On the no(M) for M of singular power

Abstract: We prove that for λ singular of cofinality $\kappa > \aleph_0$, if $(\forall \mu < \lambda)\mu^\kappa < \lambda$ *then* for some model M, $M = (M, R^M)$, R a two place predicate, $\|M\| = \lambda$ and $no(M) = \{N/\approx : N \equiv_{\infty,\lambda} M, \|N\| = \lambda\}$ is quite arbitrary e.g. any $\mu < \lambda$ and λ^κ (hence 2^λ).

See [Sh 5] for the back ground: where the result were proved for M with relations with infinitely many places. By the present paper the only problem left, if we assume $V = L$, is whether $no(M) = \lambda$, may happen for M of cardinality λ for λ singular.

§1 On γ- systems of groups.

1.1 Definition : A γ-system will mean here a model of the form $\mathcal{A} = \langle G_\alpha, h_{i,j} \rangle_{\substack{i \le j < \gamma \\ \alpha < \gamma}}$ where

(i) G_i is a group with the unit $e_i = e^{G_i} = e_i^{\mathcal{A}}$, the G_i's are pairwise disjoint.

(ii) $h_{i,j}$ is a homomorphism from G_j into G_i when $i \le j$.

(iii) $h_{i_1,i_2} \circ h_{i_2,i_3} = h_{i_1,i_3}$ when $i_1 \le i_2 \le i_3 < \gamma$.

(iv) $h_{i,i}$ is the identity. (so we sometimes ignore them).

We denote γ-systems by $\mathcal{A}, \mathcal{B}$ and for a system $\mathcal{A}$, we write $G_i = G_i^{\mathcal{A}}, \gamma = \gamma^{\mathcal{A}}$ $h_{i,j} = h_{i,j}^{\mathcal{A}}$. Let $\|\mathcal{A}\| = \sum_{i<\kappa} \|G_i\|$. We omit the $\mathcal{A}$ when there is no danger of confusion.

Let $\gamma = \gamma^{\mathcal{A}}$, for $\beta \le \gamma$ let $\mathcal{A} \restriction \beta = \langle G_\alpha^{\mathcal{A}}, h_{i,j}^{\mathcal{A}} \rangle_{i \le j < \beta, \alpha < \beta}$. The really interesting case is $\gamma =$ limit.

1.2 Definition : For a γ-system $\mathcal{A}$ let $Gr(\mathcal{A}) = \{\mathbf{a} = \langle a_{i,j} : i \le j < \gamma \rangle : a_{i,j} \in G_i,\ a_{i,i} = e^{G_i}$ and if $\alpha \le \beta \le \varepsilon < \gamma$ then

$$a_{\alpha,\varepsilon} = h_{\alpha,\beta}(a_{\beta,\varepsilon})\ a_{\alpha,\beta}$$

Let $\mathbf{a} \restriction \beta = \langle a_{i,j} : i \le j < \beta \rangle$.

1.3 Definition : For $\mathbf{a} = \langle a_i : i < \gamma \rangle \in \prod_{i<\kappa} G_i$, let fact $(\mathbf{a}) = \langle a_{i,j} : i < j < \gamma \rangle$ where $a_{i,j} = h_{i,j}(a_j)^{-1}\ a_i$. Let Fact $(\mathcal{A}) = \}\{fact(\mathbf{a}) : \mathbf{a} \in \Pi G_i^{\mathcal{A}}\}$.

1.4 Claim: The mapping $\mathbf{a} \to fact(\mathbf{a})$ is from $\prod_{i<\kappa} G_i$ into $Gr(\mathcal{A})$. So fact $(\mathcal{A})$ is a subset of $Gr(\mathcal{A})$.

Proof : Trivially $a_{i,j} \in G_i, a_{i,i} = e_i$, and if $\alpha \le \beta \le \varepsilon$;

$h_{\alpha,\beta}(a_{\beta,\varepsilon}) \circ a_{\alpha,\beta} = (h_{\alpha,\beta}(h_{\beta,\varepsilon}(a_\varepsilon)^{-1})h_{\alpha,\beta}(a_\beta))(h_{\alpha,\beta}(a_\beta)^{-1} \circ a_\alpha) =$
$(h_{\alpha,\beta}h_{\beta,\varepsilon})(a_\varepsilon)^{-1}a_\alpha = h_{\alpha,\varepsilon}(a_\varepsilon)^{-1}a_\alpha = a_{\alpha,\varepsilon}$.

1.5 Definition : 1) $Gs(\mathcal{A}) = \{\bar{a} \in Gr(\mathcal{A})$: for every $\beta < \gamma^{\mathcal{A}}$ $\langle a_{i,j} : i < j < \gamma \rangle \in Fact(\mathcal{A} \restriction \gamma)\}$. [2]

2) We define a relation $\approx_{\mathcal{A}}$ on $Gr(\mathcal{A})$ (let $\gamma = \gamma^{\mathcal{A}}$): $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$ if for some $\langle g_i : i < \gamma \rangle \in \prod_{i<\gamma} G_i^{\mathcal{A}}$, for every $i < j < \gamma$ $b_{i,j} = h_{i,j}(g_j)^{-1}a_{i,j}\ g_i$.

We shall say that $\langle g_i : i < \gamma \rangle$ exemplify $\mathbf{a} \approx_{\mathcal{A}} \mathbf{B}$.

3) $\mathcal{A}$ is called *smooth* if for every limit $\beta < \gamma$, $Gr(\mathcal{A} \restriction \beta) = Fact(\mathcal{A} \restriction \beta)$.

1.6. Claim: For a γ-system $\mathcal{A}$:

1) $\approx_{\mathcal{A}}$ is an equivalence relation on $Gr(\mathcal{A})$ (hence also on $Gs(\mathcal{A})$).

2) If $\mathbf{a},\mathbf{b} \in Gr(\mathcal{A})$, $\beta < \gamma^{\mathcal{A}}$ and $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$ then $\mathbf{b} \restriction \beta \approx_{\mathcal{A} \restriction \beta} \mathbf{b} \restriction \beta$.

3) For $\mathbf{a} \in Gr(\mathcal{A})$: $\mathbf{a} \in Fact(\mathcal{A})$ iff $\mathbf{a} \approx_{\mathcal{A}} \langle e_i^{\mathcal{A}} : i < j < \gamma^{\mathcal{A}} \rangle$ (where $e_i^{\mathcal{A}}$ is the unit of $G_i^{\mathcal{A}}$).

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[2] Really $Gs(\mathcal{A}) = Gr(\mathcal{A})$, as if $\mathbf{a} = \langle a_{i,j} : i < j < \gamma \rangle \in Gr(\mathcal{A})$ then $\langle a_{i,\beta} : i < \beta \rangle$ witness $\mathbf{a} \restriction \beta \in Gs(\mathcal{A})$; but we shall not use this.

4) For $\mathbf{a},\mathbf{b} \in Gr(\mathcal{A})$, if $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$ *then* $\mathbf{a} \in Gs(\mathcal{A}) \Longleftrightarrow \mathbf{b} \in Gs(\mathcal{A})$.

Proof: 1) Let us check the properties.
reflexivity for $\mathbf{a} \in Gr(\mathcal{A})$, $\mathbf{a} \approx_{\mathcal{A}} \mathbf{a}$: $\langle e_i^{\mathcal{A}}:i \quad j < \gamma \rangle$ exemplify this

symmetry: suppose $\bar{\mathbf{a}} \approx_{\mathcal{A}} \mathbf{b}$ and $\langle g_i:i < \gamma \rangle$ exemplify this, so for every $i \le j < \gamma$, $b_{i,j} = h_{i,j}(g_j)^{-1} a_{i,j} g_i$, hence $h_{i,j}(g_j) h_{i,j} g_i^{-1} = a_{i,j}$ but $h_{i,j}(g_j^{-1}) = (h_{i,j}(g_j))^{-1}$ (as $h_{i,j}$ is a homomorphism from G_j into G_i). So (for every $i \le j \le \gamma$)
$a_{i,j} = (h_{i,j}(g_j^{-1}))^{-1} b_{i,j} (g_i^{-1})$ so $\langle g_i^{-1}:i < \gamma \rangle$ exemplify $\mathbf{b} \approx_{\mathcal{A}} \mathbf{a}$.

transitivity: suppose $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$, $\mathbf{b} \approx_{\mathcal{A}} \mathbf{c}$ and $\langle g_i^0:i < \gamma \rangle$,$\langle g_i^1:i < \gamma \rangle$ exemplify them (resp.) So for $i \le j < \gamma$, $b_{i,j} = h_{i,j}(g_j^0)^{-1} a_{i,j}\, g_i^0$ and $c_{i,j} = h_{i,j}(g_j^1)^{-1}$ $b_{i,j}\, g_i^1$, substituting we get

$$c_{i,j} = h_{i,j}(g_j^1)^{-1}(h_{i,j}(g_j^0)^{-1} a_{i,j} g_i^0) g_i^1 =$$
$$(h_{i,j}(g_j^0)\, h_{i,j}(g_j^1))^{-1} a_{i,j} (g_i^0 g_i^1) =$$
$$h_{i,j}(g_j^0 g_j^1)^{-1}\, a_{i,j} (g_i^0 g_i^1)$$

So $\langle g_i^0 g_i^1:i < \gamma \rangle$ exemplify $\mathbf{a} \approx_{\mathcal{A}} \mathbf{c}$.

2) If $\langle g_i:i < \gamma \rangle$ exemplify $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$ then $\langle g_i:i < \beta \rangle$ exemplify $\mathbf{a} \restriction \beta \approx_{\mathcal{A} \restriction \beta} \mathbf{b} \restriction \beta$.

3) Because $\langle g_i:i < \gamma \rangle$ exemplify $\langle e_i^{\mathcal{A}}:i < j < \gamma \rangle \approx_{\mathcal{A}} \mathbf{a}$ *iff* $a_{i,j} = h_{i,j}(g_j)^{-1} g_i$ (for every $i < j < \gamma$.) i.e. iff $\langle g_i : i < \gamma \rangle$ exemplify $\mathbf{a} \in Fact(\mathcal{A})$.

4) By 3) $\mathbf{c} \in Gs(\mathcal{A})$ iff for every $\beta < \gamma^{\mathcal{A}}$, $\mathbf{c} \restriction \beta \approx_{\mathcal{A}} \langle e_i^{\mathcal{A}}:i \le j < \beta \rangle$, and by 2) for $\beta < \gamma^{\mathcal{A}}$,$\mathbf{a} \restriction \beta \approx_{\mathcal{A}} \mathbf{b} \restriction \beta$, hence (as $\approx_{\mathcal{A} \restriction \beta}$ is as an equivalence relation)

$\mathbf{a} \restriction \beta \approx_{\mathcal{A}} \langle e_i^{\mathcal{A}}:i \le j < \beta \rangle$ iff $\mathbf{b} \restriction \beta \approx_{\mathcal{A}} \langle e_i^{\mathcal{A}}:i \le j < \beta \rangle$ and the result follows.

1.7 Definition : For a γ-system $\mathcal{A}$, let $no^*(\mathcal{A})$ be the cardinality of $Gs(\mathcal{A}) / \approx_{\mathcal{A}}$(i.e. the number of non $\approx_{\mathcal{A}}$-equivalent $\mathbf{a} \in Gs(\mathcal{A})$).

1.8 Lemma : Suppose $\mathcal{A}$,$\mathcal{B}$ are γ-systems,.

(i) H_i is a homomorphism from $G_i^{\mathcal{A}}$ onto $G_i^{\mathcal{B}}$.

(ii) for $i < j < \gamma$, $H_i \circ h^{\mathcal{A}}_{i,j} = h^{\mathcal{B}}_{i,j} \circ H_j$

(iii) for every $\beta < \gamma$, $\mathbf{a},\mathbf{b} \in Fact(\mathcal{A} \restriction \beta)$, satisfying $H_i(a_{i,j}) = H_i(b_{i,j})$ for $i < j < \beta$, a member $g^i_{\mathbf{a},\mathbf{b}} \in G^{\mathcal{A}}_i$ are defined for $i < \beta$ such that :

a) if $i < \alpha < \beta$ then $g^i_{\mathbf{a}\restriction\alpha,\mathbf{b}\restriction\alpha} = g^i_{\mathbf{a},\mathbf{b}}$.

b) $b_{i,j} = h^{\mathcal{A}}_{i,j}(g^j_{\mathbf{a},\mathbf{b}})^{-1} a_{i,j}\, g^i_{\mathbf{a},\mathbf{b}}$ for $i < j < \beta$.

Then $no^*(\mathcal{A}) \leq no^*(\mathcal{B})$.

Proof: We define a function H with domain $Gr(\mathcal{A}) : H(\mathbf{a}) = H(\langle a_{i,j} : i < j < \gamma \rangle) = \langle H_i(a_{i,j}) : i < j < \gamma \rangle$. By (ii) we can check that H is into $Gr(\mathcal{B})$. We shall show later

(*) for $\mathbf{a},\mathbf{b} \in Gr(\mathcal{A})$, $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$ iff $H(\mathbf{a}) \approx_{\mathcal{B}} H(\mathbf{b})$.

Applying this to $\mathcal{A} \restriction \beta$ (for $\beta < \gamma$) and noting that $H_i(e^{\mathcal{A}}_i) = e^{\mathcal{B}}_i$. $H(\langle e^{\mathcal{A}}_i : i < j < \beta \rangle) = \langle e^{\mathcal{B}}_i : i < j < \beta \rangle$ we see that for $\mathbf{a} \in Gr(\mathcal{A})$, $\beta < \gamma$.

[$\mathbf{a} \restriction \beta \in Fact(\mathcal{A} \restriction \beta)$ iff $H(\mathbf{a}) \restriction \in Gs(\mathcal{B})$]. So by (*) H induces a one to one map from $Gs(\mathcal{A}) / \approx_{\mathcal{A}}$ into $Gs(\mathcal{B}) / \approx_{\mathcal{B}}$, so $no^*(\mathcal{A}) \leq n^*(\mathcal{B})$.

Proof of (*): First suppose $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$ and let $\langle g_i : i < \gamma \rangle$ exemplify this. So for every $i < j < \gamma$

$$b_{i,j} = h^{\mathcal{A}}_{i,j}(g_j)^{-1}\, a_{i,j}\, g_i$$

applying H_i we get $H_i(b_{i,j}) = H_i(h^{\mathcal{A}}_{i,j}(g_j)^{-1}) H_i(a_{i,j}) H_i(g_i)$

Now by (ii) $H_i(h^{\mathcal{A}}_{i,j}(g_j)^{-1}) = (H^{\mathcal{A}}_i(h_{i,j}(g_j))^{-1} = (h^{\mathcal{B}}_{i,j}(H_j(g_j)))^{-1}$, so

$$H_i(b_{i,j}) = h^{\mathcal{B}}_{i,j}(H_j(g_j))^{-1} H_i(a_{i,j}) H_i(g_i)$$

So $\langle H_i(g_i) : i < \gamma \rangle$ exemplify that $H(\mathbf{a}) \approx_{\mathcal{B}} H(\mathbf{b})$.

Next suppose $H(\mathbf{a}) \approx_{\mathcal{B}} H(\mathbf{b})$ and let $\langle g^*_i : i < \gamma \rangle$ exemplify it. As H_i is a homomorphism from $G^{\mathcal{A}}_i$ onto $G^{\mathcal{B}}_i$, there are $g_i \in G^{\mathcal{A}}_i$, such that $H_i(g_i) = g^*_i$ (for $i < \gamma$). Now $H_i(b_{i,j}) = h^{\mathcal{B}}_{i,j}(g^*_j)^{-1} H_i(a_{i,j})\, g^*_i = h^{\mathcal{B}}_{i,j}(H_j(g_j))^{-1} H_i(a_{i,j}) H_i(g_i)$

$$= H_i(h^{\mathcal{A}}_{i,j}(g_j)^{-1}) H_i(a_{i,j}) H_i(g_i) = H_i(h^{\mathcal{A}}_{i,j}(g_j)^{-1} a_{i,j}\, g_i)$$

Let us define $\mathbf{c} \in Gr(\mathcal{A})$ by $\mathbf{c}_{i,j} = b_{i,j}^{\mathcal{A}}(g_i)^{-1}\, a_{i,j}\, g_i$. It is easy to check that $\mathbf{c}$ really belongs to $Gr(\mathcal{A})$ and $\langle g_i : i < \gamma \rangle$ exemplify $\mathbf{a} \approx_{\mathcal{A}} \mathbf{c}$, and the above equation shows that $H(\mathbf{b}) = H(\mathbf{c})$, and by (iii) this implies $\mathbf{b} \approx_{\mathcal{A}} \mathbf{c}$ ($\langle g^i_{\mathbf{b},\mathbf{c}} : i < \gamma \rangle$ exemplify that). Together $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$.

So we have proved (*) hence 1.8.

1.9 Claim: If in 1.8 in addition:

(iv) H_i^+ is a homomorphism from $G_i^{\mathcal{B}}$ into $G_i^{\mathcal{A}}$.

(v) $H_i \circ H_i^+$ is the identity (on $G_i^{\mathcal{B}}$)

(vi) $h_{i,j}^{\mathcal{B}} \circ H_j^+ = H_i^+ \circ h_{i,j}^{\mathcal{A}}$ for $i < j < \gamma$.

Then $no^*(\mathcal{A}) = n^*(\mathcal{B})$.

Proof : We define a function H^+ with domain $Gr(\mathcal{B})$: $H^+(\mathbf{a}) = \langle H_i^+(a_{i,j}) : i \leq j < \gamma \rangle$. By (vi) $H^+(\mathbf{a})$ is always in $Gr(\mathcal{A})$. Clearly $H \circ H^+$ is the identity on $Gr(\mathcal{B})$, so let $\{\mathbf{c}^\zeta : \zeta < no^*(\mathcal{B})\}$ be pairwise non $\approx_{\mathcal{B}}$ equivalent members of $Gs(\mathcal{B})$, and let $\mathbf{a}^\zeta = H^+(\mathbf{c}^\zeta) \in Gr(\mathcal{A})$. So $H(\mathbf{a}^\zeta) = \mathbf{c}^\zeta$. From the proof of 1.8 we know that: $\mathbf{a}^\zeta \in Gs(\mathcal{A})$ because $\mathbf{c}^\zeta \in Gs(\mathcal{B})$, and for $\xi < \zeta < no^*(\mathcal{B})$—$\mathbf{a}^\xi, \mathbf{a}^\zeta$ are non $\approx_{\mathcal{A}}$ equivalent (because $\mathbf{c}^\zeta, \mathbf{c}^\xi$ are non $\approx_{\mathcal{B}}$ equivalent). So $no^*(\mathcal{A}) \geq no^*(\mathcal{B})$ hence we finish (by 1.8).

1.10 Claim: For a γ-system of abelian groups.

1) $Gr(\mathcal{A})$ here is the same as $Gr(\mathcal{A})$ from [Sh 5], Definition 3.4 (except that here we do not put the group structure.

2) $Fact(\mathcal{A})$ here is the same (set) as $Fact(\mathcal{A})$ from [Sh 5] Definition 3.5 .

3) For $\mathbf{a},\mathbf{b} \in Gr(\mathcal{A})$, $\mathbf{a} \approx_{\mathcal{A}} \mathbf{b}$, *iff* (in [Sh 5] notation), $\mathbf{a}-\mathbf{b} \in Fact(\mathcal{A})$.

4) $Gs(\mathcal{A})$ here is the same as $Gs(\mathcal{A})$ from [Sh 5] Definition 3.7(1).

5) $no^*(\mathcal{A})$ here is the same as the cardinality of $E'(\mathcal{A})$ (from [Sh 5] Definition 3.7(2)).

Proof : Straightforward.

1.11 Conclusion: For every regular $\kappa > \aleph_0$ and μ, for some κ-system, $\mathcal{A}$, $||\mathcal{A}|| \le \mu^\kappa$, and $no^*(\mathcal{A}) = \mu$.

1.12 Claim : Suppose $\mathcal{A}$ is a γ-system, γ limit and for $\ell = 1,2$ $\mathbf{a}^\ell = \langle a^\ell_{i,j}: i \le j < \gamma \rangle$ belongs to $Gs(\mathcal{A})$.

Suppose further $S \subseteq \gamma$ is unbounded in γ and $a^1_{i,j} = a^2_{i,j}$ when $i,j \in S$. Then $\mathbf{a}^1 \approx_{\mathcal{A}} \mathbf{a}^2$.

Proof: For every $\beta < \gamma, \ell = 1,2$, $\mathbf{a}^\ell \restriction (\beta+1) \in Fact(\mathcal{A} \restriction (\beta+1))$ hence there is $\mathbf{g}^\ell_\beta = \langle g^{\ell,\beta}_i : i \le \beta \rangle \in \prod_{i \le \beta} G^{\mathcal{A}}_i$ such that $a^\ell_{i,j} = (h^{\mathcal{A}}_{i,j}(g^{\ell,\beta}_j)^{-1})\ g^{\ell,\beta}_i$ when $i \le j \le \beta$. For $\alpha < \gamma$ let $\varepsilon(\alpha) = Min\{\beta: \alpha \le \beta \in S\}$.

We want to find $g_i \in G^{\mathcal{A}}_i$ $(i < \gamma)$ such that $a^2_{i,j} = h_{i,j}(g_j)^{-1} a^1_{i,j}\ g_i$.

Now for $\ell = 1,2$, if $i \le \varepsilon(i) \le j$

$$a^\ell_{i,j} = h^{\mathcal{A}}_{i,\varepsilon(i)}(a^\ell_{\varepsilon(i),j})\ a^\ell_{i,\varepsilon(i)} =$$
$$h^{\mathcal{A}}_{i,\varepsilon(i)}(h^{\mathcal{A}}_{\varepsilon(i),j}(a^\ell_{j,\varepsilon(j)})^{-1}\ a^\ell_{\varepsilon(i),\varepsilon(j)})a^\ell_{i,\varepsilon(i)} =$$
$$h^{\mathcal{A}}_{i,j}(a^\ell_{j,\varepsilon(j)})^{-1}\ h^{\mathcal{A}}_{i,\varepsilon(i)}(a^\ell_{\varepsilon(i),\varepsilon(j)})\ a^\ell_{i,\varepsilon(i)}$$

[apply twice Definition 1.2 first for $i, \varepsilon(i)$, j standing for α,β,ε, and second for $\varepsilon(i),\varepsilon(j)$ standing for α,β,ε].

Now if $i \le j \le \varepsilon(i)$, applying twice this equation (remembering $a^2_{\xi(i),\xi(j)} = a^2_{\xi(i),\xi(j)}$):

$$a^2_{i,j} = h_{i,j}(a^2_{j,\varepsilon(j)})^{-1} h^{\mathcal{A}}_{i,\varepsilon(i)}(a^2_{\varepsilon(i),\varepsilon(j)})\ a^2_{i,\varepsilon(i)} =$$
$$h_{i,j}(a^2_{j,\varepsilon(j)})^{-1} h^{\mathcal{A}}_{i,\varepsilon(i)}(a^1_{\varepsilon(i),\varepsilon(j)})\ a^2_{i,\varepsilon(i)}$$
$$= h_{i,j}(a^2_{j,\varepsilon(j)})^{-1}\ (h_{i,j}(a^1_{j,\varepsilon(j)})\ a^1_{i,j}\ (a^1_{i,\varepsilon(i)})^{-1})\ a^2_{i,\varepsilon(i)} =$$
$$h_{i,j}((a^2_{j,\varepsilon(j)})^{-1}\ a^1_{j,\varepsilon(j)})a^1_{i,j}((a^1_{i,\varepsilon(i)})^{-1}\ a^2_{i,\varepsilon(i)}) =$$
$$= h_{i,j}((a^1_{j,\varepsilon(j)})^{-1}\ a^2_{j,\varepsilon(j)})^{-1}\ a^1_{i,j}\ ((a^1_{i,\varepsilon(i)})^{-1}\ a^2_{i,\varepsilon(i)})$$

This suggests to show that $\langle (a^1_{i,\varepsilon(i)})^{-1}\ a^2_{i,\varepsilon(i)} : i < \kappa \rangle$ exemplify $\mathbf{a}^1 \approx_{\mathcal{A}} \mathbf{a}^2$ as required. The missing case is $i < j < \gamma$ $j < \varepsilon(i)$; so $\varepsilon(i) = \varepsilon(j)$ and so we should prove $a^2_{i,j} = h_{i,j}((a^1_{j,\varepsilon(j)})^{-1} a^2_{j,\varepsilon(j)})^{-1} a^1_{i,j}((a^1_{i,\varepsilon(i)})^{-1}\ a^2_{i,\varepsilon(i)})$.

This is equivalent to $h_{i,j}(a^2_{j,\varepsilon(j)})\ a^2_{i,j}\ (a_{i,\varepsilon(i)})^{-1} = h_{i,j}(a^1_{j,\varepsilon(j)})\ a^1_{i,j}\ (a^1_{i,\varepsilon(i)})^{-1}$. Applying twice the equation from Definition 1.2 this is equivalent to $a^2_{i,\varepsilon(j)}(a^2_{i,\varepsilon(i)})^{-1} = a^1_{i,\varepsilon(j)}(a^1_{i,\varepsilon(i)})^{-1}$. As $\varepsilon(i) = \varepsilon(j)$ we finish.

§2 On γ- systems of automorphisms

For this section we make the assumption.

2.1 Assumption: M is an L-model, $P_i \in L$ monadic predicate, $P_i^M (i < \gamma)$ are pairwise disjoint and $|M| = \bigcup_{i<\gamma} P_i^M$. For such M let $M^{[\alpha]} = M \upharpoonright \bigcup_{i \le \alpha} P_i^M$ for $\alpha < \gamma$.

2.2 Definition : 1) Let K^M be the class of L-models N such that $N = \bigcup_{i<\gamma} P_i^N$ and $N^{[\beta]} = N \upharpoonright \bigcup_{i\le\beta} P_i^N$ is isomorphic to $M^{[\beta]}$ for every $\beta < \gamma$.

2) Let G^M_α be the group of automorphisms of $M^{[\alpha]}$.

3) Let $h^M_{i,j}$ (for $i \le j < \gamma$) be the following function with domain $G^M_j : h^M_{i,j}(g) = g \upharpoonright M^{[i]}$.

4) Let $\mathcal{A} = \mathcal{A}^M = \langle G^M_\alpha, h^M_{i,j} : \alpha < \gamma, i < j < \gamma \rangle$. (i.e. as long as M is constant we can omit M).

2.3 Fact: 1) $h^M_{i,j}$ is a homomorphism from G^M_j into G^M_i.

2) $\mathcal{A}^M$ is a γ-system.

Proof : Immediate.

2.4 Definition: 1) We call $\mathbf{g} = \langle g_{i,j} : i \le j < \gamma \rangle$ a *representation* of $N \in K^M$ *if* there are isomorphism f_i from $M \upharpoonright \bigcup_{\varepsilon \le i} P^M_\varepsilon$ onto $N \upharpoonright \bigcup_{\varepsilon\le i} P^N_i$ (for $i < \gamma$) such that $g_{i,j} = (f_j^{-1} \upharpoonright N^{(\alpha)}) \circ f_i$.

2) For $\mathbf{g}$, $f_i (i < \gamma)$ as above we say that $\langle f_i : i < \gamma \rangle$ *exemplify* $\mathbf{g}$ being a representation of N.

2.5 Fact: Every $N \in K^M$ has a representation.

Proof : By the definition of K^M (definition. 2.2(1)) there are f_i as required.

2.6 Fact: If $\mathbf{g}$ is a representation of N $(N \in K^M)$ then $\mathbf{g} \in Gr(\mathcal{A})$.

Proof : Let $\langle f_i : i < \gamma \rangle$ exemplify $\mathbf{g} \in Gr(\mathcal{A})$ is a representation of M. For each $i \leq j$, as f_j is an isomorphism from $M^{[j]}$ onto $N^{[j]}$ clearly f_j^{-1} is an isomorphism from $N^{[j]}$ onto $M^{[j]}$, hence $f_j^{-1} \restriction N^{[i]}$ is an isomorphism from $N^{[i]}$ onto $M^{[i]}$ clearly $(f_j^{-1} \restriction N^{[i]}) \circ f_i$ is an isomorphism from $M^{[i]}$ onto $M^{[i]}$ so it belongs to G_i^M. So $g_{i,j} \in G_i^M$.

Easily $g_{i,i}$ is the unit of G_i^M.

We can now check that for $i \leq j \leq \beta < \alpha$, $g_{i,\beta} = h_{i,j}^M(g_{j,\beta}) \circ g_{i,j}$; remembering the definition of $h_{i,j}^M$ this means that

$$(f_\beta^{-1} \restriction N^{[i]}) \circ f_i = ((f_\beta^{-1} \restriction N^{[j]}) \circ f_j) \restriction M^{[i]}) \circ (f_j^{-1} \restriction N^{[i]}) \circ f_i$$

or equivalent by, for every $x \in M^{[i]}$,

$$f_\beta^{-1} \circ f_i(x)) = f_\beta^{-1} \ f_j \ \ f_j^{-1} f_i(x)$$

which is obvious.

2.7 Fact: Let $\mathbf{g}^0$ be a representation of $N(\in K^M)$. Then $\mathbf{g} \in Gr(\mathcal{A})$ is also a representation of N *iff* $\mathbf{g} \approx_{\mathcal{A}} \mathbf{g}^0$.

Proof : First suppose that $\mathbf{g}^0 \approx_{\mathcal{A}} \mathbf{g}$, and let $\langle k_i : i < \gamma \rangle \in \prod_{i<\gamma} G_i^M$ exemplify this (see Definition. 1.2). So $g_{i,j} = h_{i,j}^M(k_j)^{-1} g_{i,j}^0 k_i$ (for $i \leq j \leq \gamma$). Let $\langle f_i : i < \gamma \rangle$ exemplify $\mathbf{g}^0$ being a representation of N (see Definition. 2.4(2)).

So $g_{i,j}^0 = (f_j^{-1} \restriction N^{[i]}) \circ f_i$, and we get

$$g_{i,j} = h_{i,j}^M(k_j)^{-1} \circ (f_j^{-1} \restriction N^{[i]}) \circ f_i \circ k_i =$$
$$(f_j \restriction M^{[i]} \circ h_{i,j}^M(k_j))^{-1} \circ (f_i \circ k_i)$$

[Note that $(f_j \restriction M^{[i]})^{-1} = f_j^{-1} \restriction N^{[i]}$]; we would like to show that $\langle f_i \circ k_i : i < \gamma \rangle$ exemplify $\mathbf{g}_{i,j}$ is a representation of N. Clearly $f_i \circ k_i$ is an isomorphism from $M^{[i]}$ onto $N^{[i]}$. The above equality will be the only missing information provided that we shall show that

$$f_j \restriction M^{[i]} \circ h_{i,j}(k_j) = (f_j \circ k_j) \restriction M^{[i]}$$

which is easy.

Second suppose $\mathbf{g} \in Gr(\mathcal{A})$ is a representation of N and we shall prove that $\mathbf{g} \approx_{\mathcal{A}} \mathbf{g}^0$.

Let $\langle f_i^0 : i < \gamma \rangle$ exemplify $\mathbf{g}^0$ being a representation of N and $\langle f_i : i < \gamma \rangle$ exemplify $\mathbf{g}$ being a representation of N (see Definition. 2.4(2)). So

$$g_{i,j}^0 = (f_j^0 \restriction N^{[i]})^{-1} \circ f_i^0,$$
$$g_{i,j} = (f_j \restriction N^{[i]})^{-1} \circ f_i$$

(for $i \le j < \gamma$). Let $k_i \stackrel{\text{def}}{=} f_i^{-1} f_i^0$ (for $i < \gamma$). As f_i, f_i^0 are isomorphism from $M^{[i]}$ onto $N^{[i]}$ clearly k_i is an automorphism of $M^{[i]}$, i.e. it belongs to G_i^M. Now $f_i^0 = f_i k_i$ hence

$$g_{i,j}^0 = (f_j^0 \restriction M^{[i]})^{-1} \circ f_i^0 = ((f_j \circ k_j) \restriction M^{[i]})^{-1} \circ (f_i \circ h_i) =$$
$$= (k_i \restriction M^{[i]})^{-1} \circ (f_j \restriction M^{[i]})^{-1} \circ f_i \circ k_i =$$
$$(k_j \restriction M^{[i]})^{-1} \circ g_{i,j} \circ k_i$$

But easily $k_j \restriction M^{[i]} = h_{i,j}^M(k_i)$, so $\langle k_i : i < \gamma \rangle$ exemplify $\mathbf{g} \approx_{\mathcal{A}} \mathbf{g}^0$.

Fact 2.8: Suppose the models $N_1, N_2 \in K^M$ has representations $\mathbf{g}^1, \mathbf{g}^2$ respectively, then $N_1 \cong N_2$ iff $\mathbf{g}^1 \approx_{\mathcal{A}} \mathbf{g}^2$.

Proof : Let $\langle f_i^\ell : i < \gamma \rangle$ exemplify "$\mathbf{g}^\ell$ is a representation of N_ℓ" for $\ell = 1,2$. So $g_{i,j}^\ell = (f_j^\ell \restriction M^{[i]})^{-1} \circ f_i^\ell$ for $\ell = 1,2$, $i \le j < \gamma$.

First assume N^1, N^2 are isomorphic, and let H be an isomorphism from N^1 onto N^2. For each $i < \gamma$, $H \restriction N_1^{[i]}$ is an isomorphism from $N_1^{[i]}$ onto $N_2^{[i]}$, hence $k_i \stackrel{\text{def}}{=} (f_i^2)^{-1}(H \restriction N_1^{[i]}) f_i^1$ is an isomorphism from $M^{[i]}$ onto $M^{[i]}$, i.e. $k_i \in G_i^M$. So for every $i, f_i^2 = (H \restriction N_1^{[i]}) \circ f_i^1 \circ k_i^{-1}$, and let $H_i \stackrel{\text{def}}{=} H \restriction N_1^{[i]}$ (so for $i < j, H_i = H_j \restriction N_1^{[i)]}$). Now for $i \le j < \gamma$.

$$g_{i,j}^2 = (f_j^2 \restriction M^{[i]})^{-1} \circ f_i^2 =$$
$$= (H_j \circ f_j^1 \circ k_j^{-1} \restriction M^{[i]})^{-1} \circ (H_i \circ f_i^1 \circ k_i^{-1}) =$$
$$= (H_i \circ (f_j^1 \restriction M^{[i]}) \circ (k_j \restriction M^{[i]})^{-1})^{-1} \circ (H_i \circ f_i^1 \circ k_i^{-1}) =$$

$$= (k_j \restriction M^{[i]}) \circ (f_j^1 \restriction M^{[i]})^{-1} \circ H_i^{-1} \circ H_i \circ f_i^1 \circ k_i^{-1} =$$
$$= (k_j \restriction M^{[i]}) \circ (f_j^1 \restriction M^{[i]})^{-1} \circ f_i^1 \circ k_i^{-1} = (k_j \restriction M^{[i]}) \circ g_{i,j}^1 \circ k_i^{-2}$$

So $\langle k_i^{-1} : i < \gamma \rangle$ exemplify $\mathbf{g}^1 \approx_{\mathcal{A}} \mathbf{g}^2$.

Second, assume $\mathbf{g}^1 \approx_{\mathcal{A}} \mathbf{g}^2$ and let this be exemplified by $\langle k_i^{-1} : i < \gamma \rangle$. Define

$$H_i = f_i^2 \circ k_i \circ (f_i^1)^{-1}$$

It is easy to check that H_i is an isomorphism from $N_1^{[i]}$ for $i < \gamma$ and $H_i = H_j \restriction M^{[i]}$, for $i < j < \gamma$. So $\bigcup_{i<\gamma} H_i$ is an isomorphism from N_1 onto N_2.

2.9 Lemma : If $\mathbf{g}$ is a representation of $N \in K^M$ then $\mathbf{g} \in Gs(\mathcal{A})$.

Proof : Suppose not so for some $\beta < \gamma$, $\mathbf{g} \restriction \gamma \not\in Fact(\mathcal{A} \restriction \gamma)$ so $\mathbf{g} \restriction \gamma, \langle e_i^{\mathcal{A}} : i \le j < \beta \rangle$ are not $\approx_{(\mathcal{A} \restriction \beta)}$-equivalent. Apply 2.8 to $M^{[\beta]}$ instead M (and $\mathbf{g} \restriction \beta, \langle e_i^{\mathcal{A}} : i < j < \beta \rangle$, $N^{[\beta]}, M^{[\beta]}$), and get that $N^{[\beta]}, M^{[\beta]}$ are not isomorphic contradicting $N \in K^M$.

2.10 Lemma : Every $\mathbf{g} \in Gs(\mathcal{A})$ represents some $N \in K^M$.

Proof : We define by induction j

(a) an L-model N_j, such that $N_j \cong M^{[i]}$ and $N_i \subseteq N_j$ for $i \le j$.

(b) an isomorphism f_j from $M^{[j]}$ onto N_j, such that for $i \le j$, $g_{i,j} = (f_j \restriction M^{[i]}) \circ f_i$.

For $j = 0$, j successor there is no problem. For j limit $\bigcup_{i<j} N_i$ is isomorphic to $\bigcup_{i<j} M^{[i]} = M \restriction \bigcup_{i<j} P_i^M$ by 2.8, and multiplied by some $k \in Aut\ (M \restriction \bigcup_{i<j} P_i)$ it will be as required.

2.11 Conclusion: The numbers of non-isomorphic $N \in K^M$ is equal to $|Gs(\mathcal{A}) / \approx_{\mathcal{A}}|$.

Proof : By 2.5-2.10.

2.12 Lemma : If the following conditions hold, then every $N \in K^M$ is $L_{\infty,\lambda}$-equivalent to M.

a) Every function F of M are 1-place, amd for $x \in M^{[i]}$, $F_i^M(x) \in M^{[i]}$.

b) for any relation R of M for some $n < \omega$ and $i < \gamma$:

$$M \models (\forall x_1, \ldots, x_n)[R(x_1, \ldots, x_n) \to \bigwedge_{\ell=1}^{n} P_i(x_\ell)]$$

c) *if* $i < j < \gamma$, $g \in G_i^M$, g^* *a partial* automorphism of $M^{[j]}$,Dom (g^*) closed under the function of M, and $g \cup g^*$ is a partial automorphism of M and Dom (g^*) is in $\mathcal{I}_i$, (see below) *then* $g \cup g^*$ an be extended to an automorphism of $M^{[j]}$.

d) $\mathcal{I}_i$ is a family of subsets of M,$[i < j \Longrightarrow \mathcal{I}_i \subseteq \mathcal{I}_j]$ $\mathcal{I}_i$ closed under finite unions, and $[A \subseteq M,\ |A| < \lambda \Longrightarrow A \in \bigcup_{i<\gamma} \mathcal{I}_i]$.

Proof: Easy.

§3 Constructing the model.

3.1 Main Theorem: Suppose

(i) $\kappa = cf(\lambda) < \lambda$ and $(\forall \mu < \lambda)(\mu^{<\kappa} < \lambda)$.

(ii) $\mathcal{B}$ is a κ-system, and $|G_i^{\mathcal{B}}| < \lambda$ for $i < \kappa$.

Then there is a model M (with relations and functions of finitely many places only) of cardinality λ such that $no(M) = no^*(\mathcal{B})$.

3.1A Remarks: W.l.o.g. $M = (|M|, R^M)$ for some two- place relation R. (see [Sh 5], 1.4)

Notation: For $A \subseteq M$, let $cl_M(A)$ be the closure of A under the functions of M.

Proof: By 1.12 w.l.o.g. for $j < \kappa$ limit, $h_{j,j+1}^{\mathcal{B}}$ is onto $G_j^{\mathcal{B}}$, and if $x \in G_j^{\mathcal{B}}$, $x \neq e_j^{\mathcal{B}}$ then for some $i < j$, $h_{i,j}^{\mathcal{B}}(x) \neq e_i^{\mathcal{B}}$. By 1.12 w.l.o.g. $G_0^{\mathcal{B}}$ is trivial $(=\{e_0^{\mathcal{B}}\})$. Let $L = \{P_i, F_{i,j}, : i < j < \kappa\} \cup \{R_i : i < \kappa\}$, $P_i (i < \kappa)$ monadic predicates, $F_{i,j}$ one place function symbols, R_i three place predicate. Let $\boldsymbol{\lambda} = \sum_{i<\kappa} \boldsymbol{\lambda}_i$, $\boldsymbol{\lambda}_i^{<\kappa} = \lambda_i < \lambda$, $\lambda_i > ((\sum_{j<i} \lambda_j^+ + |G_i^{\mathcal{B}}|)^{\kappa})^{+5}$. We shall now define by induction on

$j < \kappa$, M_j, G_j, H_j, H_j^+, $\mathcal{P}_i$ $(i < j)$ such that :

(A) (1) M_j is an L-model,

(A) (2) M_j is the disjoint union of $P_i^{M_j}(i < j)$ and $P_i^{M_j} = (\lambda_i, \lambda_i^{+2})$ when $i < j$, $P_i^{M_j} = \phi$ when $\kappa > i \geq j$

(A) (3) $F_{\alpha,\beta}^{M_j}$ is a 1-place function from $P_\beta^{M_j}$ into $P_\alpha^{M_j}$ (and not defined otherwise) for $\alpha < \beta < \kappa$.

(A) (4) for any R_i $R_i^{M_j}$ is a (three place) relation on $P_i^{M_j}$,

(A) (5) for $i < j$, $M_i = M_j \restriction (\bigcup_{\varepsilon < i} P_\varepsilon^{M_j})$.

(B) (1) G_j is the group of automorphism of M_j if j is a successor ordinal, otherwise $G_j = \{k \in Aut(M_j)$: for some $a \in G_j^\beta$ for every $i < j$, $H_j(k \restriction M_i) = h_{i,j}(a)\}$, (see below on H_j)

(B) (2) H_j is a homomorphism from G_j onto G_j^β.

(B) (3) for $i < j$, $k \in G_j$, $h_{i,j}^\beta(H_j(k)) = H_i(k \restriction M_i)$.

(B) (4) G_j has cardinality $\leq \lambda_j^{+2}$.

(B) (5) H_j^+ is a homomorphism from G_j^β into G_j, $H_j \circ H_j^+$ is the identity (on G_j^β) and for $i < j, a \in G_j$, $H_j^+(a) \restriction M_i = H_i^+(h_{i,j}^\beta(a))$.

(C) (1) $\mathcal{P}_i^j$ is a family of subsets of $(\lambda_j, \lambda_j^{+2})$ (when $i < j$).

(C) (2) if $A \in \mathcal{P}_i^j$, $i < \alpha < j$, then $cl_M(A) \cap (\lambda_\alpha, \lambda_\alpha^{+2}) \in \mathcal{P}_i^\alpha$.

(C) (3) for $i < \alpha < j$, $\mathcal{P}_i^j \subseteq \mathcal{P}_\alpha^j$.

(C) (5) every $g \in G_{j+1}$ maps any $A \in \mathcal{P}_i^j$ to a member of $\mathcal{P}_i^j$.

(C) (6) $\mathcal{P}_i^j$ is closed under union of $\leq \kappa$, (i.e if $A_\xi \in \mathcal{P}_i^j$ for $\xi < \zeta \leq \kappa$ then $\bigcup_{\xi < \zeta} A_\xi \in \mathcal{P}_i^j$).

(C) (7) every subset of $(\lambda_j, \lambda_j^{+2})$ of power $\leq \|M_i\|$ is included in some member of $\mathcal{P}_i^j$.

(D) (1) For $i < j$ let $Q_i^j = \{A \subseteq M_j:$ for $\alpha < i$, $(\lambda_\alpha, \lambda_\alpha^{+2}) \subseteq A$ and for $\alpha \in [i,j)$, $A \cap (\lambda_\alpha, \lambda_\alpha^{+2}) \in \mathcal{P}_i^\alpha$ and $A = cl_{M_j}(A)\}$.

(D) (2) If $i < j$, $k_0, k_1 \in G_j$, $A \in Q_i^j$, k_0, k_1 are equal on $(\bigcup_{\alpha<i} P_\alpha^{M_j}) \cap A$ *then* $(k_0 \restriction A) \cup (k_1 \restriction \bigcup_{\alpha<i} P_\alpha^{M_j})$ can be extended to an automorphism k of M_j.

Moreover, if $a \in G_j^{\mathcal{B}}$, $b_{i,j}^{\mathcal{B}}(a) = H_i(k_1 \restriction M_i)$ then we can demand $H_j(k) = a$.

Clearly it suffices to carry the construction by induction, as then $M \stackrel{def}{=} \bigcup_{j<\kappa} M_j$ is as required by the previous Lemmas (i.e. by 2.12 every $N \in K_M$ is $L_{\infty,\lambda}$-equivalent to it (and clearly $[N \equiv_{\infty,\lambda} M \Longrightarrow N \in K_M]$) so $no(\mathfrak{A}) = \{N/\cong : N \in K_M\}$. But 2.11 this number is equal to $no^*(M) = |Gs(\mathfrak{A})/\approx_{\mathfrak{A}}|$ where $\mathfrak{A} = \mathfrak{A}^M$ (see Definition 2.2(4)). By 1.9 this number is $no^*(\mathcal{B})$. But $\mathcal{B}$ was chosen so that it is μ.)

Case I: $j = 0$.

Nothing to do.

Case II: j is limit.

In this case let $M_j = \bigcup_{i<j} M_i$, and there is no problem to check all the conditions. Note that in (D)(2) we can easily prove the second sentence.

