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Second-Order Quantifiers and the Complexity of Theories

J. T. BALDWIN* and S. SHELAH**

In this paper[1] we classify all theories of the form $(T, \mathcal{L})$ where $(T, \mathcal{L})$ is the collection of $\mathcal{L}$ sentences valid on the models of the complete first-order theory T and $\mathcal{L}$ is one of the following: second-order logic, permutational logic, or monadic logic. We regard any theory into which second-order logic can be interpreted (in the strong sense used here) as hopelessly complicated. We partition those theories in which such an interpretation does not exist into four classes and find a prototype for each class. One of the classes contains unstable theories; the other three are stable. In the unstable case we show the following. First, the permutational theory of the class interprets second-order logic so only the monadic theory is interesting. We show the monadic theory of the class is at least as complicated as the monadic theory of order. Now by one measure the members of this class cannot be differentiated: (under weak set theoretic hypotheses) all the monadic theories have the same Löwenheim number—that of second-order logic. In contrast we show that the Hanf number of the monadic theory of order is strictly less than the Hanf number of second-order logic. This proof requires the computations of Hanf and Löwenheim numbers of various theories of (well) orderings. These computations are interesting in their own right. If T is stable we use a Feferman-Vaught type of theorem to decompose models of these theories into trees. The nodes of these trees are small models and the height of the tree is uniformly bounded over all models of T. The other three cases arise when this tree is: (a) well-founded, (b) imbeddable in $\lambda^{<\omega}$, or (c) otherwise. This allows us to compute the Hanf and Löwenheim numbers of such theories. A more detailed exposition of the methodology and aims of the paper appears in the survey (Section 1.2) of results which follow.

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One motivation for this paper arises from a glance at the literature about second-order quantifiers. There has been a lot of work done in monadic logic (mainly about orderings and trees) and some on permutational logic but nothing on other second-order quantifiers ($Q_{\exists \text{ a 2-step nilpotent group}}$)??. What distinguishes these two quantifiers? This question was largely answered in [21]. Up to a strong notion of interpretability there are only four second-order quantifiers of which one, first-order logic, is a degenerate case, and another, second-order logic, is hopelessly complicated; this leaves monadic and permutational logic. But another glance at the literature raises another question. Why does research in monadic logic focus on theories of order and trees and why is research in permutational logic concerned entirely with the theory of equality? Much of this paper is devoted to answering this question. We show that if T is a "nontrivial" first-order theory then $(T, 1\text{-}1)$ is bi-interpretable with second-order logic, so only pure permutational logic is interesting. We find there is one class of structures which provides an interesting monadic logic that had been overlooked (the trees: $\lambda^{\leq\omega}$). However, in a rough sense made more precise in this paper, the classes of trees $\lambda^{<\omega}$, $\lambda^{\leq\omega}$ and linear orders exhaust the interesting monadic theories. We obtain some specific results about these theories.

In another way this article can be viewed as propaganda for classification theory as the present material demonstrates the importance of the earlier work on classification theory by applying it in a nontrivial context.

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1 Introduction Section 1.2 is a survey of the paper; Sections 1.3 and 1.4 elaborate upon some distinctions in the definitions of Hanf numbers and interpretations which arise in the course of the paper.

1.1 Notations and conventions Our notation is standard but we list here some slight variations and some conventions we adopt which are not fixed in general usage. We denote structures by capital letters $M, N, \ldots$ and $|M|$ is the cardinality of the universe of M. We use small letters r, s, t for second-order variables and R, S, T to name subsets (relations) on a model. A barred letter denotes a finite sequence of variables or parameters. We write $\bar{m} \in M$ instead of $\bar{m} \in M^{lg(\bar{m})}$ (lg = length). If $\phi(\bar{x}, \bar{a}, \bar{R})$ is a formula (with $\bar{a}$ and $\bar{R}$ parameters from M) $\phi(M, \bar{a}, \bar{R})$ denotes $\{\bar{m} \in M: M \models \phi(\bar{m}, \bar{a}, \bar{R})\}$.

T_∞ denotes the theory in the language of equality which has only infinite models. $Th(<)$ (or occasionally LO) denotes the first-order theory of infinite linear orders. $(T_\infty,\text{2nd})$ denotes pure second-order logic.

A language L or similarity type (τ) is a collection of relation and function symbols of specified arity. A logic $\mathcal{L}$ is a function which assigns to each similarity type τ a collection of $\mathcal{L}(\tau)$-sentences, and a truth predicate $\models_{\mathcal{L}}$ which to each pair $\langle M, \phi\rangle$, M a τ-structure, ϕ an $\mathcal{L}(\tau)$ sentence, assigns a value of true or false. When we deal with fixed similarity type we may write $\mathcal{L}$-sentence rather than $\mathcal{L}(\tau)$-sentence.

For λ a cardinal and α an ordinal, a set $I \subseteq \lambda^{\leq\alpha}$ or $\lambda^{<\alpha}$ (sequences of elements of λ with length $\leq$ or $<\alpha$ respectively) which is closed under a subsequence is a tree under the partial order of extension. We write η, τ, etc. for elements of I. If $lg(\eta)$ is a successor then η^- denotes the unique predecessor of η; $\eta^\frown j$ denotes the result of concatenating η and $\langle j\rangle$. $\langle\,\rangle$ denotes the empty sequence. If $\langle a_i: i < \alpha\rangle$ is a sequence then A_j denotes $\{a_i: i < j\}$.

1.2 Survey of the paper In the later 1950s, model theorists began to study not only the properties of first-order predicate logic (e.g., preservation theorems) but also properties of models which were not first-order expressible. Early pursuits of this theme were the study of such properties as "omitting a type" and "being a two-cardinal model". It was quickly realized that those properties could be expressed in more sophisticated languages: $L_{\omega_1\omega}$ in the first case and $L(Q_1)$ in the second. This discovery spurred investigations of these logics.

Another theme of contemporary model theory is the study of the relative complexity of various theories in an arbitrary logic $\mathcal{L}$. Of course this study began with the early interpretation results regarding decidability. The investigation of interpretations continues through the study of theories in stronger logics, e.g., the monadic theory of successor, order, etc. Another approach is to classify first-order theories by whether they satisfy various non-first-order properties, e.g., "has a prime model", or by the complexity of certain associated invariants, e.g., $n(T, \lambda)$ (the number of models of T in power λ). This kind of classification culminated in the development of stability theory.

In this paper we combine these two themes in the following program. Consider two theories T_1 and T_2 formulated in a logic $\mathcal{L}$. To compare their strengths, choose a stronger logic $\mathcal{L}'$. Now think of T_1 and T_2 as $\mathcal{L}'$-theories and investigate their properties.

We divide all countable first-order theories into five classes: The first, those which interpret second-order logic, we regard as beyond analysis. Each of the other classes is associated with a class of structures. The second is associated with the class of linear orders. In the other three cases each model of the theory with cardinality λ is associated with a tree of height $<\aleph_1$. The three cases correspond to $\lambda^{<\aleph_1}$, the full tree $\lambda^{<\omega}$ and well-founded subtrees of $\lambda^{<\omega}$. The rationale for this classification is based on the relative interpretability of theories with second-order quantifiers.

There are several notions of interpretation which play a role in this paper. Here, we only indicate when different notions are used; a careful delineation of the distinctions occurs in Section 4. We deal only with theories with no finite models. We write T_∞ for the theory in the language with only the equality symbol which has this property. For convenience of exposition we restrict discussion in the introduction to countable theories. Much of our work extends to uncountable languages but we are inconsistent in our stress on that fact.

In this paper we focus on those $\mathcal{L}$ obtained by adjoining to first-order logic those second-order quantifiers considered by Shelah in [21]. Informally, we adjoin to first-order logic quantification over arbitrary second-order variables which must satisfy a specific first-order condition $\phi(\bar{r})$ in a language whose only nonlogical symbols are the $\bar{r}$. Thus we can say "there is a permutation r such that...". We cannot say "there is an automorphism r such that...". (The assertion that r is an automorphism requires additional nonlogical symbols which are not permitted.) Formal definitions of this notion and the ones in the next few paragraphs appear in Section 2.1.

Shelah proved in [21] that if $Q_{\phi(\bar{r})}$ is any such quantifier then $(T_\infty, Q_{\phi(\bar{r})})$ is bi-interpretable with one of: $(T_\infty, \text{1st})$ (first-order logic); (T_∞, Mon) (monadic logic); $(T_\infty, \text{1-1})$ (permutation logic); $(T_\infty, \text{2nd})$ (second-order logic). (An exposition of this argument, supplemented by the methods of this paper occurs in [2].)

Our intuitive claim is that if second-order logic is not interpretable in $(T, Q_{\psi(\bar{s})})$ then $(T, Q_{\psi(\bar{s})})$ is simple. We make this precise by drawing a number of conclusions from the hypothesis of noninterpretability. We deduce a structure theory for models of T from this hypothesis. To test the value of this structure theory we use it to compute the Hanf and Löwenheim numbers relative to T for various logics.

In Section 2 we make precise our notion of interpretation and summarize the combinatorial properties of formulas which are sufficient to imply $(T_\infty, \text{2nd})$ is interpretable in (T, Mon) (written $(T_\infty, \text{2nd}) \leq (T, Mon)$). In Section 3.1 we extend the analysis of generalized sums from [22] to infinitary monadic logic and we allow the sum to be amalgamated over a "heart" N. We refer to this construction as a free union over N. If every model of T has a decomposition as a free union over a countable submodel of countable submodels we say T is strongly decomposable. If for some κ, each model of T can be decomposed as a tree (with height $<\kappa$) of countable models, we say T is tree decomposable.

Our basic classification of theories is based on whether (T, *Mon*) or (T, 1-1) interpret (T_∞, 2nd). This analysis is simplified if we first ask whether the monadic theory of order ($Th(<)$, *Mon*) is interpretable in (T, *Mon*), for ($Th(<)$, *Mon*) $\leq$ (T, *Mon*) implies (T_∞, 2nd) $\leq$ (T, 1-1). We show in Section 8.1 that T is unstable if and only if the monadic theory of order is interpretable in (T, *Mon*).

Our viewpoint on theories which interpret the monadic theory of order (i.e., unstable) is explained more fully in the introduction to Section 8. The following results indicate the situation.

We prove that for unstable T (T_∞, 2nd) $\leq$ (T, *Mon*) (cf. Definition 2.1) if T has the independence property. We show that (T_∞, 2nd) $\not\leq$ ($Th(<)$, *Mon*) by a computation of Hanf numbers. (Note that Gurevich and Shelah [7] have constructed an interpretation of second-order logic into the monadic theory of order but not one with the strong model theoretic properties we demand here.)

Consider now those theories which do not interpret ($Th(<)$, *Mon*), i.e., the stable case. Our basic intuition is that all models of such theories are "short". Complete information about a finite sequence of elements depends on only a countable number of other elements—those below (in a certain tree) the elements in question.

The basic tool in our investigation of stable theories is developed in Section 4.2. For any subset A and elements a, b of a model of a stable theory we define the fundamental equivalence relation: aE_Ab if $t(a; A \cup b)$ forks over A. In general of course, this relation is not transitive but we show that it is if (T_∞, 2nd) $\not\leq$ (T, *Mon*).

For any model M of T and any $N \prec M$, E_N decomposes M into a free union over N of the equivalence classes of E_N. (For each equivalence class X, $X \cup N \prec M$.)

There are now several classes of theories T such that T is stable and (T_∞, 2nd) $\not\leq$ (T, *Mon*).

The "simplest" class contains those T such that (T_∞, 2nd) $\not\leq$ (T, 1-1). We characterize this class in Section 7. We show that in this case the fundamental equivalence relation is actually "a is in the algebraic closure of $A \cup \{b\}$". Thus the cardinality of each equivalence class is $\leq |T|$ and T is strongly decomposable. We also show that (T, 1-1) is bi-interpretable with pure permutational logic, and that the Löwenheim number of finitary monadic logic restricted to models of T is $\aleph_0$.

Now suppose only that (T_∞, 2nd) $\not\leq$ (T, *Mon*). We show in Sections 3.2 and 4.2 that T is tree decomposable if and only if T is stable. Now there are three subcases depending on the complexity of the trees associated with T. If T is not superstable then we show in Section 7 that the class of trees $\lambda^{\leq\omega}$ can be infinitarily monadically interpreted in T. If T is superstable there are two cases depending on whether all the trees associated with T are well-founded (T is shallow) or not (T is deep). Again in Section 7 we show that for deep theories all trees of the form $\lambda^{<\omega}$ can be interpreted in T (via infinitary monadic logic). We show that this interpretation pushes up the lower bounds on the Hanf and Löwenheim numbers of infinitary monadic logic restricted to T. In Section 5 we compute exact bounds on these numbers for shallow T.

The following chart summarizes the analysis outlined above.

Assume $(T_\infty, \text{2nd}) \not\leq (T, Mon)$.

	$(T_\infty, \text{2nd}) \leq (T, \text{1-1})$	$(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$
$(Th(<), Mon) \leq (T, Mon)$ (unstable)	prototype $(Th(<), Mon)$	impossible
$(Th(<), Mon) \not\leq (T, Mon)$ (stable)	tree decomposable prototypes $\lambda^{<\omega}$, $\lambda^{\leq\omega}$	strongly decomposable

The sense of "prototype" is much stronger in the stable case than in the unstable. In the stable case we have the bi-interpretability results of Section 7. In contrast, in Section 8 (especially 8.2) we see that $(Th(<), Mon)$ is in a certain sense the simplest unstable theory which does not interpret $(T_\infty, \text{2nd})$. An investigation of the sense in which it is a "typical" such theory continues.

1.3 On the significance of Hanf and Löwenheim numbers Intuitively, we think a simple logic $\mathcal{L}$ should have low Löwenheim and Hanf numbers so the computations of those numbers serve as a test question. In the case of Löwenheim numbers it is easy to make this notion precise. If $ls_{\mathcal{L}} = \kappa$ then the collection of validities of $\mathcal{L}$ depends only on sets whose rank can be bounded in terms of κ. Thus the dependence of the validities of $\mathcal{L}$ on set theory is slight as they remain the same in any model of set theory provided only that we fix a small initial segment of the rank hierarchy.

In the case of Hanf numbers this intuition is harder to pin down. Partly this is because there are a number of variants of the notion of Hanf number which must be considered. We discuss these variations now; in Section 1.4 we show how an appropriate Hanf number can be used as a test of noninterpretability.

1.3.1 Notation For any sentence ϕ the *spectrum of* ϕ, $\text{spec}(\phi) = \{\kappa : \kappa \geq \aleph_0 \ \exists M \models \phi \ |M| = \kappa\}$. We relativize this notion to a theory: $\text{Spec}_T(\phi) = \{\kappa \geq \aleph_0 : \exists M \models T \cup \phi, |M| = \kappa\}$. We say ϕ is *bounded* if $\text{spec}(\phi)$ is a set. We say M is *a largest* model of ϕ if $N \models \phi$ implies $|N| \leq |M|$. If ϕ has a largest model of power λ we say ϕ characterizes λ. Let κ and λ be Hanf or Löwenheim numbers of some logic; we say κ *is bounded in terms of* λ and write $\kappa \quad \lambda$ if $\kappa < \beth_\omega(\lambda)$. If κ and λ are each bounded in terms of the other, we write $\kappa \sim \lambda$. (Of course, the choice of the function $\aleph_\omega(x)$ is arbitrary; it is large enough for our purposes and a correct general definition is unclear.)

Note that we have explicitly ignored finite models.

1.3.2 Definition

(a) For any logic $\mathcal{L}$ we define the following Hanf numbers:

(i) The *Hanf number of* $\mathcal{L}$ ($h_{\mathcal{L}}$) is the least κ such that for any $\mathcal{L}$-sentence ϕ: if ϕ has a model of power κ then ϕ has arbitrarily large models.

(ii) The *Hanf number of* $\mathcal{L}$ *for theories* ($H_{\mathcal{L}}$) is the least κ such that any $\mathcal{L}$-theory Φ which has a model of power κ has arbitrarily large models.

(iii) The *Hanf number of* $\mathcal{L}$ *relative to* T ($h^T_{\mathcal{L}}$) is the least κ such that for any $\mathcal{L}$ sentence ϕ in the language of T if $T \cup \{\phi\}$ has a model of power κ it has arbitrarily large models.

(iv) The *Hanf number of* $\mathcal{L}$ *for theories relative to* T ($H^T_{\mathcal{L}}$) is defined analogously to (ii).

(v) The *Hanf number for omitting types of* T ($H.O.^T$) is the least cardinal κ such that for every type p (over the empty set), if T has a model of power κ omitting p then T has arbitrarily large models omitting p.

One further complexity arises when dealing with infinitary languages. Hanf's general argument for the existence of Hanf numbers requires that the logic have a set of sentences. The languages $L_{\infty,\kappa}$ or $L^a_{\infty,\kappa}$ (cf. [8]) have a proper class of sentences if all similarity types are considered, but only a set for any fixed similarity type (cf. [10], p. 46). Thus we define

(vi) The *Hanf number of* $\mathcal{L}$ *relative to the similarity type* τ ($h^\tau_{\mathcal{L}}$) is the least κ such that for any $\mathcal{L}(\tau)$-sentence ϕ: if ϕ has a model of power κ then ϕ has arbitrarily large models.

(b) We define a similar family of Löwenheim numbers. To save space we write out only the definition of $ls_{\mathcal{L}}$ and $ls^T_{\mathcal{L}}$. However, $LS_{\mathcal{L}}$, $ls^T_{\mathcal{L}}$, and LS^T are defined analogously.

The Löwenheim number of $\mathcal{L}$($ls_{\mathcal{L}}$) is the least κ such if the $\mathcal{L}$-sentence ϕ has a model, it has a model of cardinality $\leq\kappa$.

The Löwenheim number of $\mathcal{L}$ relative to T is the least κ such that for any $\mathcal{L}$-sentence ϕ, if $T \cup \{\phi\}$ has a model it has model with cardinality $\leq\kappa$.

There is no established convention on the question, "Is the Löwenheim number the least κ such that each sentence has a model of power $<\kappa$ or the least κ such that each sentence has a model of power $\leq\kappa$?" Thus, the Hanf and Löwenheim numbers we have defined must be carefully distinguished from the following variants:

(i) $h^*_{\mathcal{L}} = \sup\{\kappa$: There is an $\mathcal{L}$-sentence characterizing $\kappa\}$.

(ii) $ls^*_{\mathcal{L}}$ is the least cardinal κ such that each sentence with a model has one of cardinality $<\kappa$.

1.3.3 Notation When we write $H^T_{L_{\omega,\omega}(Mon)}$, $h^T_{L_{\omega_1,\omega}(Mon)}$, etc., we intend L to specify that we are discussing sentences in the language of T. Of course in these examples the extension of L by a finite number of unary relation symbols would make no difference since they could be existentially quantified in finitary monadic logic. However, in either case the addition of countably many additional unary predicates is significant. Accordingly, when L is the language of T we denote by $\bar{L}$ the expansion of L obtained by adding countably many unary predicates.

The following examples illustrate two weaknesses of considering $h^T_{L_{\omega,\omega}(Mon)}$ as an invariant.

1.3.4 Example Consider the standard example [13] of a theory T with $H.O.^T \geq \beth_1$; any model of T contains two infinite sets P_0 and P_1, a unary function S and constant $0 \in P_0$ such that $(P_0, S, 0)$ is a model of the theory of integers with 0 under successor. Now let ϕ be the $L_{\omega,\omega}(Mon)$-sentence which asserts that P_0 contains only a single copy of (Z, S):

$$\forall Q\{[Q \subseteq P_0 \wedge \forall x(x \in Q \leftrightarrow Sx \in Q)] \\ \rightarrow [\forall y(y \in P_0 \rightarrow y \in Q)]\} .$$

Moreover, let E be a binary relation between P_0 and P_1 which satisfies the axiom of extensionality. Now any model of $T \cup \{\phi\}$ has cardinality $\leq \beth_1$, so $h^T_{L_{\omega,\omega}(Mon)} \geq (\beth_1)^+$.

Iterating this example exactly as in the original construction shows $h^T_{L_{\omega,\omega}(Mon)} \geq \beth_\omega$.

The difficulty here is that $L_{\omega,\omega}(Mon)$, although ostensibly a finitary language, inherently possesses some of the power of an infinitary language.

1.3.5 Example On the other hand by taking a disjoint union of the theory in the previous example with the theory of an infinite set we find a fairly complicated theory T with the same language but with $h^T_{L_{\omega,\omega}(Mon)} = \aleph_0$.

We deal simultaneously with both of these problems by using $h^T_{L_{\omega_1,\omega}(Mon)}$ as our preferred invariant. The first example showed we might as well use $L_{\omega_1\omega}$ since for some theories the complexity of $L_{\omega_1,\omega}$ is implicit in $L_{\omega,\omega}(Mon)$. By moving to $\bar{L}_{\omega_1,\omega}(Mon)$ we are able to eliminate possibilities like the second example since we are able to fix the simple part of models of the theory. For example, by adding unary predicates we are able in monadic logic to keep models of $Th(=)$ countable.

In fact, the natural closure point for our methods seems to be $\bar{L}_{\infty,\omega}(Mon)$. In Section 3.1 we prove a Feferman-Vaught theorem for our notion of generalized sum. This theorem asserts that the generalized sum preserves $L^\alpha_{\infty,\lambda}(Mon)$ equivalence for appropriate α and λ. Just as in proving results in first-order logic we add constant symbols to witness instantiations, we add unary predicate symbols to witness existential monadic quantification. This construction relies heavily on the number of variables instantiated at one time (λ) and on the number of alternations of quantifiers (α), but very little on the length of conjunctions. Thus the natural language is $\bar{L}^\alpha_{\infty,\omega}(Mon)$ (formally defined in Section 3.1).

If we move to L_{ω_1,ω_1} we are able to get the analogous results without the necessity of adding additional unary predicates.

1.4 On the complexity of theories Although much of the emphasis of this paper is on second-order quantification, we do produce a new classification of first-order theories. A classification of theories is just a partitioning of theories into classes such that the common properties of the theory in each class are of overriding importance for the problem at hand. The most naive such classification is into those theories which have or do not have one specific property. More useful classifications retain their value from one problem to another. Thus, the stability classification was designed to study the spectrum problem for first-order theories, but has ramifications for $L_{\infty,\kappa}$, elimination of generalized quantifiers, and now the study of second-order quantifiers. The stability classification suggested the importance of studying the combinatorial properties of first-order formulas. Here, we study the combinatorial properties of finitary monadic formulas. The notion of interpretability provides an organizational scheme for our classification (as opposed to the *ad hoc* nature of the stability scheme which succeeds based on its consequences, not on any a priori coherence). It is thus important to examine the notion "interpretation". We consider several abstract properties one might demand of an interpretation.

1.4.1 Definition The $\mathcal{L}_1$-theory T_1 is *syntactically interpretable* in the $\mathcal{L}_2$-theory T_2 if there is a map * assigning to each $\mathcal{L}_1$-sentence ϕ an $\mathcal{L}_2$-sentence ϕ^* such that $T_1 \vdash \phi$ iff $T_2 \vdash \phi^*$.

This is the weakest notion of interpretability. It has no effect on Hanf or Löwenheim numbers.

We will now consider several notions of interpretability which demand some model theoretic content of the interpretation. Namely, we will insist that the interpretation exert some control on the cardinality of models.

1.4.2 Definition There is a *spectrum-preserving interpretation* of the $\mathcal{L}_1$-theory T_1 into the $\mathcal{L}_2$-theory T_2 if there is a map * mapping each $\mathcal{L}_1$-sentence to an $\mathcal{L}_2$-sentence ϕ^* such that for each $\mathcal{L}_1$-sentence ϕ there is an $\mathcal{L}_2$-sentence ϕ^* such that $Spec_{T_1}(\phi) = Spec_{T_2}(\phi^*)$.

It is immediate that:

1.4.3 Lemma *If there is a spectrum preserving interpretation of the $\mathcal{L}_1$-theory T_1 into the $\mathcal{L}_2$-theory T_2 then $ls_{\mathcal{L}_1}^{T_1} \leqq ls_{\mathcal{L}_2}^{T_2}$ and $h_{\mathcal{L}_1}^{T_1} \leqq h_{\mathcal{L}_2}^{T_2}$.*

With the aid of these notions we can see that there are some weaknesses in our position that second-order logic is the most complicated logic. It is of course clear that there are first-order theories with Turing degree above that of second-order logic. However, if T is "normal" (e.g., finitely, recursively or even arithmetically axiomatizable) this is impossible and $(T, \text{2nd})$ has a spectrum preserving interpretation into $(T_\infty, \text{2nd})$. The following example (worked out with Victor Harnik) shows that some sort of "normality" assumption is necessary.

1.4.4 Example There is a first-order theory T such that there is no spectrum preserving interpretation of $(T, \text{2nd})$ into $(T_\infty, \text{2nd})$.

Proof: Let $\langle \phi_i : i < \omega \rangle$ be a list of the bounded sentences of pure second-order equality theory. Suppose ϕ_i has a model of cardinal η_i and let $\eta = sup(\eta_i)$ (η is the Hanf number of second-order logic).

Construct a theory T and a second-order sentence ϕ such that if $M \models T \cup \{\phi\}$ then for each i there is a definable subset X_i of M such that X_i is a model of ϕ_i and $M = \bigcup_{i<\omega} X_i$. Now, there is no ϕ^* in second-order logic with

$Spec_T(\phi) = Spec(\phi^*)$. For $Spec_T(\phi)$ is bounded by η while each $Spec(\psi)$ for ψ a sentence of second-order logic is bounded below η.

(This example also shows $h_{L_{\omega_1,\omega}(2\text{nd})} > h_{L_{\omega,\omega}(2\text{nd})}$.)

We will show in Section 8.2 that by adding infinitely many unary predicates we can convert one of our usual interpretations into one in $L_{\omega_1,\omega}(Mon)$ which is almost decreasing and thus preserves the Hanf number.

The normal conclusion of a classification theorem has the form, "all theories in this class have that property". In our situation we cannot get that result in some cases but we get one which is, in some sense, stronger. For example, we cannot show specifically a function f such that for every stable but not superstable theory T such that $(T_\infty, \text{2nd}) \not\leq (T, Mon)$, $h^T_{L^\alpha_{\infty,\omega}(Mon)} = f(\alpha)$. But, we can prove (Section 7.1.15) that such a T is bi-interpretable in $L_{\omega_1,\omega}(Mon)$ with a class of trees $\lambda^{\leq\omega}$. Thus we reduce the problem to a computation for a specific class of models and show that this Löwenheim number is a property not of the individual theories, but of the class. Here is a similar result (Section 7.1.14): if T is superstable and deep then T is bi-interpretable in $L_{\omega_1,\omega}(Mon)$ with the collection of trees $\lambda^{<\omega}$. We use these trees to designate the class in Tables 1 and 2 which summarize the results of this paper on the Hanf and Löwenheim numbers restricted to T for a countable stable theory T. An entry of $\geq\kappa$ or $\leq\kappa$ indicates a lower or upper bound respectively over all theories in the class. An entry of $(\leq\kappa)^*$ indicates there is no uniform improvement of the bound over all members of the class. An entry of κ indicates the bound is exact for each theory in the class. We have results for uncountable languages and $L_{\infty,\kappa}(Mon)$ for uncountable κ which will be mentioned in the relevant sections.

The question marks and lack of lower bounds in the finitary case arise for two reasons. We do not have a *finitary* monadic interpretation of $\lambda^{<\omega}$ into an

Table 1

Infinitary Logic ($L^\alpha_{\infty,\omega}(Mon)$)
(For simplicity, take $\alpha \geq \omega_1$.)

	Löwenheim Number	*Hanf Number* (for sentences)
$\lambda^{\leq\omega}$	$\kappa_0(\alpha)$	$\beth^+_\alpha$ (*)
$\lambda^{<\omega}$(deep)	$\kappa_1(\alpha)$	$\beth^+_\alpha$(**)
$\lambda^{<\omega}_{\text{depth}(T)=\beta}$(shallow)	$(\beth_\beta)^+$	$(\beth_\beta)^+$
strongly decomposable	$(\beth_1)^+$	$(\beth_1)^+$

Here $\kappa_0(\alpha)$ and $\kappa_1(\alpha)$ are functions which do *not* depend on T. We conjecture $\kappa_1(\alpha) < \kappa_0(\alpha)$.

(*)$\beth_\alpha$ if α is a limit ordinal and $\text{cf}(\alpha) > \omega$.

(**)$\beth_\alpha$ if α is a limit ordinal.

Table 2

Finitary Logic ($L_{\omega,\omega}(Mon)$)

	Löwenheim Number	*Hanf Number*
$\lambda^{\leq\omega}$	?	$(\leq\beth_\omega)^*$
$\lambda^{<\omega}$(deep)	?	$(\leq\beth_\omega)^*$
$\lambda^{<\omega}_{\text{depth}(T)=\beta}$(shallow)	$\leq\min(\beth_\beta, \beth_\omega)$	$(\leq\min(\beth_\beta, \beth_\omega))^*$
strongly decomposable	$\aleph_0$	$\aleph_0$

Some of these values can be improved by considering nicely decomposable theories (Section 6.2).

arbitrary stable but not superstable T with $(T_\infty, \text{2nd}) \neq (T, Mon)$. In the case of $\lambda^{\leq\omega}$ we find such an interpretation in Section 7.2. However, the theory of trees $\lambda^{\leq\omega}$ is still very complicated. In [20] Shelah proves (assuming $V = L$) that the Löwenheim number of this class is the same as second-order logic.

2 Interpretations and codings In this section we define precisely the second-order quantifiers and our notions of interpretation. In Section 2.2 we introduce the notion of a codable theory and use it to provide several sufficient conditions for interpretability used in the paper.

2.1 Second-order quantifiers Here are the basic definitions of this paper. This section describes the results of [23] and indicates how we vary from the notions defined there.

2.1.1 Definition Let $\phi(\bar{r})$ be a first-order sentence whose only nonlogical symbols are $\bar{r} = \langle r_0, \ldots, r_{n-1}\rangle$ and equality. For any language L, we define a generalized logic $L(Q_{\phi(\bar{r})})$ with the following semantics. Let M be an L-structure and let $L' = L(\bar{r})$.

> $M \models Q_{\phi(\bar{r})}\psi(\bar{r})$ if for some relations $R_0, R_1, \ldots, R_{n-1}$ on M (of appropriate arity)
> $M \models \phi(\bar{R}) \wedge \psi(\bar{R})$.

We will consider pairs $(T, Q_{\phi(\bar{r})})$ where T is a first-order theory and $Q_{\phi(\bar{r})}$ is a second-order quantifier. Thus, syntactically $(T, Q_{\phi(\bar{r})})$ is the collection of $L(Q_{\phi(\bar{r})})$ sentences true in each model of T.

To simplify notation we assume that for each quantifier $Q_{\phi(\bar{r})}$ all the relations symbols have the same arity (depending on ϕ). If we extended the definition of interpretation (2.1.2 below) to allow formulas $\phi(\bar{r})$ where the r_i have several arities, it would be a trivial matter to show any such $Q_{\phi(\bar{r})}$ bi-interpretable with a $Q_{\psi(\bar{s})}$ where all the s_i have the same arity.

2.1.2 Notation Let $\phi(\bar{s}) = \phi(s_0, \ldots, s_{n-1})$ be a formula of pure identity theory with s_i a k-ary free relation variable. For any set, A, $R_\phi(A)$ denotes the set of tuples $\langle S_0, \ldots, S_{n-1}\rangle$ where S_i is a k-ary relation on A and $A \models \phi(S_0, \ldots, S_{n-1})$.

Thus we can write $\bar{S} \in R_\psi(A)$ to abbreviate $A \models \psi(\bar{S})$.

By second-order logic we mean "quantification over binary relations". Of course, for infinite domains this is the same as quantification over n-ary relations (for all finite n).

We focus our attention on four cases: (T, 1st), (T, *Mon*), (T, 1-1), (T, 2nd). Here, (T, 1st) and (T, 2nd) are simply the first- and second-order theories of the models of T and (T, *Mon*) is the monadic theory of T. Thus if $T = Th(<)$ is the theory of linear order, then (T, *Mon*) is the monadic theory of order. (T, 1-1) requires somewhat more explanation. In this case $M \models Qf\phi(f)$ just if there is a permutation α of M with order two such that $M \models \phi(\alpha)$. Clearly, this is the same as quantifying over equivalence relations such that each class has ≤ 2 elements. (The equivalence classes are the orbits of the permutation.) It follows from [11], Lemma 5, that this is the same as quantifying over arbitrary permutations.

We restrict our attention to these quantifiers because Shelah showed in [21] that, up to interpretability, this is a complete list. To discuss this result we first make precise our notions of interpretability.

Instead of describing an interpretation as an effective map from the sentences in $L(Q_{\phi(\bar{r})})$ to those in $L(Q_{\psi(\bar{s})})$, we emphasize the interpretation of the quantifiers and leave the induction describing the rest of the translation to the reader. (It follows from Lemma 1 in [21].) One reason for this emphasis is that we are not dealing with first-order logic and do not have the Löwenheim-Skolem theorem at our command. Thus we spell out in the definition a uniformity which trivially follows from less-complicated definitions in the first-order case. More important distinctions were discussed in 1.4.

In [21] Shelah defined $Q_{\psi(r)}$ to be interpretable in $Q_{\phi(\bar{s})}$ if there is a formula $\chi(\bar{x}, \bar{y}, \bar{s})$ such that for any infinite A and any R such that $A \models \psi(R)$ there is an $\bar{S}$ satisfying $A \models \phi(\bar{S})$ and an $\bar{a}$ such that $A \models \forall\bar{x}(\chi(\bar{x}, \bar{a}, \bar{S}) \leftrightarrow R(\bar{x}))$.

We extend the study here to interpretations of $Q_{\psi(\bar{r})}$ into theories (T, $Q_{\phi(\bar{s})}$). We allow the range of the variable r to be any set which is a (uniformly over all models) definable subset of a model of T. The defining formula is first order but may have both first- and second-order parameters.

We denote by T_∞ the theory (in the language with no nonlogical symbols) of an infinite set.

In both our notions of interpretation we allow both individuals and unary predicates as parameters. We speak of monadic interpretation when the interpreting formulas also contain quantifiers over sets.

In the following definition θ picks out the domain of the interpretation and the χ_i define the R_i. Note that since $\phi(r_0, \ldots, r_{n-1})$ contains no nonlogical symbols, for any A, in particular one of the form $A = \phi(M, B, \bar{S})$, $A \models \phi(R_0, \ldots, R_{n-1})$ makes sense if the R_i are relations on A of the appropriate arity.

2.1.3 Definition Let $\phi(r_0, \ldots, r_{n-1}) = \phi(\bar{r})$ be a formula in m k-ary relation variables, $\psi(\bar{s})$ a formula in m' k'-ary predicates, and T a first-order theory with no finite models.

(a) $(T_\infty, Q_{\phi(\bar{r})})$ is *first-order interpretable* in $(T, Q_{\psi(\bar{s})})$ if there exist $n < \omega$ and first-order formulas

$$\pi(\bar{y}, \bar{s}_0, \ldots, \bar{s}_{n-1}),\ \theta(x, \bar{y}, \bar{s}_0, \ldots, \bar{s}_{n-1})$$

and

$$\chi_i(x_0, \ldots, x_{k-1}, \bar{y}, \bar{s}_0, \ldots, \bar{s}_{n-1})\ (i < m)\ ,$$

where $\bar{s}_i$ is an m' tuple of k'-ary relation variables such that:

(i) If $M \models T$, $\bar{a} \in M$ and $\bar{S}_0, \ldots, \bar{S}_{n-1} \in R_\psi(M)$ and $M \models \pi(\bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})$ then $\theta(M, \bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1}) = B$ is an infinite set and $R_\phi(B) = \{\langle \chi_0(M, \bar{a}', \bar{s}_0', \ldots, \bar{s}_{n-1}), \ldots, \chi_{m-1}(M, \bar{a}', \bar{s}_0', \ldots, \bar{s}_{n-1}')\rangle : \bar{a}' \in M, s_i' \in R_\psi(M)\}$.

(ii) For all $\kappa \geq \aleph_0$, there exist $M \models T$, $\bar{a} \in M$ and $\bar{s}_0, \ldots, \bar{s}_{n-1} \in R_\psi(A)$ with $|\theta(M, \bar{a}, \bar{s}_0, \ldots, \bar{s}_{n-1})| = \kappa$ and $\models \pi(\bar{a}, \bar{s}_0, \ldots, \bar{s}_{n-1})$.

We write $(T_\infty, Q_{\phi(\bar{r})}) \leq (T, Q_{\psi(\bar{s})})$.

The definition (a) applies only to interpretations of $(T_\infty, Q_{\phi(\bar{r})})$. We have required no structure on $\theta_0(M, \bar{a}, \bar{S})$ except that specified by $\phi(\bar{r})$. We will apply the following more general notion in Sections 6, 7, and 8.

We continue to employ all the notation introduced in (a).

(b) Let T_0 be a first-order theory in the relational language L_0. $(T_0, Q_{\phi(\bar{r})})$ is *first-order interpretable* in $(T_1, Q_{\psi(\bar{s})})$ if $(T_\infty, Q_{\phi(\bar{r})}) \leq (T_1, Q_{\psi(\bar{s})})$ and for each relation symbol $R \in L_0$ there is a formula $\chi_R(\bar{x}, \bar{y}, \bar{s}_0, \ldots, \bar{s}_{n-1})$ (with $lg(\bar{x})$ equal to the arity of R) such that

(i)′ if $M \models T_1$ and $M \models \pi(\bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})$ then $\chi_R(M, \bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1}) \subseteq \theta(M, \bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})$ for each R and $\langle B, \{\chi_R(M, \bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1}) : R \in L_0\}\rangle \models T_0$.

(ii)′ For every $N \models T_0$, there exist $M \models T_1$, $\bar{a} \in M$ and $\bar{S}_0, \ldots, \bar{S}_{n-1} \in R_\psi(M)$ such that $\pi(\bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})$ and $N \approx \langle \theta(M, a, \bar{S}_0, \ldots, \bar{S}_{n-1}), \{\chi_R(M, a, \bar{S}_0, \ldots, \bar{S}_{n-1}) : R \in L_0\}\rangle$.

We write: $(T_1, Q_{\phi(\bar{r})}) \leq (T_2, Q_{\psi(\bar{s})})$.

(c) If $(T_1, Q_{\phi(\bar{r})}) \leq (T_2, Q_{\psi(\bar{s})})$ and $(T_2, Q_{\psi(\bar{s})}) \leq (T_1, Q_{\phi(\bar{r})})$ we write $(T_1, Q_{\phi(\bar{r})}) \equiv (T_2, Q_{\psi(\bar{s})})$.

We will require one further generalization of the notion of interpretation. Namely, in (b) we can replace the first-order theories T_0, T_1 by arbitrary classes K_0, K_1 (and assertions $M \models T_i$ by $M \in K_i$) and/or replace the logic with quantifier $Q_{\psi(\bar{s})}$ by an arbitrary logic $\mathfrak{L}$.

(d) We then write $(K_0, \mathfrak{L}_1) \leq_{\mathfrak{L}} (K_1, \mathfrak{L}_2)$ if there is an interpretation of K_0 into K_1 (modified as noted above) with the interpreting formulas in $\mathfrak{L}$.

In order to discuss interpretations which preserve Hanf numbers we require the following definition. (The phrase "bounded in terms" is defined in Section 1.3.1.)

2.1.4 Definition Suppose $(T_1, \mathcal{L}_1) \le_{\mathcal{L}} (T_2, \mathcal{L}_2)$ as in Definition 2.1.3(d). The interpretation is *almost decreasing* if for each $M \models \Pi$, $|M|$ is bounded in terms of $|\theta(M, \bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})|$.

The following result is now easy.

2.1.5 Theorem *If there is a nondecreasing interpretation* $(T_1, \mathcal{L}_1) \le_{\mathcal{L}} (T_2, \mathcal{L}_2)$ *with* $\mathcal{L} \le \mathcal{L}_2$ *then* $h_{\mathcal{L}_1}^{T_1} \le h_{\mathcal{L}_2}^{T_2}$.

In [21] Shelah proved:

2.1.6 Theorem
(i) $(T_\infty, \text{1st}) \lneqq (T_\infty, Mon) \lneqq (T_\infty, \text{1-1}) \lneqq (T_\infty, \text{2nd})$
(ii) For any $Q_{\psi(\bar{r})}$, $(T_\infty, Q_{\psi(\bar{r})})$ is first-order bi-interpretable with one of those four theories.

Now although, $(T_\infty, \text{2nd}) \not\le (T_\infty, Mon)$, it is quite possible for a particular T that $(T_\infty, \text{2nd}) \le (T, Mon)$ (e.g., any theory with a pairing function). Such an interpretation indicates T is complex. Our aim is to demonstrate this by showing that if no such interpretation can be made then (T, Mon) is "simple".

2.2 Some sufficient conditions for interpretations In this section we collect some lemmas which provide the tools for the principal interpretation results in the later sections. There are two examples which should be kept in mind. The prototype of a first-order theory with $(T, \text{2nd}) \le (T, Mon)$ is the theory of two cross-cutting equivalence relations. The prototype of a first-order theory with $(T, \text{2nd}) \le (T, \text{1-1})$ is the theory of an equivalence relation with infinitely many infinite classes. (These examples are discussed in more detail in Sections 4.2.2 and 6.1.)

To show $(T_\infty, Q_{\phi(\bar{r})}) \le (T, Q_{\psi(\bar{s})})$ it suffices to satisfy condition (ii) of Definition 1.3(a) with π, θ, χ_i such that $M \models \pi(\bar{a}, \bar{s}_0, \ldots, \bar{s}_{n-1})$ implies

$$R_\phi(B) \subseteq \{(\chi_0(M, a', s_0', \ldots, s_{n-1}'), \ldots, \chi_{m-1}(M, a', s_0', \ldots, s_{n-1}')\colon \bar{a}' \in M\ \bar{S}_i' \in R_\psi(M)\}$$

as the other inclusion can then be obtained by an easy modification of π.

The following obvious remark illustrates the definition of first-order interpretation.

2.2.1 Definition The theory T has a *definable pairing function* if there are formulas $\phi(x, y, z)$, $\rho_0(x, y)$, $\rho_1(x, y)$ which define the graphs of functions $\rho(x, y)$, $\rho_0(x)$, $\rho_1(x)$ such that $\rho(\rho_0(x), \rho_1(x)) = x$, $\rho_0(\rho(x, y)) = x$ and $\rho_1(\rho(x, y)) = y$.

2.2.2 Lemma *If T has a definable pairing function then* $(T_\infty, \text{2nd}) \le (T, Mon)$.

We want to weaken the hypothesis of this lemma as follows.

2.2.3 Definition
(i) The theory T *admits coding* if there is a model M of T containing infinite sets B_0, B_1, C such that some first-order formula $\phi(x_0, x_1, y, \bar{z}, B_0, B_1, C)$

defines (when appropriate constants $\bar{d}$ are substituted for the $\bar{z}$) a 1-1 function from $B_0 \times B_1$ onto C.

Note that by the Löwenheim-Skolem theorem if T has one such model it has one of every infinite cardinality.

Many theories which admit coding (e.g., any nontrivial infinite group) satisfy the even stronger property that the formula ϕ does not require unary parameters.

(ii) T admits strong coding if there is a formula $\phi(x_1, x_2, \bar{y}, \bar{z})$ such that for some M there definable infinite subsets B_0, B_1, C and a sequence of parameters $\bar{d}$ such that $\phi(x_1, x_2, y, \bar{d})$ defines a 1-1 function from $B_0 \times B_1$ onto C.

We next show, essentially, that if T admits coding then T has a monadically definable pairing function (although the argument doesn't proceed that way). This theorem has content since Lachlan showed (cf. [1: Section 5]) that no superstable theory admits a first-order pairing function while there are superstable theories which admit coding.

2.2.4 Lemma *If T admits coding then* $(T_\infty, \text{2nd}) \leq (T, Mon)$.

Proof: It suffices to interpret an arbitrary relation. Thus, according to Definition 2.1.3, we must find formulas $\theta(x, \bar{z}, \bar{s})$, $\pi(\bar{z}, \bar{s})$, and $\chi(x, y, \bar{z}, \bar{s})$ such that:

(i) If $M \models T$, $\bar{a} \in M$ and $\bar{U}$ is a sequence of subsets of M such that $M \models \pi(a, \bar{U})$ then $\theta(M, \bar{a}, \bar{U}) = B$ is an infinite set and $R_\chi(B) = \mathcal{P}(B \times B)$.

(ii) For all $\kappa \geq \aleph_0$, there exist $M \models T$, $\bar{a} \in M$ and subsets $\bar{U}$ of M with $|\theta(M, \bar{a}, \bar{U})| = \kappa$ and $M \models \pi(\bar{a}, \bar{U})$.

Let $M \models T$ and let $f(x, y)$ defined by $\psi(x, y)$ be a coding of $B_1 \times B_2$ by B_0 with $|B_0| = |B_1| = \kappa$. Fix 1-1 maps α and β of B_0 into B_1 and B_2 respectively. Define P_1, P_2 mapping B_0 to B_1, respectively B_2, by $f(P_1(y), P_2(y)) = y$.

Now let $\pi(U_0, U_1, U_2)$ assert ψ is a definable coding of $U_1 \times U_2$ by U_0. Let $\theta(x, \bar{U})$ be $U_0(x)$ and let $\chi(x, y, U)$ be $U_3(f(P_1(x), P_2(x)))$. Then condition (i) is satisfied by each $R \in \mathcal{P}(B_0 \times B_0)$, interpreting U_3 as $\{b \in B_0\colon R(\alpha^{-1}(P_1(b)), \beta^{-1}(P_2(b)))\}$. Condition (ii) is immediate.

The formula ϕ only needs to be first-order to guarantee that it can be applied uniformly to arbitrarily large models. So the same proof yields:

2.2.5 Corollary If there is an $\mathcal{L}$-formula ψ for $\mathcal{L} \in \{Mon, \text{1-1}\}$ such that for arbitrarily large κ there is a model M of T with $|M| = \kappa$ B_0, B_1, $C \subseteq M$ such that ψ codes $B_0 \times B_1$ into C then $(T_\infty, \text{2nd}) \leq_{\mathcal{L}} (T, \mathcal{L})$.

We will now consider $(T, \text{1-1})$. Recall that although we are restricted to quantification over permutations of the universe we can define arbitrary subsets (e.g., by the set of fixed points of a permutation) and such notions as "V has the same cardinality as the universe". $(\exists f f^2(x) = x \wedge f(x) \neq x \wedge \forall x\, U(f(x)) \vee U(x))$.

We want to provide a similar sufficient condition as that in Lemma 2.2.4 for $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$.

2.2.6 Lemma *If there is a first-order formula $\phi(x, y)$ such that for some $M \models T$ and some $A \subseteq M$, $\phi(x, y)$ defines an equivalence relation on A with infinitely many infinite classes, then* $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$.

Proof: Note first that by the Löwenheim-Skolem theorem, if the hypothesis holds for one M it holds for a model in every infinite power.

Consider a model M of T with a subset A such that $|A| = \kappa$ and $\phi(x, y)$ defines an equivalence on A with κ classes, each with cardinality κ. We will find two subsets of power κ coded by a third and conclude the lemma at hand from Corollary 2.2.5.

Let U_0 and U_1 define a partition of A which respects the equivalence relation ϕ and such that each U_i contains κ distinct ϕ equivalence classes. Let f be a bijection between U_0 and U_1 such that for each pair of ϕ-equivalence classes $X \subseteq U_0$, $Y \subseteq U_1$ $|f(X) \cap Y| = |1|$. Let U_0', U_1' be a set of representatives for the equivalence classes in U_0, U_1 respectively. Then each element x of U_0 codes the unique pair of y, z satisfying $y \in U_0'$, $\phi(x, y)$, $z \in U_1'$ and $\phi(f(x), z)$.

The following result is known so we only sketch a proof.

2.2.7 Corollary $(T_\infty, \text{2nd}) \leq (Th(<), \text{1-1})$.

Proof: Let M be a linear order of power κ which is κ-dense, and such that κ can be embedded in M. Let U_0 pick out a set $\langle a_i : i < \kappa \rangle$. Now define $\phi(x, y)$ as $\exists x_1 x_2 (U_0(x_1) \wedge U_0(x_2) \wedge \forall z \; (U_0(z) \rightarrow z \leq x_1 \vee z \geq x_2) \wedge x_1 < x < x_2 \wedge x_1 < y < x_2)$. Then ϕ defines an equivalence relation with κ-classes of cardinality $\geq\kappa$. So, as in the proof of Lemma 2.2.6, T can be coded by use of the permutational quantifier so $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$.

3 Decompositions of models

3.1 Infinitary monadic logic and generalized sums Shelah devised in [22] a formalism for back and forth arguments in monadic logic which is particularly useful in proving Feferman-Vaught type theorems. In this section we give a somewhat more detailed account of this formalism and extend it to infinitary monadic logics. We define the notion of a free union over N of a family of models which intersect in N and prove a Feferman-Vaught theorem for this notion. In the second section we use this theorem to calculate upper bounds for Hanf and Löwenheim numbers of theories whose models can be decomposed as free unions of "small" models.

By $L_{\kappa,\lambda}(Mon)$ we mean the language built by allowing conjunctions of fewer than κ-formulas and quantifications over strings of $<\lambda$ set or individual variables. $L^{\alpha}_{\kappa,\lambda}(Mon)$ denotes the formulas of rank $\leq\alpha$ (cf. [10]). In the proof of the Feferman-Vaught type theorem which we give below it is convenient to induct not over quantifier rank directly but rather over the object $mT^{\alpha}_{\bar{k}}$ which we are about to define. Roughly speaking, $mT^{\alpha+1}_{\bar{k}}(M)$ lists all the possible complete expansions $mT^{\alpha}_{\bar{k}}(M, \bar{P})$ of M. We do not give $mT^{\alpha}_{\bar{k}}$ a specific English rendering but it can be thought of as a strong monadic theory of rank α (i.e., corresponding to formulas with quantifier rank α). The role of $\bar{k}$ is to determine the number of possible additional predicates we are considering. Specifically, $\bar{k}$ will be a function with domain containing α. The range of $\bar{k}$ will be λ

when we discuss $L_{\infty,\lambda}(Mon)$. We deal only with relational languages. Except when specified the language may have infinitely many symbols.

In specifying an infinitary language there are five parameters which may be taken into account: the number of relation symbols in the language ($|L|$), the number of individuals or relations which can be quantified over (λ), the number of variables which can appear free in a formula (β), the quantifier depth (α), and the size of Boolean combinations which are permitted (κ). The fairly standard notation $L^{\alpha}_{\kappa,\lambda}$ takes account of all of these except β. So does the definition of $mT^{\alpha}_{\bar{k}}(L)$ defined immediately below. We will later define a finer hierarchy which takes note of β.

3.1.1 Definition

(a) Fix a language L. For any function $\bar{k}$ mapping $\{-1\} \cup \gamma$ into a set of cardinals, any cardinal λ and any ordinal α, we define the following:

(i) For any L-structure M and any sequence $\bar{P}$ of subsets of M with $lg(\bar{P}) < \lambda$, $mT^{0}_{\bar{k}}(M, \bar{P}) = \{\exists\bar{x}\phi(\bar{x}, \bar{S}): M \models \exists\bar{x}\phi(\bar{x}, \bar{P}); lg(\bar{x}) \leq \bar{k}(-1)\}$ (where ϕ is a possibly infinite conjunction of quantifier-free formulas). The role of $\bar{k}(-1)$ is explained in 3.1.2.

$$mT^{\alpha+1}_{\bar{k}}(M, \bar{P}) = \{mT^{\alpha}_{\bar{k}}(M, \bar{P}, \bar{Q}): lg(\bar{Q}) < \bar{k}(\alpha)\}.$$
$$mT^{\delta}_{\bar{k}}(M, \bar{P}) = \bigcup_{\beta<\delta} mT^{\beta}_{\bar{k}}(M, \bar{P}) \text{ if } \delta \text{ is a limit ordinal.}$$

(ii) $mT^{\alpha}_{\bar{k}}(L) = \{mT^{\alpha}_{\bar{k}}(M): M \text{ an } L\text{-structure}\}$

(iii) $mT^{\alpha}_{\mu}(L) = \{mT^{\alpha}_{\bar{k}}(M): rng\,\bar{k} \subseteq (\mu + |L|)\}$.

(b) We will also use $pT^{\alpha}_{\bar{k}}(M, \bar{P})$ and $pT^{\alpha}_{\bar{k}}(L)$ which are defined as in part (a) except that in the successor stage we require that the sequence $\bar{Q}$ partitions M. For the same $\bar{k}$, $pT^{\alpha}_{\bar{k}}$ is less expressive than $mT^{\alpha}_{\bar{k}}$. Thus if $\bar{k}$ has length $\aleph_0$ with $pT^{\alpha}_{\bar{k}}$ we can only guarantee a set is infinite; with $mT^{\alpha}_{\bar{k}}$ we can show it is larger than the continuum. However, by increasing the length of $\bar{k}$ (in this case to $2^{\aleph_0}$) we can find a $pT^{\alpha}_{\bar{k}}$ which is as expressive as $mT^{\alpha}_{\bar{k}}$.

(iv) To connect these concepts with a more common notation we define by induction $L^{\alpha}_{\infty,\lambda}(Mon)$. A Boolean combination in this definition may involve arbitrarily long conjunctions and disjunctions.

$L^{0}_{\infty,\lambda}(\bar{s}, \bar{x})$ is the collection of all Boolean combinations of quantifier-free L-formulas in the unary predicate variables $\langle s_i: i < u < \lambda\rangle$ and the individual variables $\langle x_i: i < u < \lambda\rangle$.

$L^{\alpha+1}_{\infty,\lambda}(\bar{s}, \bar{x}) = \{\exists\bar{r}\exists\bar{y}\phi(\bar{r}, \bar{y}, \bar{s}, \bar{x}): \phi$ a Boolean combination of formulas in $L^{\alpha}_{\infty,\lambda}(\bar{r}\bar{s}, \bar{x}\bar{y})\}$.

$$L^{\delta}_{\infty,\lambda}(\bar{s}, \bar{x}) = \bigcup_{\alpha<\lambda} L^{\alpha}_{\infty,\lambda}(\bar{s}, \bar{x}) \text{ for } \delta \text{ a limit ordinal.}$$

We write $L^{\alpha}_{\infty,\lambda}(Mon)$ when the particular set of variables $\bar{s}$, $\bar{x}$ is not important.

3.1.2 Notes

(a) The definition of $mT^{0}_{\bar{k}}(M, \bar{P})$ involves existential formulas so that it determines the Boolean relations between any finitely many of the $\bar{P}$. In particular, $mT^0(M, \bar{P}) = mT^0(M', P')$ implies (when $\bar{k}(-1) \geq 2$) that P_i is a singleton iff P'_i is a singleton. Thus quantification on individuals as well as subsets is encoded by the $mT^\alpha_{\bar{k}}$.

(b) The possibility of $\gamma \geq \alpha$ arises only to guarantee that $mT^\beta_{\bar{k}}$ is defined for $\beta < \alpha$ when we prove a result by induction on α.

(c) The role of $\bar{k}(-1)$ is just to make the finite language finitary logic case fit into our general rubric. In that case we set $\bar{k}(-1)$ as one plus the supremum of the arities of the relation symbols in L. This guarantees that $mT^0_{\bar{k}}(M)$ is finite. Otherwise we set $\bar{k}(-1) = \aleph_0$.

(d) We define both $mT^\alpha_{\bar{k}}$ and $pT^\alpha_{\bar{k}}$ because the first corresponds more closely with the $L^\alpha_{\infty,\lambda}(Mon)$ (see 3.1.5 below) but the second is more convenient for our computations.

In our discussion of Löwenheim and Hanf numbers the cardinalities of the $MT^\alpha_{\bar{k}}$ play a crucial role. The following remark is used for Hanf number calculations; the succeeding refinement is needed for the Löwenheim numbers.

3.1.3 Lemma *If $\bar{k}$: $\alpha \to \lambda + |L|$ then $|MT^\alpha_{\bar{k}}(L)| \leq \beth_{1+\alpha+1}(\lambda + |L|)$.*

Proof: $mT^0_{\bar{k}}(L) = \{mT^0_{\bar{k}}(M)$: M and L-structure$\}$ is the collection of all (consistent) collections of existential quantifications of infinite conjunctions. There are $\leq \beth_1(\lambda + |L|)$ possible conjunctions and so $mT^0_{\bar{k}}(L)$ has cardinality $\leq \beth_2(\lambda + |L|)$. Note that the cardinality of $mT^0_{\bar{k}}(L)$ does not depend on λ.

Suppose $|mT^\beta_{\bar{k}}(L)| \leq \beth_{1+\beta+1}(\lambda + |L|)$. Then $|mT^{\beta+1}_{\bar{k}}(L)| = |\{mT^{\beta+1}_{\bar{k}}(M)$: M an L-structure$\}| = |\{mT^\beta_{\bar{k}}(M, \bar{P})$: M an L-structure, $lg(\bar{P}) < \bar{k}(\beta)\}| \leq \beth_{1+\beta+1}(\lambda + |L|)$.

Suppose β is a limit ordinal and for $\gamma < \beta$ $|mT^\gamma_{\bar{k}}(L)| \leq \beth_{1+\gamma+1}(\lambda + |L|)$. Then $|mT^\beta_{\bar{k}}(L)| = |\{mT^\beta_{\bar{k}}(M)$: M an L-structure$\}| = \beth_{1+\beta+1}(\lambda + |L|)$; for the map sending $mT^\beta_{\bar{k}}(M)$ to $\langle mT^\gamma_{\bar{k}}(M)$: $\gamma < \beta\rangle$ is a 1-1 map from $mT^\beta_{\bar{k}}(L)$ into $\left(\bigcup_{\gamma<\beta} mT^\gamma(L)\right)^\beta$. We have shown 3.1.3.

The number of elements of $mT^\alpha_{\bar{k}}(L)$ which arises as $mT^\alpha_{\bar{k}}(M)$ with M of fixed cardinality does not depend on α. That is,

3.1.4 Lemma *For every α, $\bar{k}$, L and λ, if $W^\alpha = \{mT^\alpha_{\bar{k}}(M)$: M an L-structure $|M| \leq \lambda\}$ then $|W^\alpha| \leq 2^{\lambda+|L|}$.*

Proof: The isomorphism type of M determines $mT^\alpha_{\bar{k}}(M)$.

The next result, easily established by induction, links the $mT^\alpha_{\bar{k}}$ with standard infinitary languages.

3.1.5 Lemma *For each α, $\bar{k}$, λ and L and each $t \in mT^\alpha_{\bar{k}}(L)$, if λ^* denotes $\sup_{\beta<\alpha} (\bar{k}(\beta))$, there is a Boolean combination ψ_t of $L^\alpha_{\infty,\lambda^*}(Mon)$ sentences such that $mT^\alpha_{\bar{k}}(M) = t$ iff $M \models \psi_t$.*

Note that while quantifier rank and the number of quantifiers are both important the length of conjunctions is not significant here. Thus, at least in this context, the distinction between $L_{\infty,\lambda}$ and $L_{\lambda^+,\lambda}$ is minor.

Note that we have interpreted each $mT_{\bar{k}}^{\alpha}(L)$ into an $L_{\infty,\lambda^*}^{\alpha}(Mon)$. A converse interpretation is possible but the relation between $\bar{k}$ and λ^* is not the same.

We now describe a kind of "disjoint union" of a family of structures. Disjoint union is a misnomer in two senses. First, the components are not actually required to be disjoint, but are amalgamated over a "heart" N. More importantly, in the intuitive notion of a disjoint union there are no relations between elements in different components. Here we permit some such relations but only those which depend on the components separately and not on any interaction between them. The following example will illustrate this point as well as illuminating the role of the heart.

3.1.6 Example Let $T = Th(Z; S, R)$ where Z is the set of integers, S is the successor relation, and R is an equivalence relation with two classes: the evens and the odds. Now if M is any model of T and N is an elementary submodel containing a single component (i.e., an equivalence class for the equivalence relation, "finitely far apart"), we can write M as a free union of M_i over N where each M_i is the elementary substructure with universe $N \cup C_i$ and C_i is a component. Now R divides each component into two classes. To see whether $R(a, b)$ holds for a and b on different components ask whether for some $n \in N$, $R(n, a) \wedge R(n, b)$.

The key idea for this notion of free union over N is that the holding of $R(a, b)$ "should depend" only on $t(a; N)$ and $t(b; N)$, not on $t(a^\frown b; N)$. We analyze "should depend" as follows: Let M_i be a collection of L-structures which pairwise intersect in N. The universe of the new structure will be $\bigcup M_i$.

In order to describe the collection of n-tuples from M for which an n-ary relation R holds we must consider the partitions of n. We assign to each partition a sequence of types over N such that an n-tuple partitioned in that way (by intersection with the M_i) and realizing that sequence of types will satisfy R. Since we must assign various such correct partitions to R by a map σ we proceed formally to define a free union of $\langle M_i : i < \alpha\rangle$ over N (with respect to σ) as follows:

We make the following definition for notational convenience.

3.1.7 Definition If $\langle M_i : i < \alpha\rangle$ is a sequence of L-structures such that $i \neq j$ implies $M_i \cap M_j = N$ we call the M_i *a sequence with heart* N.

3.1.8 Definition Let $\langle M_i : i < \alpha\rangle$ be a family of L-structures with heart N. To define the *free union (with respect to* σ*) over* N *of the* M_i we need first the following auxiliary notions:

(i) An n-condition τ is a pair $\langle \Theta, \bar{p}\rangle$ where Θ is a partition of n into sets $\theta_0, \ldots, \theta_{k-1}$ with $|\theta_i| = l_i$ and $\bar{p} = \langle p_0, \ldots, p_{k-1}\rangle$ where p_i is a quantifier-free l_i-type $\in S(N)$ for $i < k$ and $p_0 = \bar{b}$ is a sequence of l_0 elements of N.

(ii) σ is a map which assigns to each relation symbol R of L a collection of m-conditions (where m is the arity of R).

(iii) Let $|M| = \bigcup M_i$. Let $\bar{a} = \langle a_0, \ldots, a_{n-1}\rangle$ in M. Partition n as follows: Let $\theta_0 = \{i : a_i \in N\}$. If for some $M_{p_1}, \ldots, M_{p_{k-1}}$, with the p_i; distinct, $\bar{a} - N \subseteq \bigcup_{1\le j<k} (M_{p_j} - N)$ then $\theta_j = \{i : a_i \in M_{p_j}\}$. Now $\bar{a}$

satisfies the n-condition $(\Theta, \bar{p})$ if $\Theta = \langle \theta_0, \ldots, \theta_{k-1} \rangle$ and for each $j < k$ $t(\bar{a}_j; N) = p_j$.

Now M is the following structure:

(a) $|M| = \cup M_i$.
(b) $R^M = \{\bar{a} : \bar{a}$ satisfies an n-condition in $\sigma(R)\}$.

We denote this structure as $\bigcup_N^{\sigma} \{M_i : i \in \alpha\}$. Note that there is no relation between σ and α.

We say M is a free union over N if for some $\langle M_i : i < \alpha \rangle$ and σ, $M = \bigcup_N^{\sigma} \{M_i : i \in \alpha\}$.

The restriction that the types p_i which appear in a collection are quantifier free is not as strong as might at first appear. When we decompose models of a stable theory in Section 4 we will see that we can guarantee this condition. For other applications, it is only necessary to check that the usual device of expanding the language by definitions to produce a theory which has elimination of quantifiers does not affect the result. In those instances (e.g., Section 6) where we restrict to a finite language we can eliminate quantifiers only as specifically needed in the argument. This will keep the number of relation symbols in the language finite.

Several years into the preparation of this paper we discovered that the definition of a free union could be made more strict by replacing the p_i in the definition of an n-condition by quantifier-free finitary formulas. With this revised definition all the important results of this paper (e.g., 3.1.13) would go through and the $mT_{\bar{k}}^{\alpha}(M)$ could be redefined to start with first-order formulas rather than infinite conjunctions. This would improve the Hanf number computations in Section 3.2 for $L_{\infty,\mu}^{\alpha}(Mon)$ and $\alpha < \omega$. This relatively minor gain did not seem to justify an extensive rewriting of the paper. However, we point out here what is probably *the* correct definition of a free union and in Section 4 where the decomposition theorems are proved we indicate how to extend the result to this strong form.

We must introduce the following notions to make inductions in monadic logic about free unions with hearts.

3.1.9 Definition If $\langle M_i : i < \alpha \rangle$ is a sequence of L-structures with heart N, the family $\langle P_i : i < \alpha \rangle$ with $P_i \subseteq M_i$ is a *compatible family* if $P_i \cap N = P_j \cap N$ for all i and j. We extend this definition naturally to $\langle \bar{P}_i : i < \alpha \rangle$ where $\bar{P}_i = \langle P_i^j : j < \beta \rangle$ is a sequence of β subsets of M_i.

3.1.10 Notation If $\bar{P}_i = \langle P_i^j : j < \beta \rangle$ is a compatible family of subsets of $\langle M_i : i < \alpha \rangle$ then $\bigcup_N^{\sigma} (M_i, \bar{P}_i)$ denotes the structure obtained from $\bigcup_N^{\sigma} M_i$ by denoting $\cup P_i^j$ by P^j for $j < \beta$.

3.1.11 Context We work with a sequence $\langle M_i : i < \alpha \rangle$ of L-structures with a heart N with $|N| = |L|$ and we assume L contains names for the elements of N.

Note that if $\bar{a}_i \in M_i - N$, $\bar{a}_j \in M_j - N$, $M = \bigcup_N^{\sigma} \{M_l : l \in \alpha\}$ and $mT_{\bar{k}}^0(M_i, \bar{a}_i) = mT_{\bar{k}}^0(M_j, a_j)$ for some $\bar{k}$ then $t_{qf}(\bar{a}_i; N) = t_{qf}(\bar{a}_j; N)$.

3.1.12 Notation In the following $\mathfrak{M} = \langle\langle M_i, \bar{P}_i\rangle : i \in I\rangle$ and $\mathfrak{M}' = \langle\langle M_j', \bar{P}_j'\rangle : j \in J\rangle$ denote compatible families with heart N. Each $\bar{P}_i$ denotes a sequence $\langle P_{i,l} : l < \lambda\rangle$ of subsets of M_i (similarly for M_i'). For any α and $\bar{k}$, we construct structures with universes I, J respectively for the language with $\beth_{\alpha+1}$ unary predicates by defining appropriate subsets of I and J. Let $t \in mT_\lambda^\alpha(L)$:

$$Q_t = \{i \in I : mT_{\bar{k}}^\alpha(M_i, \bar{P}_i) = t\}$$

and

$$Q_t' = \{j \in J : mT_{\bar{k}}^\alpha(M_j', \bar{P}_j') = t\} \ .$$

$$\bar{Q} = \langle Q_t : t \in mT_\lambda^\alpha(L)\rangle; \ \bar{Q}' = \langle Q_t' : t \in mT_\lambda^\alpha(L)\rangle \ .$$

For $\beta < \alpha$ let L^β denote L supplemented with $\bar{k}(\beta)$ unary predicates and

$$W^{\beta,\bar{k}} = \{s : s \in mT_{\bar{k}}^\beta(L^\beta) \text{ and } \exists i \in I \text{ and } \bar{R} \subseteq M_i \text{ with } lg(\bar{k}) = \bar{k}(\beta) \text{ with } s = mT_{\bar{k}}^\beta(M_i, \bar{P}_i, \bar{R}_i)\} \ .$$

In the following, $\bar{k}$ is fixed so we shorten $W^{\beta,\bar{k}}$ to W^β.

3.1.13 Theorem *For any α, σ, $\bar{k}$ and $\bar{r}$ such that for each $\beta < \alpha$, $\bar{r}(\beta) \geq |W^\beta|$, $pT_{\bar{r}}^\alpha(I, \bar{Q}) = pT_{\bar{r}}^\alpha(J, Q')$ implies $mT_{\bar{k}}^\alpha\left(\bigcup_N^\sigma \mathfrak{M}\right) = mT_{\bar{k}}^\alpha\left(\bigcup_N^\sigma \mathfrak{M}'\right)$.*

Proof: The proof is by induction on α. If $\alpha = 0$, let $\exists\bar{x}\phi(\bar{x}, \bar{P}) \in mT_{\bar{k}}^0\left(\bigcup_N^\sigma \mathfrak{M}\right)$ where ϕ is an infinite conjunction of quantifier-free formulas. Then for some $\bar{a} \in \bigcup_N^\sigma \mathfrak{M}$, $\models\phi(\bar{a}, \bar{P})$. Fix a partition of $\bar{a} - N$ into sequences $\bar{a}_l = \langle a_{l,j} : j < k_l\rangle$ for l less than some m with $\bar{a}_l$ a maximal subset of $M_l - N$ contained in $\bar{a}$. Thinking of the $a_{l,j}$ as singleton subsets, let $s_l = mT_{\bar{k}}^0(M_l, \bar{P}_l, \bar{a}_l)$ and $t_l = mT_{\bar{k}}^0(M_l, \bar{P}_l)$. For $s \in mT_{\bar{k}}^0(L')$ (when L' is the result of adding $lg(\bar{a})$ unary predicates to L), let $U_s = \{i \in I : mT_{\bar{k}}^0(M_i, \bar{P}_i, \bar{a}_i) = s\}$. For each $l < m$, $Q_{t_l} \neq \emptyset$. Since the Q_t partition I, the Q_t' partition J, and $pT_{\bar{r}}^0(I, \bar{Q}) = pT_{\bar{r}}^0(J, \bar{Q}')$, $|Q_{t_l}| = |Q_{t_l}'|$ $(mod\ \aleph_0)$. For each l since $l \in U_{s_l} \cap Q_{t_l}$ it is possible to choose $j \in Q_{t_l}'$, $\bar{a}_j' \in M_j'$ such that $mT_{\bar{k}}^0(M_j', P_j', a_j') = s_l$. By the definition of free decomposition $\bigcup_N^\sigma \mathfrak{M}' \models \exists\bar{x}\phi(\bar{x}, \bar{P}')$.

Limit ordinals are routine; we turn to the successor case: $\alpha = \gamma + 1$ for some γ. If $t \in mT_{\bar{k}}^{\gamma+1}\left(\bigcup_N^\sigma \mathfrak{M}\right)$ then for some $\bar{R}$ with $lg(\bar{R}) < \bar{k}(\gamma)$, $t = mT_{\bar{k}}^\gamma\left(\left\langle \bigcup_N^\sigma \mathfrak{M}, \bar{R}\right\rangle\right)$. Let $\bar{R}_i$ be the sequence obtained by intersecting the members of $\bar{R}$ with M_i. Then $t = mT_{\bar{k}}^\gamma\left(\bigcup_N^\sigma (M_i, \bar{P}_i, \bar{R}_i)\right)$. Now for $s \in mT_{\bar{k}}^\gamma(L, \bar{R})$ let

$$U_s = \{i \in I : mT_{\bar{k}}^\gamma(\bar{M}_i, \bar{P}_i, \bar{R}_i) = s\}; \ \bar{U} = \{U_s : s \in W^\gamma\} \ .$$

Note that the U_s partition I. Let $r = pI_{\bar{r}}^\gamma(\bar{I}, \bar{Q}, \bar{U})$. Since $pT_{\bar{r}}^{\gamma+1}(I, \bar{Q}) = pT_{\bar{r}}^{\gamma+1}(J, \bar{Q}')$ and $\bar{r}(\gamma) \geq |W^\gamma| = lg(\bar{U})$ there is a partition $\bar{U}'$ of J such that $r = mT^\gamma(J, \bar{Q}', \bar{U}')$. Now if $j \in U_s' \cap Q_t'$, $mT_{\bar{k}}^{\gamma+1}(M_j', P_j') = t$ and $s \in t$ so there exists an $\bar{R}_j$ such that $mT_{\bar{k}}^{\gamma}(M_j', \bar{P}_j', \bar{R}_j') = S$. Now by induction $mT_{\bar{k}}^{\gamma}\left(\bigcup_N^{\sigma}(M_i, \bar{P}_i, \bar{R}_i)\right) = mT_{\bar{k}}^{\beta}\left(\bigcup_N^{\sigma}(M_j', \bar{P}_j, \bar{R}_j)\right) = t$ so $t \in mT_{\bar{k}}^{\gamma+1}\left(\bigcup_N^{\sigma} M'\right)$ as required.

Observe that while we treated I as a set with no structure, the identical argument works if we replace I by some structure $\mathcal{I}$ (e.g., a linear order). This observation is exploited extensively in Section 8.

In order to apply this result to compute Hanf and Löwenheim numbers we need an explicit characterization of when $mT_{\bar{r}}^{\alpha}(I, \bar{Q}) = mT_{\bar{r}}^{\alpha}(J, \bar{Q}')$.

3.1.14 Definition Let $\bar{Q}$ be a partition of I, and $\bar{Q}'$ a partition of J with the same ordinal length. Then $(I, \bar{Q}) \equiv_u (J, \bar{Q}')$ if for each i either $|Q_i| = |Q_i'| < u$ or both $|Q_i|, |Q_i'|$ are $\geq u$.

3.1.15 Lemma *Let $\bar{Q}$, $\bar{Q}'$ partition I and J respectively with $lg(\bar{Q}) = lg(\bar{Q}') < \lambda$. If $\gamma = (\sup \bar{r}(\beta)) + \lambda$ and $(I, \bar{Q}) \equiv_{\gamma^+} (J, \bar{Q}')$ then $pT_{\bar{r}}^{\alpha}(I, \bar{Q}) = pT_{\bar{r}}^{\alpha}(J, \bar{Q}')$.*

Proof: $\alpha = 0$. All that can be said by an $mT_{\bar{k}}^{0}(I, \bar{U})$ is that each Q_i has a certain finite cardinality or is infinite. $\alpha = \beta + 1$: If $t \in mT_{\bar{r}}^{\alpha}(I, \bar{Q})$, $t = mT_{\bar{r}}^{\beta}(I, \bar{Q}, \bar{R})$ with $lg(\bar{R}) = \bar{r}(\beta)$. For each i $(Q_i) \equiv_{\gamma^+} (Q_i')$, so we can choose R_j' so that $|R_j \cap Q_i| \equiv_{\bar{r}(\beta)} |R_j' \cap Q_i'|$ for each j and i, so by induction $t = mT_{\bar{r}}^{\beta}(J, \bar{Q}', \bar{R}') \in mT_{\bar{r}}^{\alpha}(J, \bar{Q}')$. Reversing the process we finish the induction stage. For limit ordinals the result is immediate.

3.1.16 Notation For any α and $\bar{k}$, if $\bar{r}$ is chosen to satisfy 3.1.13 when the Q_t are interpreted as in 3.1.12, we say $pT_{\bar{r}}^{\alpha}(I, \bar{Q})$ *determines* $mT_{\bar{k}}^{\alpha}(M)$.

Our notion of free union is analogous to the algebraic concept of an "external" direct sum. We want now to define the corresponding "internal" notion, an L-congruence. In Section 4 we decompose a model M by constructing a sequence of L-congruences and obtaining from them a representation of M as a free union.

We are going to construct equivalence relations E on a model M such that the type of a finite sequence $\bar{b}$ over a set $A \subseteq M$ depends only on the types over A of the subsequences of $\bar{b}$ which are in the same E-class.

Considering the theory T, $Th(Z, s)$ of integers under a successor function will make the following definitions clearer. Any model M of T decomposes into countable components (the component of $x = \{s^n(x) : n \in z\}$). To know the type over the empty set of an n-tuple of elements you need only know which elements are infinitely far apart (i.e., in different components), and for those elements in the same component, how far apart they are. These components naturally correspond to the equivalence relation on M: "x and y are finitely far apart". To completely describe an n-tuple, it suffices to give the type of those subsequences which are in the same component and the restriction of the equivalence relation to the sequence. In giving a general account of this phenomena we sometimes must replace the equivalence relation by a more complicated structure. We also consider one further factor: we have to replace types over the empty set by types over a small set. This leads to some ambiguity in

Definitions 3.1.19 and 4.2.1 below. We define an L congruence on M over A as an equivalence relation on $M - A$ with certain properties. In Section 4 we define the fundamental equivalence relation E_A on all of M (where the elements of A form singleton classes). These two notions are harmonized by restricting E_A to $M - A$.

The following definitions provide a convenient notation for generalizing the last example. A similar, but more detailed development occurs in [15], VII.

3.1.17 Definition The *model* N is *indexed* by the index model (I, R) (where R is a binary relation) if there is 1-1 function f from N onto I. Letting $a_\alpha = f^{-1}(\alpha)$, we write $N = \{a_\alpha : \alpha \in I\}$. We say $\bar{a}$ and $\bar{b}$ in N are *similar* if $f(\bar{a})$ and $f(\bar{b})$ are partially isomorphic in (I, R).

In the particular case where we index elements by themselves (i.e., $N = X$) and R is an equivalence relation we obtain the following more concrete notion of two sequences being similar.

3.1.18 Notation Two sequences $\bar{a} = \langle a^i : i < n\rangle$ and $\bar{b} = \langle b^i : i < n\rangle$ are *similar* for the equivalence relation E if there is a partition of n into, say, k sets $J_0, \ldots, J_{k-1}$ such that $b^i E b^j$ if and only if $a^i E a^j$ if and only if i and j are in the same member of the partition. We write $\bar{a} = (\bar{a}_0, \ldots, \bar{a}_{k-1})$ where $\bar{a}_j = \{a^i : i \in J_j\}$.

Below, Δ denotes a collection of formulas closed under conjunction and disjunction, e.g., the quantifier-free formulas, the Σ_1^0-formulas. In particular, Δ_0 denotes the quantifier-free formulas.

3.1.19 Definition The equivalence relation E on $M - A$ is a (Δ_1, Δ_2)-congruence over A if for any two similar sequences from $M - A$, $(\bar{a}_0, \ldots, \bar{a}_{k-1})$ and $(\bar{a}'_0, \ldots, \bar{a}'_{k-1})$, $t_{\Delta_1}(\bar{a}_j; A) = t_{\Delta_1}(\bar{a}_j; A)$ for $j < k$ implies $t_{\Delta_2}(\bar{a}; A) = t_{\Delta_2}(\bar{a}'; A)$. If $\Delta_1 = \Delta_2 = \Delta$ we write Δ- for (Δ_1, Δ_2)-; if $\Delta_1 = \Delta_2$ is the set of all formulas we write: L-congruence. We may say M is E-decomposable over A (with respect to Δ).

Note that to establish that E is a Δ-congruence over A it suffices to show that if $\bar{a} = \bar{a}_0 {}^\frown \ldots {}^\frown \bar{a}_{k-1}$ and $\bar{a}' = \bar{a}'_0 {}^\frown \bar{a}'_1 {}^\frown \ldots {}^\frown \bar{a}'_{k-1}$ are similar sequences as in 3.1.18 and $t_\Delta(\bar{a}_0; A) = t_\Delta(\bar{a}'_0; A)$, then $t_\Delta(\bar{a}_0; A \cup \{\bar{a}_1, \ldots, \bar{a}_{k-1}\}) = t_\Delta(\bar{a}'_0; A \cup \{\bar{a}_1, \ldots, \bar{a}_k\})$.

Observe also that if M is E-decomposable over M_0 then truth in M is not affected by the order in which the components of E are listed. We will exploit this observation to simplify the statements of some of our lemmas.

We can extend an E satisfying 3.1.19 to an equivalence relation on all of M by putting each element of A into a singleton equivalence class. We may have occasionally slipped into this viewpoint.

We establish now that the notion of Δ-congruence is even stronger than it appears at first sight. In fact, we will show that an $L_{\infty,\omega}$-definable Δ_0-congruence is, in fact, an $L_{\infty,\omega}$-congruence. This result is proved by a straightforward induction. It is not used in the main line of the paper.

3.1.20 Theorem *Let E be an $L^\gamma_{\infty,\omega}$-definable Δ_0-congruence on M over A. Then E is an $L^\alpha_{\infty,\omega}$-congruence for each $\alpha \geq \gamma$.*

Proof: Without loss of generality, suppose the language L contains names for the elements of A. We show by induction on α that E is an $(L^{\gamma+\alpha}_{\infty,\omega}, L^{\alpha}_{\infty,\omega})$-congruence. When $\alpha \geq \gamma\cdot\omega$ we have the theorem. That is, we show by induction on the quantifier rank α of ϕ that for every $\phi(\bar{x})$ and every k (with $lg(\bar{x}) = k$) if $\bar{a}$ and $\bar{a}'$ are E-similar sequences of length k such that for each i, $tp^{\gamma+\alpha}_{\infty,\omega}(a_i; A) = tp^{\gamma+\alpha}_{\infty,\omega}(a_i'; A)$ then $M \models \phi(\bar{a})$ iff $M \models \phi(a')$. If $\alpha = 0$ the result is given and limit stages in the induction are easy. The interesting case is $qr(\phi) = \alpha + 1$ and $\phi(\bar{a}) = \exists x\, \psi(x, \bar{a})$ where $qr(\psi) \leq \alpha$. Then for some b, $M \models \psi(b, \bar{a})$. Let $\chi(y)$ be the $L^{\gamma+\alpha}_{\infty,\omega}$-formula $\wedge\, tp^{\alpha}_{\infty,\omega}(b, A)$.

Now, if for some i, $\models E(a_i, b)$, $(\exists y)(\chi(y) \wedge E(y, z))$ is in $tp^{\gamma+\alpha+1}_{\infty,\omega}(a_i; A)$ and so in $tp^{\gamma+\alpha+1}_{\infty,\omega}(a_i'; A)$. Choose b' to satisfy this formula. If $\neg E(a_i, b)$ for each $i < k$ then for each i, $M \models \exists y(\chi(y)) \wedge (\forall y)(E(y, a_i') \to \neg\chi(y))$ (since this sentence has quantifier rank $\gamma + \alpha + 1$) so $M \models \exists y\left[\chi(y) \wedge \bigwedge_{i<k} \neg E(y, a_i)\right]$. Choose b' to witness this sentence. Now $\bar{a}\frown b$ and $\bar{a}\frown b'$ are similar sequences which component by component realize the same $L^{\gamma+\alpha}_{\infty,\omega}$-type so by induction they realize the same $L^{\gamma+\alpha}_{\infty,\omega}$-type. In particular, $M \models \psi(b', \bar{a}')$, i.e., $M \models \phi(\bar{a}')$ as required.

By a similar proof (not by propositional logic, since in Theorem 3.1.19 we assume the components are $L_{\infty,\omega}$-equivalent) we can show:

3.1.21 Theorem *If E is an $L_{\omega,\omega}$-definable Δ_0-congruence on M over A then the E is an $L_{\omega,\omega}$-congruence.*

3.2 Tree-decomposable theories In this section we define the notion of a tree-decomposable theory and divide the tree-decomposable theories into three species which must be investigated by divergent means. Then we prove that the principal properties of an arbitrary tree-decomposable first-order theory, namely T, is stable and we can compute upper bounds on $h^T_{\mathfrak{L}}$ for various $\mathfrak{L}$. In Section 5 we improve these bounds for special kinds of tree-decomposable theories.

By a "tree" throughout this paper we mean a subset I of the collection of functions from an initial segment of κ to λ (written $\lambda^{\leq\kappa}$) such that if $\tau \in I$ and $\sigma \subseteq \tau$ then $\sigma \in I$.

We write $\rho * \tau$ if every proper initial segment of ρ is a proper initial segment of τ.

3.2.1 Definition The model M is decomposed by the tree $I \subseteq |M|^{\leq\kappa}$ if there exist models $\{\langle M_\eta, N_\eta\rangle : \eta \in I\}$ such that:

(i) $|N_\eta| = |T|$ for every η.
(ii) $\eta \subseteq \rho$ then $N_\eta \subseteq N_\rho \subseteq M_\rho \subseteq M_\eta$.
(iii) For $\tau \in I$, such that for some j $\tau\frown j \in I$, there are index sets J and functions σ such that:
 (a) $M_\tau = \bigcup^{\sigma}_{N_\tau} \{M_{\tau\frown j} : j \in J\}$ and
 (b) $M = \bigcup^{\sigma}_{N_\tau} (\{M_{\tau\frown j} : j \in J\} \cup \{M_\rho \cup N_\tau : \rho \neq \tau \text{ but } \rho * \tau\})$.
(iv) $M_{\langle\,\rangle} = M$; if $lg(\eta)$ is a limit ordinal $N_\eta = \bigcup_{\tau\subseteq\eta} N_\tau$ and $M_\eta = \bigcap_{\tau\subseteq\eta} M_\tau$.
(v) $M = \cup\{N_\tau : \tau \in I\}$.

We say M is κ-tree decomposable if it can be decomposed by some tree of height κ.

In fact, (iii) $\rightarrow$ (ii). If we omit (iii)(b) the decomposition is like that in [9] and [23]. Because of the stronger assumptions in this paper we do not need to discuss prime models and can recover M at each stage of the construction.

We associate with each model M decomposed by a tree I a family $\langle E_\rho : \rho \in I\rangle$ of equivalence relations. Namely, E_ρ is the equivalence relation on $M - N_\rho$ which induces the partition of $M - N_\rho$ given in (iii)(b).

3.2.2 Definition

(o) Let λ be a cardinal and α an ordinal then a *tree* of height α and width λ is a collection of functions I such that for each $\sigma \in I$ $dom\ \sigma \subsetneq \alpha$, $rng\ \sigma \subseteq \lambda$, if $\sigma \in I$ and $\tau \subseteq \sigma$, $\tau \in I$ and $\alpha = \sup_{\sigma \in I}((dom\ \sigma) + 1)$.

(i) For a cardinal κ, the theory T is *κ-tree decomposable* if every model of T admits a decomposition by a tree of height $\alpha \leq \kappa$.

(ii) If T is a 1-tree decomposable we say T is *strongly decomposable*.

(iii) If every model of T is decomposed by a tree with no infinite path then T is *shallow*.

(iv) Note that if T is shallow then we can assign to each model of T an ordinal *depth* (M) $(dp(M))$ which is the least ordinal assigned by the Kleene-Brouwer ordering to a tree which decomposes M.

(v) For an ordinal α, T is *α-tree decomposable* if every model of T has depth $\leq \alpha$.

Differing slightly from standard notation, we take the Kleene-Brouwer ordering to assign ordinals to the nodes of a well-founded tree $I \subseteq \lambda^{<\omega}$ as follows. If σ is an endpoint then $ord(\sigma) = 0$; otherwise $ord(\sigma) = \sup_{i<\lambda}(ord(\sigma^\frown i)) + 1$.

The ordinal of the tree is ordinal of the empty node. Thus every model of a strongly decomposable theory has both height and depth 1. The height of the tree is relevant only in computing Hanf numbers while the depth affects both Löwenheim number and the Hanf number.

Note that if T is strongly decomposable all the constituent models have cardinality $\leq |T|$.

3.2.3 Definition Any tree-decomposable theory which is not shallow is *deep.*

Deep theories divide according to whether their height is greater than $\omega + 1$ (i.e., whether or not each tree associated with a model can be imbedded in $\lambda^{<\omega}$). The impact of this distinction will be investigated in the future.

The next few results provide evidence for the intuition that tree-decomposable theories lack complexity.

3.2.4 Theorem *If for some κ, T is κ-tree decomposable, then* $(Th(<), Mon) \not\leq_{Mon} (T, Mon)$.

Proof: Suppose to the contrary that there is a formula $\phi(x, y, \bar{z}, \bar{r})$, a model M, and sequences $\bar{a}$, $\bar{R}$ such that $\phi(x, y, \bar{a}, \bar{R})$ defines a linear order B of length greater than $sup(\beth_\omega, \kappa)$. (If $(Th(<), Mon)$ were interpretable in (T, Mon) we could find an arbitrarily long order.)

Assign a node τ_b to each $b \in B$ by letting τ_b be the least σ such that $b \in N_\sigma$. Now partition the two-element subsets of B into κ classes by $\{x, y\} \in C_\alpha$ if $\tau_x \restriction \alpha = \tau_y \restriction \alpha$ but $\tau_x(\alpha) \neq \tau_y(\alpha)$ and $\{xy\} \in C_\infty$ if $\tau_x = \tau_y$. By the Erdos-Rado theorem there is a subset $B_0 \subseteq B$ with $|B_0| > \beth_\omega(|T|, 2^\kappa)$ which is homogeneous for this partition. Since for each τ, $|N_\tau| \leq |T|$, $B_0^{(2)}$ is not a subset of C_∞. Fix α with $B_0^{(2)} \subseteq C_\alpha$. Let τ be the largest common segment shared by all τ_b for $b \in B_0$. Then each $b \in B_0$ is in $M_{\tau\frown i}$ for some i and no two are in the same $M_{\tau\frown i}$.

For some n and $\bar{k}$ the truth of $\phi(b_i, b_j, \bar{a}, \bar{R})$ depends only on $mT_{\bar{k}}^n(M_{\tau\frown i}, b_i, \bar{a}, \bar{R})$ and $mT_{\bar{k}}^n(M_{\tau\frown j}, b_j, \bar{a}, \bar{R})$. But since $|B_0| \geq \beth_\omega(|T|) = \beth_\omega(|N_\tau|)$ we can find distinct b_i and b_j with the same $n - \bar{k}$ theory and neither equivalent to any $\bar{a}$. Thus $M \models \phi(b_i, b_j, \bar{a}, \bar{R})$ if and only if $M \models \phi(b_j, b_i, \bar{a}, \bar{R})$ contrary to the choice of ϕ.

The previous argument shows there is no $L_{\omega,\omega}(Mon)$ interpretation of the monadic theory of order into a κ-decomposable theory. A similar argument would show there is no $L^\alpha_{\infty,\lambda}(Mon)$ interpretation, although the bound on maximal linearly ordered sequences would significantly increase.

3.2.5 Corollary *If for some κ, T is κ-tree decomposable, then* $(T_\infty, \text{2nd}) \not\leq (T, Mon)$.

Proof: This is immediate since $(Th(<), Mon) \leq (T_\infty, \text{2nd})$. We will prove a strong converse to this corollary in Section 4.

3.2.6 Corollary *If for some κ, T is κ-tree decomposable then T is stable.*

Proof: This can be shown by directly computing the number of types or applying Theorem 8.1.6; if T is unstable, $(Th(<), Mon) \leq (T, Mon)$.

We could also derive Corollary 3.2.5 from the following results.

3.2.7 Theorem *If for some κ, T is κ-tree decomposable then* $(T_\infty, \text{1-1}) \not\leq (T, Mon)$.

Proof: Suppose to the contrary that there is a formula $\phi(x, y, \bar{z}, s, \bar{r}) \in L_{\omega\omega}(Mon)$, a model M and sequences $\bar{a}$, $\bar{R}$ and a set B such that $|S| > sup(\beth_\omega, 2^x)^+$ and every permutation π of B is defined by $\phi(x, y, \bar{a}_\pi, S, \bar{R}_\pi)$ for appropriate choice of $\bar{a}_\pi$ and $\bar{R}_\pi$. Exactly as in 3.2.4 there exists a $B_0 \subseteq B$ and an N_τ such that no elements of B_0 are in the same $N_{\tau\frown i}$. Now fix a permutation π of B such that if $b \in N_{\tau\frown i} - N_\tau$ then $\pi(b) \in B_0 - N_{\tau\frown i}$. Choose $\bar{a}_\pi$, $\bar{R}_\pi$ to define π. Since $|B_0| \geq \beth_\omega(|N_\tau|)$ there exist $b_i, b_j \in B_0$ such that $mT_{\bar{k}}^n(M_i, b_i, \bar{a}_\pi, S, \bar{R}_\pi) = mT_{\bar{k}}^n(M_j, b_j, \bar{a}_\pi, S, \bar{R}_\pi)$. Now if $b_i = \pi(b_0)$ we have $\models \phi(b_0, b_i, \bar{a}_\pi, S, \bar{R}_\pi)$. But then by the definition of free union we also have $\models \phi(b_0, b_j, \bar{a}_\pi, S, \bar{R}_\pi)$. This contradicts the assumption that ϕ defines a permutation.

Now we compute an upper bound on $h^{T_\alpha}_{L_{\infty,\mu}(Mon)}$ for an arbitrary $\aleph_1$-tree decomposable theory T. We will improve this result for shallow theories in Section 5. We show that it is best possible (for arbitrary tree-decomposable T) in the strongest possible sense in Section 7.

The idea of the argument here is that α determines a bound on the total number of possible $mT_{\bar{k}}^\alpha(M)$ so if we have a tree decomposition of a model M

so that "many" of the nodes N have the same theory $mT_{\bar{k}}^{\alpha}(N)$, then we use the Feferman-Vaught type theorems of Section 1 to blow up M to an arbitrarily large model.

There are a number of variants on the basic theme. The most important is the distinction between $h_{\mathfrak{L}}^{T}$ (the Hanf number for sentences) and $H_{\mathfrak{L}}^{T}$ (the Hanf number for theories). Other parameters which affect the computation are $|L|$ and α, μ where we consider $L_{\infty,\mu}^{\alpha}(Mon)$.

As a warmup to suppress the cardinal computations we compute upper bounds on $h_{L_{\omega,\omega}(Mon)}^{T}$ and $H_{L_{\omega,\omega}(Mon)}^{T}$ (3.2.8) for T an $\aleph_0$-decomposable theory. In 3.2.9 we compute an upper bound $h_{L_{\infty,\mu}^{\alpha}(Mon)}^{T}$. In 3.2.10 we apply this result to compute an upper bound on $h_{L_{\infty,\mu}^{\alpha}(Mon)}^{T}$. In 3.2.11 and 3.2.12 we show that if α is a limit ordinal this bound can be improved. Finally, in 3.2.13 we show how to improve the bound $H_{L_{\infty,\mu}^{\alpha}(Mon)}^{T}$ when α is a limit ordinal.

As they are stated these bounds are not best possible for $\alpha < \omega$. This is essentially an accident of our notation. We report the remedy after 3.2.12.

We state the results for countable languages. The extension to uncountable languages is routine and just clutters up the notation.

3.2.8 Theorem *If T is a countable $\aleph_0$-decomposable theory then*

(i) $h_{L_{\omega,\omega}(Mon)}^{T} \leq (\beth_\omega)$

(ii) $H_{L_{\omega,\omega}(Mon)}^{T} \leq (\beth_\omega)^+$.

Proof: (i) Let $|M| = \beth_\omega$ and write $M = \bigcup_{N}^{\sigma} \langle M_i : i \in I\rangle$ with $I \subseteq (\beth_\omega)^{<\omega}$. Then there exists a τ such that $M = \bigcup_{N_\tau}^{\sigma} \{\overline{M}_k : k \in K\}$ where (a) $K = J \cup \{\rho : \rho^- \subseteq \tau^-$ and $\rho \neq \tau\}$, (b) $\overline{M}_k = M_{\tau^\frown k}$ if $k \in J$, (c) $\overline{M}_k = M_k \cup N_\tau$ if $k \in I$ and $k^- \subseteq \tau^-$ but $k^- \neq \tau^-$ and (d) $|K| \geq |M| = \beth_\omega$. (Details on the choice of N_τ can be found in the proof of 3.2.9.)

Now if $\phi \in L_{\omega,\omega}(Mon)$ the truth value of ϕ is determined by $mT_{\bar{k}}^{n}(M)$ for some $n < \omega$. But if $\lambda = |mT_{\bar{k}}^{n}(L)|$, then $\lambda^+ < \beth_\omega$ so there exists $L \subseteq K$ with $|L| > \lambda$ and $mT_{\bar{k}}^{n}(M_l) = mT_{\bar{k}}^{n}(\overline{M}_m)$ for every l, $m \in L$. For any λ form M_λ as $\bigcup_{N_\tau}^{\sigma} \{N_i : i \in K \cup \lambda\}$ with $N_i = M_i$ if $i \in K$ and $N_i \approx M_l$ for $i < \lambda$ (and l any fixed member of L).

(ii) Let $|M| = (\beth_\omega)^+$ and decompose M as above, now guaranteeing $|K| = (\beth_\omega)^+$. For each $n < \omega$ $|mT_{\bar{k}}^{n}(L)| < \beth_\alpha$. So if X_n is the set of j such that $mT_{\bar{k}}(\overline{M}_j)$ occurs less than $\beth_\omega$ times, $|X_n| < \beth_\omega$. Thus there exists an $m \in K$ such that for every n $|\{l : mT_{\bar{k}}^{n}(\overline{M}_m) = mT_{\bar{k}}^{n}(\overline{M}_l)\}| \geq \beth_\omega^+$. Now we can find arbitrarily large models M' with $mT_{\bar{k}}^{n}(M) = mT_{\bar{k}}^{n}(M')$ for all n and $\bar{k}$ by adding additional copies of $\overline{M}_l$ over N.

3.2.9 Theorem *Let $\alpha \geq \omega$ and M a model of the countable $\aleph_1$-decomposable theory T. If $|M| = \kappa > \beth_{1+\alpha+1}(\mu)$ then for every $\lambda \geq \kappa$ there exists a model M_λ with $|M_\lambda| = \lambda$ and $M_\lambda \equiv_{L_{\infty,\mu}^{\alpha}} M$.*

That is $H_{L_{\infty,\mu}^{\alpha}(Mon)}^{T} \leq (\beth_{1+\alpha+1}(\mu))^+$.

Proof: We know that M is decomposed by some tree I with height $<\aleph_1$. Then some node $\tau \in I$ has $\beth_{1+\alpha+1}^+$ successors. For, if each has $\mu \leq \beth_{1+\alpha+1}$ successors then $\mu^{\aleph_0} \leq (2^{\beth_\alpha})^{\aleph_0} \leq \beth_{\alpha+1}$ contrary to hypothesis. So for some τ, we can decompose M to satisfy the following conditions: $M = \bigcup^{\sigma}_{N_\tau} \{\bar{M}_k : k \in K\}$ where:

(i) $K = J \cup \{\rho : \rho^- \subseteq \tau^- \text{ and } \rho \neq \tau\}$
(ii) $\bar{M}_k = M_{\tau^\frown k}$ if $k \in J$
(iii) $\bar{M}_k = M_k \cup N_\tau$ if $k \in I$ and $k^- \subseteq \tau^-$
(iv) $|K| \geq |M| > \beth_{\alpha+1}$.

We want to choose M_λ with cardinality λ such that $M_\lambda \equiv_{L^{\alpha}_{\infty,\omega}(Mon)} M$. We will make $M_\lambda = \bigcup^{\sigma}_{N_\tau} \{A_l : l \in K \cup \lambda\}$ for appropriate choice of the A_l. Letting for $\tau \in mT^{\alpha}_{\bar{k}}(L)$ $Q_t = \{k \in K: mT^{\alpha}_{\bar{k}}(M_k) = t\}$ and $Q'_t = \{l \in K \cup \lambda: mT^{\alpha}_{\bar{k}}(A_l) = t\}$, we must choose the A_l so that $mT^{\alpha}_{\bar{r}}(K, \bar{Q}) = mT^{\alpha}_{\bar{r}}(K \cup \lambda; \bar{Q}')$ where $\bar{r}$ is chosen from $\bar{k}$ as in 3.1.13. To apply that theorem we see that W^γ must be taken as $mT^{\gamma}_{\bar{k}}(L)$ so $\bar{r}(\beta) = \beth_{1+\beta+1}(\lambda + L) \leq \beth_{1+\alpha+1}$. By 3.1.15 it suffices to choose the A_l so that $(K, \bar{Q}) \equiv_{\beth_{1+\alpha+1}} (K \cup \lambda, \bar{Q})$ and this is guaranteed by (iv).

We can obtain the result for h^T as follows:

3.2.10 Corollary *Let T be an $\aleph_1$-decomposable countable theory. Then $h^T_{L^{\alpha}_{\infty,\mu}(Mon)} \leq (\beth_{1+\alpha}(\mu))$.*

Proof: If $\phi \in L^{\alpha}_{\infty,\mu}(Mon)$, then by Definition 3.1.1 $\phi = \exists \bar{x}\psi$ where it is a Boolean combination of formulas in $L^{\beta}_{\infty,\mu}(Mon)$ for some $\beta < \alpha$. Let $M \models \phi$ and $(M, \bar{P}) \models \phi(\bar{P})$. By 3.2.9 applied to $L(\bar{P})$ since $|M| \geq \beth^{+}_{1+\beta+1}$ we can find $M_\lambda = (M_\lambda, \bar{P}')$ such that $(M_\lambda, \bar{P}') \equiv_{L^{\beta}_{\infty,\mu}(Mon)} (M, \bar{P})$. Thus $M_\lambda \models \phi$.

Note that this particular argument depends essentially on our rather arbitrary refusal to close $L^{\alpha}_{\infty,\mu}(Mon)$ under finite Boolean operations. To recover this theorem when such combinations are allowed the proof of 3.1.3 must be revised to deal with finite Boolean combinations of theories (i.e., of $mT^{\alpha}_{\bar{k}}(M)$).

We pointed out after the Definition 3.1.10 that the definition of free union could be made to demand that conditions be finite quantifier-free formulas. In that case, Theorem 3.1.13 holds when the $mT^{\alpha}_{\bar{k}}$ hierarchy is replaced by an $nT^{\alpha}_{\bar{k}}$ hierarchy which is defined in the same way except $nT^{0}_{\bar{k}}(M, \bar{P})$ is the collection of first-order formulas $\exists \bar{x}\psi(\bar{x}, \bar{P})$ true in $(M, \bar{P})$. Theorem 3.1.13 can be proved for this hierarchy. Theorem 3.1.13 is improved to $|nT^{\alpha}_{\bar{k}}(L)| \leq \beth_\alpha(|L|)$. This allows one to conclude $H^T_{L^{\alpha}_{\infty,\mu}(Mon)} \leq \beth_\alpha(\mu)$ for all α.

Now, by an argument like that for 3.2.8(i), we can improve the upper bound on $h^T_{L^{\alpha}_{\infty,\mu}(Mon)}$ for limit α. If T is not $\aleph_0$-decomposable this argument works only for α with $cf(\alpha) > \omega$. In 3.2.12 we see this hypothesis on α can be avoided for $\aleph_0$-decomposable theories.

3.2.11 Theorem *If T is a countable $\aleph_1$-decomposable theory then $h^T_{L^{\delta}_{\infty,\mu}(Mon)} \leq \beth_\delta(\mu)$ where δ is a limit ordinal with $cf(\delta) > \omega$.*

Proof: Let $M \models T$ with $|M| \geq \beth_\delta$. We know that M is decomposed by some tree I with height $<\aleph_1$. Some node τ of I has at least $\beth_\delta$ successors. For, if each node has $\mu < \beth_\delta$ successors then $|M| \leq \beth_\delta(\mu)^{\aleph_0}$ $(= \beth_\delta(\mu)$ since $cf(\delta) > \omega$

[11, Theorem 6.17, p. 239]. Thus, we can find τ to decompose M as follows. $M = \bigcup^{\sigma}_{N_\tau} \{\overline{M}_k : k \in K\}$ where

(i) $K = J \cup \{\rho^- \subseteq \tau^- \text{ and } \rho \neq \tau\}$
(ii) $\overline{M}_k = M_{\tau\frown k}$ if $k \in J$
(iii) $\overline{M}_k = M_k \cup N_\tau$ if $k \in I$ and $k^- \subseteq \tau^-$
(iv) $|K| \geq |M| > \beth_\delta(\mu)$.

Now if $\phi \in L^{\delta}_{\infty,\mu}$ then for some $\beta < \delta$ the truth value of ϕ is determined by $mT^{\beta}_{k}(M)$. If $\lambda = |mT^{\beta}_{k}(L)|$, $\lambda^+ < \beth_\delta(\mu)$ so there exists a subset L of K such that for every l, $m \in L$ $mT^{\beta}_{k}(M_l) = mT^{\beta}_{k}(M_k)$. For any λ form M_λ as $\bigcup^{\sigma}_{N_\tau} \{N_i : i \in K \cup \lambda\}$ with $N_i = M_i$ if $i \in K$, $N_i \approx M_l$ (over N_τ) for $i < \lambda$ and l any fixed member of L. Then $mT^{\beta}_{k}(M) = mT^{\beta}_{k}(M_\lambda)$ and we finish.

If T is $\aleph_0$-decomposable the decomposition of M in 3.2.11 does not require $cf(\delta) > \omega$ so we can conclude:

3.2.12 Theorem *If T is a countable $\aleph_0$-decomposable theory then for any limit ordinal δ, $h^{T}_{L^{\delta}_{\infty,\mu}(Mon)} \leq \beth_\delta(\mu)$.*

Shelah has later shown that for limit δ, $H^{T}_{L^{\delta}_{\infty,\mu}(Mon)} \leq \beth_\delta(\mu)$, for an $\aleph_1$-decomposable theory.

Note that if $|T|$ unary predicates are adjoined to L, they are absorbed by the computation of the $\beth_\alpha$. Thus $h^{T}_{L^{\alpha}_{\infty,\mu}(Mon)} = h^{T}_{L^{\alpha}_{\infty,\mu}(Mon)}$ and $H^{T}_{L^{\alpha}_{\infty,\mu}(Mon)} = H^{T}_{L^{\alpha}_{\infty,\mu}(Mon)}$ for infinite α. This observation is only required to show that some of the computations in Section 8.2 which require expansion to $\bar{L}$ are best possible.

4 Decomposition in stable theories The main result in the section asserts that T is tree-decomposable if and only if T is stable and $(T_\infty, \text{2nd}) \not\leq (T, Mon)$. This result depends on the apparatus of stability theory so we list in Section 4.1 the main properties of forking which we invoke here. In Section 4.2 we introduce the fundamental equivalence relation aE_Ab if and only if $t(a; A \cup b)$ forks over A, and we show how it induces a tree-decomposition of every model of a stable theory T satisfying $(T_\infty, \text{2nd}) \not\leq (T, Mon)$. In Section 4.3 we show the fundamental equivalence relation is sometimes monadically definable. This result is essential for Section 7, but is not used elsewhere in this paper.

4.1 Properties of forking We list here the salient properties of the forking relation which hold in a stable theory and on which we rely in the next section. See [12] and [15] for further definitions and proofs.

Whenever we write $t(B; A)$ we refer to the type of some fixed enumeration of B.

4.1.1 (Finite character)
(i) $t(A; B \cup C)$ forks over B if and only if for some finite $A_0 \subseteq A$, $C_0 \subseteq C$, $t(A_0, B \cup C_0)$ forks over B.
(ii) Moreover, if $t(\bar{a}; A \cup \bar{b})$ forks over A there is a formula $\phi(\bar{x}, \bar{y})$ (over A)

such that for any $\bar{b}'$ and $\bar{a}'$ such that $t(\bar{b}; A) = t(b'; A)$ if $\models \phi(\bar{a}'; \bar{b}')$ then $t(\bar{a}'; A \cup \bar{b}')$ forks over A.

4.1.2 (Monotonicity) $t(A; B \cup C)$ does not fork over B if and only if $t(A; B \cup C)$ does not fork over $B \cup C_0$ and $t(A; B \cup C_0)$ does not fork over B (for any $C_0 \subseteq C$).

4.1.3 (Symmetry) $t(A; B \cup C)$ forks over B if and only if $t(C; A \cup B)$ forks over B.

4.1.4 (Extension) If the type p over B does not fork over A and $B \subseteq C$ there is a $q \in S(C)$ with $p \subseteq q$ such that q does not fork over A.

4.1.5 (Existence of non-forking types) For any a and A, $t(a; A)$ does not fork over A.

The following two important properties can be deduced from the preceding properties.

4.1.6 $t(a \frown b; A \cup B)$ does not fork over A if and only if $t(a; A \cup B)$ does not fork over A and $t(b; A \cup B \cup \{a\})$ does not fork over $A \cup \{a\}$.

4.1.7 If $t(a \frown b; A \cup B)$ does not fork over A then $t(a; A \cup b)$ forks over A if and only if $t(a; B \cup b)$ forks over B.

We also refer from time to time to the $stp(a; A)$ (see [15], III). The key fact we use is:

4.1.8 Theorem *Suppose $\langle a_i : i < \alpha \rangle$ is a sequence such that for all i, j if $stp(a_i; A) = stp(a_j; A)$ and $t(a_i; A \cup A_i)$ does not fork over A, then $\{a_i : i < \alpha\}$ is an indiscernible set over A.*

We frequently write d.n.f. for "does not fork".

4.2 The fundamental equivalence relation Throughout this section we deal with a stable theory such that $(T_\infty, \text{2nd}) \not\leq (T, Mon)$. For any set $A \subseteq N$ a model of T we make the following definition.

4.2.1 Definition $xE_A y$ if $t(x; A \cup y)$ forks over A, or $x = y$.

If T is stable, for any A, then E_A is symmetric and reflexive. If we omitted the clause, "or $x = y$" it would only be reflexive on the elements of N which are not algebraic over A. The following examples show that when $(T, \text{2nd}) \leq (T, Mon)$, E_A may not be transitive. For any equivalence relation $\mathcal{E}$ we denote by $[a\mathcal{E}]$ the $\mathcal{E}$-equivalence class of a.

4.2.2 Example The language L has two binary relations, E_0, E_1. The theory T_0 asserts that each E_i equivalence class is split into infinitely many E_{1-i} equivalence classes. Now if $a_0E_0a_1$, $a_1E_1a_2$ and $\neg a_0E_0a_2$, $\neg a_0E_1a_2$, $E_\emptyset$ is not an equivalence relation since it is not transitive.

4.2.3 Example The language L has one unary relation U and one binary relation R. In any model M of the theory T_1 each element of $\neg U(M)$ is linked to exactly two elements of $U(M)$ by R (and every pair from U is linked to a unique element from $\neg U(M)$). Choosing $a, b, c \in \neg U(M)$ such that a "codes" $\{a_0, a_1\}$, b codes $\{a_1, a_2\}$, and c codes $\{a_2, a_3\}$ we have another counterexample to transitivity.

It is easy to code an arbitrary binary relation on models of either of these theories. Lemma 4.2.6 generalizes such a coding. We will show that in our situation $((T_\infty, \text{2nd}) \not\leq (T, Mon))$, E_A is an equivalence relation and if $A = M$ is the universe of a model of T then E_M is an L-congruence.

In the remainder of this section we require only that T be stable and not admit coding (although we state this hypothesis as $(T_\infty, \text{2nd}) \not\leq (T, Mon)$). From these two hypotheses we deduce a number of properties of the relation E_A.

4.2.4 Lemma *Suppose there exist* a, $B = \langle b_i : i < \omega\rangle$, *and* $C = \langle c_j : j < \omega\rangle$ *such that*:
(i) B *is a set of indiscernibles*,
(ii) C *is a set of indiscernibles over* B *and there is a* $\phi(x, y, z)$ *such that* $\phi(a, b_i, c_j)$ *if and only if* $i = j = 0$. *Then* T *admits coding.*

Proof: For each i, j there is an elementary map f_{ij} interchanging b_0 with b_i and c_0 with c_j and fixing all other elements of $B \cup C$. Thus $\models \phi(f_{ij}(a), b_k, c_l)$ if and only if $k = i$ and $j = l$. If $A = \{f_{ij}(a); i, j < \omega\}$, A codes $B \times C$.

Naturally, this lemma remains true if B and C are replaced by $B_0, B_1, \ldots, B_{n-1}$ satisfying analogous conditions.

If $\{\bar{a}_i : i < \alpha\}$ is a sequence, we let $A_j = \{\bar{a}_i : i < j\}$.

4.2.5 Remark The next lemma shows that if T does not admit coding, then in T, forking defines an equivalence relation on single elements and is totally trivial. That is, an element a cannot depend on a sequence $\bar{b}$ unless it depends on a single element of $\bar{b}$. This notion of triviality has been exploited in numerous papers since this one was begun, e.g., [3].

4.2.6 Lemma *If* T *is stable and either*
(i) $\exists A, a, b, c$ *such that* $t(a; A \cup b)$ *forks over* A, $t(a; A \cup c)$ *forks over* A *but* $t(b; A \cup c)$ *d.n.f. over* A, *or*
(ii) *there exist* $A, a, b_1, \ldots, b_n$ *such that the* b_i *are independent, for each* i, $t(a; b_i)$ *does not fork over* A *but* $t(a; \{b_1, \ldots, b_n\} \cup A)$ *forks over* A, *or*
(iii) *there exist* $A, a, b_1, \ldots, b_n$ *such that* $t(a; A \cup b_i)$ *d.n.f. over* A *but* $t(a; A \cup \{b_1, \ldots, b_n\})$ *forks over* A
then T *admits coding.*

Proof: (i) We will construct for $i, j < \omega$, b_i, c_j such that $t(b_i {}^\frown c_j; A) = t(b {}^\frown c; A)$ but $t(a; A \cup b_i {}^\frown c_j)$ d.n.f. over A unless $b_i = b$ and $c_j = c$. Having done so, we can choose $\langle a_{i,j} : i, j < \omega\rangle$ such that $t(a_{i,j}; \{b_k, b_l\})$ forks over $\emptyset$ just if $k = i$ and $j = l$. (Let a_{ij} be the image of a under an elementary map which takes $b {}^\frown c$ to $b_i {}^\frown c_j$). Then choosing $\phi(x, y, z)$ such that $\models \phi(a, b, c)$ and $\phi(x, b, c)$ forks over $\emptyset$, ϕ defines a coding of $\langle b_i : i < \omega\rangle \times \langle c_j : j < \omega\rangle$ by $\{a_{ij}: i, j < \omega\}$.

For the construction, fix $stp(b, A)$ and $stp(c, A)$. Choose $\{b_i : 1 \leq i < \omega\}$ so that $t(b_i; A \cup B_i \cup \{a, c\})$ d.n.f. over A and b_i realizes $stp(b, A)$. Let $b_0 = b$ and $B = \{b_i : i < \omega\}$. Then choose $\{c_j : 1 \leq j < \omega\}$ such that $t(c_j;$

$A \cup B \cup \{a\} \cup C_j\}$ d.n.f. over A and c_j realizes $stp(c; A)$. Let $c_0 = c$ and $C = \{c_i : i < \omega\}$. (Setting $b_0 = b$ and $c_0 = c$ is permissible because $t(c; A \cup \{b\})$ d.n.f. over A.) Now suppose $i \neq 0$ and consider $t(a; A \cup b_i \cup c_j)$. By the construction $t(c_j : A \cup \{b_i, a\})$ d.n.f. over A and since $i \neq 0$, $t(b_i; A \cup \{a\})$ d.n.f. over A. Thus, by 4.1.6 and monotonicity $t(c_j \frown b_i ; A \cup \{a\})$ d.n.f. over A. By symmetry, $t(a; A \cup \{c_j, b_i\})$ d.n.f. over A. Similarly if $j \neq 0$, $t(a; A \cup \{c_j, b_i\})$ d.n.f. over A. It remains to show $t(b_i \frown c_j; A) = t(b \frown c; A)$. But this is immediate since $t(b_i; A) = t(b; A)$, $t(c; A \cup \{b\})$ d.n.f. over A, $t(c_j; A \cup \{b_i\})$ d.n.f. over A, $stp(c_i, A) = stp(c, A)$, and strong types are stationary.

(ii) Without loss of generality we may assume $n = 2$ and $A = \emptyset$. Let $\bar{c}_0 = b_1^0 b_2^0$ and for $i < \omega$, choose $\bar{c}_i$ so that $t(\bar{c}_i; C_i)$ does fork over $\emptyset$ and $b_1^i(b_2^i)$ realizes $stp(b_1; \emptyset)$ $(stp(b_2; \emptyset))$. Note that for any i, j $t(b_1^i, b_2^j; \emptyset) = t(b_1^0, b_2^0; \emptyset)$. Thus for each i, j there is an a_{ij} such that $t(a_{ij}; b_1^i, b_2^j; \emptyset) = t(a; b_1, b_2; \emptyset)$. In particular $t(a_{ij}; \{b_1^i, b_2^j\})$ forks over the empty set. By the extension lemma we can assume $t(a_{ij}; C_\omega)$ does not fork over $\{b_1^i, b_2^j\}$. Since $t(C_\omega - \{b_1^i, b_2^j\}; \{b_1^i, b_2^j\})$ does not fork over the empty set transitivity of nonforking yields $t(a_{ij}; C_\omega - \{b_1^i, b_2^j\})$ does not fork over the empty set. As in part (i), choosing a formula $\phi(x, y, z)$ so that $\phi(a, b_1, b_2)$ witnesses the forking of $t(a; \{b_1, b_2\})$ over $\emptyset$, we contradict 4.2.4.

(iii) Without loss of generality, we may assume $n = 2$ and $A = \emptyset$. For ease of notation in the argument we rename a, b_1, b_2 as a, b, c. Thus we have $t(a; b)$ and $t(a; c)$ do not fork over the empty set but $t(a; b, c)$ forks over the empty set. By (ii) we see that $t(b; c)$ forks over the empty set.

Now choose formulas $\phi(y, z)$ and $\psi(x, y, z)$ to witness this forking. That is, for any b' realizing $p = t(b; \emptyset)$ and c' realizing $q = t(c; \emptyset)$, $\models \phi(b', c')$ implies $t(b'; c')$ forks over $\emptyset$. Moreover, if in addition $t(a' \frown b'; \emptyset) = t(a \frown b; \emptyset)$ and $t(b' \frown c'; \emptyset) = t(b \frown c; \emptyset)$ then $\psi(a', b', c')$ implies $t(a'; \{b', c'\})$ forks over b'. Since $\models \psi(a, b, c) \wedge \phi(b, c)$ we can assume $\psi(x, y, z) \to \phi(y, z)$. Now choose a set $A = \{a_i : i < \kappa\}$ of realizations of $stp(a; \emptyset)$ and a set $B = \{b_j; j < \kappa\}$ of realizations of $stp(b; \emptyset)$ such that if $e \in A \cup B$, $t(e; A \cup B - \{e\})$ does not fork over e.

We now code $A \times B$ by a set C of realizations of q. Note that for any $a' \in A$, $b' \in B$, $t(a', b'; \emptyset) = t(a, b; \emptyset)$. Thus for each i, j there exists a c_{ij} with $t(a_i, b_j, c_{ij}; \emptyset) = t(a, b, c; \emptyset)$. We claim $\models \psi(a_i, b_j, c_{kl})$ if and only if $i = k$ and $j = l$.

The "if" is immediate by the choice of c_{ij}. Now suppose $\models \psi(a_i, b_j, c_{kl})$. In particular, we have $\phi(b_j, c_{k,l}) \wedge \phi(b_l, c_{k,l})$. Thus $t(b_l; c_{k,l})$ forks over $\emptyset$ and (by symmetry) $t(c_{k,l}; b_j)$ forks over $\emptyset$. By (i) $t(b_l, b_j)$ forks over $\emptyset$. By the construction of B as an independent set, $j = l$. Using this, we have $\psi(a_i, b_l, c_{k,l}) \wedge \psi(a_k, b_l, c_{k,l})$. Thus $t(a_i; \{b_l, c_{k,l}\})$ forks over b_l and $t(a_k, \{b_l, c_{k,l}\})$ forks over b_l. Applying (i) with $\{b_l\}$ as A we have $t(a_i; \{b_l, a_k\})$ forks over b_l. By (ii) this implies $a_i = a_k$ and we finish.

We now show that if $M \models T$, E_M is not only an equivalence relation, but an L-congruence. Thus if T is stable and $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ for any $M \models T$ and any $N \prec M$ we can decompose M over N. We will in this case conclude the even stronger result that if X is an equivalence class of E_N then $N \cup X$ is the universe of a model of T.

4.2.7 Lemma *Assume T does not admit coding. If for every $\bar{b}$, $t(\bar{b}, A)$ is stationary then E_A is an L-congruence.*

Proof: Let $\bar{a} = \bar{a}_0 \ldots \bar{a}_{k-1}$ be the decomposition of $\bar{a}$ by E_A. We prove by induction on $lg(\bar{a}_0)$ that $t(\bar{a}_0; A \cup \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{k-1}\})$ d.n.f. over A. Since $t(\bar{a}_0; A)$ is stationary this suffices. Let $\bar{a}_0 = \langle a_0^0, \ldots, a_0^l \rangle$; $\bar{c}$ denote $\langle a_0^0, \ldots, a_0^{l-1} \rangle$; $\bar{b}$ denote $\{\bar{a}_1, \ldots, \bar{a}_{k-1}\}$. By induction we have $t(\bar{c}; A \cup \bar{b})$ d.n.f. over A. We finish by 4.1.6 if we show $t(a_0^l; A \cup \bar{c} \cup \bar{b})$ d.n.f. over $A \cup \bar{c}$. But if $t(a_0^l; A \cup \bar{c} \cup \bar{b})$ forks over $A \cup \bar{c}$, applying 4.2.6 (with $A \cup \bar{c}$ playing the role of A) there is some i, $1 \leq i \leq k-1$ and some $d \in rng\, \bar{a}_i$ such that $t(a_0^l; A \cup \bar{c} \cup d)$ forks over $A \cup \bar{c}$. By 4.1.6 this implies $t(d; A \cup \bar{c} \cup a_0^l)$ forks over A. Now since d is in a different E_A-equivalence class than each member of $\bar{a}_0$, this contradicts 4.2.6 (with A playing the role of A).

The following example shows that some restriction on A is necessary if E_A is to be an L-congruence.

4.2.8 Example Let N be a structure with a universe of all functions from the natural numbers into the integers. Let the language L contain binary relations E_i and F_i and define $\sigma, \tau \in N$: $E_i(\sigma, \tau)$ if and only if $\sigma \upharpoonright i = \tau \upharpoonright i$ and $F_i(\sigma, \tau)$ if and only if $\sigma \upharpoonright i = \tau \upharpoonright i$ and $\tau(i) < 0$, if and only if $\sigma(i) < 0$. (Thus the E_i and the F_i are infinitely decreasing sequences of equivalence relations. Each E_i class contains infinitely many E_{i+1} classes and two F_i classes.) Now T is stable $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ but if A is finite, E_A is not an L-congruence.

4.2.9 Theorem *If T is a stable theory and* $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ *then for any* $N \prec M \models T$, *M is decomposable over N.*

Proof: Since $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ T does not admit coding (Lemma 2.2.4). Thus, by Lemma 4.2.6(i) E_N is an equivalence relation. As every type over a model is stationary, by Lemma 4.2.7, E_N is an L-congruence. The result is now immediate from Definition 3.1.18.

4.2.10 Theorem *If T is stable,* $(T_\infty, \text{2nd}) \not\leq (T, Mon)$, $M \models T$ *and* $N \prec M$, *then for each equivalence class X of* E_N, $N \cup X$ *is the universe of a model of T.*

Proof: We show $X \cup N$ satisfies the Tarski-Vaught criterion. Suppose $\bar{a} \in X \cup N$ and $M \models \exists v \phi(v, \bar{a})$. Then for some $b \in M$, $M \models \phi(b, \bar{a})$. If $b \in X \cup N$, we're done, if not, $t(b; X \cup N)$ does not fork over N and hence by [15], III.4.10, is finitely satisfiable in N. That is, for some $b' \in N$, $\models \phi(\bar{a}, b')$ as required.

In the remainder of this section we compile some useful facts about the relation E_A.

4.2.11 Lemma *If* $a \in acl(A)$ *then* $\{a\}$ *is an* E_A*-equivalence class.*

Proof: Algebraic types do not fork.

Note that this means no nontrivial equivalence class of E_A intersects A.

4.2.12 Lemma *If* $A \subseteq B$ *then* E_B *refines* E_A.

Proof: Suppose $t(a; A \cup b)$ d.n.f. over A. We want to show $t(a; B \cup b)$ d.n.f.

over B. Write $B = B_1 \cup B_2 \cup A$ where B_2 contains exactly the elements of B which are E_A equivalent to a or b. Since T does not admit coding, by 4.2.6(ii) $t(\{a, b\} \cup B_2; B_1 \cup A)$ d.n.f. over A. But then $t(\{a, b\}; B_1 \cup B_2 \cup A)$ d.n.f. over $A \cup B_2$. Now write B_2 as $C_1 \cup C_2$ where C_1 is $B \cap [aE_A]$, and $C_2 = [bE_A] \cap B$. By the proof of 4.2.7 we have $t(a \cup C_1; A \cup b \cup C_2)$ d.n.f. over A. In particular this implies $t(a; A \cup C_1 \cup C_2 \cup b)$ d.n.f. over $A \cup B_2$. Now applying 4.1.7 we have the result.

4.2.13 Lemma *If $A' = A \cup C$ and C is contained in the E_A class D then $E_{A'}$ agrees with E_A outside D.*

Proof: Let $a \in D$ and suppose $\neg aE_Ac$, $\neg aE_Ab$ but bE_Ac. We show $bE_{A'}c$. In conjunction with the previous lemma this yields the result. By 4.2.6(ii), $t(C; A \cup bc)$ d.n.f. over A; i.e., $t(bc; A \cup C)$ d.n.f. over A. Since $t(b; A \cup c)$ forks over A, 4.1.7 yields $t(b; A \cup C \cup a)$ forks over $A \cup C$.

Our next result does not figure directly in the sequel, but is a further illustration of the simplicity of stable theories with $(T_\infty, \text{2nd}) \not\leq (T, Mon)$. Regular types, introduced in [15], V, are types on which a notion of dimension is well defined. A major difficulty in extending results from superstable to stable theories is the paucity of regular types over models of an arbitrary theory. The next shows there is no such problem in our situation.

4.2.14 Theorem *If $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ and T is stable, then for every A, any $p \in S(A)$ is regular.*

Proof: We use the characterization of regularity given by V.1.9(3) in [15]. Thus, suppose $p \in S(A)$, $B = A \cup J$ where each member of J realizes p, a and c realize p. Suppose further that $t(a; B)$ forks over A and $t(c; B \cup a)$ forks over A. We must show $t(c; B)$ forks over A. By the finite character of forking we can assume J is finite. So let $J = \{b_i : i < n\}$. $t(a; B)$ forks over A, so by Lemma 4.2.6(ii) for some $i < n$ we have aE_Ab_i. By the same token, since $t(c; B \cup \{a\})$ forks over A, either cE_Aa or cE_Ab_i for some $i < n$. Since E_A is transitive, cE_Ab_i for some $i < n$, i.e., $t(c; A \cup \{b_i\})$, forks over A, whence $t(c; B)$ forks over A by monotonicity.

We want to show that the fundamental equivalence relation determines a decomposition of a model into a free union as discussed in Section 3. For this we need to show E_M is a Δ_0-congruence. The following lemma is the key.

4.2.15 Lemma *If M is a model of a stable theory and $\phi(\bar{x}, \bar{y})$ is a formula then there is an n such that for any $\bar{a} \notin M$ there exist $\bar{b}_0, \dots, \bar{b}_{2n} \in M$ such that for any $\bar{m} \in M \models \phi(\bar{a}, \bar{m})$ if and only if $\bigvee\left\{ \bigwedge_{i \in W} \phi(\bar{b}_i, m) : w \subseteq 2n + 1\ |w| = n\right\}$.*

Proof: Choose by [15], II.2.20, a finite Δ and n such that if $\langle \bar{b}_i : i < \omega\rangle$ is a Δ-n-indiscernible set of sequences then for any $\bar{c}$ either $|\{i : \models \phi(\bar{b}_i, \bar{c})\}| < n$ or $|\{i : \models \phi(\bar{b}_i, \bar{c})\}| < n$. Now choose in M a sequence $\langle \bar{c}_i : i < \omega\rangle$ such that $\bar{c}_i$ realizes $p_i = t_{\Delta^*}(\bar{a}; C_i)$ (where Δ^* is chosen as in [15], II.2.17). This is possible since Δ^* and $\bar{c}_i$ are finite. Since there is no infinite decreasing sequence of natural numbers for some k $R(p_m, \Delta^*, 2) = R(p_k, \Delta^*, 2)$ for all $m \geq k$. Now, noting that for each $\phi \in \Delta^*$, p_i is definable over $\bar{c}_i$ by the same formula that $t(\bar{a}; C_i)$ is, we conclude from [15], II.2.17, that $\{\bar{c}_i : k \leq i\} \cup \{\bar{a}\}$ is a Δ-indiscernible sequence. Letting $\bar{b}_i = \bar{c}_{k+i}$ we have the result by [15], II.2.20.

We will use the following sharper statement of the result: For each formula $\phi(\bar{x}; \bar{y}, \bar{m})$ there is a quantifier-free formula $\psi_\phi(\bar{y}, \bar{m}; \bar{z})$ such that for each $p = t_{qf}(\bar{a}; M)$ for some $\bar{a} \notin M$, there is a sequence $\bar{d}_p \in N$ such that for all $\bar{n} \in M \models \psi_\phi(\bar{n}, \bar{m}; \bar{d}_p)$ if and only if $\phi(\bar{x}; \bar{n}, \bar{m}) \in p$.

4.2.16 Lemma *If M is decomposed over N by the fundamental equivalence relation E_N and $\langle X_i : i \in I\rangle$ lists the equivalence classes of E_N then, setting $M_i = X_i \cup N$, for some σ $M = \bigcup_N^\sigma \{M_i : i \in I\}$.*

Proof: We must define a map σ such that $\sigma(R)$ is a collection of conditions $\langle \theta, \bar{p}\rangle$ such that $\bar{a}$ satisfies one of these conditions if and only if $M \models R(\bar{a})$. The central tool for this result is the definability of types. This tool will apply directly if $\bar{a}$ is partitioned into exactly two sequences by the fundamental equivalence relation. This fact is somewhat obscured by the inductive procedure which allows us to reduce consideration of an arbitrary partition to consideration of a sequence of two-element partitions.

For any quantifier-free formula $\phi(\bar{x}, \bar{y}, \bar{n})$ with $\bar{n} \in N$ let $\sigma(\phi)$ be the collection of conditions $\langle \theta, p\rangle$ such that for some $\bar{a} \in M$ with $M \models R(\bar{a})$, $\bar{a}$ is decomposed by E_N as $\bar{a}_0 \ldots \bar{a}_{k-1}$ with $\bar{a}_{k-1} \in N$, $\theta = \langle lg(\bar{a}_0), \ldots, lg(\bar{a}_{k-1})\rangle$, for $i < k$, $p_i = t_{qf}(\bar{a}_i; N)$ and $p_{k-1} = \bar{a}_{k-1}$.

Now we show by induction on the length of a partition that for any $\bar{c} \in M$ and any quantifier-free formula $M \models \phi(\bar{c})$ if and only if $\bar{c}$ satisfies some condition in $\sigma(R)$.

If $\bar{c}$ is entirely in one component $\phi(\bar{c}) \in t(\bar{c}; N)$ and we finish. Suppose $\bar{c} = \langle \bar{c}_0, \ldots, \bar{c}_{k-1}\rangle$ with $\bar{c}_{k-1} \in N_x$ and $\bar{c}$ satisfies the condition $\langle \Theta, \bar{p}\rangle \in \sigma(\phi)$. Then there exists $\bar{a} \in M$ with $\bar{a}$ similar to $\bar{c}$. We apply Lemma 4.2.15 to the formula $\phi(\bar{x}_0, \ldots, \bar{x}_{k-1})$ regarding $\bar{x}_0$ as the free variable. Then there is a formula $\theta_\phi(\bar{y}, \bar{d}_0)$ which defines q_0 over N. Since $M \models \phi(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k-1})$ $M \models \theta_\phi(\bar{a}_1, \ldots, \bar{a}_{k-1}, \bar{d}_0)$. Since $\langle \bar{a}_1, \ldots, \bar{a}_{k-2}, \bar{a}_{k-1}, \bar{d}_0\rangle$ is similar to $\langle \bar{c}_1, \ldots, \bar{c}_{k-2}, \bar{c}_{k-1}, \bar{d}_0\rangle$, the induction hypothesis yields $M \models \theta_\phi(\bar{c}_1, \ldots, \bar{c}_{k-1}, \bar{d}_0)$. Since $t(\bar{c}_0; N \cup \{\bar{c}_1, \ldots, \bar{c}_{k-2}\})$ does not fork over N, this implies $M \models \phi(\bar{c})$ as required.

We remarked in Section 3 that we could prove Theorem 4.2.15 in an improved form where the conditions are single formulas. We indicate here the necessary revision of $\sigma(R)$ in the proof of Theorem 4.2.15. We rely on the following fact about the definability of ϕ-types pointed out explicitly in [8] and as Corollary III.1.23 of [1]. For each formula $\phi(\bar{x}, \bar{y})$ and each $p \in S(A)$ there are formulas $\theta_p^\phi(\bar{x}, \bar{z})$ and $\chi_p^\phi(\bar{y}, \bar{z})$ and a sequence $\bar{a}_p \in A$ such that

(i) For some $\bar{a} \in A$, $\theta_p^\phi(\bar{x}, \bar{a}) \in p$.
(ii) If $\theta_p^\phi(\bar{x}, \bar{a}_p) \in p$ then for all $\bar{b} \in A$, $\chi_p^\phi(\bar{b}, \bar{a}_p)$ if and only if $\phi(\bar{x}, \bar{b}) \in p$.

For any pair of types p, q let $\lambda_{p,q}^\phi(\bar{x})$ be $\theta_p^\phi(\bar{x}, \bar{a}_p) \wedge \chi_q^\phi(\bar{x}, \bar{a}_q)$.

Now we define $\sigma(\phi(\bar{x}))$ by induction. $\sigma_0(\phi(\bar{x}))$ is the set all pairs of formulas $\langle \lambda^{\phi}_{p,q}(\bar{x}), \lambda^{\phi}_{p,q}(\bar{y})\rangle$ such that for some sequence $\bar{a} ^\frown \bar{b}$ with $t(\bar{a}; N \cup \bar{b})$ not forking over N, $t(\bar{a}; N) = p$ and $t(\bar{b}; N) = q$. Let $\sigma_{i+1}(\phi(\bar{x}))$ be $\bigcup_{\gamma \in \sigma_i(\phi(R))} \sigma_0(\gamma)$. The sequence of free variables in $\sigma_{i+1}(\phi(\bar{x}))$ is shorter than that in σ_i since each is a subset of such a sequence for $\sigma_i(\phi)$. Thus, the construction of the σ_i must cease at some stage n. $\sigma(R) = \bigcup_{i<n} \sigma_i(R)$ is the required set of conditions.

In 4.2.15 the conditions are quantifier-free types; in the variation just discussed they are a finite set of existential formulas. Is it possible to form a decomposition whose conditions are finite sets of quantifier-free formulas?

4.2.17 Theorem *Let T be a stable theory such that $(T_\infty, \text{2nd}) \not\leq (T, \mathit{Mon})$. If N is a model of T with $|N| = \lambda$ then there exist $M < N$, $|M| \leq |T|$ and $I \subseteq \lambda^{\kappa(T)}$ such that N is $(I, \subseteq)$ decomposable over M.*

Proof: We define $\{\langle M_\eta, N_\eta\rangle : \eta \in I\}$ by induction on $lg(\eta)$. Let $M_{\langle\rangle} = N$ and $N_{\langle\rangle}$ be any submodel of N with $|N_{\langle\rangle}| = |T|$. Let $\langle X_i : i < \mu\rangle$ be a list of the equivalence classes of $E_{N_{\langle\rangle}}$ (for appropriate $\mu \leq \lambda$). Then $M_{\langle i\rangle} = X_i \cup N_{\langle\rangle}$. Suppose for some β we have defined a tree decomposition of N $\{\langle M_\eta, N_\eta\rangle : \eta \in I_\beta\}$ where $I_\beta \subseteq \lambda^{\beta+1}$. (Note that if $lg(\eta) = \beta + 1$, M_η is defined, but N_η is not.) Let N_η be any elementary submodel of M_η with $|N_\eta| = |T|$ and $N_\eta \supseteq N_{\eta^-}$. For appropriate $\mu \leq \lambda$, let $\langle X_i : i < \mu\rangle$ be a list of the equivalence classes of E_{N_η}. Note that by Lemma 4.2.13 for any ν with $\nu \neq \eta$, $\nu^- = \eta^-$, $M_\nu - N_{\eta^-} = X_i$ for some i. Let $M_{\tau^\frown i} = X_i \cup N_\tau$. Then $M_\tau = \bigcup^\sigma_{N_\tau} \{M_{\tau^\frown i} : X_i \subseteq M_\tau\}$ and $M = \bigcup^\sigma_{N_\tau} \{M_{\tau^\frown i} : i < \mu\}$ for appropriate σ (by Lemma 4.2.15). We must show $N = \bigcup\{M_\tau : lg(\tau) < \kappa(T)\}$. If not, for some $\nu \in \lambda^{\kappa(T)}$ there is an $x \in N$ such that for every $\alpha < \kappa(T)$, $xE_{M_{\nu|\alpha}} a_{\nu|\alpha+1}$ (where $a_{\nu|\alpha}$ is any element of $N_{\nu|\alpha}$). This contradicts the definition of $\kappa(T)$ (cf. [15], III.3).

Note that the role of $\kappa(T)$ in this theorem is to hold down the height of the tree. Many families $\{E_A : A \subseteq N\}$ induce similar decompositions, possibly of greater height.

We will invoke in Section 7 the following generalization of 4.2.10.

4.2.18 Lemma *If T is stable and $M \models T$ is tree-decomposed by $\langle\langle M_\eta, N_\eta\rangle : \eta \in I\rangle$ with $I \subseteq \lambda^\omega$, then for each $n < \omega$, $\bar{N}_n = \bigcup\{N_\eta : lg(\eta) = n\} < M$.*

Proof: By the usual induction on formulas and the Tarski-Vaught criteria it suffices to show that if $\bar{a} \in \bar{N}_n$, and for all $\bar{a}, c \in \bar{N}_n$, $\bar{N}_n \models \phi(\bar{a}, c)$ if and only if $M \models \phi(\bar{a}, c)$, then if $M \models \exists\, x\, \phi(\bar{a}, x)$ (with $\bar{a} \in \bar{N}_n$) then there is a $c \in \bar{N}_n$ such that $M \models \phi(\bar{a}, c)$. For this, fix c' such that $M \models \phi(\bar{a}, c')$. For some ν, $c' \in M_\nu$. If $lg(\nu) \leq n$, we finish; if not, let $\eta = \nu|n$ and let $\bar{a} = \bar{a}_0 {}^\frown \bar{a}_1$ with $\bar{a}_0 \in N_\eta$, $\bar{a}_1 \in \bigcup\{N_\tau : lg(\tau) = n, \tau \neq \eta\} - N_{\langle\rangle}$. Then $t(c'; N_\eta \cup \bar{a}_1)$ does not fork over N_η so there is $c \in N_\eta \subseteq \bar{N}_n$ such that $M \models \phi(c, \bar{a}_0, \bar{a}_1)$ (by the "coheir" definition of nonforking).

4.3 Monadic definability of the fundamental equivalence relation We show here that if T is a stable theory $(T_\infty, \text{2nd}) \not\leq (T, \mathit{Mon})$ and M is $|M|^+$ saturated then for $M \prec N$ the fundamental equivalence relation is monadically definable. This result is applied in Section 7 to compute lower bounds (in the strong sense) on Hanf and Löwenheim numbers. It provides a tool for finding prototypes for each variety of tree-decomposable theory.

In this section monadic definability means definability in $L_{\omega,\omega}(Mon)$, L is a finite language, and $r(L)$ is the sup of the arities of symbols in L.

Recall that since T is stable for every formula $\phi(x, y)$, every A and every $p \in S(A)$ there is a formula $d_p\phi(y)$ with parameters from A such that $\phi(x, \bar{a}) \in p$ if and only if $d_p\phi(\bar{a})$. Moreover, if $M \models T$ and $d_p\phi(\bar{y})$, $d'_p\phi(\bar{y})$ define $p \in S(M)$ in this sense then $\models \forall \bar{y} d_p\phi(\bar{y}) \leftrightarrow d'_p\phi(\bar{y})$. In fact, the defining formula can be chosen as $d\phi(\bar{y}, \bar{a}_p)$. When the reliance on p is not essential we write $d\phi$ as ψ_ϕ (cf. [15], II.2).

4.3.1 Definition Let N be an $\aleph_1$-saturated model of T and fix $M_0 \prec N$, $|M_0| = \aleph_0$.

(i) $\mathfrak{F}_0 = \mathfrak{F}_0(M_0)$ is the set of A such that $M_0 \subseteq A \subseteq N$ and for every pair of formulas $\phi(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{y})$: $\forall \bar{y} \in A[(\forall \bar{x} \in A\phi(\bar{x}, \bar{y}) \leftrightarrow \psi(\bar{x}, \bar{y}) \rightarrow \forall \bar{x}(\phi(\bar{x}, \bar{y}) \leftrightarrow \psi(\bar{x}, \bar{y}))]$.

Note that this condition is immediate if A is a model.

(ii) For each $A \in \mathfrak{F}_0$ if $p \in S(A)$, fix a definition d_p of p over A. If p d.n.f. over M_0 choose d_p to define p over M_0.
(iii) Fix Δ as the set of quantifier-free formulas of T. (Note that Δ includes $\phi(\bar{x}, \bar{y})$ for each possible placement of the semicolon if ϕ is q.f.)
(iv) For $M_0 \subseteq A \subseteq B$, B is *formally good* over A if for every $\bar{c} \in N - B$ with $l(\bar{c}) < r(L)$, if $q = t_\Delta(\bar{c}; A)$ then $t_\Delta(\bar{c}, B)$ is defined by d_q.

We hold M_0 fixed for the remainder of this section. A and M will range through $\mathfrak{F}_0$ while B will satisfy $M_0 \subseteq B \subseteq N$.

The definition of $\mathfrak{F}_0$ guarantees that the choice of d_p to define p in (ii) is unique. Thus, "B is formally good over A" is a monadically definable predicate of A, B, and M_0.

4.3.2 Definition We say $xE_A^{\text{for}}y$ (x *is formally equivalent* to y), if for every B which is formally good over A, $x \in B \leftrightarrow y \in B$.

Clearly E_A^{for} is monadically definable. We would like to show $E_A^{\text{for}} = E_A$. However, we have only been able to establish this result when $A = M$ is a model. We show $E_M^{\text{for}} \leq E_M$ and $E_M \leq E_M^{\text{for}}$. One direction is easy.

4.3.3 Lemma *If M is a model $E_M^{\text{for}} \leq E_M$.*

Proof: Suppose $\neg aE_Mb$. Let B be the E_M equivalence class of b. Then, for any $\bar{c} \notin B$, $t(\bar{c}; B \cup M)$ d.n.f. over M and realizes the definable extension of $t(\bar{c}; M)$. In particular, $t_\Delta(\bar{c}; B \cup M)$ realizes the definable extension of $t_\Delta(\bar{c}; M)$ so B is formally good over M. Thus $\neg aE_M^{\text{for}}b$.

To show $E_M \leq E_M^{\text{for}}$, we actually show the result for a more general concept—a protomodel—and then show each model is a protomodel. This procedure highlights the properties of a model necessary to show $E_M \leq E_M^{\text{for}}$. Moreover, since the class of protomodels is monadically definable they may be useful for further investigations in this area.

4.3.4 Definition A is a *protomodel* if E_A^{for} is a Δ-congruence. (Recall we fixed Δ as the set of atomic formulas.)

The following lemma is the crucial step in showing that $E_A \leq E_A^{\text{for}}$ if A is a protomodel.

4.3.5 Lemma (the automorphism lemma) *Let A be a protomodel. Suppose f is a permutation of N such that*:

(i) $f \restriction A = 1_A$

(ii) $bE_A^{\text{for}} C$ *if and only if* $f(b) E_A^{\text{for}} f(c)$

(iii) *For any E_A^{for} equivalence class, B, if $\bar{a} \in B$ then* $t_\Delta(\bar{a}; A) = t_\Delta(f(\bar{a}); A)$.

Then f is an automorphism of N.

Proof: We need only check that for a relation symbol R, $M \models R(\bar{a})$ if and only if $M \models R(f(a))$. But this is immediate from the definition of Δ-congruence and protomodel.

4.3.6 Theorem *If A is a protomodel, $|A| < |N|$, and N is saturated then E_A refines E_A^{for}.*

Proof: Suppose $t(a; A \cup b)$ forks over A. Then $t(a; A)$ is not algebraic. We must show $aE_A^{\text{for}} b$. Let $a_0 = a$. Choose a_1, a_2 so that $t(a_i; A) = t(a; A)$ but $t(a_i; A_i \cup A \cup b)$ d.n.f. over A.

Suppose $a_0 E_A^{\text{for}} a_1$. Since $t(a_0; A) = t(a_1; A)$, there is an automorphism g of N which maps a_0 to a_1. Since E_A^{for} is monadically definable, this automorphism fixes $C_0 = [a_0 E_A^{\text{for}}]$. Let f be the identity on $N - C$ and agree with g on C. By Lemma 4.3.5 f is an automorphism of N. But this is impossible as $t(a_0; A \cup b)$ forks over A and $t(a_1; A \cup b)$ d.n.f. over A.

Suppose $\neg a_0 E_A^{\text{for}} a_1$. Then (since there is an automorphism fixing A and mapping $a_0 a_1$ to $a_1 a_2$) $\neg a_1 E_A^{\text{for}} a_2$. So we may choose from $\{a_0, a_1, a_2\}$ two elements, say a_0 and a_1, such that $\neg a_0 E_A^{\text{for}} a_1$, $\neg b E_A^{\text{for}} a_0$ and $\neg b E_A^{\text{for}} a_1$. Let h be an automorphism of N which fixes A and maps a_0 to a_1. Denoting $[a_0 E_A^{\text{for}}]$ by C_0 and $[a_1 E_A^{\text{for}}]$ by C_1 define f as follows:

$$f(x) = \begin{cases} h(x) & x \in C_0 \\ h^{-1}(x) & x \in C_1 \\ x & x \notin C_0 \cup C_1 \end{cases} .$$

Again, the conditions of the automorphism lemma are clear. Thus f is an automorphism. But, as in the previous case, this is impossible and the theorem holds.

Now we show every model is a protomodel.

4.3.7 Definition If $A \subseteq B \subseteq N$ then B' denotes $(N\text{-}B) \cup A$.

Clearly, the collection of all sets $A \subseteq B \subseteq N$ forms a complete Boolean algebra with operations of set intersection and complement. We now show $\mathfrak{F}$, the class of sets which are formally good over M also form a Boolean algebra.

In the first paragraph of the proof we spell out the fact that formally good sets are closed under intersection. In the second we give the more complicated argument for complementation. Only this second argument depends on M being a model. It is just a variant of the proof of the symmetry lemma. The distinguishing property of a model which is used here is that if a type is defined over a model then it is defined over every superset.

We rely in the next lemma on another version of the definability lemma. For each formula $\phi(\bar{x}; \bar{y})$ there is a formula $d\phi(\bar{y}, \bar{z})$ such that for each set A and each type p over A there is a sequence $\bar{d}_p \in A$ such that for all $\bar{a} \in A$, $\phi(\bar{x}; \bar{a}) \in p$ if and only if $d\phi(\bar{a}; \bar{d}_p)$. The specification of the formula ϕ for this result includes the placement of the semicolon. In this theorem we will consider formulas which are only distinguished by the placement of the semicolon.

4.3.8 Theorem *The set $\mathfrak{F} = \{B : B \supseteq M, B$ is formally good over $M\}$ forms a complete Boolean algebra with operations $\{\cap, \cup, '\}$.*

Proof: Let $\{A_i : i \in I\}$ be a sequence of elements from $\mathfrak{F}$. Let $A_0 = \cap\{A_i : i \in I\}$.

Fix $i \in I$ and a partition of $\bar{b}$ as $\bar{b}_1 \bar{b}_2$ so that $\bar{b}_1 \in A_i$ and for some $j \in I$, $\bar{b}_1 \cap A_j = \emptyset_1$.

Let $\phi_0 = \phi(\bar{x}; \bar{y}_1, \bar{y}_2)$, and $\phi_1 = \phi(\bar{x}, \bar{y}_1; y_2)$. (The difference is in the placement of the semicolon.) Since $\bar{a} \in A_0 \subseteq A_i$ and A_i is formally good we have $p = t(\bar{b}_2; M)$. Let $q = t(\bar{b}_1; M)$ and $r = t(\bar{b}_1 {}^\frown \bar{b}_2; M)$. We must show $\phi_0(\bar{a}; \bar{b}_1, \bar{b}_2)$ if and only if $d\phi_0(\bar{a}; \bar{d}_r)$. Since $\bar{b}_1 \cap A_j = \emptyset$, $d\phi_1(\bar{a}, \bar{b}_1; \bar{d}_p)$ if and only if (letting χ be the defining formula associated with $d\phi_1(\bar{x}, \bar{y}, \bar{z})$ for free variable $\bar{y}$), $\chi(\bar{a}, \bar{d}_p; \bar{d}_q)$.

Now we claim that for any $\bar{m} \in M$, $\phi_0(\bar{m}, \bar{b}_1, \bar{b}_2)$ is equivalent to both $d\phi_0(\bar{m}, \bar{d}_r)$ and to $\chi(\bar{m}, \bar{d}_p; \bar{d}_q)$. The first equivalence is immediate from the definition of a defining type. The second follows since $\chi(\bar{m}, \bar{d}_p; \bar{d}_q)$ holds if and only if $d\phi_1(\bar{m}, \bar{b}_1, \bar{d}_p)$ which in turn is equivalent to $\phi_1(\bar{m}, \bar{b}_1, \bar{b}_2)$.

To finish the proof of the theorem, we now have $d\phi_0(\bar{a}, \bar{d}_r)$ if and only if $\chi(\bar{a}, \bar{d}_p; \bar{d}_q)$ (as these formulas were equivalent on a model) if and only if $d\phi_1(\bar{a}, \bar{b}_1; \bar{d}_q)$, since A_j is formally good, if and only if $\phi(\bar{a}, \bar{b}_1, \bar{b}_2)$, since A_i is formally good.

Suppose B_0 is formally good over M. We must show B_0' is formally good.

Let $\bar{\phi}(\bar{x}, \bar{y}) = \phi(\bar{y}, \bar{x})$. Suppose d defines $t_\phi(\bar{a}; M)$; i.e., for each $\bar{m} \in M$ $\phi(\bar{a}, \bar{m})$ if and only if $d\phi(\bar{m})$. Similarly, suppose d' defines $t_{\bar{\phi}}(\bar{c}; M)$; i.e., for each $m \in M$ $\phi(\bar{m}, \bar{c})$ if and only if $d'\bar{\phi}(\bar{m})$. We have d' defines $t_{\bar{\phi}}(\bar{c}; M \cup \bar{a})$ and assume for contradiction that d does not define $t_\phi(\bar{a}; M \cup \bar{c})$, i.e., $\models \phi(\bar{a}, \bar{c}) \wedge \neg d\phi(\bar{c})$. Now let $\bar{e}_0 = \bar{a}$, $\bar{f}_0 = \bar{c}$ and define $\langle \bar{e}_i : i < \omega \rangle$ $\langle f_i : i < \omega \rangle$ such that

(i) $\bar{e}_i$ realizes the d'-definable extension of $t(\bar{a}; M)$ on $M \cup E_i \cup F_i$.
(ii) $\bar{f}_i$ realizes the d-definable extension of $t(\bar{c}; M)$ on $M \cup E_{i+1} \cup F_i$.

Since $\bar{e}_i$ realizes $t(\bar{a}; M)$, $\models d'\bar{\phi}(\bar{e}_i)$ for all i. Similarly, since $\bar{f}_i$ realizes $t(\bar{c}; M)$, $\models \neg d\phi(\bar{f}_i)$ for all i. Now, (i) implies for $i > j$ $\phi(\bar{e}_i, \bar{f}_j)$ if and only if $d\phi(\bar{f}_i)$, i.e., $\neg\phi(\bar{e}_i, \bar{f}_j)$ while (ii) implies that for $i \leq j$ $\phi(\bar{e}_i, \bar{f}_j)$ if and only if $d\psi(\bar{e}_i)$, i.e., $\phi(\bar{e}_i, \bar{f}_j)$. Thus we have $\phi(\bar{e}_i, \bar{f}_j)$ if and only if $i \leq j$ contrary to stability.

4.3.9 Corollary *If M is a model then every equivalence class of E_M^{for} is formally good.*

Proof: Each equivalence class is a Boolean combination of sets which are formally good over M.

4.3.10 Corollary *If M is a model then M is a protomodel.*

Proof: Suppose $\bar{a} = \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k-1}$ and $\bar{a}' = \bar{a}_0', \bar{a}_1, \ldots, \bar{a}_{k-1}$ are similar sequences for E_M^{for} and $t_\Delta(\bar{a}_0; M) = t_\Delta(\bar{a}_0'; M)$. We must show $t_\Delta(\bar{a}_0; M \cup \{\bar{a}_1 \ldots \bar{a}_{k-1}\}) = t_\Delta(\bar{a}_0'; M \cup \{\bar{a}_1 \ldots \bar{a}_{k-1}\})$. Let B be the E_M^{for} equivalence class which contains $\bar{a}_0, \bar{a}_0'$. By Theorem 4.3.9 B is formally good over M and 4.3.8 then implies B' is also. But then both $\bar{a}_0$ and $\bar{a}_0'$ realize the definable extension of $t_\Delta(\bar{a}_0; M)$.

Combining 4.3.3, 4.3.6, and 4.3.10 we have:

4.3.11. Theorem *If $M \prec N$ is a model and N is $|M|^+$-saturated then $E_M = E_M^{\text{for}}$, thus E_M is monadically definable.*

5 Shallow theories Let T be a theory such that every model is decomposed by a well-founded tree. We called such a theory shallow in 3.2.2. Each such theory determines an ordinal, β, the "depth", namely the sup of depths of models of T. We show that the upper bounds on the Hanf and Löwenheim numbers in $L_{\infty,\omega}(Mon)$ of such theory depends only on β (and not, as in 3.2.11, on the complexity of the sentence).

The results in this section stand midway between those in Sections 3.2 and 6.3. In Section 3.2 we computed upper bounds on Hanf numbers for $L^{\alpha}_{\infty,\omega}(Mon)$ for arbitrary tree-decomposable theories but had no result on Löwenheim numbers. Here, we improve the upper bound on Hanf numbers and find upper bounds on the Löwenheim number $L^{T}_{\infty,\omega}(Mon)$ for shallow T. In Section 7.3 we show that under still stronger hypotheses (nicely shallow) we can improve these upper bounds if we restrict to $L_{\omega,\omega}(Mon)$. In Section 7.1 we compute lower bounds on Hanf and Löwenheim numbers to show our results are the best possible.

The notions of this section should be compared with the discussions of shallow theories in [7] and [19]. In particular, the upper bounds on $n(\lambda, T)$ transfer to this situation.

We defined the Löwenheim number $ls^{T}_{\mathfrak{L}}$ for individual sentences (i.e., analogously to $h^{T}_{\mathfrak{L}}$). The following results are actually stronger, showing that there are "$\mathfrak{L}$-elementary submodels" of the appropriate power.

5.1.1 Notation

(i) For any language L and any ordinal β, K^β denotes the class of β-decomposable L-structures.

(ii) $mT^{\alpha}_{K}(K^\beta) = \{mT^{\alpha}_{K}(M) : M \in K^\beta\}$.

5.1.2 Theorem *Fix a language L.*

(i) *If M is β-tree decomposable (i.e., $M \in K^\beta$) then there exists an $M' \subseteq M$*

with $|M'| \leq (\beth_\beta(\lambda + |L|))^+$ *such that for every* α *and* $\bar{k} : \alpha \to \lambda$, $mT^\alpha_{\bar{k}}(M) = mT^\alpha_{\bar{k}}(M')$.
(ii) $|mT^\alpha_{\bar{k}}(K^\beta)| \leq \beth_{\beta+1}(|L|)$.

Proof: We prove (i) and (ii) simultaneously by induction on β. First we show (i) for $\beta = 1$, i.e., M is strongly decomposable. Write $M = \bigcup^\sigma_N \langle M_i : i \in I\rangle$ for appropriate N, σ and I. There are at most $\beth_1(|L|)$ isomorphism types over N of L-structures. Let, for each isomorphism type ρ, M_ρ be a representative of ρ and $Q_\rho = \{i \in I : M_i \approx_N M_\rho\}$. Now choose $I' \subseteq I$ such that setting $Q'_\rho = \{i \in I' : M_i \approx_N M_\rho\}$, $\langle I, \bar{Q}\rangle \equiv_{(\beth_1(|L|+\lambda))} \langle I', \bar{Q}'\rangle$. We will now apply Theorem 3.1.13. By Lemma 3.1.4 for each α, $W^{\alpha,\bar{k}} \leq \beth_1(\lambda + |L|)$. Then by 3.1.13 and 3.1.15 for every α and $\bar{k}$, $mT^\alpha_{\bar{k}}(M) = mT^\alpha_{\bar{k}}(M')$ where $M' = \bigcup^\sigma_N \langle M_i : i \in I'\rangle$.

Now we show (ii) for $\beta = 1$; i.e., there are only $\beth_2(|L|)$ equivalence classes modulo $L_{\infty,|L|}(Mon)$ of models in K_β. By 3.1.13 an arbitrary strongly decomposable model $M = \bigcup^\sigma_N \{M_i : i \in \alpha\}$. There are $\beth_2(|L|)$ possibilities for σ, $\beth_1(|L|)$ possibilities for N, and $\beth_1(|L|)$ possibilities for the isomorphism type of M_i over N. For each possible isomorphism type over N we must specify either that it occurs $> \beth_1(|L|)$ times or that it occurs λ times where λ is a cardinal $\leq \beth_1(|L|)$. This gives us $\beth_2(|L|)$ possibilities when we classify strongly decomposable structures according to $L_{\infty,\lambda}(Mon)$-equivalence.

To continue the proof, suppose we have shown both (i) and (ii) for $\gamma \leq \beta$. To show (i) at the next stage consider a model M with $dp(M) = \beta + 1$ and decompose M as $\bigcup^\sigma_N \langle M_i : i \in I\rangle$ where for each i, $dp(M_i) \leq \beta$. By the induction hypothesis (i), for each i there is a structure M'_i with $|M'_i| \leq (\beth_\beta(\lambda + |L|))^+$ such that for all α and $\bar{k} : \alpha \to \lambda$ $mT^\alpha_{\bar{k}}(M_i) = mT^\alpha_{\bar{k}}(M'_i)$.

Let $Q_\rho = \{i : M'_i \approx M_\rho\}$ as ρ ranges over the $L_{\infty,|L|}(Mon)$ equivalence types of β-decomposable structures. By (ii) of the induction hypothesis, $\langle I, \bar{Q}\rangle$ is a set with $\beth_{\beta+1}(\lambda + |L|)$ unary predicates. Choose $(I', \bar{Q}')$ with $(I', \bar{Q}') \equiv_{(\beth_{\beta+1}(\lambda+|L|))^+} (I, \bar{Q})$ so that $Q'_\rho = \{i \in I' : M''_i \approx M_i\}$. If $M'' = \bigcup_{N^\sigma} \{M''_i : i \in I'\}$, then $|M''| \leq (\beth_{\beta+1}(\lambda + |L|))^+$ and by 3.1.15 $mT^\alpha_{\bar{k}}(M) = mT^\alpha_{\bar{k}}(M'')$ for all α and $\bar{k}$.

To see if (ii) holds for $\beta + 1$, note that any $\beta + 1$-decomposable model can be written as $\bigcup^\sigma_N \{M_i : i \in I\}$ where the M_i are all β-decomposable. There are $\beth_2(|L|)$ possibilities for σ, $\beth_1(L)$ possibilities for N, and (applying the induction hypothesis) $\beth_{\beta+1}(|L|)$ possibilities for the $L_{\infty,\omega}(Mon)$ theory of each M_i. Now the $L_{\infty,\omega}(Mon)$ theory of M is determined by the number ($\equiv \beth_{\beta+1}(L)^+$) of M_i which have each of these theories, that is, by a function from $\beth_{\beta+1}(|L|)$ into the number of cardinals $\leq \beth_{\beta+1}(|L|)^+$. There are $\leq \beth_{\beta+1}(|L|)$ such functions and we finish.

The limit stage of each induction remains. For (i), suppose $dp(M) = \delta$ and we have proved (i) and (ii) for each ordinal less than the limit ordinal δ. Then $M = \bigcup^\sigma_N \{M_i : i \in I\}$ where, for each i, $dp(M_i) < \delta$. Thus by induction we can replace each M_i by an M'_i with $|M'_i| < \beth_\delta(|L|)$ and M'_i has the same $L_{\infty,\omega}(Mon)$ theory as M_i. Now choose $(I', \bar{Q}')$ as in the proof of the successor stage so that $(I, \bar{Q}) \equiv_{\beth_\delta(|L|)^+}(I', \bar{Q}')$. By induction, for each α, $\bar{k}$, $|W^{\alpha,\bar{k}}| \leq \beth_\delta(|L|)$, so by 3.1.13 and 3.1.15 we finish.

To see that there are only $\beth_{\delta+1}(|L|)$ $L_{\infty,\omega}(Mon)$ equivalence types of δ-decomposable models, write a δ-decomposable M as $\bigcup_N^\sigma \{M_i : i \in I\}$. Once again there are $\beth_2(L)$ possibilities for σ, $\beth_1(L)$ possibilities for N, and $\beth_\delta(L)$ possibilities for $L_{\infty,\omega}(Mon)$ theory of each M_i. As before, the $L_{\infty,\omega}(Mon)$ theory of M is determined by a function from $\beth_\delta(|L|)$ into the set of cardinals $\leq (\beth_\delta)^+$ and there are, at most, $\beth_{\delta+1}$ such, so we finish.

[The referee pointed out the necessity of the dual induction to obtain this result.]

A slight variant of this argument yields an upward Löwenheim-Skolem theorem. If M is β-tree decomposable and $|M| > \beth_\beta(\lambda + |L|)$, the I found in the proof of Theorem 5.1.2 will have cardinality $> \beth_\beta(\lambda + |L|)$ and so for some i_0, $|\{i : M_i \equiv_N M_{i_0}\}| > \beth_\beta(\lambda + |L|)$. Thus we can choose J with $|J| = \kappa$ for any $\kappa > \beth_\beta(\lambda + |L|)$. With this modification we prove:

5.1.2′ Theorem *If M is β-tree decomposable and $|M| \geq (\beth_\beta(L))^+$ for all α and $\bar{k}$ and all $\kappa \geq \beth_\beta(L)^+$ then there is a model M_κ with $|M_\kappa| = \kappa$ and $mT^\alpha_{\bar{k}}(M_\kappa) = m^\alpha_{\bar{k}}(M)$.*

5.1.3 Corollary *If T is β-tree decomposable,*
(i) $ls^T_{L_{\infty,\lambda}(Mon)} \leq \beth_\beta(|L| + \lambda)$.
(ii) $H^T_{L_{\infty,\lambda}(Mon)} \leq (\beth_\beta(|L| + \lambda))^+$.

We will see in Section 7 that if $dp(T) = \beta$ there is a sentence $\phi \in L^\beta_{\infty,\omega}(Mon)$ such that $T \cup \{\phi\}$ has only models with cardinality $\beth^+_\beta$.

6 Nicely decomposable theories The main notion of this chapter—that of a nice decomposition—is generalized from properties of strongly decomposable theories.

Before introducing this notion we discuss some properties of strongly decomposable theories. In Section 6.1 we prove a theory T is strongly decomposable if and only if T is stable and $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$ and derive some properties of such theories.

In Section 6.2 we define the notion of a nice decomposition and such derivative concepts as nicely shallow. We show that if T is nicely shallow then $ls^T_{L_{\omega,\omega}(Mon)}$ is $\aleph_0$. Further, in 6.2.7 we show the restriction to nice theories is essential to lower the Löwenheim number to $\aleph_0$. In Section 6.3 we study the effect of restricting permutational logic to strongly decomposable theories. Our main result asserts that for any theory T either $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$ or $(T, \text{1-1})$ is interpretable in $(T_\infty, \text{1-1})$. (Our notion of interpretation in Section 6 is that in [13] so this notion is defined (cf. note after 2.1.4)). This allows us to compute the Hanf and Löwenheim numbers of $(T, \text{1-1})$ then $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$.

6.1 Characterizations of strongly decomposable theories First, suppose T is a stable theory and $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$. Then we know (as certainly $(T_\infty, \text{2nd}) \not\leq (T, Mon)$) from Section 4.2 that, for any A, E_A is an equivalence relation. In fact, we will now show that $E_A(a, b)$ means exactly that $a \in$

$cl(A \cup \{b\})$. Thus, in this case we have combined the desirable properties of forking (symmetry) with those of algebraic closure (transitivity and reflexitivity) to obtain the equivalence relation E_A. One example of such a theory is the class of all unary algebras satisfying $f^3(x) = x$.

A straightforward compactness argument shows that if T is strongly decomposable then for every N and M, E_N is defined by aE_Nb if and only if $a \in acl(N \cup b) - acl(N)$. The next lemma and corollary show the converse.

