERUSALEM