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ON FINITE RIGID STRUCTURES

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Abstract. The main result of this paper is a probabilistic construction of finite rigid structures. It yields a finitely axiomatizable class of finite rigid structures where no $L^{\omega}_{\infty,\omega}$ formula with counting quantifiers defines a linear order.

§1. Introduction. In this paper, structures are finite and of course vocabularies are finite as well. A class is always a collection of structures of the same vocabulary which is closed under isomorphisms.

An r-ary *global relation* on a class K is a function ρ that associates an r-ary relation ρ_A with each structure $A \in K$ in such a way that every isomorphism from A onto whatever structure B extends to an isomorphism from the structure (A, ρ_A) onto the structure (B, ρ_B) [2].

Recall that a structure is *rigid* if it has no nontrivial automorphisms. If a binary global relation $<$ defines a linear order in a class K (that is, on each structure in K) then every structure in K is rigid. Indeed, suppose that θ is an automorphism of a structure $A \in K$ and let a be an arbitrary element of A. Since

$$A \models \theta(x) < \theta(a) \iff A \models x < a,$$
$$A \models \theta(x) > \theta(a) \iff A \models x > a,$$

the number of elements preceding $\theta(a)$ in the linear order $<_A$ equals the number of elements preceding a. Hence $\theta(a) = a$.

Conversely, if every structure in a class K is rigid then some global relation ρ defines a linear order on each structure in K. The question arises how easy it is to define such an order. Alex Stolboushkin constructed a finitely axiomatizable class of rigid structures such that no first-order formula defines a linear order in K [4]. Anuj Dawar conjectured that, for every finitely axiomatizable class K of rigid structures, some formula in the fixed-point extension FO+FP of first-logic defines a linear order in K [1]. Using the probabilistic method, we refute Dawar's conjecture and construct a finitely axiomatizable class of rigid structures where no linear order is definable even in the extension $L^{\omega}_{\infty,\omega} + C$ of logic $L^{\omega}_{\infty,\omega}$ with counting quantifiers; see Theorem 4.1. (It is well-known that every global relation definable in FO+FP is definable in $L^{\omega}_{\infty,\omega}$.) At the end of Section 4, we answer a question of Scott Weinstein [5] related to rigid structures.

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To make this paper self-contained, we provide a reminder on $L^{\omega}_{\infty,\omega} + C$ in the rest of this section. As in a popular version of first-order logic, $L^{\omega}_{\infty,\omega}$ formulas are built from atomic formulas by means of negations, conjunctions, disjunctions, the existential quantifier and the universal quantifier. The only difference is that $L^{\omega}_{\infty,\omega}$ allows one to form the conjunction and the disjunction of an arbitrary set S of formulas provided that the total number of variables in all S-formulas is finite. $L^{\omega}_{\infty,\omega} + C$ is the extension of $L^{\omega}_{\infty,\omega}$ by means of *counting quantifiers* $(\exists 2x)$, $(\exists 3x)$, etc. The semantics is obvious. $L^{k}_{\infty,\omega}$ (resp. $L^{k}_{\infty,\omega} + C$) is the fragment of $L^{\omega}_{\infty,\omega}$ (resp. $L^{\omega}_{\infty,\omega} + C$) where formulas use at most k variables. The counting quantifiers are useful because of the restriction on the number of variables.

There is a pebble game $G^k(A, a_1, \dots, a_l; B, b_1, \dots, b_l)$ appropriate to $L^{k}_{\infty,\omega} + C$ [3]. Here A and B are structures of the same purely relational vocabulary, $l \le k$ and each a_i (respectively b_i) is an element of A (respectively B). Often $l = 0$. For explanatory purposes, we pretend that $(A, a_1, \dots, a_l)$ is located on the left and $(B, b_1, \dots, b_l)$ is located on the right, but in fact A and B may be the same structure.

The game is played by Spoiler and Duplicator. For each $i = 1, \dots, k$, there are two pebbles marked by i: the left i-pebble and the right i-pebble. Initially the left (respectively the right) i-pebble with $i \le l$ covers a_i (respectively b_i), and the other pebbles are off the board. After any number of rounds, for every i, either both i-pebbles are off the board or else the left i-pebble covers an element of A and the right i-pebble covers an element of B; the pebbles *define a partial isomorphism* if (a) the left i-pebble and the left j-pebble cover different elements if and only if the right i-pebble and the right j-pebble cover different elements, and (b) the map that takes any left-pebble-covered element of A to the element of B covered by the right pebble of the same number is a partial isomorphism. A round of $G^k(A, B)$ is played as follows.

1. If the pebbles do not define a partial isomorphism, then the game is over; Spoiler has won and Duplicator has lost. Otherwise Spoiler chooses a number i; if the i-pebbles are on the board, they are taken off the board. Then Spoiler chooses *left* or *right* and a nonempty subset X of the corresponding structure.
2. Duplicator chooses a subset Y on the other side such that $\|Y\| = \|X\|$. If such Y does not exist, then the game is over; Spoiler has won and Duplicator has lost.
3. Spoiler puts an i-pebble on an element $y \in Y$. It is the right i-pebble if Spoiler has chosen *left*, and the left i-pebble otherwise.
4. Duplicator puts the other i-pebble on an element $x \in X$.

Duplicator wins a play of the game if the number of rounds in the play is infinite.

THEOREM 1.1 ([3]). *No sentence $\varphi(v_1, \dots, v_l)$ in $L^{k}_{\infty,\omega} + C$ distinguishes between $(A, a_1, \dots, a_l)$ and $(B, b_1, \dots, b_l)$ if Duplicator has a winning strategy in the game $G^k(A, a_1, \dots, a_l; B, b_1, \dots, b_l)$.*

It is not hard to prove the theorem by induction on φ. The converse implication is true too [3] but we will not use it.

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§2. Hypergraphs.