6.1.1 Lemma *If T is stable and if there exist a, b such that $t(a; b \cup A)$ forks over A but $t(a; b \cup A)$ is not algebraic, then there is a subset of a model of T on which there is definable (with additional unary predicates) an equivalence relation with infinitely many infinite classes.*

Proof: Suppose $t(a; A \cup b)$ forks over A but $t(a; A \cup b)$ is not algebraic. Construct $\langle b_i : i < \omega \rangle$ such that $t(b_i; B_i \cup A)$ d.n.f. over A and $t(b_i; A) = t(b; A)$. WOLOG $b_0 = b$. Fix f_i, an automorphism of the monster model, which fixes A pointwise and maps b_0 to b_i. Choose $i, j < \omega$, $a_{i,j}$ such that $t(a_{i,j} : A \cup B)$ d.n.f. over $b_i \cup A$ and extends $f_i(t(a; b \cup A))$. Then for each i, j, k, $t(a_{i,j}; b_k \cup A)$ forks over A if and only if $i = k$. Choose $\phi(x, y) \in F(A)$ such that $\phi(a, b)$ holds and $\phi(x, b)$ forks over A. Then we have $\phi(a_{i,j}; b_k)$ if and only if $i = k$. Now, adding a unary predicate U to pick out the b_i's, we define an equivalence relation on the $a_{i,j}$ by $\phi(x, y) : \exists z U(z) \wedge [\phi(x, z) \leftrightarrow \phi(y, z)]$.

6.1.2 Corollary *If $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$ and T is stable, then for every A the relation aE_Ab if and only if $t(a; A \cup b)$ forks over A satisfies:*
(i) *aE_Ab if and only if a is algebraic over A and $a = b$ or a is not algebraic over A and a is algebraic over $A \cup b$.*
(ii) *aE_Ab is an equivalence relation.*
(iii) *Each equivalence class of E_A has $\leq |T| + |A|$ elements.*

Proof: (i) follows from Lemmas 6.1.1 and 2.2.6. But if (i) holds, then (ii) holds as "forking" is symmetric and reflexive and "algebraic in" is transitive. Moreover, (iii) is immediate since the algebraic closure of a singleton has $\leq |T| + |A|$ elements.

6.1.3 Corollary *If T is stable and $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$ then T is strongly decomposable.*

Proof: Let $M \models T$ and let N be an arbitrary elementary submodel of M with power $|T|$. By Lemma 6.2.7 E_N is an L-congruence and by Corollary 6.1.2 each equivalence class of E_N has cardinality $\leq |T|$. So T is strongly decomposable.

Very little can be interpreted in $(T, \text{1-1})$ if T is strongly decomposable, even if we allow a very powerful logic for the interpretation.

6.1.4 Theorem *Let T be strongly decomposable and let $\mathfrak{L}$ be any logic such that truth is preserved by isomorphism. Then $Th(<)$ is not interpretable in the $\mathfrak{L}$-theory of T.*

Proof: Let κ be much larger than $|T|$ ($\beth_2(|T|)$ would do) and let M be an $|L|^+$-saturated model of T. Suppose some $\mathfrak{L}$-formula $\phi(x, y, \bar{a})$ defined a linear order $(A, <)$ of length κ. Let M be decomposed over N and suppose $M = N \bigcup_{\alpha<|M|} X_\alpha$. Further, fix an ordering of each $X_\alpha = \{x_i^\alpha : i < |T|\}$. Now since A is so large we can invoke the Erdos-Rado theorem and (possibly replacing A by a subset) find a set of X_α's such that $A \cap X_\alpha = a_\alpha$ and for $\alpha \neq \beta$ there is an automorphism of M fixing N and mapping a_α to a_β. But then A is a set of $\mathcal{L}$-indiscernibles contradicting the hypothesis.

Since permutational logic preserves isomorphism we can combine 6.1.3 and 6.1.4 as:

6.1.5 Theorem *Let T be a first-order theory. Then T is strongly decomposable if and only if T is stable and* $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$.

Proof: The "if" direction is 6.1.3. "Only if" follows from the next claim.

Claim Let T be strongly decomposable and T' be the extension of T obtained by adding finitely many function symbols and axioms asserting each is a permutation of the universe. Then T' is strongly decomposable.

Proof of Claim: Without loss of generality we can assume T' contains names for the inverse to each of the added functions. It suffices to show that, for any $a \in M$, $[a]E_{N'} \subseteq acl_{L'}(a \cup N')$ since the latter set has cardinality less than $|N| + |T'|$. But note that any L-formula $\phi(x, y)$ which witnesses that $t(a; b \cup N)$ forks over N' can be written as $\hat{\phi}(\bar{x}, \bar{y})$ where $\hat{\phi}$ is an L-formula and each $x_i(y_i)$ is a term $t_i(x)(t_i(y))$. Thus $t(\bar{a}; N \cup \bar{b})$ forks over N where $a_i = t_i(a)$ $(b_i = t_i(b))$. By the triviality of forking for some i and j, $t(a_i; N \cup b_j)$ forks over N. As we remarked before (Lemma 6.1.1), $a_i \in acl_L(n \cup b_j)$, $b_j \in acl_{L'}(N \cup b)$, and $a \in acl_{L'}(a_i \cup N)$ (since we added inverses) so $a \in acl_{L'}(b \cup N)$.

Now we can deduce Theorem 6.1.5 from Theorem 6.1.4 and the claim. For, if T is unstable then we can define in T arbitrarily long linear orderings of n-tuples for some n. Applying the proof of 6.1.4 to n-tuples we can see that (even adding a finite number of permutations because of the claim) we cannot define such an order in $(T, \text{1-1})$. Certainly if we cannot define a linear order we cannot define arbitrary relations so $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$.

We show in 8.1.7 that if T is unstable then $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$. Combined with 6.1.3 this yields: $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$ implies T is strongly decomposable.

Note that in proving $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$ implies T is strongly decomposable, we relied on only two consequences of the noninterpretability: (a) T does not admit coding, and (b) there is no definable equivalence relation on a subset of a model of T with infinitely many classes. Using this observation, we can also characterize this class of theories via a concept introduced by Buechler [4].

6.1.6 Definition

(i) A type $p \in S^n(M)$ is strongly nonalgebraic if $\bar{a}$ realizes p implies $\bar{a} \cap M = \emptyset$.

(ii) The first-order theory T is *bounded* if there exists a cardinal $\beta(T)$ such that

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for every $M \models T$, the set of strongly nonalgebraic complete types over M has cardinality less than $\beta(T)$.
(iii) A definable equivalence relation E on C^n is *nontrivial* if infinitely many equivalence classes of E contain an infinite set of n-tuples which are pairwise disjoint.

6.1.7 Lemma *If T is bounded, then*
(a) *T does not admit a strong coding.*
(b) *For every n, every definable equivalence relation on n-tuples from a model of T is trivial.*

Proof: Both of these are easy. Suppose, for example that $\phi(x, y_0, \dots, y_{n-1})$ defined a coding of $B_0 \times B_1$ by C, where $|B|$ and $|C| > \kappa$. Then, fixing $b_2, \dots, b_n$ and c, the types $p_i = \{\phi(x, b_1^i, c)\} \cup \{x \neq m : m \in M\}$ as b_1^i ranges through B_1 define more than κ strongly nonalgebraic types.

6.1.8 Corollary *The following are equivalent*:
(i) $(T_\infty, \text{2nd}) \not\leq (T, \text{1-1})$.
(ii) *T is bounded.*
(iii) *T is superstable and for every n every definable equivalence relation on M^n for some model M of T is trivial.*
(iv) *T is strongly decomposable.*

Proof: We have shown (i) $\leftrightarrow$ (iv). Clearly (iv) implies (ii) since if M is a model of T and $p \in S(M)$ is a strongly nonalgebraic type and p is realized by $\bar{a} \in M_1 \succ M$ where M_1 is strongly decomposed over M_0 then $\bar{a}$ must be in an E_{M_0}-class which does not intersect M_0 but then $t(a; M)$ is determined by $t(a; M_0)$. From the previous lemma and observation we deduce (ii) $\rightarrow$ (i), and the following variation of the treatment in Buechler [4] (also suggested by Buechler) shows (ii) $\leftrightarrow$ (iii).

To show (iii) $\rightarrow$ (ii), suppose $\phi(\bar{x}, \bar{y})$ defines a nontrivial equivalence relation. For any cardinal k, choose a model M_k containing $\langle \bar{b}_i : i < k \rangle$ which are pairwise inequivalent and such that $[\bar{b}_i]E$ contains infinitely many disjoint n-tuples. Now for each i, the type p_i which asserts $\phi(\bar{x}, \bar{b}_i)$ but for each $j < n$, x_j is not in M is a consistent strongly nonalgebraic type. Thus T is unbounded.

For (iii) $\rightarrow$ (ii), suppose that T is unbounded and let M be a saturated model which has more than $2^{\aleph_0}$ n-types. Since there are only $2^{\aleph_0}$ equivalence classes in the fundamental order there exists a set P with $|P| > 2^{\aleph_0}$ of types over M which are equivalent in the fundamental order. Since M is saturated, all members of P are conjugate over the empty set (cf. [9]). Let $P = \langle p_i : i < (2^{\aleph_0})^+ \rangle$ and for each i choose $I_i = \langle \bar{a}_j^i : j < \aleph_0 \rangle$ a sequence of indiscernibles in M with $A \cup (I_i; M) = P_i$. Since the p_i are conjugate so are the I_i. Since T is superstable there is an integer k such that for any k-tuple $\bar{b}$ from I_i, p_i is definable over $\bar{b}$.

We can cover the set of two-element subsets of P by the sets $C_\phi = \{\langle p, q \rangle : p_\phi \neq q_\phi\}$. By the Erdos-Rado theorem for some ϕ there is an uncountable subset $\hat{P}$ of P such that for all $p \neq q \in \hat{P}$, $p_\phi \neq q_\phi$. Without loss of generality we call $\hat{P}$, P. Let for $\bar{b} \subseteq I_0$ with $|\bar{b}| = k$, $d\phi(\bar{y}, \bar{b})$ define $p_0|\phi$. Now let $\psi(\bar{w}, \bar{v})$ be $\forall \bar{y}(d\phi(\bar{y}, \bar{v}) \leftrightarrow d\phi(\bar{y}, \bar{w}))$. Clearly ψ defines an equivalence relation. To see it is nontrivial note that if $\bar{b}$, $\bar{c}$ are k-element subsets from I_i, I_j with

$i \neq j$, we have $\neg\psi(\bar{b}, \bar{c})$. For each i, we can choose disjoint sequences $\langle \bar{b}^i_j : j < \omega\rangle$ such that $d\phi(\bar{y}; \bar{b}^i_j)$ defines p_i for each j. Thus $\psi(\bar{w}, \bar{v})$ is nontrivial and we finish.

Here is a sample of another sort of computation that can be made in this situation.

6.1.9 Corollary If T is a countable strongly decomposable theory then $n(T, \aleph_\alpha) \leq (\alpha + \aleph_0)^{2^{\aleph_0}}$.

Proof: Fix a countable model N of T and consider models M with $|M| = \aleph_\alpha$ and $N \prec M$. Then, since T is strongly decomposable M can be viewed as a free union over N of countable structures M_i. There are only $2^{|\aleph_0|}$ possibilities for the isomorphism type of M_i. The isomorphism type of M is determined by the number of times each countable isomorphism types occur as a factor of M. That is, for each N the number of models of T of power $\aleph_0$ which contain N is $\leq (\alpha + \aleph_0)^{2^{|\aleph_0|}}$. Since there are only $2^{\aleph_0}$ choices for N, $n(T, \aleph_\alpha) \leq (\alpha + \aleph_0)^{2^{\aleph_0}}$.

6.2 Nice decompositions and $ls^T_{L_{\omega,\omega}(Mon)}$ This section is devoted to characterizing those theories T such that $ls^T_{L_{\omega,\omega}(Mon)} = \aleph_0$. We complete this task if T is superstable and shallow. If T is superstable and deep, or stable but not superstable, Shelah considers the Löwenheim number of (T, Mon) in [16]. If T is unstable we show in Section 8 that $ls^T_{L_{\omega,\omega}(Mon)} \geq$ Löwenheim number of the monadic theory of order.

Consider now T which are superstable and shallow. If $dp(T) = \beta$, we know from Section 5 that $ls^T_{L_{\omega,\omega}(Mon)} \leq (\beth_\beta)^+$. We define two notions of nice decomposition which imply that if T admits a nice decomposition then $ls^T_{L_{\omega,\omega}(Mon)} = \aleph_0$. Unfortunately the stronger and simpler of these two notions (an extremely nice decomposition) does not produce a useful dichotomy; we can obtain no strong result from its negation. The second notion (a nice decomposition) is more fruitful. If some model of T does not admit a nice decomposition then second-order logic on the continuum is interpretable in T.

6.2.1 Definition Let $\langle M_i : i \in I\rangle$ be a family of L-structures with heart N. We say M is an *extremely nice free union* of the M_i over (N, H) if there is a finite subset $H \subseteq N$ such that defining a nice m-condition as in (i)′, (ii)′ holds.
(i)′ An extremely nice m-condition over H is defined like a condition in Section 2, but with the additional requirement that each p_i for $i < k$ is a quantifier-free type over H.
(ii)′ For any $\bar{a} \in M\text{-}H$ and any partition of $\bar{a}$ into $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k-1})$ where each $\bar{a}_i = \bar{a} \cap (M_j - N)$ for some $j \in I$ and $\bar{a}_{k-1} \in N$, $M \models R(\bar{a})$ if and only if $\bar{a}$ satisfies an m-condition in $\sigma(R)$ where $\sigma(R)$ is a set of extremely nice m-conditions.

We first show that strong decompositions are even stronger than we thought.

6.2.2 Theorem *If T is a strongly decomposable theory in a finite language L, then every model M of T has an extremely nice decomposition.*

Proof: We know that the fundamental equivalence relation decomposes M as $\bigcup_N^\sigma M_i$ for appropriate M_i, σ and N. We must show conditions (i)′ and (ii)′ from Definition 6.2.1 hold.

Note first that it suffices to show:

(*) For every relation symbol $R \in L$, $\exists$ a finite subset H_R of N such that for all elementary extensions M_1 of N and any sequence $\bar{b} \in N$ and $\bar{a}$ in a single equivalence class of E_N, $t_{qf}(\bar{a}; H_R) \cup t_{qf}(\bar{b}; H_R)$ determines the truth value of $R(\bar{a}, \bar{b})$.

For, this implies that $t_{qf}(\bar{a}; H) \models t_{qf}(\bar{a}; N)$ where $H = \cup\{H_R : R \in L\}$.

Using this fact it is easy to see that the decomposition is extremely nice.

If (*) does not hold we can find an $M = \bigcup_N^\sigma M_i$, an R, an i, and a sequence H_j of finite subsets of N such that for each j there exist $\bar{a}_j, \bar{a}_j' \in M_i - N$, $\bar{b}_j, \bar{b}_j' \in N$ with $t_{qf}(a_j; H_j) = t_{qf}(a_j'; H_j)$ and $t_{qf}(\bar{b}_j; H_j) = t_{qf}(\bar{b}_j'; H_j)$ but $M_i \models R(\bar{a}_j; \bar{b}_j)$ and $M_i \models \neg R(\bar{a}_j'; \bar{b}_j')$.

Now let $\bar{a}_\omega$, $\bar{b}_\omega$, $\bar{a}_\omega'$, $\bar{b}_\omega'$ realize an ultraproduct of $\langle \bar{a}_j \rangle$, $\langle \bar{b}_j \rangle$, $\langle \bar{a}_j' \rangle$, $\langle \bar{b}_j' \rangle$, respectively. Then $t_{qf}(\bar{a}_\omega; N) = t_{qf}(\bar{a}_\omega'; N)$ and $t_{qf}(\bar{b}_\omega; N) = t_{qf}(\bar{b}_\omega'; N)$. Since for all $j < \omega$ no element of $\bar{a}_j$ is in the algebraic closure of $\bar{b}_j \cup H_j$, by 6.1.2, no component of $\bar{a}_\omega$ is in the same E_N class as any component of $\bar{b}_\omega$; similarly for $\bar{a}_\omega'$ and $\bar{b}_\omega'$. Thus by the definition of free union we have $R(\bar{a}_\omega, \bar{b}_\omega) \leftrightarrow R(\bar{a}_\omega', \bar{b}_\omega')$ but by the construction $\models R(\bar{a}_\omega, \bar{b}_\omega) \wedge \neg R(\bar{a}_\omega', \bar{b}_\omega')$.

As mentioned above we can deduce no quotable result from the assumption that some model of T does not admit an extremely nice decomposition. Thus we weaken the notion as follows.

We want to define a notion of nice free union so that if a model M is decomposed as a nice free union of countable structures over a heart N, then M has a countable $L_{\omega,\omega}(Mon)$-elementary submodel. To describe the decomposition of a model M we will consider an expansion L^* of the language L. The choice of the language L^* will depend on the choice of a countable heart N for M.

For a finite subset H of N and each n an n-assignment ρ is a map from $S_\Delta^n(H)$ into $S_\Delta^n(N)$.

To form L^* add to L unary predicates $\boldsymbol{N}$, $\boldsymbol{I}$, $\boldsymbol{V}_{n,\rho}$ for $n < \omega$ and each n-assignment ρ, $\boldsymbol{H}_{m,i}$ for $m < \omega$ and $i < \omega$, a binary relation $\boldsymbol{S}$, and names for the elements of N.

As an aid to understanding the following definitions we indicate the intended expansion of an L-structure $M = \bigcup_N^\sigma M_i$ to an L^* structure M^*. $\boldsymbol{N}$ should denote N, $\boldsymbol{I}$ should denote the index set I. $\boldsymbol{S}(m, i)$ holds if $m \in M_i$. For each n, $H_{n,i}$ is an enumeration of the finite subsets of N. The index j is in the denotation of $\boldsymbol{H}_{n,i}$ if $H_{n,i}$ is a minimal subset of N such that for each n-tuple $\bar{a} \in M_i - N$, $t(\bar{a}; H_{n,i})$ determines $t(\bar{a}; N)$. (The existence of such a finite subset will be guaranteed by Lemma 6.2.10 below.) $\boldsymbol{V}_{n,\rho}$ denotes the set of i such that for any n-tuple $\bar{a} \in M_i - N$, $t(\bar{a}; N) = \rho(t(\bar{a}; H_{n,j}))$ if $\boldsymbol{H}_{n,j}(i)$ holds.

Before we give the formal definition of a nice free union, we must describe the appropriate notion of a condition.

A nice n-condition is a triple $\langle \Theta, \bar{p}, \bar{\rho} \rangle$ such that $\Theta = \langle \theta_0, \ldots, \theta_{k-1} \rangle$ is a partition of n, for $j < k - 1$, $p_j \in S_\Delta^{\theta_j}(H_{n,j})$ for some $H_{n,j} \subseteq N$, $\bar{p}_{n-1} \in N$ and ρ_j is a θ_j-assignment of $S_\Delta^{\theta_j}(H_{n,j})$ into $S_\Delta^{\theta_j}(N)$.

A sequence $\bar{a} \in M$ with $\bar{a} \cap M_{ij} = \bar{a}_j$ for $j < k - 1$ and $\bar{a} \cap N = \bar{a}_{k-1}$ *satisfies the nice n-condition* $\langle \Theta, \bar{p}, \bar{\rho} \rangle$ under the following circumstances:

i. For $j < k - 1$ if $H_{\theta_j,l}(i_j)$ then $p_j \in S_\Delta^{\theta_j}(H_{\theta_j,l})$ and $\bar{a}_j$ realizes p_j.
ii. $\bar{p}_{k-1} = \bar{a}_{k-1}$.
iii. For $j < k - 1$ $i_j \in V_{\theta_j,\rho_j}$.
iv. For $j < k - 1$, $\bar{a}_j$ realizes $\rho_j(p_j)$.

6.2.3 Definition Let $\langle M_i : i \in I \rangle$ be a family of L-structures with heart N. Then M is a *nice free union* of the M_i over N if:
(i) For some σ, $M = \bigcup_N^\sigma M_i$ and $S(M^*, i) = M_i$.
(ii) $H_{n,i} \subseteq I$ for $n < \omega$, $i < \omega$.
(iii) Each $V_{n,\rho}$ is a subset of I.
(iv) For each relation symbol $R \in L$ and any $\bar{a} \in M$, $M \models R(\bar{a})$ if and only if $\bar{a}$ satisfies a nice n-condition $\langle \Theta, \bar{p}, \bar{\rho} \rangle$ with $\langle \Theta, \rho_0(p_0), \ldots, \rho_{k-2}(p_{k-2}) \rangle \in \sigma(R)$. In this situation, we write $M^* = \bigcup_{N^*}^{\sigma^*} M_i$.

Now we have a notion of free union about halfway between Shelah's original version ([22], 2.3) and that in Section 3. Our conditions are quantifier-free types in a finite language over a finite set; thus they are formulas. On the other hand, the infinite heart comes into play via the H_i. Let $L(H)$ be the language obtained by adding names for the elements of H to L. Then each condition is a collection of $L(H)$ sentences for some finite set $H \subseteq N$. Now analogously to 3.1.13 (or [22]) one can prove:

6.2.4 Theorem For any σ, n, $\bar{k}$ we can find an $\bar{r}$ such that if $M = \bigcup_{N^*}^{\sigma^*} M_i$, $t \in mT_{\bar{k}}^n(L(H))$ and $Q_{n,t} = \{i \in N^* : mT_{\bar{k}}^n(M_i, h)_{h \in H} = t\}$ then from $pT_{\bar{r}}^n(I, \bar{V}, \bar{Q})$ we can compute $mT_{\bar{k}}^n(M)$.

Note that $Q_{n,t}$ as defined in the previous theorem also depends on H but we suppress this parameter to ease readability.

We briefly indicate the difference in the proof of this theorem from that of 3.1.13. The program of the induction and the method of refining the partition is just as in the proof of 3.1.13. The difference is in the method of checking the truth of the formula. For example, just as in the $\alpha = 0$ case of Theorem 3.1.13, we can find $j' \in Q'_{t_l}$ and $\bar{a}'_j \in M'_j$ such that $mT_{\bar{k}}^0(\langle M'_j, h, P'_j, a'_j \rangle_{h \in H_j}) = S_l$. Since $pT_{\bar{r}}^0(I, \bar{P}, \bar{Q}) = pT_{\bar{r}}^0(J, \bar{P}, \bar{Q}')$ we also know that $V_{lg(a_j),\rho_j}(l)$ if and only if $V_{lg(\bar{a}),\rho_j}(j')$.

But then $t(\bar{a}_l H_l^{\theta_j}) = t(\bar{a}'_j H_{j'}^{\theta_j})$ and both $V_{\theta_j,\rho_j}(l)$ and $V_{\theta_j,\rho_j}(j')$ hold. So $t_\Delta(\bar{a}_l; N) = t_\Delta(\bar{a}'_j; N)$. Checking this for each l and associated j', we see $M' \models \phi(\bar{a}', \bar{P}')$ as required.

Note that the nice free union is indexed by a set I with countably many unary predicates, the $\boldsymbol{V}_{n,\rho}$ and the $\boldsymbol{H}_{n,i}$. Thus each index set has a countable $L_{\omega,\omega}(Mon)$-elementary submodel.

6.2.5 Definition The model M is *nicely tree decomposed* by the tree I if M

is tree decomposed by I and each of the free unions in Definition 3.2.1(iii) is a nice free union.

In complete analogy with Definition 3.2.2 we define the notions of a theory being *nicely* κ*-tree decomposable* and *nicely shallow*. (We could define nicely strongly decomposable but Lemma 3.5 shows that, at least in our context, this notion is redundant.)

Note that if T is nicely shallow we can attach to each model M of T an ordinal, the Kleene-Brouwer ordinal of the tree which nicely decomposes M. We will call this ordinal the depth (M) and ignore non-nice decompositions (which a priori) might have lower depth.

6.2.6 Theorem *If each finite reduct of the countable first-order theory is nicely shallow, then any model M of T has a countable* $L_{\omega,\omega}(Mon)$*-elementary submodel.*

Proof: First suppose $L(T)$ is finite. We work by induction on $dp(M)$. If $dp(M) = 1$, then for some I, σ^*, and $\langle M_i : i \in I\rangle$, $M = \bigcup^{\sigma^*}_{N^*} \{M_i : i \in I\}$ and each M_i is countable. By 6.2.4 we can replace I by a countable subset I_0 and N^* by a countable elementary submodel N_0^*, so that $\bigcup^{\sigma^*}_{N_0^*} \{\overline{M}_i : i \in I_0\}$ is as required. If $dp(M) = \alpha + 1$, $M = \bigcup^{\sigma^*}_{N^*} \{\overline{M}_i : i \in I\}$ for appropriate σ^*, I, M_i. By induction each $\overline{M}_i$ can be replaced by an $L_{\omega,\omega}(Mon)$ equivalent countable model and applying the $dp(M) = 1$ argument we obtain the theorem.

The proof for countable languages can proceed by applying the above argument, but noticing that for each finite sublanguage we obtain a closed unbounded subset of elementary models and the diagonal intersection yields the required L-elementary submodel.

For superstable deep theories, the concept of nice does not suffice to settle the size of the Löwenheim number. The monadic theory of single unary function has Löwenheim number $\aleph_0$. But in 1.16 of [20] an example is given of a nice superstable theory with large Löwenheim number (i.e., the same as second-order logic if $V = L$).

We now show that the requirement that the decomposition be nice is essential.

6.2.7 Theorem *If T is a superstable theory in a finite language which is not nicely decomposable then second-order logic on the continuum is monadically interpretable in* (T, Mon).

This yields, in particular, that if T is superstable then $ls^T_{L_{\omega,\omega}(Mon)}$ is $\aleph_0$ if T is nice and otherwise at least $2^{\aleph_0}$.

To prove 6.2.7 we will first define "second-order logic on the continuum" explicitly, then find a more tractable theory which is bi-interpretable with it and then interpret this theory in (T, Mon).

6.2.8 Notation

(i) By *second-order logic on the continuum* we mean (T, Mon) where T is the first-order theory of the structure $\mathcal{B} = (\omega, 2^\omega, \in, \langle\,\rangle)$ where $\in$ is the membership relation between elements of ω and subsets of ω and $\langle\,\rangle$ is a fixed pairing.

(ii) Let $\mathfrak{A} = \langle \omega \times 2^\omega, E_1, E_2 \rangle$ where E_1 and E_2 are two crosscutting equivalence relations; E_1 has ω classes, E_2 has 2^ω classes, and all classes are infinite. (Explicitly $E_1(\langle n, \sigma \rangle, \langle m, \tau \rangle)$ iff $n = m$; $E_2(\langle n, \sigma \rangle, \langle m, \tau \rangle)$ iff $\sigma = \tau$.)

6.2.9 Lemma $(Th(\mathfrak{A}), Mon) \equiv_{MON} (Th(\mathfrak{B}), Mon)$.

Proof: $(\mathfrak{A}, Mon) \leq (\mathfrak{B}, Mon)$ is obvious. For the converse, let $[\eta]E_2$, $\eta \in 2^\omega$ and $[n]E_1$, $n < \omega$ enumerate the equivalence classes of E_2 and E_1, respectively. Let $R = \bigcup\{[n]E_1 \cap [\eta]E_2 : \eta(n) = 0, \eta < 2^\omega, n < \omega\}$. Now, via R, each E_2 class codes a set of natural numbers, and every subset of ω is coded by an E_2-class. In particular, we can code the pairing function on ω. Then we extend this coding pointwise to code arbitrary relations on the continuum.

Thus we want to define from a model M of a superstable theory T which does not have a nice decomposition a structure $\mathfrak{A} = (A, E_1, E_2)$ as in 6.2.8(ii). Let M be decomposed over N by the fundamental equivalence relation.

6.2.10 Lemma *If T is stable, $M = \bigcup_N^\sigma M_i$, $|N|^+$-saturated and some M_i realizes infinitely many distinct quantifier-free m-types over N, then $(Th(\mathfrak{B}), Mon) \leq_{Mon} (T, Mon)$.*

Without loss of generality, we may assume M is $|N|^+$-saturated. Choose a finite sequence $\bar{b} \in M_k$ with m minimal among all pairs $(l, \bar{c})$ such that infinitely many quantifier-free l-types over $N \cup \bar{c}$ are realized in M_k. Let $\langle \bar{c}_i : i < \omega \rangle$ witness these distinct types. If $i \neq j$ then $\bar{c}_i$ and $\bar{c}_j$ are disjoint. For, if not, by a weak use of the Δ-system lemma we could find an infinite subsequence of $\bar{c}_i$ which have a common initial segment $\bar{b}'$. But if $lg(\bar{b}') = k$, there are infinitely many $(m\text{-}k)$-types over $N \cup \bar{b} \cup \bar{b}'$ contradicting the choice of m and $\bar{b}$. Let $C^0 = \{\bar{c}_i : i < \omega\}$. Choose an independent sequence $\{\bar{b}^\alpha \cup C^\alpha : \alpha < \alpha^\omega\}$ such that $t(\bar{b}^\alpha \cup C^\alpha; N) = t(\bar{b} \cup C^0; N)$.

If the length of each $\bar{c}_i^\alpha$ is one, we can easily define the required pair of equivalence relations by setting $E_1(c_i^\alpha, c_j^\beta)$ if and only if $tp_{qf}(c_i^\alpha; N \cup \bar{b}) = tp_{qf}(c_j^\beta; N \cup \bar{b})$ if and only if $i = j$ and $E_2(c_i^\alpha, c_j^\beta)$ if and only if $t(c_i^\alpha; N \cup c_j^\beta)$ forks over N if and only if $\alpha = \beta$. If not, write $\bar{c}_i^\alpha$ as $\bar{e}_i^\alpha d_i^\alpha$ for each α and i. We will define the required equivalence relations on $\{d_i^\alpha : 0 < \alpha < 2^{\aleph_0}, i < \omega\}$. By the construction $t(d_i^\alpha; N \cup d_j^\beta)$ forks over N if and only if $\alpha = \beta$. Since the fundamental equivalence relation over N is monadically definable we have an E_2 meeting the requirements of 6.2.8(ii).

Thus, we finish if we show $E_1(d^\alpha, d_j^\beta)$ if and only if $i = j$ is also monadically definable. We require some further notation to establish this result. Let $lg(\bar{b}) = l$ and choose subsets $B_0, \ldots, B_{l-1}$ such that for $i < l$, $B_i \cap [d_i^\alpha]E_2$ is the i'th member of $\bar{b}^\alpha$. Thus, using E_2 and the B_i, $\bar{b}^\alpha$ is the definable closure of d_i^α. Let D denote $\{d_i^\alpha : \alpha < 2^{\aleph_0}, i < \omega\}$. For $d_i^\alpha \in D$, let $\mathcal{P}(d_i^\alpha) = \{t_{qf}(d_i^\alpha {}^\frown b^\alpha {}^\frown \bar{c}; N) : \bar{c} \in [d_i^\alpha]E_2$ with $lg(\bar{c}) = m - 1\}$. By the choice of m, each $\mathcal{P}(d_i^\alpha)$ is finite; by the construction $\mathcal{P}(d_i^\alpha) = \mathcal{P}(d_i^\beta)$ for all α and β. Since for each α, $\bigcup_{i<\omega} \mathcal{P}(d_i^\alpha)$ is infinite we can, possibly by thinning, assume that if $i \neq j$ $\mathcal{P}(d_i^\alpha) \neq \mathcal{P}(d_j^\beta)$. Remembering that $\bar{b}^\alpha$ is definable from d_i^α by some function $\bar{b}(d_i^\alpha)$ we can define $E_1(x, y)$ by:

$$\bigvee_{\phi\in\Delta} \begin{array}{l}(\forall \bar{z} \in [x]E_2)(\exists \bar{w} \in [y]E_2)\\ \forall v \in N((\phi(x, \bar{b}(x), \bar{z}, \bar{v}) \leftrightarrow \phi(y, \bar{b}(y), \bar{w}, \bar{v})))\end{array}$$

In the last formula $lg(\bar{z}) = lg(\bar{w}) = lg(\bar{e}_i^\alpha)$.

The preceding proof was suggested by A. H. Lachlan.

Finally, to finish the proof of Theorem 6.2.7 we need only show that if some model M of T does not admit a nice decomposition, then letting $M = \bigcup_N^\sigma M_i$, some M_i realizes infinitely many distinct quantifier-free types over N. But this is immediate since if not we could easily choose a finite set H so that for p, q any two of finitely many quantifier-free types over N realized in M_i, $q \neq p$ implies $q|H \neq p|H$. Since T is stable for each finite Δ, there are only countably many Δ-n-types over N. As each of the finitely many Δ-n-types realized in M_i are determined by a finite subset $H_{n,i}$ of N, we can choose for each n on n-assignment ρ to decompose M as a nice free union over N.

6.2.11 Corollary *Let T be a superstable theory with $(T_\infty, \text{2nd}) \not\leq (T_\infty, \text{Mon})$ and suppose T is shallow.*

If T is nicely decomposable $ls^T_{L_{\omega,\omega}(Mon)} = \aleph_0$. Otherwise $ls^T_{L_{\omega,\omega}(Mon)} = 2^{\aleph_0}$.

Proof: We know that every reduct of T to a finite language satisfies the dichotomy. If second-order logic on the continuum is interpretable in some finite reduct of T it is certainly interpretable in T. If not, by 6.2.6 $ls^T_{L_{\omega,\omega}(Mon)} = \aleph_0$.