Case III: $j + 1$ (assuming we have defined for j).

We shall define by induction on $\xi < \lambda_j^{+2}$, a group $G_{j,\xi}$, an ordinal $\alpha(\xi)$, an action of the group $G_{j,\xi}$ on $M_j \cup (\lambda_j, \alpha(\xi))$ and $H_{j,\xi}, P_{i,\xi}^j, F_{\alpha,j}^\xi, R^\xi$ such that

(i) for $\zeta < \xi$, $G_{j,\zeta}$ is a subgroup of $G_{j,\xi}$ and the action of $g \in G_{j,\zeta}$ on $M_j \cup (\lambda_j, \alpha(\zeta))$ is extended too, and for $k \in G_{j,\xi}$, $k \restriction M_j \in G_j$.

(ii) $\alpha(\xi) \in (\lambda_j^{+1}, \lambda_j^{+2})$ and $\alpha(\xi)$ is increasing and continuous.

(iii) for ξ limit $G_{j,\xi} = \bigcup_{\zeta<\xi} G_{j,\zeta}$.

(iv) $H_{j,\xi}$ is a homomorphism from $G_{j,\xi}$ onto G_j^β.

(v) $F^\xi_{\alpha,j}$ is a one-place function from $(\lambda_j, \alpha(\xi))$ into $P^{M_j}_\alpha$ increasing and continuous in ξ.

(vi) $P^j_{i,\xi}$ is a family of subsets of $(\lambda_j, \alpha(\xi))$ such that $A^{[i]} \stackrel{def}{=} \{F^\xi_{\alpha,j}(x) : \alpha < j, x \in A\} \in Q^j_i$ for each $A \in P^j_{i,\xi}$ $i < j$.

(vii) if $A \in P^j_{i,\xi}$, $g \in G_{j,\xi}$ then $g(A) \in P^j_{i,\xi}$.

(viii) $P^j_{i,\xi}$ is closed under union of $\leq \kappa$ members and it is increasing with ξ and if $cf\ \xi > \kappa$ then $P^j_{i,\xi} = \bigcup_{\zeta<\xi} P^j_{i,\zeta}$.

(ix) we can choose for every $\alpha(\xi)$ an increasing sequence $B^\xi_\varepsilon (\varepsilon < \lambda_j^+)$ such that $(\lambda_j, \alpha(\xi)) = \bigcup_{\varepsilon<\lambda_j^+} B^\xi_\varepsilon$, and B^ξ_ε has cardinality $\leq \lambda_j$. We shall guarantee that for any $\xi < \lambda_j^{++}$, $\varepsilon < \lambda_j^+, i < j$ and $A \in Q^j_i$ for some ξ_1, $\xi < \xi_1 < \lambda_j^{+2}$, and $B \in P^j_{i,\xi_1}$, $B^\xi_\varepsilon \subseteq B$.

(x) if $k_0, k_1 \in G_{j,\xi}$, $A \in Q^j_i$ k_0, k_1 are equal on A, $a \in G^\beta_{j+1}$, $h^\beta_{j+1}(a) = H_j(k_1 \restriction M_i)$ *then* $(k_0 \restriction A) \cup (k_1 \restriction M_j)$ can be extended in some $G_{j,\zeta}(\xi \leq \zeta < \lambda_j^{+2})$ to k, $H_{j,\zeta}(k) = a$.

(xi) R^ξ is a three place relation on $(\lambda_j, \alpha(\xi))$, increasing with ξ, but for $\zeta < \xi$, $R^\zeta = R^\xi \restriction (\lambda_j, \alpha(\zeta))$.

(xii) each $g \in G_{j,\xi}$ preserves R^i and $F^\xi_{\alpha,j}$.

(xiii) if $cf\ \xi = \lambda_j^+$, then $R(\alpha(\xi) -, -)$ define on $(\lambda_j, \alpha(\xi))$ a well-ordering [so *if* $g \in G_{j,\zeta}, \zeta > \xi$, g maps $(\lambda_j, \alpha(\xi))$ on itself *then*, $g \restriction (\lambda_j, \alpha(\xi))$ is determined by $g(\alpha(\xi))$].

(xiv) no $\alpha \neq \beta \in (\lambda_j, \alpha(\xi))$ realize the same quantifiers free, R_ξ-type over (λ_j, λ_j^+). (So together with (xiii) we have a strict control over the automorphism of M_{j+1}).

There is no problem to carry the induction on ξ hence on j, hence to finish the proof of 3.1.

REFERENCES

[C] C.C. Chang, Some remarks on the model theory of infinitary languages:, pp. 36-63 in: *The Syntax and Semantics of Infinitary Languages*, *Lecture Notes in Mathematics*, 72, ed., J. Barwise, Springer, 1968.

[HS] H.L. Hiller, and S. Shelah, "Singular cohomology in L", *Israel Journal of Mathematics*, vol. 26 (1977), pp. 313-319.

[NS] M. Nadel and J. Stavi, " $L_{\infty\lambda}$-equivalence, isomorphism and potential isomorphism ," *Transactions of the American Mathematical Society, vol. 236 (1978)., pp.* 51-74.

[P] E. A. Palyutin, "Number of models in L_{∞,ω_1} theories III," pp. 443-456 in *Algebra I Logika*, vol. 16, no. 4. (1977): English translation in *Algebra and Logic*, vol. 16, no. 4 (1977), pp. 299-309.

[S] D. Scott, "Logic with denumerably long formulas and finite strings of quantifiers," pp. 329-341 in *The Theory of Models*, North Holland, 1965.

[Sh1]
S. Shelah, "On the number of non-isomorphic models of cardinality $\lambda, L_{\infty\lambda}$-equivalent to a fixed model", *Notre Dame J. of Formal Logic* 22 (1981), 5-10.

[Sh2]
--------, "On the number of non-isomorphic models of power $\lambda \equiv_{\infty\lambda}$ to a model of power λ, when λ is weakly compact," *Notre Dame J. of Formal Logic*, 23 (1982), 21-26.

[Sh3]
--------, "The consistency of $Ext(G,\mathbb{Z}) = \mathbb{Q}$" *Israel Journal of Mathematics*, 39 (1981), 283-288.

[Sn4]
---------, "A pair of non-isomorphic $\equiv_{\infty,\lambda}$, Models of Power λ for λ singular with $\lambda^{\omega} = \lambda$". *Notre Dame J. of Formal Logic, 25 (1984), 97-104.*

[Sh5]
--------, "On the possible number $no(M) =$ the numbers of non isomorphic models $L_{\infty,\lambda}$-equivalent of power λ for λ singular, *Notre Dame J. Formal Logic*, in press.