2.1. Preliminaries. In this paper, a *hypergraph* is a pair $H = (U, T)$ where $U = |H|$ is a nonempty set and T is a collection of 3-element subsets of U; elements of U are *vertices* of H, and elements of T are *hyperedges* of H. H can be seen as a structure with universe U and irreflexive symmetric ternary relation $\{(x, y, z) : \{x, y, z\} \in T\}$.

Every nonempty subset X of U gives a *sub-hypergraph*

$$H|X = \Big(X, \{h : h \in T \wedge h \subseteq X\}\Big)$$

of H. The number of hyperedges in $H|X$ will be called the *weight* of X and denoted $[X]$. As usual, the number of vertices of X is called the cardinality of X and denoted $\|X\|$.

Vertices x, y of a hypergraph H are *adjacent* if there is a hyperedge $\{x, y, z\}$; the vertex z *witnesses* that x and y are adjacent.

DEFINITION 2.1.1. A vertex set X is *dense* if $\|X\| \leq 2[X]$. A hypergraph is *l-meager* if it has no dense vertex sets of cardinality $\leq 2l$. ⊣

LEMMA 2.1.1. *In a 2-meager hypergraph, the intersection of any two distinct hyperedges contains at most one vertex.*

PROOF. If $\|h_1 \cap h_2\| = 2$ then $h_1 \cup h_2$ is dense. ⊣

DEFINITION 2.1.2. A vertex set X is *super-dense* or *immodest* if $\|X\| < 2[X]$. A hypergraph is *l-modest* if it has no super-dense sets of cardinality $\leq 2l$. ⊣

It follows that if X is a dense vertex set of cardinality $\leq 2l$ in an l-modest hypergraph then $\|X\| = 2[X]$ and in particular $\|X\|$ is even.

2.2. Cycles.

DEFINITION 2.2.1. A sequence $x_1, \ldots, x_k$ of $k \geq 3$ distinct vertices is a *weak cycle* of length k if

1. each x_i is adjacent to x_{i+1}, and
2. either $k > 3$ or else $k = 3$ but $\{x_1, x_2, x_3\}$ is not a hyperedge

where the subscripts are numbers modulo k. ⊣

We will index elements of a weak cycle of length k with numbers modulo k.

DEFINITION 2.2.2. Let $k \geq 3$. A weak cycle $x_1, \ldots, x_k$ is a *cycle* of length k if no triple x_i, x_{i+1}, x_{i+2} forms a hyperedge. A corresponding *witnessed cycle* of length k is a vertex sequence $x_1, \ldots, x_k, y_1, \ldots, y_k$ where each y_i witnesses that x_i is adjacent to x_{i+1}. ⊣

DEFINITION 2.2.3. A vertex sequence x_1, x_2 is a *cycle* of length 2 if there are distinct vertices y_1, y_2 different from x_1, x_2 such that $\{x_1, x_2, y_1\}$ and $\{x_2, x_1, y_2\}$ are hyperedges; the sequence x_1, x_2, y_1, y_2 is a corresponding *witnessed cycle* of length 2. ⊣

LEMMA 2.2.1. *Every weak cycle includes a cycle. More exactly, some (not necessarily contiguous) subsequence of a weak cycle is a cycle. Thus, an acyclic hypergraph (that is, a hypergraph without any cycles) has no weak cycles.*

PROOF. We prove the lemma by induction on the length of a weak cycle. Let $x_1, \dots, x_k$ be a weak cycle that is not a cycle, so that some x_i, x_{i+1}, x_{i+2} is a hyperedge; without loss of generality, $i = 1$. Then the sequence $x_1, x_3, \dots, x_k$ of length $k - 1$ is a weak cycle or a hyperedge. In the first case, use the induction hypothesis. In the second, $k = 4$ and x_1, x_3 form a cycle witnessed by x_2 and x_4. ⊣

THEOREM 2.2.1. *In any l-modest graph,*

- *every minimal dense set of cardinality $2k \leq 2l$ forms a witnessed cycle of length k, and*
- *the vertices of every witnessed cycle of length $k \leq l$ form a minimal dense set of cardinality $2k$.*

The theorem clarifies the structure of minimal dense sets of cardinality $\leq 2l$ which play an important role in our probabilistic construction. However the theorem itself will not be used and can be skipped. The rest of this subsection is devoted to proving the theorem.

PROOF. The case $l = 1$ is trivial: there are no dense sets of cardinality 2 and there are no cycles of length 1. Fix some number $l \geq 2$ and restrict attention to l-modest hypergraphs.

LEMMA 2.2.2. *For every vertex set X, the following statements are equivalent:*

1. *X is a dense set of cardinality 4.*
2. *X is a minimal dense set of cardinality 4*
3. *The vertices of X form a witnessed cycle of length 2.*

PROOF. A set of cardinality ≤ 3 cannot be dense. Thus 1 is equivalent to 2. It is easy to see that 3 implies 1. It remains to check that 1 implies 3. Suppose 1. By l-modesty $[X] = 2$. Thus, X includes two hyperedges h_1 and h_2. Clearly, $h_1 \cup h_2 = X$ and $\|h_1 \cap h_2\| = 2$. It is easy to see that the vertices of $h_1 \cap h_2$ form a cycle and the vertices of X form a corresponding witnessed cycle. ⊣

In the rest of this subsection, $3 \leq k \leq l$.

LEMMA 2.2.3. *Every witnessed cycle $x_1, \dots, x_k, y_1, \dots, y_k$ forms a dense set of cardinality $2k$.*

PROOF. Let $W = \{x_1, \dots, x_k, y_1, \dots, y_k\}$. It suffices to check that the k hyperedges $\{x_i, x_{i+1}, y_i\}$ are all distinct. For then, using l-modesty, we have

$$2k \leq 2[W] \leq \|W\| \leq 2k.$$

If $i \neq j$ but $\{x_i, x_{i+1}, y_i\} = \{x_j, x_{j+1}, y_j\}$ then either $x_j = x_{i+1}$ or else $x_j = y_i$ in which case $x_{j+1} = x_i$. Without loss of generality, $x_j = x_{i+1}$ and therefore $j = i+1$ modulo k. If also $x_{j+1} = x_i$ then $i = j + 1 = i + 2$ modulo k which contradicts the fact that $k > 2$. Thus $x_{j+1} = y_i$, so that $y_i = x_{i+2}$ and therefore $\{x_i, x_{i+1}, x_{i+2}\}$ is a hyperedge which contradicts the definition of cycles. ⊣

LEMMA 2.2.4. *Every minimal dense vertex set of cardinality* $2k$ *forms a witnessed cycle of length* k.

PROOF. Without loss of generality, the given minimal dense vertex set contains all vertices of the given hypergraph H; if not, restrict attention to the corresponding sub-hypergraph of H.

It suffices to prove that H includes a weak cycle of length $\leq k$. For then, by Lemma 2.2.1, H includes a cycle of length $\leq k$. If a witnessed version of the cycle contains less than $2k$ vertices then, by the previous lemma, H contains a proper dense subset, which is impossible.

By contradiction suppose that H does not include a weak cycle of length k.

CLAIM 2.2.1. *A hypergraph of cardinality* $2k$ *is acyclic if no proper vertex set is dense and there is no weak cycle of length* $\leq k$.

PROOF. By contradiction suppose that there is a cycle of length $m > k$ and choose the minimal possible m. Consider a witnessed cycle $x_1, \dots, x_m, y_1, \dots, y_m$.

Since the hypergraph has $< 2m$ vertices, some y_i occurs in $x_1, \dots, x_m$. Without loss of generality, $y_1 = x_j$ for some j, so that $\{x_1, x_2, x_j\}$ is a hyperedge and therefore j differs from 1, 2 and 3. But then the sequence $x_2, \dots, x_j$ is a weak cycle and therefore includes a cycle of length $< m$. This contradicts the choice of m. ⊣

It follows that H is acyclic.

CLAIM 2.2.2. *Any acyclic hypergraph of positive weight contains a hyperedge* Y *such that at most one vertex of* Y *belongs to any other hyperedge.*

PROOF. Let $s = (x_1, \dots, x_k)$ be a longest vertex sequence such that (i) for every $i < k$, x_i is adjacent to x_{i+1}, and (ii) if $k > 2$ then, for no $i < k - 1$, the triple x_i, x_{i+1}, x_{i+2} forms a hyperedge. Since the hypergraph has hyperedges, $k \geq 2$. If $k = 2$ then all hyperedges are disjoint and the claim is obvious. Suppose that $k \geq 3$.

Since x_{k-1} and x_k are adjacent, there is a vertex y such that $Y = \{x_{k-1}, x_k, y\}$ is a hyperedge. Since there are no cycles of length 2, y is defined uniquely. We prove that neither x_k nor y belongs to any other hyperedge. Vertex y does not occur in $x_1, \dots, x_k$; otherwise the segment $[y, x_{k-1}]$ of s is a weak cycle. Notice that y can replace x_k in s. Thus it suffices to prove that x_k does not belong to any other hyperedge.

By contradiction, suppose that a hyperedge $Z \neq Y$ contains x_k and let $z \in Z - Y$. By the maximality of s, it contains z; otherwise s can be extended by z. But then the final segment $[z, x_k]$ of s forms a weak cycle. ⊣

CLAIM 2.2.3. *No acyclic hypergraph is dense.*

PROOF. Induction on the cardinality of the given hypergraph I. The claim is trivial if $[I] = 0$. Suppose that $[I] > 0$. By the previous claim, I has a hyperedge $X = \{x, y, z\}$ such that neither y nor z belongs to any other hyperedge. Let J be the sub-hypergraph of I obtained by removing vertices y and z. Using the induction hypothesis, we have

$$\|I\| = \|J\| + 2 > 2[J] + 2 = 2([I] + 1) = 2[I]. \qquad \dashv$$

By Claim 2.2.3, H is not dense which gives the desired contradiction. Lemma 2.2.4 is proved $\dashv$

LEMMA 2.2.5. *Every witnessed cycle of length k forms a minimal dense set.*

PROOF. Let W be the set of the vertices of the given witnessed cycle of length k. By Lemma 2.2.3, W is a dense set of cardinality $2k$. By the l-modesty of the hypergraph, W contains precisely k hyperedges. It is easy to see now that every proper subset X of W is acyclic; by Claim 2.2.3, X is not dense. $\dashv$

Lemmas 2.2.2–2.2.5 imply the theorem. $\dashv$

2.3. Green and red vertices. Fix $l \geq 2$. For brevity, we use the following terminology. A minimal dense vertex set of cardinality $\leq 2l$ is a *red block*. A vertex is *red* if it belongs to a red block; otherwise it is *green*. A hyperedge is *green* if it consists of green vertices. The *green sub-hypergraph* is the sub-hypergraph of green vertices.

Consider a sufficiently modest hypergraph. More precisely, we require that the hypergraph is $(2l+2)$-modest. It follows that, for every dense set V of cardinality $\leq 4l+4$, $\|V\| = 2[V]$.

LEMMA 2.3.1. *Distinct red blocks are disjoint.*

PROOF. We suppose that distinct red blocks X and Y have a nonempty intersection Z and prove that the union $V = X \cup Y$ is immodest. Indeed, Z is a proper subset of X; otherwise Y is not a minimal dense set. Therefore Z is not dense and

$$\begin{aligned} \|V\| &= \|X\| + \|Y\| - \|Z\| = 2[X] + 2[Y] - \|Z\| \\ &< 2[X] + 2[Y] - 2[Z] = 2([X] + [Y] - [Z]) \leq 2[V]. \end{aligned}$$ $\dashv$

LEMMA 2.3.2. *Adjacent red vertices belong to the same red block.*

PROOF. Suppose that adjacent red vertices x and y belong to different red blocks X and Y respectively, and let h be a hyperedge containing x and y. We show that the set $V = X \cup Y \cup h$ is immodest. Indeed,

$$\|V\| \leq \|X\| + \|Y\| + 1 = 2[X] + 2[Y] + 1 < 2([X] + [Y] + 1) \leq 2[V].$$ $\dashv$

LEMMA 2.3.3. *No green vertex is adjacent to two (or more) red vertices.*

PROOF. By contradiction suppose that a green vertex b is adjacent to distinct red vertices x and x'. Let X, X' be the red blocks of x, x' respectively, h be a hyperedge containing b and x, and h' be a hyperedge containing b and x'. We show that the set $V = X \cup X' \cup h \cup h'$ is immodest. By the previous lemma, $h = h'$ implies $X = X'$.