6.3 The permutational theory of strongly decomposable theories This section shows why all study of permutational logic has been restricted to "pure permutational logic": Any permutational theory in a finite language either interprets second-order logic or is bi-interpretable $(\leq_{1\text{-}1})$ in the sense of 2.1.3 with $(T_\infty, 1\text{-}1)$ (i.e., pure permutational logic). If $(T_\infty, \text{2nd}) \not\leq (T, 1\text{-}1)$ then T is strongly decomposable (6.1.3, 8.1.7) and, in fact, extremely nicely decomposable (6.2.2). Thus it suffices to find formulas (possibly with permutational quantifiers) $\theta(x, \bar{y}, \bar{f})$, $\pi(\bar{y}, \bar{f})$, and for each relation symbol $R(\bar{x}) \in L(T)$ a formula $\theta_R(\bar{x}, \bar{y}, \bar{f})$ such that for any set A, elements $\bar{a}$, and permutations $\bar{F}$ of A if $A \models \pi(\bar{a}, \bar{F})$ then $(\theta(A, \bar{a}, \bar{F}), \langle\theta_k(A; \bar{a}, \bar{F})\rangle) \models T$ and for every model M of T there exist A, $\bar{a}$, $\bar{F}$ such that $M \approx (\theta(A, \bar{a}, \bar{F}), \langle\theta_k(A, \bar{a}, \bar{F}\rangle)$. We do not need to worry about the questions of interpreting the quantifiers (which are described in Definition 2.1.2) since both of the logics we are discussing admit permutational quantification.

6.3.1 Theorem *If T is a first-order theory in a finite language $(T_\infty, \text{2nd}) \not\leq (T, 1\text{-}1)$, then $(T_\infty, 1\text{-}1) \equiv_{1\text{-}1} (T, 1\text{-}1)$.*

Proof: Of course $(T_\infty, 1\text{-}1) \leq_{1\text{-}1} (T, 1\text{-}1)$. For the converse, note that by 6.2.2 every model of T is nicely decomposable. Since monadic logic is interpretable in permutational logic we will use monadic formulas and parameters in our argument whenever convenient. Give a model M of T with cardinality κ we will define the formulas $\theta_R(\bar{x}, \bar{y}, \bar{f})$ then choose parameters $\bar{a}_M$, $\bar{F}_M$ such that $(\kappa, \langle\theta_R(\kappa, \bar{A}_M, \bar{F}_M)\rangle) \approx M$.

Examination of the argument will then show that only the choice of $\bar{A}_M$

and $\bar{F}_M$ but not the definition of the formulas θ_R depend on M. Thus the interpretation is as uniform as required.

Let M be extremely nicely decomposed over the infinite elementary submodel N as $\bigcup_N^\sigma \langle M_i : i \in I\rangle$ with H as the heart of the heart. Add unary predicates H_1 and H_2 to be interpreted on N and H respectively. (Fix a permutation F of M such that (N, F) and each (M_i, F) is isomorphic to (Z, S). If any of M_i are finite let (M_i, F) form a finite cycle under S.) Call a subset X closed if $x \in X$ implies $F(x)$ and $F^{-1}(x) \in X$. Now define an equivalence relation on M by $x \sim$ y if and only if they are contained in the same closed subset. Finally, let $\theta_R(\bar{x}, H_1, H_2, F)$ be the formalization of the m-conditions which define R as $\bigcup_N^\sigma \langle M_i : i \in I\rangle$. That is, for each partition of $\bar{x}$, σ assigns a finite set of sequences $\langle p_1^i, \ldots, p_k^i\rangle$ of quantifier-free types over H such that $R(\bar{a}_1, \ldots, \bar{a}_k)$ if and only if for some i, each $\bar{a}_j$ satisfies $p_j^i (1 \le j \le k)$. Clearly this can be expressed by a single formula.

Shelah has shown that 6.3.1 can be improved by requiring the interpreting formulas to be first-order formulas (with a finite number of function parameters). This requires about two more pages of argument.

7 Some prototypes In this section we investigate stable theories T with $(T_\infty, \text{2nd}) \not\le (T, Mon)$. Of course, this condition applied as well in Sections 5 and 6, but there we discussed even stronger additional conditions. For the moment, suppose T is countable. Then by 4.2.16 and 3.2.11 we have established upper bounds for the Hanf number of (T, Mon) (in various languages). The main result of this section describes prototypes for theories satisfying our condition. We will explicitly describe the prototypes only for the superstable and deep case ($\lambda^{<\omega}$) and the strictly stable case. Similar arguments will produce similar results for shallow and in particular n-tree decomposable theories.

In Section 7.1, we describe the prototypes up to infinitary monadic equivalence. In Section 7.2 (which was completed later) we show that in the strictly stable case we can describe the interpretation in finitary monadic logic and in fact we can refine the description of the prototypes.

7.1 Lower bounds for Hanf and Löwenheim numbers The major result of this section, Theorem 7.1.14, establishes that structures of the form $\lambda^{<\omega}$, $\lambda^{\le\omega}$ are prototypes for (models of) superstable, respectively stable theories T such that $(T_\infty, \text{2nd}) \not\le (T, Mon)$. The same reasoning establishes lower bounds for the Hanf and Löwenheim numbers of infinitary monadic logic restricted to such theories. We have organized the section around the computation of these bounds. Thus, we first show by example in 7.1.3 that if T is $\aleph_0$-tree decomposable that our computation $h^T_{L_{\omega,\omega}(Mon)} = \beth_\omega$ is best possible by finding theories T_n and sentences ϕ_n which are bounded by $\beth_n$. Then we improve the counterexamples to theorems (in the infinitary case). That is, for every deep T, $h^T_{L_{\omega_1,\omega}(Mon)} \ge \beth_{\omega_1}$. (This is an immediate consequence of Theorem 7.1.10.)

7.1.1 Example For $n \geq 1$ let L_n contain unary predicates $\langle P_i : i \leq n \rangle$ unary function symbols f and g, and a constant symbol 0.

Let T_n assert that the P_i are disjoint, f is a 1-1, onto function with no finite cycles, and that f and g commute. Moreover, let T_n assert f preserves each P_i while g maps P_i to P_{i-1} if $i > 0$, g fixes P_0 pointwise, for each a $g^{-1}(a)$ is infinite, and $P_0(0)$ holds.

It is now easy to establish:

7.1.2 Lemma *Each T_n is a complete ω-stable theory and* $(T_\infty, \text{2nd}) \not\leq (T_n, Mon)$.

Proof: It is easy to see that T_n is complete, ω-stable, and $n + 1$ tree decomposable so the result follows from Corollary 7.3.15 (below).

7.1.3 Theorem *For each n, there are sentences in finitary monadic logic λ_n and η_n such that*
(i) $M_n \models T_n \cup \{\lambda_n\}$ *implies* $|M_n| \geq \beth_n$ *and* $T_n \cup \{\lambda_n\}$ *has a model.*
(ii) $M_n \models T_n \cup \{\eta_n\}$ *implies* $|M_n| \leq \beth_n$ *and* $T_n \cup \{\eta_n\}$ *has a model of power* $\beth_n$.

Now (i) guarantees the Löwenheim number of $(\mathfrak{T}, Mon) \geq \beth_\omega$ and (ii) guarantees the Hanf number of $(\mathfrak{T}, Mon) \geq \beth_\omega$ for $\mathfrak{T}$ the class of superstable, $\aleph_0$-tree decomposable theories.

In the proof of this theorem we require the following formulas which are easily expressible in monadic logic: comp(y, x) (x is in the f-component of y), $\gamma(y)$ ($g^k(y) = 0$ for some $k < n$).

7.1.4 Definition Let $M_n \models T_n$. We define by induction on l a predicate $Set_l(x, Y)$. It is defined for any $Y \subseteq M_n$ and any $x \in P_{n-l}(M_n) \cap \gamma(M_n)$. If $l = 0$ and $x \in P_n(M_n) \cap \gamma(M_n)$ then $Set_0(x, Y) = \{n : f^n(x) \in Y\}$. For $x \in P_{n-l-1}(M_n) \cap \gamma(M_n)$ then $Set_{l+1}(x, Y) = \{Set_l(x', Y) : g(x') = x\}$.

Note that each $Set_l(x', Y)$ is a subset of $\beth_l(\aleph_0)$ and is *not* contained in M_n. Nevertheless, we are able to define in M_n the relation between $Set(x, Y_1)$ and $Set(y, Y_2)$.

7.1.5 Lemma *For each l, there is a monadic formula $\phi_l(u, v, Z_1, Z_2)$ such that for $x, y \in P_{n-l}(M_n)$, $Y_1, Y_2 \subseteq M_n$,*

$$M_n \models \phi_l(x, y, Y_1, Y_2) \text{ if and only if } Set_l(x, Y_1) = Set_l(y, Y_2) \ .$$

Proof: The proof is by induction on l. If $l = 0$, $\phi_0(u, v, Z_1, Z_2)$ is $\forall w_1 \forall w_2[(comp(u, w_1) \wedge comp(v, w_2) \wedge g^n(w_1) = g^n(w_2)) \rightarrow (Z_1(w_1) \leftrightarrow Z_2(w_2))]$. The formula $\phi_{l+1}(u, v, Z_1, Z_2)$ is

$$(\forall w_1\, g(w_1) = u \rightarrow (\exists w_2\, g(w_2) = V \wedge Set_l(w_1, Z_1) = Set_l(w_2, Z_2))) \wedge$$
$$(\forall w_1\, g(w_1) = v \rightarrow (\exists w_2\, g(w_2) = U \wedge Set_l(w_1, Z_1) = Set_l(w_2, Z_2))) \ .$$

Now let $\mathcal{P}_0 = Z$ and $\mathcal{P}_{l+1}$ be the power set of $\mathcal{P}_l$. Again these are not subsets of M_n, but we will show they are representable in M_n in the following precise sense.

7.1.6 Lemma *There is a formula $\psi_l(u, Z)$ such that for $x \in \gamma(M_n) \cap P_{n-l}(M_n)$ and $Y \subseteq M_n$, $M_n \models \psi_l(x, Y)$ if and only if $Set_l(x, Y) = \mathcal{P}_l$.*

Proof: We require one preliminary notation. Let $\psi_l^*(u, Z)$ abbreviate $\forall z_1 z_2 u'(g(z_1) = g(z_2) = u' \wedge g(u') = u) \wedge z_1 \neq z_2 \rightarrow Set_{l-2}(z_1, Z) \neq Set_{l-2}(z_2, Z)$. This expresses that for each $u' \in g^{-1}(u)$, $Set_l(u', Z)$ contains each subset of $\mathcal{P}_{l-2}$ at most once.

Now we define $\psi_l(u, Z)$ by induction for $l \geq 1$:

$l = 1$: Let $\psi_1(u, Z)$ be $P_{n-1}(u) \wedge \gamma(u) \wedge \psi_1^*(u, Z) \wedge \forall u' \forall Y(g(u') = u \rightarrow \exists u'' g(u'') = u \wedge Set_0(u', Y) = Set_0(u'', Z))$.

$l + 1$: Let $\psi_{l+1}(u, Z)$ be $P_{n-l-1}(u) \wedge \psi_{l+1}^*(u, Z) \wedge \forall u \forall Y(g(u') = u \rightarrow (\exists u'' g(u'') = u \wedge Set_l(u', Y) = Set_l(u'', Z))) \wedge \exists u' \exists Y \psi_l(u', Y) \wedge g(u') = u$.

Now, it is easy to prove by induction on $l \geq 1$.

7.1.7 Lemma *If $l \geq 1$, and $M_n \models \exists Z \psi_l(x, Z)$ then $|\{m \in M_n : g(m) = x\}| \geq \beth_l$.*

Moreover, it is clear that:

7.1.8 Lemma *If $l \geq 0$ and $M_n \models \psi_l^*(x, Z)$, then $|\{m \in M_n : g(m) = x\}| \leq \beth_l$.*

Thus, setting η_n as $\exists x \exists Z \psi_n(x, Z)$ and λ_n as $\bigwedge_{l<n} \forall x \forall Z \psi_n(x, Z)$ we have the required sentences.

Now we use infinitary monadic logic and the results of Section 4.3 to extend this result to an exact computation of $h^T_{L^{\alpha}_{\infty,\omega}(Mon)}$ for shallow T in a finite language. We require one technical lemma. (Allen Mekler noted that this result is true in an arbitrary stable theory.)

7.1.9 Lemma *If T is superstable, $|M - N| < \omega$ and $a \in \mathfrak{C} - M$ then $t(a; M)$ does not fork over N.*

Proof: Assume for contradiction that $M - N = \bar{b} = \langle b_0, \ldots, b_{n-1} \rangle$. There exists finite $\bar{a} \subseteq N$ $t(\bar{b}; N)$ d.n.f. over $\bar{a}$. If $t(a; M)$ forks over N then (possibly enlarging $\bar{a}$ slightly) there exists a formula $\phi(x, \bar{b}, \bar{a})$ such that for any q with $\phi(x, \bar{b}, \bar{a}) \in q$, q forks over $\bar{a}$. Now since $M \prec \mathfrak{C}$, for some $c \in M \models \phi(c, \bar{b}, \bar{a})$. Then $t(c; N)$ forks over $\bar{a}$ and $c \in M - N$, contrary to the choice of $\bar{a}$.

We also rely on the observation that '$t(a; A \cup b)$ forks over A' is easy to define in $L_{|T|^+,\omega}(Mon)$ by adding a monadic predicate to name A. The definition of forking in [11] is directly expressible in this language.

7.1.10 Theorem *Suppose T is superstable, $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ but T is deep. Then for every $\alpha < \aleph_1$ there is a sentence $\bar{\psi}_\alpha \in L^{-\alpha}_{\infty,\omega}(Mon)$ such that $T \cup \{\psi_\alpha\}$ has models only of cardinality $\beth_\alpha$.*

Proof: We begin by defining a model M_α. Then we will define a sentence $\bar{\psi}_\alpha$ whose only model is M_α. Let $I_\alpha \subseteq {}^{<\omega}\beth_\alpha$ be a tree with depth α. Since T is deep, repeated application of the preceding lemma produces a sequence of models $\langle N_i : i < \omega \rangle$ such that $N_i \prec N_{i+1}$, all members of $N_{i+1} - N_i$ are E_N equivalent,

$t(N_{i+2}; N_{i+1})$ forks over N_i, and $|T| \geq |N_{i+1} - N_i| \geq \aleph_0$. Now let I be a subtree of ${}^{<\omega}\beth_\alpha$ whose Kleene-Brouwer ordinal is α and define $\langle N_\eta : \eta \in I \rangle$ by choosing $N_\eta \approx N_j$ if $lg(\eta) = j$, so that for each η, i, j, $N_{\eta \frown i} \cap N_{\eta \frown j} = N_\eta$, and $t(N_{\eta \frown i}; N_{\eta \frown j})$ d.n.f. over N_η if $i \neq j$. Let $M_\alpha = \bigcup\{N_\eta : \eta \in I\}$. Now if $M_\eta = \bigcup\{N_\rho : \eta \leq \rho\}$, M_α is tree decomposed by $\langle\langle M_\eta, N_\eta\rangle : \eta \in I\rangle$. We now show how to define a copy of I in M_α by a sentence in $L^\alpha_{\infty,\omega}(Mon)$. We add unary predicates $\langle P^l_i : l < \omega, i < |\alpha|\rangle$ and $\langle P^l : l < \omega\rangle$ to L and interpret them in M_α as follows. Fix an enumeration of each N_η as $\langle a^\eta_i : i < \mu_\eta \leq |T|\rangle$ such that if $\eta \leq \nu$ and a^η_i is defined then $a^\eta_i = a^\nu_i$. Interpret P^l_i as $\{a^\eta_i : \eta \in I, lg(\eta) = l\}$ and P^l as $\bar{N}_l = \bigcup\{N_\eta : lg(\eta) = l\}$. As remarked before the theorem we can monadically define E_{P^l} in $L_{|T|^+,\omega}(Mon)$. Then we monadically define a predicate $C(x, y)$ such that $C(x, y)$ holds just if $y \in N_\eta - N_\eta$ for the shortest η such that $x \in N_\eta$: $C(x, y)$ is $\bigwedge_{l<\omega} P^l(x) \leftrightarrow P^l(y) \wedge \bigwedge_{l<\omega} ((P^{l+1}(x) \wedge \neg P^l(x)) \rightarrow E_{P^l}(x, y))$. Note that $C(x, y)$ is an equivalence relation and we can define a transitive reflexive relation on M_α which induces a partial order isomorphic to I when we form M_α/C. Define $x \leq y$ to be $\bigvee_{l<\omega} P^{l+1}(x) \wedge \neg P^l(y) \wedge \neg P^l(x) \wedge E_{P^{l-1}}(x, y)$ and for convenience let $x < y$ be $x \leq y \wedge \bigvee_{l<\omega} P^l(x) \wedge P^{l+1}(y) \wedge \neg P^l(y) \wedge \neg P^{l-1}(y)$.

The remainder of the argument follows that in Theorem 7.1.3 where we found one theory with Hanf number $\beth_n$. We define a collection of sets $Set_\beta(x, Y)$ for $\alpha < |T|^+$ and formulas $\phi(x, Y, x_1, Y_1)$ and $\psi_\beta(x, Y)$ for $\beta < |T|^\alpha$ such that

(i) $M_\alpha \models \phi_\beta(x, Y, x_1, Y_1)$ iff $Set_\beta(x, Y) = Set_\beta(x_1, Y_1)$ for $\beta \leq \alpha$.
(ii) $M_\alpha \models \psi_\alpha(x, Y)$ iff $Set_l(x, Y) = \mathcal{P}_l$.

Here, $\mathcal{P}_0 = \aleph_0$, $P_{\gamma+1}$ is the power set of $\mathcal{P}_\gamma$, and $\mathcal{P}_\delta = \bigcup_{\alpha<\delta} \mathcal{P}_\alpha$ if δ a limit.

$$Set_0(x, Y) = \{i : \exists y C(x, y) \wedge \bigwedge_{l<\omega} (P^{l+1}(x) \wedge \neg P^l(x) \rightarrow P^{l+1}_i(y))\}$$
$$Set_{l+1}(x, Y) = \{Set_l(y, Y) : x < y\}$$
$$Set_\delta(x, Y) = \bigcup\{Set_\alpha(x, Y) : \alpha < \delta\} \text{ if } \delta \text{ is a limit ordinal.}$$

Let $\phi_0(x, Y, x', Y')$ be $\bigwedge_{i<\omega; l<\omega} (\exists y C(x, y) \wedge y \in Y \wedge P^l_i(y) \leftrightarrow \exists y C(x', y) \wedge y \in Y' \wedge P^l_i(y))$ and let $\phi_{l+1}(x, Y, x', Y')$ be $(\forall y(x < y \rightarrow \exists z y < z \wedge \phi_l(y, Y, z, Y'))) \wedge (\forall y(x' < y \rightarrow \exists z\, y < z \wedge \phi_l(y, Y, z, Y')))$. For limit δ let $\phi_\delta(x, Y, x', Y')$ be $\bigwedge_{\alpha<\delta} \phi_\alpha(x, Y, x', Y')$.

Finally, define ψ_α by induction for $\alpha \geq 1$. Let $\psi_{\alpha+1}$ be $\exists y(x < y \wedge \forall Z \exists z(x < z \wedge \phi_\alpha(y, Z) \leftrightarrow \phi_\alpha(z, Y)))$.

Let ψ_δ be $\bigwedge_{\alpha<\delta} \psi_\alpha(x, Y)$. Taking $\bar{\psi}_\alpha$ as $\exists x \exists Y \psi_\alpha(x, Y)$ we have the theorem.

7.1.12 Notes

(a) The role of Lemma 7.1.9 is to guarantee that each difference $N_{i+1} - N_i$ is infinite. The construction of ψ_α actually guarantees that $|M_\alpha| \geq \beth_\alpha(\gamma)$ if γ is the $inf\{|N_{i+1} : N_i : i < \omega\}$. Since $\beth_\alpha(1) = \beth_\alpha(\aleph_0)$ for $\alpha > \omega\cdot\omega$, Lemma 7.1.9 is only important for small values of α.

(b) In fact, if T is shallow the argument for 7.1.10 yields: If $dp(T) = B < |T|^+$ there is a sentence $\psi_\beta \in L_{|T|^+,T}(Mon)$ which has models only of cardinality $\beth_\beta$.

The proof of the preceding theorem also allows us to find prototypical theories for the class of superstable deep theories.

7.1.13 Definition Let K_0 be the class of all subtrees of trees in set: $\{\lambda^{<\omega} : \lambda \in ord\}$ and K_1 be the class of all subtrees of trees in the set: $\{\lambda^{<\aleph_1} : \lambda \in ord\}$.

Note that the language L of K_0 and K_1 contains only one binary relation (partial order). In the following theorem, bi-interpretability is meant in the sense of 2.1.3(d).

7.1.14 Theorem
(i) *If T is a countable superstable deep theory with* $(T_\infty, \text{2nd}) \not\leq (T, Mon)$, $(T, \bar{L}_{\omega_1,\omega}(Mon)) \equiv_{\bar{L}_{\omega_1,\omega}(Mon)} (K_0, \bar{L}_{\omega_1,\omega}(Mon))$.

Proof: The interpretation of K_0 into the $\bar{L}_{\omega_1,\omega}(Mon)$-theory of T is contained in the first half of the proof of Theorem 7.1.10.

The converse is a reformulation of the decomposability of T. We must find a sentence π involving $<$ and infinitely many unary predicate symbols $\bar{P}$, $\bar{Q}$ such that: (i) if a tree $(I, c, \bar{P}, \bar{Q}) \models \pi$, a specified definable subset of I with relations defined in terms of $<$ and the $\bar{P}$, $\bar{Q}$ is a model of T and (ii) every model of T has such a representation.

To prove this, fix a countable model N of T. Let M be a saturated model of T. We will describe an interpretation of $<$ and countably many unary predicates $\langle P_i : i < \omega \rangle$ on the universe of M so that M can be defined by $L_{\omega_1,\omega}$-sentences in terms of $<$ and the $\bar{P}$. Since every model of T is a submodel of a saturated model using one more unary predicate, we achieve (ii). Afterward, we will describe the construction of π to satisfy condition (i).

Fix a countable model N of T and decompose M as a free union $\bigcup_N \{N_\tau : \tau \in I\}$ for some $I \subseteq |M|^{<\omega}$. For each $\tau \in I$ enumerate $M_\tau - M_{\tau^-}$ as $\{a_i^\tau : i < p \leq \omega\}$. For a finite sequence $\bar{k} = \langle k_0, \ldots, k_{l-1} \rangle$ of natural numbers we write $\bar{a}_{\bar{k}}^\tau$ for $\langle a_{k_0}^\tau, \ldots, a_{k_{l-1}}^\tau \rangle$.

The universe of our tree will be I. There are two important families of unary predicates which we will define on I. The first allows us to describe the types of finite sequences which come from any particular M_τ. For any formula $\phi(\bar{x})$ with $lg(\bar{x}) = l$ and any partition $\bar{k} = \langle \bar{k}_0, \ldots, \bar{k}_n \rangle$ of l let

$$P_{\bar{k}_0,\ldots,\bar{k}_n,\phi} = \{\eta \in I : M \models \phi(\bar{a}_{\bar{k}_0}^{\eta|0}, \ldots, \bar{a}_{\bar{k}_n}^{\eta})\} .$$

The second family of predicates allows us to recover the truth on sequences which intersect independent models. Recall that for any quantifier-free formula $\phi(\bar{x}; \bar{y})$ there is a quantifier-free formula $d\phi(\bar{y}, \bar{z})$ such that for any A and any $p \in S(A)$ such that for all $\bar{b} \in A$, $\phi(\bar{x}, \bar{b}) \in p$ if and only if $d\phi(\bar{b}, \bar{a}_p)$. For each quantifier-free $\phi(\bar{x}; \bar{y})$, associated with $d\phi(\bar{y}, \bar{z})$ where $lg(\bar{x}) = l_0$, $lg(\bar{y}) = l_1$ and $lg(\bar{z}) = k_0 + k_1 + \ldots + k_{n-1}$ let for $\bar{k}_0, \ldots, \bar{k}_{n-1}$, $\bar{L}_0 \in \omega^{<\omega}$, and $lg(\bar{L}_0) = l_0$,

$$Q_{\bar{k}_0,\ldots,\bar{k}_{n-1},\phi,\bar{L}_0}(\eta) = \{\eta : d\phi(\bar{y}; \bar{a}_{\bar{k}_0}^{\eta|0}, \ldots, \bar{a}_{\bar{k}_{n-1}}^{\eta|n-1}) \text{ defines } t_\phi(\bar{a}_{\bar{L}_0}^{\eta}; M_{\eta^-})\} .$$

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To recover M from I, we will think of M as a subset of $I \times I$. More precisely, we will code a_i^η by $\langle \eta, \tau \rangle$ where $lg(\tau) = i$. Of course in $L_{\omega_1,\omega}$ we can recover $lg(\tau)$ from τ so this is permissible. For simplicity of notation we will henceforth regard the elements of our intended copy, M^*, of M as having the form $\langle \eta, k \rangle$. Thus we regard the domain as $I \times \omega$. We will define for each quantifier-free formula $\phi(\bar{x})$ a formula ϕ^* such that $M^* \models \phi^*(\langle \eta_0, k_0 \rangle, \ldots, \langle \eta_l, k_l \rangle)$ if and only if $M \models \phi(\bar{a}_{k_0}^{\eta_0}, \ldots, \bar{a}_{k_l}^{\eta_l})$.

We first define ϕ^* on sequences of the form $\bar{a}_{\bar{k}_0}^{\eta\,i}$, $\bar{a}_{\bar{k}_1}^{\eta\,j}$ with $lg(\bar{k}_0) = l$ and $lg(\bar{k}_1) = m$ which are independent over M_η. For such sequences,

$$\phi^*(\langle \eta ^\frown i, k_0^1 \rangle, \ldots, \langle \eta ^\frown i, k_0^l \rangle, \langle \eta ^\frown j, k_1^1 \rangle, \ldots, \langle \eta ^\frown j, k_1^m \rangle)$$

holds if $t_\phi(a_{\bar{k}_1}^{\eta\,j}; M_\eta)$ is defined by $d\phi(\bar{y}, \bar{a}_{\bar{p}_0}^{\eta|0}, \ldots, \bar{a}_{\bar{p}_{n-1}}^{\eta^-})$ and this defining formula is satisfied by $\bar{a}_{\bar{k}_0}^{\eta\,i}$. That is, we need

(i) $P_{\bar{m}_0, \ldots, \bar{m}_r, \phi}(\eta ^\frown j)$ if and only if $M \models \phi(a_{\bar{k}_1}^{\eta}, \bar{a}_{\bar{m}_0}^{\eta|0}, \ldots, \bar{a}_{\bar{m}_r}^{\eta^-})$ for each $\bar{m}_0, \ldots, \bar{m}_r$ which partition $lg(\bar{y})$ and

(ii) $Q_{\bar{p}_0, \ldots, \bar{p}_{n-1}, d\phi, \bar{k}_0}(\eta ^\frown i)$.

To see that we can recover the entire structure of M from this coding, note first that if $\bar{a} = \bar{a}_0, \ldots, \bar{a}_k$ where $\bar{a}_i = M_{\tau j_i} \cap \bar{a}$ for $i \leq k$, an induction on k determines the truth of $\phi(\bar{a}_0, \ldots, \bar{a}_k)$ from the information specified in the previous paragraph.

Now to settle $\phi(\bar{a}_{\bar{k}_0}^{\tau_0}, \ldots, \bar{a}_{\bar{k}_n}^{\tau_n})$ we induct on the maximum length of the τ_i. We define all ϕ^* on $I_n \times \omega$ where $I_n = \{\tau : lg(\tau) < n\}$ by induction on n.

Let each of $\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_m, k_m \rangle$ have $lg(\tau_i) = 1$. Now we induct on m. If $m = 1$, the definition is accomplished by the $\bar{P}$. If $m = l + 1$ and we want to define ϕ^* we have by induction defined $d\phi^*(\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_{m-1}, k_{m-1} \rangle)$ and we say $\phi^*(\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_m, k_m \rangle)$ holds if and only if, letting $d\phi(\bar{y}_0, \ldots, \bar{y}_{m-1}, \bar{m})$ define $t(\bar{a}_{\bar{k}_m}^{\tau}; M_{\tau_{\bar{m}}})$, we have $d\phi^*(\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_{m-1}, k_{m-1} \rangle, \bar{m}^*)$ where $\bar{m}^*$ are the pairs of sequences attached to $\bar{m}$ and $P_{\bar{m}_0, \ldots, \bar{m}_r, \phi}(\tau_m)$ if and only if $\phi(\bar{a}_{k_m}^{\tau m}, \bar{a}_{\bar{m}_0}^{\tau m/0}, \ldots, \bar{a}_{\bar{m}_r}^{\tau \bar{m}})$ for all instances of ϕ in $M_{\tau_{\bar{m}}}$.

Now assume by induction that we have defined all formulas $\phi^*(\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_m, k_m \rangle)$ with $lg(\tau_i) \leq n$ for all $i < m$. Let each of $\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_m, k_m \rangle$ have $lg(\tau_i) = n + 1$. Again we induct on m. Note that by the definition of a free decomposition

$$M = \bigcup_{M_{\tau_{\bar{m}}}} \{M_{\tau_{\bar{m}} j} : j \in J\} \cup \{M_\rho \cup N_{\tau_{\bar{m}}} : \rho \neq \tau \text{ but } \rho^- \subseteq \tau^-\} \ .$$

Thus $M \models \phi(\bar{a}_{\bar{k}_0}^{\tau_0}, \ldots, \bar{a}_{\bar{k}_m}^{\tau m})$ if and only if letting $d\phi(\bar{y}; \bar{m})$ define $t(\bar{a}_{\bar{k}_m}^{\tau m}; M_{\tau_{\bar{m}}})$, $M \models d\phi(\bar{a}_{\bar{k}_0}^{\tau_0}, \ldots, \bar{a}_{\bar{k}_{m-1}}^{\tau m-1}, \bar{m})$. Thus we define $\phi^*(\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_m, k_m \rangle)$ to hold just if $P_{\bar{m}_0, \ldots, \bar{m}_r, \phi}(\tau_m)$ if and only if $\phi(\bar{a}_{k_m}^{\tau m}, \bar{a}_{\bar{m}_0}^{\tau m/0}, \ldots, \bar{a}_{\bar{m}_r}^{\tau \bar{m}})$ for all instances of ϕ in $M_{\tau_{\bar{m}}}$ and $d\phi^*(\langle \tau_0, k_0 \rangle, \ldots, \langle \tau_{m-1}, k_{m-1} \rangle)$.

This reduces the calculation of Hanf and Löwenheim numbers for $(T, \bar{L}_{\omega_1,\omega}(Mon))$ to that of K_0. We have established lower bounds. A forthcoming paper of Gurevich and Shelah will investigate upper bounds for $(K_1, \bar{L}_{\omega_1,\omega})$. $(K_1 = \{\lambda^{\leq\omega} : \lambda \in Card\})$.

The situation for strictly stable theories (i.e., stable but not superstable) is a little more complicated. We can associate with each theory, the class K_T of trees, which can represent the forking relation in some model of T. Then the proof of 7.1.14 yields us bi-interpretation in $L_{\omega_1,\omega}(Mon)$. However, we have no precise description of the class K_T. The following theorem summarizes our present knowledge.

7.1.15 Theorem For each countable stable but not superstable theory T there is a class K_T of trees which contains K_0, is closed under subtree, and is contained in the collection of subtrees of trees $\{\lambda^{<\aleph_1}: \lambda \in Card\}$, such that

$$(K_T, \bar{L}_{\omega_1,\omega}(Mon)) \equiv_{\bar{L}_{\omega_1,\omega}(Mon)} (T, \bar{L}_{\omega_1,\omega}(Mon)) .$$

The standard example of a stable but not superstable theory (infinitely many refining equivalence relations such that each E_i is split into infinitely many E_{i+1}-classes) satisfies $(T_\infty, \text{2nd}) \not\leq (T, Mon)$. However, an infinite language is not required to obtain such an example.

7.1.16 Example Let L have unary predicates P and Q and a binary relation R. Define a structure M with universe $\lambda^{\leq\omega} \cup \{\langle\eta, \eta|n, k\rangle : \eta \in \lambda^\omega, n < \omega, 0 \leq k \leq n\}$. Let $P^M = \lambda^{<\omega}$, $Q^M = \lambda^{\leq\omega}$ and $R^M = \{\langle\eta, \langle\eta, \eta|n, 1\rangle\rangle : \eta \in \lambda^\omega, n \in \omega\} \cup \{\langle\langle\eta, \eta|n, l\rangle, \langle\eta, \eta|n, l+1\rangle\rangle : \eta \in \lambda^\omega, l < n, n \in \omega\} \cup \{\langle\langle\eta, \eta|n, n-1\rangle, \eta|n\rangle : \eta \in \lambda^\omega, n \in \omega\}$.

The following chart indicates how these results extend to uncountable languages.

T uncountable: infinitary monadic logic $(L^\alpha_{\infty,|T|}(Mon))$.
For simplicity assume $\alpha > |T|$.

