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ON INCOMPACTNESS FOR CHROMATIC NUMBER OF GRAPHS

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Abstract. We deal with incompactness. Assume the existence of non-reflecting stationary set of cofinality κ. We prove that one can define a graph G whose chromatic number is $> \kappa$, while the chromatic number of every subgraph $G' \subsetneqq G$, $|G'| < |G|$ is $\leqq \kappa$. The main case is $\kappa = \aleph_0$.

§ 0. Introduction

§ 0(A). The questions and results. During the Hajnal conference (June 2011) Magidor asked me on incompactness of "having chromatic number $\aleph_0$"; that is, there is a graph G with λ nodes, chromatic number $> \aleph_0$ but every subgraph with $< \lambda$ nodes has chromatic number $\aleph_0$ when:

$$(*)_1 \quad \begin{cases} \lambda \text{ is regular } > \aleph_1 \text{ with a non-reflecting stationary } S \subsetneqq S^\lambda_{\aleph_0}, \\ \text{possibly though better not, assuming some version of GCH.} \end{cases}$$

Subsequently also when:

$$(*)_2 \qquad \lambda = \aleph_{\omega+1}.$$

Such problems were first asked by Erdős–Hajnal, see [1]; we continue [4].

First answer was using BB, see [3, 3.24] so assuming

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⊞ (a) $\lambda = \mu^+$
(b) $\mu^{\aleph_0} = \mu$
(c) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ is stationary not reflecting

or just

⊞′ (a) $\lambda = \mathrm{cf}(\lambda)$
(b) $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$
(c) as above.

However, eventually we get more: if $\lambda = \lambda^{\aleph_0} = \mathrm{cf}(\lambda)$ and $S \subseteq S^\lambda_{\aleph_0}$ is stationary non-reflective then we have λ-incompactness for $\aleph_0$-chromatic. In fact, we replace $\aleph_0$ by $\kappa = \mathrm{cf}(\kappa) < \lambda$ using a suitable hypothesis.

Moreover, if $\lambda^\kappa > \lambda$ we still get $(\lambda^\kappa, \lambda)$-incompactness for κ-chromatic number. In §2 we use quite free family of countable sequences.

In subsequent work we shall solve also the parallel of the second question of Magidor, i.e.

$$(*)_2 \quad \begin{cases} \text{for regular } \kappa \geqq \aleph_0 \text{ and } \varepsilon < \omega \text{ there is a graph } G \text{ of chromatic} \\ \text{number} > \kappa \text{ but every sub-graph with} < \aleph_{\kappa\cdot\varepsilon+1} \text{ nodes has} \\ \text{chromatic number} \leqq \kappa. \end{cases}$$

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§ 0(B). Preliminaries. Definition 0.1. For a graph G, let $\mathrm{ch}(G)$, the chromatic number of G be the minimal cardinal χ such that there is colouring $\mathbf{c}$ of G with χ colours, that is $\mathbf{c}$ is a function from the set of nodes of G into χ or just a set of of cardinality $\leqq \chi$ such that $\mathbf{c}(x) = \mathbf{c}(y) \Rightarrow \{x, y\} \notin \mathrm{edge}(G)$.

Definition 0.2. 1) We say "we have λ-incompactness for the $(< \chi)$-chromatic number" or $\mathrm{INC}_{\mathrm{chr}}(\lambda, < \chi)$ *when:* there is a graph G with λ nodes, chromatic number $\geqq \chi$ but every subgraph with $< \lambda$ nodes has chromatic number $< \chi$.

2) If $\chi = \mu^+$ we may replace "$< \chi$" by μ; similarly in 0.3.

We also consider

Definition 0.3. 1) We say "we have (μ, λ)-incompactness for $(< \chi)$-chromatic number" or $\mathrm{INC}_{\mathrm{chr}}(\mu, \lambda, < \chi)$ *when* there is an increasing continuous sequence $\langle G_i : i \leqq \lambda \rangle$ of graphs each with $\leqq \mu$ nodes, G_i an induced subgraph of G_λ with $\mathrm{ch}(G_\lambda) \geqq \chi$ but $i < \lambda \Rightarrow \mathrm{ch}(G_i) < \chi$.

2) Replacing (in part (1)) χ by $\bar{\chi} = (< \chi_0, \chi_1)$ means $\mathrm{ch}(G_\lambda)) \geqq \chi_1$ and $i < \lambda \to \mathrm{ch}(G_i) < \chi_0$; similarly in 0.2 and parts 3), 4) below.

3) We say we have incompactness for length λ for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number *when* we fail to have (μ, λ)-compactness for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number for some μ.

4) We say we have $[\mu, \lambda]$-incompactness for $(<\chi)$-chromatic number or $\mathrm{INC}_{\mathrm{chr}}[\mu, \lambda, <\chi]$ *when* there is a graph G with μ nodes, $\mathrm{ch}\,(G) \geqq \chi$ but $G^1 \subseteq G \wedge |G^1| < \lambda \Rightarrow \mathrm{ch}\,(G^1) < \chi$.

5) Let $\mathrm{INC}^+_{\mathrm{chr}}(\mu, \lambda, <\chi)$ be as in part (1) but we add that even the $\mathrm{cl}\,(G_i)$, the colouring number of G_i is $<\chi$ for $i < \lambda$, see below.

6) Let $\mathrm{INC}^+_{\mathrm{chr}}[\mu, \lambda, <\chi]$ be as in part (4) but we add $G^1 \subseteqq G \wedge |G^1| < \lambda \Rightarrow \mathrm{cl}\,(G^1) < \chi$.

7) If $\chi = \kappa^+$ we may write κ instead of "$<\chi$".

DEFINITION 0.4. 1) For regular $\lambda > \kappa$ let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}\,(\delta) = \kappa\}$.

2) We say C is a $(\geqq \theta)$-closed subset of a set B of ordinals when: if $\delta = \sup\,(\delta \cap B) \in B$, $\mathrm{cf}\,(\delta) \geqq \theta$ and $\delta = \sup\,(C \cap \delta)$ then $\delta \in C$.

DEFINITION 0.5. For a graph G, the colouring number $\mathrm{cl}\,(G)$ is the minimal κ such that there is a list $\langle a_\alpha : \alpha < \alpha(*)\rangle$ of the nodes of G such that $\alpha < \alpha(*) \ \Rightarrow\ \kappa > |\{\beta < \alpha : \{a_\beta, a_\alpha\} \in \mathrm{edge}\,(G)\}$.

§ 1. From non-reflecting stationary in cofinality $\aleph_0$

CLAIM 1.1. *There is a graph G with λ nodes and chromatic number $> \kappa$ but every subgraph with $< \lambda$ nodes have chromatic number $\leqq \kappa$* when:

⊞ (a) *λ, κ are regular cardinals*

(b) $\kappa < \lambda = \lambda^\kappa$

(c) *$S \subseteqq S^\lambda_\kappa$ is stationary, not reflecting.*

PROOF. *Stage A:* Let $\bar{X} = \langle X_i : i < \lambda\rangle$ be a partition of λ to sets such that $|X_i| = \lambda$ or just $|X_i| = |i+2|^\kappa$ and $\min\,(X_i) \geqq i$ and let $X_{<i} = \cup\{X_j : j < i\}$ and $X_{\leqq i} = X_{<(i+1)}$. For $\alpha < \lambda$ let $\mathbf{i}(\alpha)$ be the unique ordinal $i < \lambda$ such that $\alpha \in X_i$. We choose the set of points = nodes of G as $Y = \{(\alpha, \beta) : \alpha < \beta < \lambda,\ \mathbf{i}(\beta) \in S$ and $\alpha < \mathbf{i}(\beta)\}$ and let $Y_{<i} = \{(\alpha,\beta) \in Y : \mathbf{i}(\beta) < i\}$.

Stage B: Note that if $\lambda = \kappa^+$, the complete graph with λ nodes is an example (no use of the further information in ⊞). So without loss of generality $\lambda > \kappa^+$.

Now choose a sequence satisfying the following properties (exists by [2, Ch. III]):

⊞ (a) $\bar{C} = \langle C_\delta : \delta \in S\rangle$

(b) $C_\delta \subseteqq \delta = \sup\,(C_\delta)$

(c) $\mathrm{otp}\,(C_\delta) = \kappa$ such that $(\forall \beta \in C_\delta)(\beta + 1, \beta + 2 \notin C_\delta)$

(d) $\bar{C}$ guesses[1] clubs.

Let $\langle \alpha^*_{\delta,\varepsilon} : \varepsilon < \kappa\rangle$ list C_δ in increasing order.

[1] The guessing clubs are used only in Stage D.

For $\delta \in S$ let Γ_δ be the set of sequence $\bar\beta$ such that:

$\boxplus_{\bar\beta}$ (a) $\bar\beta$ has the form $\langle \beta_\varepsilon : \varepsilon < \kappa \rangle$

(b) $\bar\beta$ is increasing with limit δ

(c) $\alpha^*_{\delta,\varepsilon} < \beta_{2\varepsilon+i} < \alpha^*_{\delta,\varepsilon+1}$ for $i < 2, \varepsilon < \kappa$

(d) $\beta_{2\varepsilon+i} \in X_{<\alpha^*_{\delta,\varepsilon+1}} \backslash X_{\leqq \alpha^*_{\delta,\varepsilon}}$ for $i < 2, \varepsilon < \kappa$

(e) $(\beta_{2\varepsilon}, \beta_{2\varepsilon+1}) \in Y$ hence $\in Y_{<\alpha^*_{\delta,\varepsilon+1}} \subseteqq Y_{<\delta}$ for each $\varepsilon < \kappa$

(can ask less).

So $|\Gamma_\delta| \leqq |\delta|^\kappa \leqq |X_\delta| \leqq \lambda$ hence we can choose a sequence $\langle \bar\beta_\gamma : \gamma \in X'_\delta \subseteqq X_\delta \rangle$ listing Γ_δ.

Now we define the set of edges of G: $\mathrm{edge}\,(G) = \Big\{ \big\{ (\alpha_1, \alpha_2), \big(\min\,(C_\delta), \gamma\big) \big\} : \delta \in S,\ \gamma \in X'_\delta$ hence the sequence $\bar\beta_\gamma = \langle \beta_{\gamma,\varepsilon} : \varepsilon < \kappa \rangle$ is well defined and we demand $(\alpha_1, \alpha_2) \in \big\{ (\beta_{\gamma,2\varepsilon}, \beta_{\gamma,2\varepsilon+1}) : \varepsilon < \kappa \big\} \Big\}$.

Stage C: Every subgraph of G of cardinality $< \lambda$ has chromatic number $\leqq \kappa$.

For this we shall prove that:

$$\oplus_1 \qquad \mathrm{ch}\,(G \restriction Y_{<i}) \leqq \kappa \quad \text{for every} \quad i < \lambda.$$

This suffices as λ is regular, hence every subgraph with $< \lambda$ nodes is included in $Y_{<i}$ for some $i < \lambda$.

For this we shall prove more by induction on $j < \lambda$:

$$\oplus_{2,j} \quad \begin{cases} \text{if } i < j, i \notin S, \mathbf{c}_1 \text{ a colouring of } G \restriction Y_{<i}, \mathrm{Rang}\,(\mathbf{c}_1) \subseteqq \kappa \text{ and} \\ u \in [\kappa]^\kappa \textit{ then} \text{ there is a colouring } \mathbf{c}_2 \text{ of } G \restriction Y_{<j} \text{ extending } \mathbf{c}_1 \text{ such} \\ \text{that } \mathrm{Rang}\,\big(\mathbf{c}_2 \restriction (Y_{<j} \backslash Y_{<i})\big) \subseteqq u. \end{cases}$$

Case 1: $j = 0$. Trivial.

Case 2: j successor, $j - 1 \notin S$. By the induction hypothesis without loss of generality $j = i + 1$, but then every node from $Y_j \backslash Y_i$ is an isolated node in $G \restriction Y_{<j}$, because if $\{ (\alpha, \beta), (\alpha', \beta') \}$ is an edge of $G \restriction Y_j$ then $\mathbf{i}(\beta), \mathbf{i}(\beta') \in S$ hence necessarily $\mathbf{i}(\beta) \neq j - 1 = i$, $\mathbf{i}(\beta') \neq j - 1 = i$ hence both (α, β), (α, β') are from Y_i.

Case 3: j successor, $j - 1 \in S$. Let $j - 1$ be called δ so $\delta \in S$. But $i \notin S$ by the assumption in $\oplus_{2,j}$ hence $i < \delta$. Let $\varepsilon(*) < \kappa$ be such that $\alpha^*_{\delta,\varepsilon(*)} > i$.

Let $\langle u_\varepsilon : \varepsilon \leqq \kappa \rangle$ be a sequence of subsets of u, a partition of u to sets each of cardinality κ; actually the only disjointness used is that $u_\kappa \cap \big(\bigcup_{\varepsilon<\kappa} u_\varepsilon \big) = \emptyset$.

We let $i_0 = i$, $i_{1+\varepsilon} = \cup \big\{ \alpha^*_{\delta,\varepsilon(*)+1+\zeta} + 1 : \zeta < 1 + \varepsilon \big\}$ for $\varepsilon < \kappa$, $i_\kappa = \delta$, $i_{\kappa+1} = \delta + 1 = j$.

Note that:

- $\varepsilon < \kappa \Rightarrow i_\varepsilon \notin S_j$.

[Why? For $\varepsilon = 0$ by the assumption on i, for ε successor i_ε is a successor ordinal and for i limit clearly $\mathrm{cf}\,(i_\varepsilon) = \mathrm{cf}\,(\varepsilon) < \kappa$ and $S \subseteqq S^\lambda_\kappa$.]

We now choose $\mathbf{c}_{2,\zeta}$ by induction on $\zeta \leqq \kappa + 1$ such that:

- $\mathbf{c}_{2,0} = \mathbf{c}_1$
- $\mathbf{c}_{2,\zeta}$ is a colouring of $G \restriction Y_{<i_\zeta}$
- $\mathbf{c}_{2,\zeta}$ is increasing with ζ
- Rang $\big(\mathbf{c}_{2,\zeta} \restriction (Y_{<i_{\xi+1}} \backslash Y_{<i_\xi})\big) \subseteqq u_\xi$ for every $\xi < \zeta$.

For $\zeta = 0, \mathbf{c}_{2,0}$ is $\mathbf{c}_1$ so is given.

For $\zeta = \varepsilon + 1 < \kappa$: use the induction hypothesis, possible as necessarily $i_\varepsilon \notin S$.

For $\zeta \leqq \kappa$ limit: take union.

For $\zeta = \kappa + 1$, note that each node b of $Y_{<i_\zeta} \backslash Y_{<i_\kappa}$ is not connected to any other such node and if the node b is connected to a node from $Y_{<i_\kappa}$ then the node b necessarily has the form $\big(\min(C_\delta), \gamma\big)$, $\gamma \in X'_\delta$, hence $\bar\beta_\gamma$ is well defined, so the node $b = \big(\min(C_\delta), \gamma\big)$ is connected in G, more exactly in $G \restriction Y_{\leqq\delta}$ exactly to the κ nodes $\big\{(\beta_{\gamma,2\varepsilon}, \beta_{\gamma,2\varepsilon+1}) : \varepsilon < \kappa\big\}$, but for every $\varepsilon < \kappa$ large enough, $\mathbf{c}_{2,\kappa}\big((\beta_{\gamma,2\varepsilon}, \beta_{\gamma,2\varepsilon+1})\big) \in u_\varepsilon$ hence $\notin u_\kappa$ and $|u_\kappa| = \kappa$ so we can choose a colour.

Case 4: j limit. By the assumption of the claim there is a club e of j disjoint to S and without loss of generality $\min(e) = i$. Now choose $\mathbf{c}_{2,\xi}$ a colouring of $Y_{<\xi}$ by induction on $\xi \in e \cup \{j\}$, increasing with ξ such that Rang $(\mathbf{c}_{2,\xi} \restriction (Y_{<\varepsilon} \backslash Y_{<i})) \subseteqq u$ and $\mathbf{c}_{2,0} = \mathbf{c}_1$

- For $\xi = \min(e) = i$ the colouring $\mathbf{c}_{2,\xi} = \mathbf{c}_{2,i} = \mathbf{c}_1$ is given,
- for ξ successor in e, i.e. $\in \mathrm{nacc}(e) \backslash \{i\}$, use the induction hypothesis with ξ, $\max(e \cap \xi)$ here playing the role of j, i there recalling $\max(e \cap \xi) \in e$, $e \cap S = \emptyset$,
- for $\xi = \sup(e \cap \xi)$ take union.

Lastly, for $\xi = j$ we are done.

Stage D: $\mathrm{ch}(G) > \kappa$. Why? Toward a contradiction, assume $\mathbf{c}$ is a colouring of G with set of colours $\subseteqq \kappa$. For each $\gamma < \lambda$ let $u_\gamma = \big\{\mathbf{c}\big((\alpha,\beta)\big) : \gamma < \alpha < \beta < \lambda$ and $(\alpha,\beta) \in Y\big\}$. So $\langle u_\gamma : \gamma < \lambda\rangle$ is $\subseteqq$-decreasing sequence of subsets of κ and $\kappa < \lambda = \mathrm{cf}(\lambda)$, hence for some $\gamma(*) < \lambda$ and $u_* \subseteqq \kappa$ we have $\gamma \in \big(\gamma(*), \lambda\big) \Rightarrow u_\gamma = u_*$.

Hence $E = \big\{\delta < \lambda : \delta$ is a limit ordinal $> \gamma(*)$ and $(\forall\alpha < \delta)\big(\mathbf{i}(\alpha) < \delta\big)$ and for every $\gamma < \delta$ and $i \in u_*$ there are $\alpha < \beta$ from (γ, δ) such that $(\alpha,\beta) \in Y$ and $\mathbf{c}\big((\alpha,\beta)\big) = i\big\}$ is a club of λ.

Now recall that $\bar C$ guesses clubs hence for some $\delta \in S$ we have $C_\delta \subseteqq E$, so for every $\varepsilon < \kappa$ we can choose $\beta_{2\varepsilon} < \beta_{2\varepsilon+1}$ from $(\alpha^*_{\delta,\varepsilon}, \alpha^*_{\delta,\varepsilon+1})$ such that $(\beta_{2\varepsilon}, \beta_{2\varepsilon+1}) \in Y$ and $\varepsilon \in u_* \Rightarrow \mathbf{c}\big((\beta_{2\varepsilon}, \beta_{2\varepsilon+1})\big) = \varepsilon$. So $\langle \beta_\varepsilon : \varepsilon < \kappa\rangle$ is well defined, increasing and belongs to Γ_δ, hence $\bar\beta_\gamma = \langle \beta_\varepsilon : \varepsilon < \kappa\rangle$ for some $\gamma \in X_\delta$, hence $(\alpha^*_{\delta,0}, \gamma)$ belongs to Y and is connected in the graph to $(\beta_{2\varepsilon}, \beta_{2\varepsilon+1})$ for $\varepsilon < \kappa$. Now if $\varepsilon \in u_*$ then $\mathbf{c}\big((\beta_{2\varepsilon}, \beta_{2\varepsilon+1})\big) = \varepsilon$ hence $\mathbf{c}\big((\alpha^*_{\delta,0}, \gamma)\big) \neq \varepsilon$ for ev-

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ery $\varepsilon \in u_*$, so $\mathbf{c}\big((\alpha^*_{\delta,0},\gamma)\big) \in \kappa\backslash u_*$. But $u_* = u_{\alpha^*_{\delta,0}}$ and $\mathbf{c}\big((\alpha^*_{\delta,0},\gamma)\big) \in \kappa\backslash u_*$, so we get contradiction to the definition of $u_{\alpha^*_{\delta,0}}$. $\square_{1.1}$

Similarly

CLAIM 1.2. *There is an increasing continuous sequence* $\langle G_i : i \leqq \lambda\rangle$ *of graphs each of cardinality* λ^κ *such that* $\mathrm{ch}\,(G_\lambda) > \kappa$ *and* $i < \lambda$ *implies* $\mathrm{ch}\,(G_i) \leqq \kappa$ *and even* $\mathrm{cl}\,(G_i) \leqq \kappa$ when:

⊞ (a) $\lambda = \mathrm{cf}\,(\lambda)$

(b) $S \subseteq \{\delta < \lambda : \mathrm{cf}\,(\delta) = \kappa\}$ *is stationary not reflecting.*

PROOF. Like 1.1 but the X_i are not necessarily $\subseteq \lambda$ or use 2.2. $\square_{1.2}$

§ 2. From almost free

DEFINITION 2.1. Suppose $\eta_\beta \in {}^\kappa\mathrm{Ord}$ for every $\beta < \alpha(*)$ and $u \subseteq \alpha(*)$, and $\alpha < \beta < \alpha(*) \Rightarrow \eta_\alpha \neq \eta_\beta$.

1) We say $\{\eta_\alpha : \alpha \in u\}$ is free *when* there exists a function $h : u \to \kappa$ such that $\big\langle \big\{\eta_\alpha(\varepsilon) : \varepsilon \in \big[h(\alpha),\kappa\big)\big\} : \alpha \in u\big\rangle$ is a sequence of pairwise disjoint sets.

2) We say $\{\eta_\alpha : \alpha \in u\}$ is weakly free *when* there exists a sequence $\langle u_{\varepsilon,\zeta} : \varepsilon,\zeta < \kappa\rangle$ of subsets of u with union u, such that the function $\eta_\alpha \mapsto \eta_\alpha(\varepsilon)$ is a one-to-one function on $u_{\varepsilon,\zeta}$, for each $\varepsilon,\zeta < \kappa$.

CLAIM 2.2. 1) *We have* $\mathrm{INC}_{\mathrm{chr}}(\mu,\lambda,\kappa)$ *and even* $\mathrm{INC}^+_{\mathrm{chr}}(\mu,\lambda,\kappa)$, *see Definition* 0.3(1), (5) *when*:

⊞ (a) $\alpha(*) \in [\mu,\mu^+)$ *and* λ *is regular* $\leqq \mu$ *and* $\mu = \mu^\kappa$

(b) $\bar\eta = \langle \eta_\alpha : \alpha < \alpha(*)\rangle$

(c) $\eta_\alpha \in {}^\kappa\mu$

(d) $\langle u_i : i \leqq \lambda\rangle$ *is a* $\subseteq$*-increasing continuous sequence of subsets of* $\alpha(*)$ *with* $u_\lambda = \alpha(*)$

(e) $\bar\eta \restriction u_\alpha$ *is free* iff $\alpha < \lambda$ iff $\bar\eta \restriction u_\alpha$ *is weakly free.*

2) *We have* $\mathrm{INC}_{\mathrm{chr}}[\mu,\lambda,\kappa]$ *and even* $\mathrm{INC}^+_{\mathrm{chr}}[\mu,\lambda,\kappa]$, *see Definition* 0.3(4) *when*:

$⊞_2$ (a), (b), (c) *as in* ⊞ *from* 2.2

(d) $\bar\eta$ *is not free*

(e) $\bar\eta \restriction u$ *is free when* $u \in \big[\alpha(*)\big]^{<\lambda}$.

PROOF. We concentrate on proving part (1) the chromatic number case; the proof of part (2) and the colouring number are similar. For $\mathscr{A} \subseteq {}^\kappa\mathrm{Ord}$, we define $\tau_{\mathscr{A}}$ as the vocabulary $\{P_\eta : \eta \in \mathscr{A}\} \cup \{F_\varepsilon : \varepsilon < \kappa\}$ where P_η is a unary predicate, F_ε a unary function (will be interpreted as possibly partial).

Without loss of generality for each $i < \lambda$, u_i is an initial segment of $\alpha(*)$ and let $\mathscr{A} = \{\eta_\alpha : \alpha < \alpha(*)\}$ and let $<_{\mathscr{A}}$ be the well ordering $\{(\eta_\alpha,\eta_\beta) : \alpha < \beta < \alpha(*)\}$ of $\mathscr{A}$.

We further let $K_{\mathscr{A}}$ be the class of structures M such that (pedantically, $K_{\mathscr{A}}$ depend also on the sequence $\langle \eta_\alpha : \alpha < \alpha(*)\rangle$:

$\boxplus_1$ (a) $M = (|M|, F^M_\varepsilon, P^M_\eta)_{\varepsilon<\kappa,\eta\in\mathscr{A}}$

(b) $\langle P^M_\eta : \eta \in \mathscr{A}\rangle$ is a partition of $|M|$, so for $a \in M$ let $\eta_a = \eta^M_a$ be the unique $\eta \in \mathscr{A}$ such that $a \in P^M_\eta$

(c) if $a_\ell \in P^M_{\eta_\ell}$ for $\ell = 1, 2$ and $F^M_\varepsilon(a_2) = a_1$ then $\eta_1(\varepsilon) = \eta_2(\varepsilon)$ and $\eta_1 <_{\mathscr{A}} \eta_2$.

Let $K^*_{\mathscr{A}}$ be the class of M such that

$\boxplus_2$ (a) $M \in K_{\mathscr{A}}$

(b) $\|M\| = \mu$

(c) if $\eta \in \mathscr{A}, u \subseteqq \kappa$ and $\eta_\varepsilon <_{\mathscr{A}} \eta, \eta_\varepsilon(\varepsilon) = \eta(\varepsilon)$ and $a_\varepsilon \in P^M_{\eta_\varepsilon}$ for $\varepsilon \in u$ *then* for some $a \in P^M_\eta$ we have $\varepsilon \in u \Rightarrow F^M_\varepsilon(a) = a_\varepsilon$ and $\varepsilon \in \kappa\backslash u \Rightarrow F^M_\varepsilon(a)$ not defined.

Clearly

$\boxplus_3$ there is $M \in K^*_{\mathscr{A}}$.

[Why? As $\mu = \mu^\kappa$ and $|\mathscr{A}| = \mu$.]

$\boxplus_4$ for $M \in K_{\mathscr{A}}$ let G_M be the graph with:

- set of nodes $|M|$
- set of edges $\big\{\{a, F^M_\varepsilon(a)\} : a \in |M|, \varepsilon < \kappa$ when $F^M_\varepsilon(a)$ is defined$\big\}$.

Now

$$\boxplus_5 \begin{cases} \text{if } u \subseteqq \alpha(*),\ \mathscr{A}_u = \{\eta_\alpha : \alpha \in u\} \subseteqq \mathscr{A} \text{ and } \bar\eta \restriction u \text{ is free, and} \\ M \in K_{\mathscr{A}} \textit{ then } G_{M,\mathscr{A}_u} := G_M \restriction (\cup\{P^M_\eta : \eta \in \mathscr{A}_u\}) \text{ has} \\ \text{chromatic number} \leqq \kappa\text{; moreover has colouring number} \leqq \kappa. \end{cases}$$

[Why? Let $h : u \to \kappa$ witness that $\bar\eta \restriction u$ is free and for $\varepsilon < \kappa$ let $\mathscr{B}_\varepsilon := \{\eta_\alpha : \alpha \in u$ and $h(\alpha) = \varepsilon\}$, so $\mathscr{B} = \cup\{\mathscr{B}_\varepsilon : \varepsilon < \kappa\}$, hence it is enough to prove for each $\varepsilon < \kappa$ that $G_{\mu,\mathscr{B}_\varepsilon}$ has chromatic number $\leqq \kappa$. To prove this, by induction on $\alpha \leqq \alpha(*)$ we choose $\mathbf{c}^\varepsilon_\alpha$ such that:

$\boxplus_{5.1}$ (a) $\mathbf{c}^\varepsilon_\alpha$ is a function

(b) $\langle \mathbf{c}_\beta : \beta \leqq \alpha\rangle$ is increasing continuous

(c) $\mathrm{Dom}(\mathbf{c}^\varepsilon_\alpha) = B^\varepsilon_\alpha := \cup\{P^M_{\eta_\beta} : \beta < \alpha$ and $\eta_\beta \in \mathscr{B}_\varepsilon\}$

(d) $\mathrm{Rang}(\mathbf{c}^\varepsilon_\alpha) \subseteqq \kappa$

(e) if $a, b, \in \mathrm{Dom}(\mathbf{c}_\alpha)$ and $\{a, b\} \in \mathrm{edge}(G_M)$ then $\mathbf{c}_\alpha(a) \neq \mathbf{c}_\alpha(b)$.

Clearly this suffices. Why is this possible?

If $\alpha = 0$ let $\mathbf{c}^\varepsilon_\alpha$ be empty, if α is a limit ordinal let $\mathbf{c}^\varepsilon_\alpha = \cup\{\mathbf{c}^\varepsilon_\beta : \beta < \alpha\}$ and if $\alpha = \beta + 1 \wedge \alpha(\beta) \neq \varepsilon$ let $\mathbf{c}_\alpha = \mathbf{c}_\beta$.

Lastly, if $\alpha = \beta + 1 \wedge h(\beta) = \varepsilon$ we define $\mathbf{c}^\varepsilon_\alpha$ as follows for $a \in \mathrm{Dom}(\mathbf{c}^\varepsilon_\alpha)$, $\mathbf{c}^\varepsilon_\alpha(a)$ is:

Case 1: $a \in B^\varepsilon_\beta$. Then $\mathbf{c}^\varepsilon_\alpha(a) = \mathbf{c}^\varepsilon_\beta(a)$.

Case 2: $a \in B^\varepsilon_\alpha \backslash B^\varepsilon_\beta$. Then $\mathbf{c}^\varepsilon_\alpha(a) = \min\big(\kappa\backslash\{\mathbf{c}^\varepsilon_\beta(F^M_\zeta(a)) : \zeta < \varepsilon$ and $F^M_\zeta(a) \in \mathrm{Dom}(\mathbf{c}^\varepsilon_\beta)\}\big)$. This is well defined as:

$\boxplus_{5.2}$ (a) $B^\varepsilon_\alpha = B^\varepsilon_\beta \cup P^M_{\eta_\beta}$

(b) if $a \in B^{\varepsilon}_{\beta}$ then $\mathbf{c}^{\varepsilon}_{\beta}(a)$ is well defined (so case 1 is O.K.)

(c) if $\{a,b\} \in \mathrm{edge}\,(G_M), a \in P^M_{\eta_\beta}$ and $b \in B^{\varepsilon}_{\alpha}$ *then* $b \in B^{\varepsilon}_{\beta}$ and $b \in \{F^M_{\zeta}(a) : \zeta < \varepsilon\}$

(d) $\mathbf{c}^{\varepsilon}_{\alpha}(a)$ is well defined in Case 2, too

(e) $\mathbf{c}^{\varepsilon}_{\alpha}$ is a function from B^{ε}_{α} to κ

(f) $\mathbf{c}^{\varepsilon}_{\alpha}$ is a colouring.

[Why? Clause (a) by $\boxplus_{5.1}(c)$, clause (b) by the induction hypothesis and clause (c) by $\boxplus_1(c) + \boxplus_4$. Next, clause (d) holds as $\{\mathbf{c}^{\varepsilon}_{\beta}(F^M_{\zeta}(a)) : \zeta < \varepsilon$ and $F^M_{\zeta}(a) \in B^{\varepsilon}_{\beta} = \mathrm{Dom}\,(\mathbf{c}^{\varepsilon}_{\beta})\}$ is a set of cardinality $\leqq |\varepsilon| < \kappa$. Clause (e) holds by the choices of the $\mathbf{c}^{\varepsilon}_{\alpha}(a)$'s. Lastly, to check that clause (f) holds assume (a,b) is an edge of $G_M \restriction B^{\varepsilon}_{\alpha}$, for some $\zeta < \kappa$ we have $b = F^M_{\zeta}(a)$, hence $\eta^M_a <_{\mathscr{A}} \eta^M_b$. If $a,b \in B^{\varepsilon}_{\beta}$ use the induction hypothesis. Otherwise, $\zeta < \varepsilon$ by the definition of "h witnesses $\bar\eta \restriction u$ is free" and the choice of B^{ε}_{α} in $\boxplus_{5.1}(c)$. Now use the choice of $\mathbf{c}^{\varepsilon}_{\alpha}(a)$ in Case 2 above.]

So indeed $\boxplus_5$ holds.]

$\boxplus_6$ $\mathrm{chr}\,(G_M) > \kappa$ if $M \in K^*_{\mathscr{A}}$.

Why? Toward contradiction assume $\mathbf{c} : G_M \to \kappa$ is a colouring. For each $\eta \in \mathscr{A}$ and $\varepsilon < \kappa$ let $\Lambda_{\eta,\varepsilon} = \{\nu : \nu \in \mathscr{A},\ \nu <_{\mathscr{A}} \eta,\ \nu(\varepsilon) = \eta(\varepsilon)$ and for some $a \in P^M_{\nu}$ we have $\mathbf{c}(a) = \varepsilon\}$.

Let $\mathscr{B}_{\varepsilon} = \{\eta \in \mathscr{A} : |\Lambda_{\eta,\varepsilon}| < \kappa\}$. Now if $\mathscr{A} \neq \cup\{\mathscr{B}_{\varepsilon} : \varepsilon < \kappa\}$ then pick any $\eta \in \mathscr{A} \setminus \cup\{\mathscr{B}_{\varepsilon} : \varepsilon < \kappa\}$ and by induction on $\varepsilon < \kappa$ choose $\nu_{\varepsilon} \in \Lambda_{\eta,\varepsilon} \setminus \{\nu_{\zeta} : \zeta < \varepsilon\}$, possible as $\eta \notin \mathscr{B}_{\varepsilon}$ by the definition of $\mathscr{B}_{\varepsilon}$. By the definition of $\Lambda_{\eta,\varepsilon}$ there is $a_{\varepsilon} \in P^M_{\nu_{\varepsilon}}$ such that $\mathbf{c}(\nu_{\varepsilon}) = \varepsilon$. So as $M \in K^*_{\mathscr{A}}$ there is $a \in P^M_{\eta}$ such that $\varepsilon < \kappa \Rightarrow F^M_{\varepsilon}(a) = a_{\varepsilon}$, but $\{a, a_{\varepsilon}\} \in \mathrm{edge}\,(G_M)$ hence $\mathbf{c}(a) \neq \mathbf{c}(a_{\varepsilon}) = \varepsilon$ for every $\varepsilon < \kappa$, contradiction. So $\mathscr{A} = \cup\{\mathscr{B}_{\varepsilon} : \varepsilon < \kappa\}$.

For each $\varepsilon < \kappa$ we choose $\zeta_{\eta} < \kappa$ for $\eta \in \mathscr{B}_{\varepsilon}$ by induction on $<_{\mathscr{A}}$ such that $\zeta_{\eta} \notin \{\zeta_{\nu} : \nu \in \Lambda_{\eta,\varepsilon} \cap \mathscr{B}_{\varepsilon}\}$. Let $\mathscr{B}_{\varepsilon,\zeta} = \{\eta \in \mathscr{B}_{\varepsilon} : \zeta_{\eta} = \zeta\}$ for $\varepsilon, \zeta < \kappa$ so $\mathscr{A} = \cup\{\mathscr{B}_{\varepsilon,\zeta} : \varepsilon, \zeta < \kappa\}$ and clearly $\eta \mapsto \eta(\varepsilon)$ is a one-to-one function with domain $\mathscr{B}_{\varepsilon,\zeta}$, contradiction to "$\bar\eta = \bar\eta \restriction u_{\lambda}$ is not weakly free". $\square_{2.2}$

Observation 2.3. 1) *If $\mathscr{A} \subseteq {}^{\kappa}\mu$ and $\eta \neq \nu \in \mathscr{A} \Rightarrow (\forall^{\infty}\varepsilon < \kappa)\ (\eta(\varepsilon) \neq \nu(\varepsilon))$ then $\mathscr{A}$ is free iff $\mathscr{A}$ is weakly free.*

2) *The assumptions of 2.2(2) hold when: $\mu \geqq \lambda > \kappa$ are regular, $S \subseteq S^{\mu}_{\kappa}$ stationary, $\bar\eta = \langle \eta_{\delta} : \delta \in S\rangle$, η_{δ} an increasing sequence of ordinals of length κ with limit δ such that $u \subseteq [\lambda]^{<\lambda} \Rightarrow \langle \mathrm{Rang}\,(\eta_{\delta}) : \eta \in u\rangle$ has a one-to-one choice function.*

Conclusion 2.4. *Assume that for every graph G, if $H \subseteq G \wedge |H| < \lambda \Rightarrow \mathrm{chr}\,(H) \leqq \kappa$ then $\mathrm{chr}\,(G) \leqq \kappa$.*

Then:

(A) *if $\mu > \kappa = \mathrm{cf}\,(\mu)$ and $\mu \geqq \lambda$ then $\mathrm{pp}\,(\mu) = \mu^+$*

(B) *if* $\mu > \mathrm{cf}(\mu) \geqq \kappa$ *and* $\mu \geqq \lambda$ *then* $\mathrm{pp}(\mu) = \mu^+$, *i.e. the strong hypothesis*

(C) *if* $\kappa = \aleph_0$ *then above* λ *the SCH holds.*

PROOF. *Clause (A):* By 2.2 and [2, Ch. II], [2, Ch. IX, §1].

Clause (B): Follows from (A) by [2, Ch. VIII, §1].

Clause (C): Follows from (B) by [2, Ch. IX, §1]. $\square_{2.4}$

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