If $h = h'$ then

$$\|V\| = \|X\| + 1 = 2[X] + 1 < 2([X] + 1) \leq [V].$$

If $h \neq h'$ but $X = X'$ then

$$\|V\| \leq \|X\| + 3 = 2[X] + 3 < 2([X] + 2) \leq 2[V].$$

If $X \neq X'$ then

$$\|V\| \leq \|X\| + \|X'\| + 3 = 2[X] + 2[X'] + 3 < 2([X] + [X'] + 2) \leq 2[V].$$ $\dashv$

DEFINITION 2.3.1. A hypergraph is *odd* if, for every nonempty vertex set X, there is a hyperedge h such that $\|h \cap X\|$ is odd. ⊣

For future reference, some assumptions are made explicit in the following theorem.

THEOREM 2.3.1. *Suppose that a hypergraph H of cardinality n satisfies the following conditions where $n' < n$.*

- *H is $(2l + 2)$-modest.*
- *The number of red vertices is $< n'$.*
- *Every vertex set of cardinality $\geq n'$ includes a hyperedge.*
- *For every nonempty vertex set X of cardinality $< n'$, there exist a vertex $x \in X$ and distinct hyperedges h_1, h_2 such that $h_1 \cap X = h_2 \cap X = h_1 \cap h_2 = \{x\}$.*

Then the green sub-hypergraph of H is an odd, l-meager hypergraph of cardinality $> n - n'$.

PROOF. Since the green sub-hypergraph G is obtained from H by removing all dense vertex sets of cardinality $\leq 2l$, G is l-meager. By the second condition, $\|G\| > n - n'$. To check that G is odd, let X be a nonempty set of green vertices. If $\|X\| \geq n'$, use the third condition. Suppose that $\|X\| < n'$ and let x, h_1, h_2 be as in the fourth condition; both $\|h_1 \cap X\|$ and $\|h_2 \cap X\|$ are odd. If at least one of the two hyperedges is green, we are done. Otherwise x is adjacent to different red vertices which, by Lemma 2.3.3, contradicts the first condition. ⊣

2.4. Attraction.

DEFINITION 2.4.1. In an arbitrary hypergraph, a vertex set X *attracts* a vertex y if there are vertices x_1, x_2 in X such that $\{x_1, x_2, y\}$ is a hyperedge. X is *closed* if it contains all elements attracted by X. As usual, the *closure* $\bar{X}$ of X is the least closed set containing X. ⊣

LEMMA 2.4.1. *In an l-meager hypergraph, if X is a vertex set of cardinality $k \leq l$ then $\|\bar{X}\| < 2k$.*

PROOF. Construct sets $X_0, \dots, X_m$ as follows. Set $X_0 = X$. Suppose that sets $X_0, \dots, X_i$ have been constructed. If X_i is closed, set $m = i$ and terminate the construction process. Otherwise pick a hyperedge h such that $\|h \cap X_i\| = 2$ and let $X_{i+1} = h \cup X_i$. We show that $m < k$.

By contradiction suppose that $m \geq k$. Check by induction on i that $\|X_i\| = k + i$ and $[X_i] \geq i$. Since the hypergraph is l-meager, we have: $2[X_k] < \|X_k\| = 2k \leq 2[X_k]$. This gives the desired contradiction. ⊣

LEMMA 2.4.2. *Suppose that Y is a vertex set of cardinality $\leq k$ in a k-meager hypergraph and $p = \|\bar{Y} - Y\|$. Then $p < k$ and there is an ordering $z_1, \dots, z_p$ of $\bar{Y} - Y$ such that each z_j is attracted by $Y \cup \{z_i : i < j\}$.*

PROOF. By the previous lemma, $\|\bar{Y}\| < 2\|Y\|$. Hence $p = \|\bar{Y} - Y\| < \|Y\| \leq k$. Choose elements z_j by induction on j. Suppose that $1 \leq j \leq p$ and all elements z_i with $i < j$ have been chosen. Since $\|\bar{Y}\| = \|Y\| + p$, the set $Z_{j-1} = Y \cup \{z_i : i < j\}$ is not closed. Let z_j be any element in $\bar{Y} - Y$ attracted by Z_{j-1}. ⊣

THEOREM 2.4.1. *Suppose that X is a vertex set of cardinality $< k$ in a k-meager hypergraph, $z_0 \notin \bar{X}$, $Y = \bar{X} \cup \{z_0\}$, $Z = \bar{Y}$ and $p = \|Z - Y\|$. Then $p < k$ and there is an ordering $z_1, \dots, z_p$ of $Z - Y$ such that, for every $j > 0$, z_j is attracted by $Y \cup \{z_i : 1 \le i < j\}$ and there is a unique hyperedge h_j witnessing the attraction.*

PROOF. The set $X \cup \{z_0\}$ is of cardinality $\le k$ and its closure includes Y and therefore includes Z. Using Lemma 2.4.2, we have

$$p = \|Z - Y\| \le \|Z - (X \cup \{z_0\})\| < k.$$

Construct sequence $z_1, \dots, z_p$ as in the proof of the previous lemma. For any $j > 0$, let h_j be a hyperedge witnessing that $Z_{j-1} = Y \cup \{z_i : 1 \le i < j\}$ attracts y_j. By contradiction suppose that, for some positive $j \le p$, some hyperedge $h_j' \ne h_j$ witnesses that z_j is attracted by Z_{j-1}. Let $S = \{h_1, \dots, h_j, h_j'\}$. We show that $V = \bigcup S$ is a dense set of cardinality $\le 2k$ which contradicts the k-meagerness of the hypergraph.

Since V contains all hyperedges in S, $[V] \ge j + 1$. Since none of the vertices $z_1, \dots, z_j$ is attracted by $\bar{X}$, $\|h \cap \bar{X}\| \le 1$ for all $h \in S$ and thus $\|V \cap \bar{X}\| \le j + 1$. We have

$$\|V\| \le \|(V \cap \bar{X}) \cup \{z_0, \dots, z_j\}\| \le (j+1) + (j+1) \le 2 \cdot [V].$$

Thus V is a dense set of cardinality $\|V\| \le 2(j+1) \le 2(p+1) \le 2k$. ⊣

§3. Existence.

THEOREM 3.1. *For any integers $l \ge 2$ and $N > 0$, there exists an odd l-meager hypergraph of cardinality $> N$.*

In fact, for every $l \ge 2$ and every sufficiently large N, there exists an odd l-meager hypergraph of cardinality precisely N but we do not need the stronger result.

PROOF. Fix $l \ge 2$ and $N > 0$ and choose a positive real $\varepsilon < 1/(2l + 3)$. Let n range over integers $\ge 2N$ divisible by 4 and U be the set of positive integers $\le n$. For each 3-element subset a of U, flip a coin with probability $p = n^{-2+\varepsilon}$ of heads, and let T be the collection of triples a such that the coin comes up heads. This gives a random hypergraph $H = (U, T)$.

We will need the following simple inequality. In this section, $\exp \alpha = e^\alpha$ and $\log \alpha = \log_e \alpha$.

CLAIM 3.1. *For all positive reals q, r, s such that $p^r < 1/2$,*

$$\exp(-2qn^{s-2r+r\varepsilon}) < (1 - p^r)^{qn^s} < \exp(-qn^{s-2r+r\varepsilon}). \tag{1}$$

PROOF. Suppose that $0 < \alpha < 1/2$. By Mean Value Theorem applied to function $f(t) = -\log(1 - t)$ on the interval $[0, \alpha]$, there is a point $t \in (0, \alpha)$ such

$$f(\alpha) - f(0) = -\log(1 - \alpha) = (\alpha - 0)f'(t) = \alpha/(1 - t).$$

Since $\alpha < \alpha/(1-t) < \alpha/(1-\alpha) < \alpha/(1-1/2) = 2\alpha$, we have $\alpha < -\log(1-\alpha) < 2\alpha$ and therefore $e^{-2\alpha} < 1 - \alpha < e^{-\alpha}$. Now let $\alpha = p^r$ and raise the terms to power qn^s. ⊣

Call an event $E = E(n)$ *almost sure* if the probability $\mathbf{P}[E]$ tends to 1 as n grows to infinity. We prove that, almost surely, H satisfies the conditions of Theorem 2.3.1 with $n' = n/4$ and therefore the green subgraph of H is an odd l-meager graph of cardinality $> N$.

LEMMA 3.1. *Almost surely, H is $(2l+2)$-modest.*

PROOF. Since l is fixed, it suffices to prove that, for each particular $m \leq 4l+4$, the probability q_m that there is a super-dense vertex set of cardinality m is $o(1)$. A vertex set X of cardinality m is super-dense if $m < 2[X]$, that is, if X includes more than $m/2$ hyperedges. Let k be the least integer that exceeds $m/2$. Then $m \leq 2k-1$ and therefore $n^{m-2k} \leq n^{-1}$. Also $2k-2 \leq m \leq 4l+4$, so that $k \leq 2l+3$ and $k\varepsilon < 1$. Let $M = \binom{m}{3}$ and $c = \binom{M}{k}$. We have

$$q_m < \binom{n}{m} \cdot c \cdot p^k < c \cdot n^m \cdot n^{(-2+\varepsilon)k} = c \cdot n^{m-2k+k\varepsilon} \leq c \cdot n^{-1+k\varepsilon} = o(1). \qquad \dashv$$

LEMMA 3.2. *Almost surely, the number of red vertices is $< n/4$.*

PROOF. It suffices to prove that the expected number of red vertices is $o(n)$. Indeed, let r be the number of red vertices and s ranges over the integer interval $[n/4, n]$. Then

$$\mathbf{E}[r] \geq \sum_s s \cdot \mathbf{P}[r = s] \geq \frac{n}{4} \sum_s \mathbf{P}[r = s] = \frac{n}{4}\mathbf{P}[r \geq \frac{n}{4}]$$

and thus $\mathbf{P}[r \geq \frac{n}{4}]$ tends to 0 if $\mathbf{E}[r] = o(n)$.

Furthermore, it suffices to show that, for each particular $m \leq 2l$, the expected number $f(m)$ of vertices v such that v belongs to a dense set X of cardinality m is $o(n)$. Let $k = \lceil m/2 \rceil$. Then $m \leq 2k$ and therefore $n^{m-2k} \leq 1$. Also, $2k \leq m+1 < 2l+1$ and therefore $k < l+1$ and $k\varepsilon < 1$. Let $M = \binom{m}{3}$ and $c = \binom{M}{k}$. We have

$$\begin{aligned} f(m) &\leq n \cdot \binom{n-1}{m-1} c p^k < n \cdot n^{m-1} c p^k \\ &= c \cdot n^m p^k = c \cdot n^{m-2k+k\varepsilon} \leq c \cdot n^{k\varepsilon} = o(n). \end{aligned} \qquad \dashv$$

LEMMA 3.3. *Almost surely, every vertex set of cardinality $\geq n/4$ includes a hyperedge.*

PROOF. Chose a real $c > 0$ so small that $cn^3 \leq \binom{n/4}{3}$ and let q be the probability that there exists a vertex set of cardinality $\geq n/4$ which does not include any hyperedges. Using inequality (1), we have

$$q < 2^n \cdot (1-p)^{\binom{n/4}{3}} < e^n \cdot (1-p)^{cn^3} < e^n \cdot \exp(-cn^{1+\varepsilon}) = o(1). \qquad \dashv$$

LEMMA 3.4. *Almost surely, for every nonempty vertex set X of cardinality $< n/4$, there exist a vertex $x \in X$ and hyperedges h_1, h_2 such that*

$$h_1 \cap X = h_2 \cap X = h_1 \cap h_2 = \{x\}.$$

PROOF. Let X range over nonempty vertex sets of cardinality $< n/4$, Y be the collection of even numbers $y \in U - X$, and Z be the collection of odd numbers $z \in U - X$. Clearly, $\|Y\| \geq n/4$ and $\|Z\| \geq n/4$.

Let x range over X, $\sigma(x, X)$ mean that there exist vertices $y_1, y_2 \in Y$ such that $\{x, y_1, y_2\}$ is a hyperedge, and $\tau(x, X)$ mean that there exist vertices $z_1, z_2 \in Z$ such that $\{x, z_1, z_2\}$ is a hyperedge. Call X *bad* if the conjunction $\sigma(x, X) \wedge \tau(x, X)$ fails for all x. We prove that, almost surely, there are no bad vertex sets.

Choose a real $c > 0$ so small that $cn^2 < \binom{n/4}{2}$. For given X and x,

$$\mathbf{P}[\neg\sigma(x, X)] = (1-p)^{\binom{\|Y\|}{2}} \leq (1-p)^{\binom{n/4}{2}} < (1-p)^{cn^2} < \exp[-cn^\varepsilon].$$

The last inequality follows from inequality (1). Similarly, $\mathbf{P}[\neg\tau(x, X)] < \exp[-cn^\varepsilon]$. Hence

$$\mathbf{P}[\neg\sigma(x, X) \vee \neg\tau(x, Y)] \leq \mathbf{P}[\neg\sigma(x, X)] + \mathbf{P}[\neg\tau(x, X)] < 2\exp[-cn^\varepsilon] = \exp[\log 2 - cn^\varepsilon].$$

If $\|X\| = m$ then

$$\mathbf{P}[X \text{ is bad }] < \left(\exp[\log 2 - cn^\varepsilon]\right)^m = \exp[m(\log 2 - cn^\varepsilon)].$$

For each $m < n/4$, let q_m be the probability that there is a bad vertex set of cardinality m. For sufficiently large n, $\log 2n - cn^\varepsilon < 0$ and therefore $\exp(\log 2n - cn^\varepsilon) < 1$. Thus

$$q_m \leq n^m \cdot \exp[m(\log 2 - cn^\varepsilon)] = \exp[m(\log 2n - cn^\varepsilon)] \leq \exp[\log 2n - cn^\varepsilon].$$

Finally, let q be the probability of the existence of a bad set. We have

$$q < \tfrac{n}{4}\exp[\log 2n - cn^\varepsilon] = o(1). \qquad \dashv$$

Theorem 3.1 is proved. $\dashv$

§4. Multipedes. The domain $\{x : \exists y(xEy)\}$ and the range $\{y : \exists x(xEy)\}$ of a binary relation E will be denoted $D(E)$ and $R(E)$ respectively.

DEFINITION 4.1. A 1-*multipede* is a directed graph (U, E) such that $D(E) \cap R(E) = \emptyset$, $D(E) \cup R(E) = U$, every element in $D(E)$ has exactly one outgoing edge and every element in $R(E)$ has exactly two incoming edges. $\dashv$

If xEy holds then x is a *foot* of y and y is the *segment* $S(x)$ of x. We extend function S as follows. If x is a segment then $S(x) = x$. If X is a set of segments and feet then $S(X) = \{S(x) : x \in X\}$.

DEFINITION 4.2. A 2^--*multipede* is a structure (U, E, T) such that (U, E) is a 1-multipede and (U, T) is a hypergraph where each hyperedge h satisfies the following conditions:

- Either all elements of h are segments or else all elements of h are feet.
- If h is a foot hyperedge then $S(h)$ is a hyperedge as well. $\dashv$

If $X = \{x, y, z\}$ is a segment hyperedge then every 3-element foot set A with $S(A) = X$ is a *slave* of X. A slave A of X is *positive* if A is a hyperedge; otherwise it is *negative*. Two slaves of X are *equivalent* if they are identical or one can be obtained from the other by permuting the feet of two segments. In other words, if

a, a' are different feet of x and b, b' are different feet of y and c, c' are different feet of z then the eight slaves of X split into the following two equivalence classes

$$\{a,b,c\}, \{a,b',c'\}, \{a',b,c'\}, \{a',b',c\}$$

and

$$\{a',b,c\}, \{a,b',c\}, \{a,b,c'\}, \{a',b',c'\}.$$

DEFINITION 4.3. A *2-multipede* is a 2^--multipede where, for each segment hyperedge X, exactly four slaves of X are positive and all four positive slaves are equivalent. $\dashv$

A 2-multipede (U, E, T) is *odd* if the segment hypergraph $(R(E), T)$ is so.

LEMMA 4.1. *If an automorphism θ of an odd 2-multipede does not move any segment then it does not move any foot either.*

PROOF. By contradiction suppose that θ moves a foot a of a segment x. Clearly, $\theta(a)$ is the other foot of x. Let X be the collection of segments x such that θ permutes the feet of x. Since the multipede is odd, there exists a segment hyperedge h such that $\|h \cap X\|$ is odd. It is easy to see that θ takes positive slaves of X to negative ones and thus is not an automorphism. $\dashv$

LEMMA 4.2. *Let M be a k-meager 2-multipede and Υ be the extension of the vocabulary of M by means of individual constants for every segment of M. No Υ-formula $\varphi(v)$ in $L^k_{\infty,\omega} + C$ distinguishes between the two feet of any segment of M.*

PROOF. Call a collection X of segments and feet *replete* if $a \in X \leftrightarrow S(a) \in X$ for every foot a. The least replete set that includes X is the *repletion* of X. Call X *closed* if it is replete and the segments of X form a closed set in the sense of Definition 2.4.1. The least closed set that includes X is the *closure* $\bar{X}$ of X.

A *partial automorphism* over M is a partial isomorphism from M to M. A partial automorphism α is *regular* if α leaves segments intact and takes any foot to a foot of the same segment. The domain of α will be denoted $D(\alpha)$. α is *safe* if there is a regular extension of α to the closure $\overline{D(\alpha)}$.

CLAIM 4.1. *Suppose that α is a safe partial automorphism over M, $X = D(\alpha)$ is replete and $\|S(X)\| \leq k$. Then there is a unique regular extension of α to $\bar{X}$.*

PROOF. Suppose that β and γ are regular extensions of α to $\bar{X}$. Let $Y = S(X)$ and $Z = S(\bar{Y})$. By Lemma 2.4.2, there exists a linear order $z_1, \dots, z_p$ of the elements of $Z - Y$ such that each z_j is attracted by the set $Z_{j-1} = Y \cup \{z_i : i < j\}$. We need to prove that, for every j, either both β and γ leave the feet of z_j intact or else both of them permute the feet. We proceed by induction on j. Suppose that β and γ coincide on the feet of every z_i with $i < j$ and let h witness that Z_{j-1} attracts z_j. Let $\{a, b, c\}$ be any positive slave of h where c is a foot of z_j. By the induction hypothesis, $\beta(a) = \gamma(a)$ and $\beta(b) = \gamma(b)$; let $a' = \beta(a)$ and $b' = \beta(b)$. Since β and γ are partial automorphisms, both $\{a', b', \beta(c)\}$ and $\{a', b', \gamma(c)\}$ are hyperedges. Since M is a 2-multipede, $\beta(c) = \gamma(c)$. $\dashv$

The unique regular extension of α will be denoted $\bar{\alpha}$.

CLAIM 4.2. *Suppose that α is a safe partial automorphism over M, $X = D(\alpha)$ is replete and $\|S(X)\| < k$. For every element $a \in |M| - \bar{X}$, there is a safe extension of α to the repletion of $X \cup \{a\}$ which leaves a intact.*

PROOF. We construct a regular extension β of $\bar{\alpha}$ to $\overline{X \cup \{a\}}$. Let z_0 be the segment of a, $Y = S(\bar{X}) \cup \{z_0\}$, $Z = S(\bar{Y})$ and $p = \|Z - Y\|$. By Theorem 2.4.1, there is a linear ordering $z_1, \dots, z_p$ on the vertices of $Z - Y$ such that, for every $j > 0$, z_j is attracted by $Y \cup \{z_i : 1 \le i < j\}$ and there is a unique hyperedge h_j witnessing the attraction.

The desired β leaves intact all segments in Z and the feet of z_0. It remains to define β on the feet of segments z_j, $1 \le j \le p$. We do that by induction on j. Suppose that β is defined on the feet of all z_i with $i < j$ and let h_j be as above. Let d be a foot of z_j and pick a positive slave $\{b, c, d\}$ of h_j; β is already defined at b and c. The slave $\{\beta(b), \beta(c), \beta(d)\}$ of h_j should be positive. This defines uniquely whether $\beta(d)$ equals d or the other foot of z_j.

We need to check that β is a partial automorphism over M. The only nontrivial part is to check that if A is a slave of a segment hyperedge h then A is positive if and only if $\beta(A)$ is positive. Without loss of generality, $A \not\subseteq \bar{X}$. Let j be the least number such that $S(\bar{X}) \cup \{z_0, \dots, z_j\}$ includes h. Since $\bar{X}$ does not attract z_0, $\bar{X}$ includes all hyperedges in $S(\bar{X}) \cup \{z_0\}$; thus $j > 0$. By the uniqueness property of h_j, $h = h_j$. By the construction of β, A is positive if and only if $\beta(A)$ is positive. ⊣

Now we are ready to finish the proof of Lemma 4.2. Let x be any segment of M and a, b are the two feet of x. By Theorem 1.1, it suffices to prove that Duplicator has a winning strategy in $G^k(M, a; M, b)$. Clearly, the pebbles define a safe partial automorphism in the initial state. The desired winning strategy is to ensure that, after each round, pebbles still define a safe partial automorphism. This is doable. Indeed, suppose that pebbles define a safe partial isomorphism η and Spoiler starts a new round. Without loss of generality, $\|D(\eta)\| < k$. (If $\|D(\eta)\| = k$ then Spoiler starts with removing a pair of pebbles; the remaining pebbles define a safe partial automorphism η' with $\|D(\eta')\| < k$). Let X be the repletion of $D(\eta)$. Since η is safe, there is a safe extension α of η to X. Without loss of generality, Spoiler chooses *left* and a set V (on the left). Duplicator chooses the set $\{f(y) : y \in V\}$ (on the right) where f is as follows. If $y \in \bar{X}$ then $f(y) = \bar{\alpha}(y)$; otherwise $f(y) = y$. If Spoiler chooses $f(y)$, then Duplicator chooses y. It remains to check that the pebbles define a partial automorphism in the resulting state. The case $y \in \bar{X}$ is obvious. In the other case, use Claim 4.2. ⊣

DEFINITION 4.4. A 3-*multipede* is a structure $(U, E, T, <)$ where (U, E, T) is a 2-multipede and $<$ is a linear order on the set of segments of (U, E, T). ⊣

DEFINITION 4.5. A 4-*multipede* is a structure $(U \cup V, E, T, <, \varepsilon)$ satisfying the following conditions.

1. $E, T, <$ are relations over U (in other words, the elements of any tuple in E, T or $<$ belong to U), and $(U, E, T, <)$ is a 3-multipede.
2. ε is a binary relation with domain U and range V, and $U \cap V = \emptyset$.
3. for every set X of segments of the 3-multipede $(U, E, T, <)$, there exists a unique $y \in V$ such that $x \in X \leftrightarrow x \varepsilon y$ for all segments x in $(U, E, T, <)$. ⊣

Intuitively, elements of V are sets of segments of the 3-multipede $(U, E, T, <)$ and ε is the corresponding containment relation. Elements of V are called *super-segments*. A 4-multipede is *odd* if the hypergraph of segments is so.

LEMMA 4.3. *The collection of odd 4-multipedes is finitely axiomatizable.*

PROOF. It is obvious that Conditions 1 and 2 are expressed by finitely many axioms. The following three axioms express Condition 3.

- There is a super-segment Y such that there is no segment x with $x \varepsilon Y$.
- For every super-segment Y and every segment x, there exists a super-segment Y' such that $y \varepsilon Y' \leftrightarrow (y \varepsilon Y \vee y = x)$ for every segment y.
- Super-segments Y and Y' are equal if $x \varepsilon Y \leftrightarrow x \varepsilon Y'$ for all segments x.

Using universal quantification over super-segments, it is easy to express the oddity condition in first-order way. ⊣

LEMMA 4.4. *Every odd 4-multipede is rigid.*

PROOF. Let θ be an automorphism of a 4-multipede M. Because of the linear order on segments, θ leaves intact all segments. Therefore it leaves intact all super-segments. By Lemma 4.1, it leaves intact all feet as well. ⊣

A 4-multipede is l-meager if the hypergraph of segments is so.

LEMMA 4.5. *Let M be a k-meager 4-multipede. No formula $\varphi(v)$ in $L^k_{\infty,\omega} + C$ distinguishes between the two feet of any segment of M.*

PROOF. The proof is similar to that of Lemma 4.2. We use the terminology and notation of the proof of Theorem 4.1. If X is a collection of segments, feet and super-segments and X' is the set of segments and feet in X, define $S(X) = S(X')$ and call X *replete* (respectively *closed*) if X' is replete (respectively closed). Claim 4.1 remains true. Claim 4.2 remains true as well; if a is a super-segment, then $\bar{X} \cup \{a\}$ is closed and the desired β is the extension of $\bar{\alpha}$ by means of $\beta(a) = a$. The remainder of the proof is as above. ⊣

LEMMA 4.6. *No $L^k_{\infty,\omega} + C$ formula $\varphi(v_1, v_2)$ defines a linear order in any k-meager 4-multipede.*

PROOF. By contradiction suppose that an $L^k_{\infty,\omega} + C$ formula $\varphi(v_1, v_2)$ defines a linear order in a k-meager 4-multipede M. It is easy to see that φ cannot be quantifier-free. Let v_3 be any bound variable of φ. The formula

$$\psi(v_1) = (\exists v_2)[\varphi(v_1, v_2) \wedge (\exists v_3)(E(v_1, v_3) \wedge E(v_2, v_3))]$$

asserts than v_1 is the first of the two feet of some segment in the order defined by φ. It follows that $\psi(v_1)$ distinguishes between the feet of any segment, which contradicts Lemma 4.5. ⊣

THEOREM 4.1. *There exists a finitely axiomatizable class of rigid structures such that no $L^\omega_{\infty,\omega} + C$ sentence defines a linear order in every structure of that class.*

PROOF. Consider the class K of odd 4-multipedes. By Lemmas 4.3 and 4.4, K is a finitely axiomatizable class of rigid structures. By Lemma 4.6, no $L^k_{\infty,\omega} + C$ sentence φ defines a linear order in any k-meager 4-multipede.

Finally, we check that, for every l, K contains an l-meager 4-multipede. By Theorem 3.1, there exists an odd l-meager 4-hypergraph H. Extend H to a 4-multipede by attaching two feet to each vertex of H, choosing positive slaves in any way consistent with the definition of 2-multipedes, ordering the segments in an arbitrary way and finally adding representations of all sets of segments. The result is an l-meager 4-multipede. ⊣

Call two structures k-equivalent if there is no $L^k_{\infty,\omega}$ sentence which distinguishes between them. The notion is k-equivalence is explored in [1]. We answer negatively a question of Scott Weinstein [5].

THEOREM 4.2. *There exist k and a structure M such that every structure k-equivalent to M is rigid but not every structure k-equivalent to M is isomorphic to M.*

Theorem remains true even if $L^k_{\infty,\omega}$ is replaced with $L^k_{\infty,\omega} + C$ in the definition of k-equivalence.

PROOF. By Lemma 4.3, there exists k such that a first-order sentence φ with k variables axiomatizes the class of odd 4-multipedes. By Lemma 4.4, every model of φ is rigid. By Theorem 3.1, there exists a k-meager odd hypergraph, and therefore there exists a k-meager odd 4-multipede M. Every structure that is k-equivalent to M satisfies φ and therefore is rigid. By Lemma 4.5, there is a structure that is k-equivalent to M (even if counting quantifiers are allowed) but not isomorphic to M. ⊣

REFERENCES

[1] ANUJ DAWAR, *Feasible computation through model theory*, ***Ph.D. thesis***, Institute for Research in Cognitive Science University of Pennsylvania, Philadelphia, 1993.

[2] YURI GUREVICH, *Logic and the challenge of computer science*, ***Current trends in theoretical computer science*** (E. Börger, editor), Computer Science Press, 1988, pp. 1–57.

[3] NEIL IMMERMAN and E. S. LANDER, *Describing graphs: A first-order approach to graph canonization*, ***Complexity theory retrospective*** (Alan Selman, editor), Springer Verlag, 1990, pp. 59–81.

[4] ALEXEI STOLBOUSHKIN, *Axiomatizable classes of finite models and definability of linear order*, ***Proceedings of the 7th IEEE Annual Symposium on Logic in Computer Science*** (1992), pp. 64–70.

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