	Löwenheim Number	Hanf Number
$\lambda^{\leq\lvert T\rvert}$	(*)	$(\beth_\alpha(\lvert T\rvert))^+$
$\lambda^{<\lvert T\rvert}$(deep)	(*)	$\{(\beth_\alpha(\lvert T\rvert))^+$
$\lambda^{<\lvert T\rvert}$(shallow) $dp(T) = \beta \geq \omega\cdot\omega$	$(\beth_\beta(\lvert T\rvert))^+$	$(\beth_\beta(\lvert T\rvert))^+$
strongly decomposable	$(\beth_1(\lvert T\rvert))^+$	$(\beth_1(\lvert T\rvert))^+$

(*) See [16].

7.2 Strictly stable theories In this section we show that in one direction the arguments of 7.1 can be improved to yield interpretations in $L_{\omega,\omega}(Mon)$ if T is strictly stable (stable but not superstable) in a finite language.

Thus, the main aim of this section is to prove (with interpretation as in the sense of 2.1.3(d)):

7.2.1 Theorem *If T is a stable but not superstable in a finite language L and $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ then $(T, Mon) \geq_{L_{\omega,\omega}(Mon)} (K_2, Mon)$ (where K_2 is the class defined in 7.2.2).*

Fix a cardinal λ and let $I = \lambda^{\leq\omega}$, $J = \lambda^{\omega}$ and $K = \lambda^{<\omega} = I - J$. For $\eta, \tau \in I$, $\eta \wedge \tau$ is the maximal common initial segment of η and τ.

7.2.2 Definition K_2 is the class of subtrees of $\langle \lambda^{\leq\omega}, \wedge \rangle$.

7.2.3 Notation We fix a finite language L and denote by $r(L)$ the maximum arity of a relation symbol in L. We set $\Delta = \{\phi(\bar{x}; \bar{y})\}$: ϕ a quantifier free and $lg(\bar{x}) + lg(\bar{y}) < r(L)$.

7.2.4 Construction Since T is not superstable we can construct a sequence of models $\langle M_i : i \leq \omega \rangle$ and an $a_\omega \in M_\omega - \bigcup\{M_\eta : n < \omega\}$ such that for each i, $t(a_\omega; M_{i+1})$ forks over M_i. Define for $\eta \in J$ and $k \leq \omega$ isomorphisms $f_{\eta|k}$ with $f_{\eta|k} \subseteq f_{\eta|l}$ if $k < l$ and models $M_{\eta|k} = f_{\eta|k}(M_k)$ such that

$$t(M_{\eta|k}; \bigcup\{M_{\eta|p} : p < k\} \cup \{M_\nu : \nu(k) < \eta(k)\})$$

does not fork over $\bigcup\{M_{\eta|p} : p < k\}$.

7.2.5 More notation Let M denote $\bigcup\{M_\eta : \eta \in K\}$ and M^* denote $\bigcup\{M_\eta : \eta \in I\}$. For $\eta \in J$ let $\overline{M}_\eta$ denote $\{M_{\eta|k} : k < \omega\}$. Since this is a nonforking tree, M and M^* are models of T and $M \prec M^*$. (This can be checked using the coheir definition of nonforking.) Let $E = E_{M_{\langle\rangle}} = E^{\text{for}}_{M_{\langle\rangle}}$ be the fundamental equivalence relation over $M_{\langle\rangle}$. Note that for $\eta \in J$, $M_\eta - M$ is the E equivalence class of a_η which we denote as $[a_\eta]E$.

The rough plan of the interpretation is now evident; for each $\eta|k$ we want to choose an element $a_{\eta|k}$ which first appears in $M_{\eta|k}$ to represent the node $\eta|k$. We expect to recover the tree from the forking relation amongst the $a_{\eta|k}$. That is, we will define the tree order from $E_{M_{\langle\rangle}}$. Unfortunately $E_{M_{\langle\rangle}}$ is not always monadically definable. However, we can embed M into a saturated M' where, by 4.3.11, $E_{M_{\langle\rangle}}$ is monadically definable as $E^{\text{for}}_{M_{\langle\rangle}}$. The tree of $a_{\eta|k}$ will be a subtree of the tree induced by decomposing M' over $M_{\langle\rangle}$.

For the details of our interpretation we need some more technical apparatus. The following definition is made for an arbitrary formula $\psi(\bar{x}, \bar{y})$; our chief application will be to take ψ as $\Delta^k(\bar{y}; \bar{z})$, the formula which defines the quantifier-free type of a tuple. We can code all quantifier-free formulas by a single formula $\Delta(\bar{x}; \bar{y})$ since L is finite. As T is stable, there is a formula $\Delta^k(\bar{y}, \bar{z})$ such that for each A and each $p \in S(A)$ there is an $\bar{a}_p$ such that $\Delta(\bar{x}; \bar{a}) \in p$ if and only if $\Delta^k(\bar{a}, \bar{a}_p)$.

7.2.6 Definition If $\psi(\bar{x}, \bar{y}) = \psi$ is a formula and $lg(\bar{y}) = lg(\bar{u}) = lg(\bar{v})$ then $E_\psi(\bar{u}, \bar{v})$ is the equivalence relation defined by $E_\psi(\bar{u}, \bar{v}) \leftrightarrow \forall x(\psi(\bar{x}, \bar{u}) \leftrightarrow \psi(\bar{x}, \bar{v}))$.

If R is an equivalence relation on m-tuples and $\bar{b}$ is an m-tuple, we write $[\bar{b}]R$ for the R-equivalence class of $\bar{b}$.

Now we make the key technical definition for this argument. In it we relate the fundamental equivalence relation E to the equivalence relations $E_\Delta k$ arising from Definition 7.2.6 and the previous discussion.

7.2.7 Definition For each $\eta \in J$, $A_\eta = \bigcup_{k<r(L)} \{[\bar{b}]E_{\Delta^k}$: $\Delta^k(\bar{y}, \bar{b})$ defines $t_\Delta(\bar{a}; M)$ for some $\bar{a} \subseteq [a_\eta]E$ with $lg(\bar{a}) = k\}$.

The remainder of the argument is most easily understood in terms of $\mathbf{C}^{eq}$. For background on this notion see [15], III.6, or [8]. Intuitively, if R is an equivalence relation definable over M the equivalence class A of R is in M^{eq} iff $A \cap M \neq \emptyset$. So for our purposes here we could restate the following lemma as: for each $\bar{a} \in [a_\eta]E$ with $lg(\bar{a}) = k < r(L)$, if $\Delta^k(\bar{y}, \bar{b})$ defines $t_\Delta(\bar{a}; M)$ then $[\bar{b}]E_{\Delta^k} \cap \overline{M}_\eta \neq \emptyset$. But this is clear since by construction $t(M_\eta; M)$ does not fork over $\overline{M}_\eta$.

7.2.8 Lemma $\quad A_\eta \subseteq \overline{M}_\eta^{eq}\left(= \bigcup_{k<\omega} M_{\eta|k}^{eq}\right).$

Now the crucial fact is that A_η is not contained in any initial segment of $\overline{M}_\eta^{eq}$.

7.2.9 Lemma $\quad$ *If $\eta \in J$ and $k < \omega$ then $A_\eta \not\subseteq M_{\eta|k}^{eq}$.*

Proof: We must show that for some $\bar{a} \in [a_\eta]E$ with $lg(\bar{a}) < r(L)$, if $\Delta^k(\bar{y}, \bar{b})$ defines $t(\bar{a}, M)$ then $[\bar{b}]E_{\Delta^k} \cap M_{\eta|k} = \emptyset$. If not, letting $B = (M - \overline{M}_\eta) \cup M_{\eta|k}$, B is a formally good set in $M \cup [a_\eta]E$ over $M_{\eta|k}$. But then $[a_\eta]E$ is a union of $E_{M_{\eta|k}}^{\text{for}}$ equivalence classes and by 4.3.11 $t([a_\eta]E; M)$ d.n.f. over $M_{\eta|k}$ contrary to the construction.

With this information we will show how to monadically interpret the class of subtrees of $K_2^* = \{\langle \lambda^{\leq\omega}; S(x, y, z)\rangle : \lambda \in Card\}$ into (T, Mon) where $S(\tau, \sigma, \eta)$ means $\tau \wedge \sigma \leq \sigma \wedge \eta$ for $\tau, \sigma, \eta \in \lambda^\omega$.

7.2.10 Lemma $\quad$ *For $\tau, \sigma \in J$, $A_\tau \cap A_\sigma = A_\tau \cap M_{\tau\wedge\sigma}^{eq}$.*

Proof: Let $k = lg(\sigma \wedge \tau)$ and let $\bar{0}$ denote the all-zero sequence. Note that $A_\tau = f_\tau(A_{\bar{0}})$ and $A_\sigma = f_\sigma(A_{\bar{0}})$ so

$$A_\tau \cap A_\sigma = f_\tau(A_{\bar{0}}) \cap f_\sigma(A_{\bar{0}}) = f_{\tau\restriction k}(A_{\bar{0}|k}) = A_\tau \cap M_{\sigma|k}^{eq} = A_\tau \cap M_{\tau\wedge\sigma}^{eq} .$$

Now interpret η as a_η and $S(a_\tau, a_\sigma, a_\eta)$ as $A_\sigma \cap A_\tau \subseteq A_\sigma \cap A_\eta$. The previous two lemmas show the resulting structure is isomorphic to $(\lambda^{\leq\omega}, S)$. Since by the construction A_η is monadically definable from a_η we can now complete the proof of Theorem 2.1.

Proof of Theorem 2.1: We must construct $L_{\omega,\omega}(Mon)$ formulas $\chi(x)$, $\phi(x, y)$ and a sentence Π such that if $M \models T \cup \{\Pi\}$, $\phi(x, y)$ defines on $\chi(M)$ a member of K_2 and every member is so represented.

It suffices to show that trees of the form $\lambda^{\leq\omega}$ with $\lambda \geq \aleph_1$ can be so represented since we can then fix any subtree with an additional unary predicate.

The interpretation of $(\lambda^{\leq\omega}, S)$ (cf. before 7.2.10) into the associated model M^{eq} is described in 7.2.4–7.2.10. Let $\chi(x)$ pick out the elements $\{a_\tau : \tau \in k\}$. We saw how to define S in M^{eq}. Since the monadic formula defining S uses only a finite number of additional sorts from M^{eq} and M^{eq} is a definitional extension of M, this formula can be translated into a formula of M and from it we can easily construct a ϕ defining $\wedge$.

In this case the choice of Π is easy since there is little difficulty in writing a monadic sentence asserting $\langle \chi(M), \phi(M)\rangle$ is isomorphic to a substructure of $\lambda^{\leq\omega}$ for some λ.

8 Unstable theories In this section we discuss $(T, \text{1-1})$ and (T, Mon) when T is unstable. We first show that the monadic theory of order is interpretable in (T, Mon) for any unstable theory T. From this, we conclude by 2.2.7 that $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$ so we need only investigate (T, Mon). We show that if T has the independence property then $(T_\infty, \text{2nd}) \leq (T, Mon)$. Thus we are left with unstable theories without the independence property. The prototype of such theories is the monadic theory of order. In Section 8.2 we study the relationship between Hanf and Löwenheim numbers of second-order logic, the monadic theory of order, the monadic theory of well order, and the monadic theory of an arbitrary stable theory without the independence property.

8.1 From n-tuples to 1-tuples The relationship between the theory of order and unstable theories is expected since, roughly speaking, T is unstable just if it admits a linear ordering of n-tuples. More precisely, we recall the following definitions and basic facts.

8.1.1 Definition
(i) T has the *order property* if there is a first-order formula $\phi(\bar{x}, \bar{y})$ and a sequence $\langle \bar{a}_i : i < \omega \rangle$ in a model M of T such that $M \models \phi(\bar{a}_i, \bar{a}_j)$ if and only if $i < j$.
(ii) T has the independence property if there exists a first-order formula $\phi(\bar{x}, \bar{y})$ and sequences $\langle \bar{a}_i : i < \omega \rangle$, $\langle b_\sigma : \sigma \in 2^\omega \rangle$ contained in a model M of T such that $M \models \phi(\bar{a}_i, b_\sigma)$ if and only if $\sigma(i) = 0$.
(iii) T has the *strict order property* if there is a first-order formula $\phi(\bar{x}, \bar{y})$ and a sequence $\langle a_i : i < \omega \rangle$ contained in a model M of T such that $M \models \forall x(\phi(x, a_l) \rightarrow \phi(x, a_k)$ if and only if $k \geq l$.

If a formula ϕ with monadic parameters satisfies condition (i), (ii), or (iii) we say ϕ has, e.g., the order property.

For variants of these definitions and proof of the following consult [11], II.4.

8.1.2 Fact [11], II.4.7 T is unstable if and only if T has the independence property or the strict order property.

Note that these properties are defined on n-tuples. Our first task is to show that using monadic predicates we can define similar relations on individuals. Our procedure is to extract from a set of order indiscernible sequences satisfying the desired relation, a subset which satisfies the same relation on one coordinate.

Note that if a formula $\phi(x, y)$ has the order property (on singletons) it is an easy matter to obtain with an additional unary predicate a formula with the strict order property. Thus, the "strict" in the conclusion of the following lemma is a bonus.

8.1.3 Lemma *If the first-order theory of T has the order property then some first-order formula of* 2 *variables* $\phi(x, y)$ *(with monadic parameters) has the strict order property.*

Proof: If T has the order property then there is an integer k such that for every λ there is a λ-dense ordered set I, a sequence $\langle \bar{a}^s : s \in I \rangle$ contained in a model of T and a formula $\phi(\bar{x}, \bar{y})$ with $lg(\bar{x}) = lg(\bar{y}) = lg(\bar{a}^s) = k$ such that:

$\phi(\bar{a}^s, \bar{a}^t)$ if and only if $s < t$. We prove by induction on k that this implies the existence of a linear order, which is definable in an expansion of T by finitely many unary predicates, which is λ-dense on a subset of M. (The λ-density is used only to guarantee uniformity in the ordering.)

If $k = 1$, the result is evident.

If $k > 1$, we make the following simplifying assumptions. First, by induction, we can assume that no such formula $\psi(\bar{x}', \bar{y}')$ exists if $lg(\bar{x}') = lg(\bar{y}') < k$. Second, by compactness, and the standard Ehrenfeucht-Mostowski technique we can assume the $\langle \bar{a}^s : s \in I\rangle$ are order indiscernible. We write $\bar{a}^s = \bar{b}^s \frown c^s$ where $lg(\bar{b}^s) = k - 1$. We further suppose there exist $s < u < v < t$ such that $\models \phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v)$. (This is by symmetry; otherwise we work with $\neg\phi$.)

Our aim is to show there is a sequence of k-1-tuples which is linearly ordered by a formula. Specifically, we will find u_0, v_0 such that if $\{s, t\} \cap \{u_0, v_0\} = \emptyset$, $\phi(\bar{b}^s \frown c^{u_0}, \bar{b}^t \frown c^{v_0})$ if and only if $s < t$. In fact, any u_0, v_0 will do if we establish:

(*) For any u, v if $\{s, t\} \cap \{u, v\} = \emptyset$ and $|\{s, t\}| = |\{u, v\}| = 2$ then $s < t$ if and only if $\phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v)$.

We will establish (*) in several stages by showing that various other alternatives contradict the induction hypothesis.

Notation We write $u \sim v \; mod(s, t)$ to indicate $s < t$, $u \neq v$, $u < s \leftrightarrow v < s$, and $u < t \leftrightarrow v < t$. We say u and v are in the same cut determined by (s, t).

8.1.4 Lemma *If $u \sim v \; mod(s, t)$ then $\phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v) \leftrightarrow \phi(\bar{b}^s \frown c^v, \bar{b}^t \frown c^u)$.*

Proof: If not, for some u', v' with $u' \sim v' \; mod(s, t)$ $\phi(\bar{b}^s \frown c^{v'}, \bar{b}^t \frown c^{u'}) \wedge \neg\phi(\bar{b}^s \frown c^{u'}, \bar{b}^t \frown c^{v'})$. Thinking of $\phi(\bar{b}^s \frown c^{u'}, \bar{b}^t \frown c^{v'})$ as $\psi(\bar{b}^s \frown c^s, \bar{b}^t \frown c^t, \bar{b}^{v'} \frown c^{v'}, \bar{b}^{u'} \frown c^{u'}) = \psi(\bar{a}^s, \bar{a}^t, \bar{a}^{v'}, \bar{a}^{u'})$ we have by the indiscernibility of the $\langle \bar{a}^s : s \in I\rangle$ that for any $u < v$ in the same interval determined by (s, t) as u', v' (call this interval J) $\phi(\bar{b}^s c^u, \bar{b}^t c^v) \wedge \neg\phi(\bar{b}^s c^v, \bar{b}^t c^u)$. That is, $\{c^u : u \in J\}$ is linearly ordered by $\phi(\bar{b}^s x, \bar{b}^t y)$ contrary to the induction hypothesis.

We will now show that if all $u < v$ in a cut determined by (s, t) satisfy $\phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v)$ then this is also true when one of u or v is moved out of the cut. Technically, there are four cases:

$$u < v < t \rightarrow u < t < v$$
$$t < u < v \rightarrow u < t < v$$
$$u < v < s \rightarrow u < s < v$$
$$s < u < v \rightarrow u < s < v.$$

Since the proofs are similar we prove only one case formally.

8.1.5 Lemma *If $s < u < v < t$ implies $\phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v)$ then for $s < u < t < v$, $\phi(\bar{b}^s c^u, \bar{b}^t c^v)$.*

Proof: If not, by indiscernibility, for any s and t $\phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v)$ if $s < u < v < t$ and $\neg\phi(\bar{b}^s \frown c^u, \bar{b}^t \frown c^v)$ if $s < u < t < v$. Fix u_0, $s < u_0 < t$, let Q be a new predicate picking out $\{c^v : v > u_0\}$, and let $\phi'(\bar{x}', \bar{y}')$ be:

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$$\forall z[(Q(z)) \wedge \phi(b^s \frown c^{u_0}, \bar{x}' \frown z)) \to \phi(b^s \frown c^{u_0}, \bar{y}' \frown z)] \ .$$

Now $\phi'(\bar{b}^p, \bar{b}^q)$ if and only if $p < q$ holds for $u_0 < p$, $q < s$. For, if $v < p$ then $v < q$ so $\models \phi(\bar{b}^t c^{u_0}, \bar{b}^p c^v) \wedge \phi(\bar{b}^t c^{u_0}, \bar{b}^q c^v)$. But if $q < v < p$ then $\models \phi(\bar{b}^t c^{u_0}, \bar{b}^p c^v) \wedge \neg\phi(\bar{b}^t c^{u_0}, \bar{b}^q c^v)$. Thus we have defined a linear order on $\{\bar{b}^p : u_0 < p < s\}$ contrary to the induction hypothesis.

Now by the assumption that there exist $s < u < v < t$ such that $\phi(b^s \frown c^u, b^t \frown c^v)$, and by the last two lemmas (and the analogs of the last), we have (*) and thus Lemma 8.1.3.

We can now immediately conclude:

8.1.6 Theorem *If T is unstable then* $(Th(<), Mon) \leq (T, Mon)$.

This yields, via Corollary 2.2.7 which asserts $(T_\infty, \text{2nd}) \leq (Th(<), \text{1-1})$,

8.1.7 Corollary *If T is unstable then* $(T_\infty, \text{2nd}) \leq (T, \text{1-1})$.

We now turn to theories with the independence property.

8.1.8 Theorem *If T has the independence property then there is a first-order formula $\phi(x, y)$ with monadic parameters which has the independence property (for appropriate interpretation of the monadic parameters).*

Proof: If T has the independence property then there is a formula $\phi(x; \bar{y})$ and a sequence $\langle \bar{a}_i : i < \omega \rangle$ such that for each $\sigma \in 2^\omega$ there is a b_σ satisfying $\phi(b_\sigma, \bar{a}_i)$ if and only if $\sigma(i) = 0$. We prove by induction on $k = lg(\bar{y})$ that this implies the existence of a first-order formula $\psi(x; y)$ with additional unary predicates which has the independence property.

If $k = 1$, the result is evident.

By compactness and Ramsey's theorem we can find b and $\langle \bar{a}_n : n < \omega \rangle$ such that

(i) $\phi(b;\bar{a}_{2n}) \wedge \neg\phi(b; \bar{a}_{2n+1})$
(ii) $\langle b \frown \bar{a}_{2n} \frown \bar{a}_{2n+1} : n < \omega \rangle$ is a sequence of order indiscernibles.
(iii) $\langle \bar{a}_n : n < \omega \rangle$ is a sequence of order indiscernibles.

We write $\bar{a}_n$ as $\bar{c}_n \frown d_n$ and $\psi(x; \bar{y})$ as $\psi(x; \bar{z}, w)$ where $lg\,\bar{z} = lg(\bar{c}_n) = k - 1$. To reduce the number of cases in the following argument let

$$\Psi = \left\{ \begin{matrix} \psi(x; \bar{z}, w), & \neg\psi(x; \bar{z}, w) \\ \psi(x; \bar{y}), & \neg\psi(x; \bar{y}) \end{matrix} \right\} .$$

(Proving a statement for some $\psi \in \Psi$ and $m < n$ is the same as proving it for ψ if $m < n$ and $\neg\psi$ if $m \geq n$.)

Our intention is to translate the independence property on the $\bar{a}_n$ to the independence property on the $\bar{c}_n$. In fact, we proceed through a number of cases, most of which contradict the induction hypothesis that the $\langle \bar{c}_{2n} : n < \omega \rangle$ do not witness the independence property for any formula $\chi(\bar{x}; \bar{z})$ with monadic parameters. The conclusion of these arguments is that

(*): $\psi(b; \bar{c}_{2n}, d_l)$ holds for all l.

The same argument applied to the set $\langle \bar{c}_{2n+1} : n < \omega \rangle$ yields:

(**): $\neg\psi(b; \bar{c}_{2n+1}, d_l)$ holds for all l.

We now set out to prove (*). This also breaks into two parts, each of which has two cases.

(even) $\psi(b; \bar{c}_{2n}, d_{2m})$ all m.
(odd) $\psi(b; \bar{c}_{2n}, d_{2m+1})$ all m.

In each of the next four cases we define a function α from $\{c_{2n} : n < \omega\}$ into the d's (with even or odd subscript depending on the case) such that for an appropriate choice of a monadic predicate U and a first-order formula ψ^*:

(***) $\alpha(\bar{c}_{2n})$ is the unique member x of U satisfying $\psi^*(b; \bar{c}_{2n}, x)$.

Clearly, whenever we achieve such a situation, the formula $\exists w \psi^*(b, \bar{z}, w) \wedge \psi(x, \bar{z}, w) \wedge U(w)$ has the independence property.

Case $(even)_{\mathrm{a}}$: $\models \psi(b; \bar{c}_{2n}, d_{2m})$ if and only if $m = n$. To satisfy (***), let $\psi^* = \psi$, $\alpha(\bar{c}_{2n}) = d_{2n}$ and $U = \{d_{2n} : n < \omega\}$.

Case $(even)_{\mathrm{b}}$: For some $\chi(x; \bar{z}, w) \in \Psi$, $\models \chi(b; \bar{c}_{2n}, d_{2m})$ if and only if $m < n$. This case (as case $(\mathrm{odd})_b$ below) has two subcases. To distinguish them, note that by construction, $\{d_{2n} : 2 < n < \omega\}$ is a sequence of order indiscernibles over $b \frown \bar{a}_0 \frown \bar{a}_1$. It may or may not be a set of indiscernibles.

Subcase $(\mathrm{even})_{\mathrm{b(i)}}$: $\{d_{2n} : 2 < n < \omega\}$ is a set of indiscernibles over $b \frown \bar{a}_0 \frown \bar{a}_1$. Taking d_{2n} as e_n and c_{2n} as f_n in the following lemma shows that this case contradicts the induction hypothesis.

8.1.9 Lemma *If* $\{e_n : n < \omega\}$ *is a set of indiscernibles over b and for each n there is an* $\bar{f}_n$ *such that* $\phi(b; \bar{f}_n, e_m)$ *holds, if and only if* $m < n$ *then* $\phi(b, \bar{z}, w)$ *has the independence property.*

Proof: This is a slight variant on [11], II.4.13.

Subcase $(\mathrm{even})_{\mathrm{b(ii)}}$: The sequence $\{d_{2i} : i < \omega\}$ is order indiscernible but *not* a set of indiscernibles. By an argument of Morley, [15], II.2.13(7) → (5), there is a first-order formula with parameters $\lambda(x, y)$ which linearly orders an infinite subset of $\{d_{2i} : i < \omega\}$. (By reindexing, if necessary, we may assume $\lambda(x, y)$ linearly orders $\{d_{2n} : n < \omega\}$.)

Now let $\psi^*(x, \bar{z}, w)$ be $\sim\chi(x, \bar{z}, w) \wedge \forall w_1[(U(w_1) \wedge \lambda(w, w_1)) \rightarrow \chi(x, \bar{z}, w)]$, $U = \{d_{2n} : n < \omega\}$ and $\alpha(c_{2n}) = d_{2n}$. ($\psi^*(x, \bar{z}, w)$ says w is the least member of U not satisfying $\chi(x, \bar{z}, w)$.) This satisfies (***).

We now have proved the (even) case: $\models \psi(b; c_{2n}; d_{2m})$ for all m.

Case $(odd)_{\mathrm{a}}$: $\models \psi(b, \bar{c}_{2m}, d_{2n+1})$ if and only if $m = n$. To satisfy (***), let $\psi^* = \psi$, $\alpha(\bar{c}_{2n}) = d_{2n+1}$ and $U = \{d_{2n+1} : l < \omega\}$.

Case $(odd)_{\mathrm{b}}$: For some $\chi(x; \bar{z}, \omega) \in \Psi$, $\models \chi(b, c_{2n}, d_{2m+1})$ if and only if $m < n$. As in case $(\mathrm{even})_{\mathrm{b}}$, there are two subcases.

Subcase $(\mathrm{odd})_{\mathrm{b(i)}}$: $\{d_{2m+1} : 2 < m < \omega\}$ is a set of indiscernibles over $b \frown \bar{a}_0 \frown \bar{a}_1$. This case is identical with subcase $(\mathrm{even})_{\mathrm{b(i)}}$ (using Lemma 8.1.9).

Subcase $(\text{odd})_{b(ii)}$: $\{d_{2m+1} : 2 < m < \omega\}$ is a sequence of order indiscernibles but is not a set of indiscernibles over $\bar{a}_0 {}^\frown \bar{a}_1 {}^\frown b$. Choose ψ^* exactly as in case $(\text{even})_{b(ii)}$ but let $U = \{d_{2m+1} : m < \omega\}$ and $\alpha(\bar{c}_{2n}) = d_{2n+1}$.

This concludes the odd case and we have $\models \psi(b, \bar{c}_{2n}, d_{2m+1})$ for all m. Thus we have proved (*).

The same argument applied to $\neg\psi(x, \bar{z}, w)$ and $\{c_{2n+1} : n < \omega\}$ yields (**). It remains to derive the theorem from (*) and (**). But this is immediate for the formula $\psi^*(x, \bar{z})$:

$$\forall w(U(w) \rightarrow \psi^*(x, \bar{z}, w)) \text{ if we interpret } U \text{ as } \{d_i : i < \omega\} .$$

We deduce immediately from 8.1.8:

8.1.10 Theorem *If T has the independence property $(T_\infty, \text{2nd}) \leq (T, Mon)$.*

Proof: It suffices to show T admits coding. Since T has the independence property on singletons for any κ we can find sets X and Y of power κ such that for every subset A of X there is an element $a \in Y$ such that for all $b \in X$, $\phi(b, a)$ if and only if $b \in A$. Now let U_0 and U_1 partition X into two sets of power κ. Let U_2 be the subset of Y containing these points of Y which code sets $A \subseteq X$ with $|A \cap U_i| = 1$ for $i = 0, 1$. Now $\phi(x, z) \wedge \phi(y, z)$ defines a coding of U_0, U_1 by U_2 so T admits coding.

8.2 Hanf and Löwenheim numbers of $(Th(<), Mon)$ One of the major goals of this paper is to justify the study of monadic theory of order. This justification rests on three claims: (i) any monadic theory "simpler" than $(Th(<), Mon)$ is "almost trivial"; (ii) $(Th(<), Mon)$ is simpler than second-order logic (i.e., $(T_\infty, \text{2nd})$); (iii) any theory "simpler" than second-order logic is "at least as simple" as $(Th(<), Mon)$. In the case of claims (i) and (ii) we make our notion of simple precise by T_1 is "simpler" than T_2 if $(T_2, Mon) \not\leq (T, Mon)$. Sections 4 through 7 show that if $(Th(<), Mon) \not\leq (T, Mon)$ and $(T_\infty, \text{2nd}) \not\leq (T, Mon)$ then (T, Mon) is at least extremely manageable even if "almost trivial" overstates the case. In this section we show $h^{Th(<)}_{L_{\omega,\omega}(Mon)} < h_{L_{\omega,\omega}(\text{2nd})}$ and thus that $(T_\infty, \text{2nd}) \not\leq (Th(<), Mon)$, thereby verifying claim (ii). We adduce one argument in favor of claim (iii) by showing that if T is unstable then $h^{T}_{L_{\omega_1,\omega}(Mon)} \geq h^{Th(<)}_{L_{\omega_1,\omega}}$. Further arguments for claim (iii) are rehearsed in Shelah's paper [15]. Our rather complicated notations for Hanf and Löwenheim numbers are explained in Section 1.2.

In this section we somewhat extend the notion of logic by considering various "applied logics". Thus if one regards a logic as a function which attaches to each similarity type a set of sentences and a semantics for sentences, then $(Th(<), Mon)$ is apparently not a logic. But we can regard $(Th(<), Mon)$ as a logic in the same general spirit by assuming that part of the semantics is to make one binary relation a linear order and the others trivial.

The task of this section is to establish claim (ii). This task is complicated because showing the Hanf number of the finitary theory of order less than the Hanf number of second-order logic does not immediately yield: There is no interpretation of second-order logic into the monadic theory of order. Here there are two difficulties. First, one must be clear on the definition of interpretation, as Gurevich et al. [6] have shown, that (under extremely weak set theoretic hypotheses) there is a syntactic interpretation of second-order logic into the monadic theory of order. Second, even fixing on the strong notion of interpretation used here (3.1.3), this kind of interpretation need not preserve Hanf number. Thus our argument that $(T_\infty, \text{2nd}) \not\leq (Th(<), Mon)$ follows a rather convoluted path. We first show (via a series of lemmas culminating in 8.2.16) that not only is the Hanf number of the monadic theory of order less than that of a second-order logic in finitary logic, but that it is less in infinitary logic with additional predicates $((\bar{L}_{\infty,\omega}(Mon))$. Then we show that if $(T_\infty, \text{2nd}) \leq (Th(<), Mon)$ then $(T_\infty, \bar{L}^\alpha_{\infty,\omega}(Mon)) \leq (Th(<), \bar{L}^\alpha_{\infty,\omega}(Mon))$ and this interpretation preserves Hanf number. The contradiction between these two assertions yields the required results.

Although claim (ii) is the theme of this section, we will establish along the way a number of results that are important in their own right. The following table summarizes one family of such results.

As in Section 1.3 $\alpha \sim \beta$ means $\alpha \leq \beth_\omega(\beta)$ and $\beta \leq \beth_\omega(\alpha)$. If $\alpha(T)$ and $\beta(T)$ are cardinal-valued functions of theories, we write $\alpha \sim \beta^*$ (abbreviating $\lambda T\alpha(T) \sim {}^*\lambda T\beta(T)$) if for each T_0, $\alpha(T_0) \sim \beta(T_0)$.

Finitary Logic

	$(Well\ order, Mon)$	$(Th(<), Mon)$	$(T_\infty, \text{2nd})$
Löwenheim number	α_0	β_0	γ_0
Hanf number	α_1	β_1	γ_1

8.2.1 Theorem $\alpha_0 \leq \beta_0 \leq \gamma_0$; $\alpha_1 \leq \beta_1 \leq \gamma_1$; $\alpha_0 = \alpha_1 \sim {}^*\beta_1$; $\gamma_0 < \gamma_1$. *It is consistent with ZFC that* $\alpha_0 = \beta_0 = \gamma_0$.

(Here is a more precise formulation of the last sentence of 8.2.1.) Gurevich et al. [5] have shown under the weak set theoretic hypothesis "there is a proper class of λ such that $\lambda^{<\lambda} = \lambda$" that $\beta_0 = \gamma_0$.

The proof of Theorem 8.2.1 constitutes most of this section. At the same time we will prove the infinitary version of the result. All of the results except one (8.2.16 and 8.2.17) hold in L (i.e., in the given similarity type). For 8.2.16 and 8.2.17 we must pass to $\bar{L}$, the extension of L by countably many unary predicates. For uniformity, and indeed so as to establish claim (ii), we will prove all the results for $\bar{L}$. We have a similar table on the following page for the infinitary case; here T is a countable first-order theory which is unstable but $(T_\infty, \text{2nd}) \not\leq (T, Mon)$. (Strictly speaking, each α_i', β_i', etc. should be α_i^α, β_i^α, etc. to indicate the reliance on $\bar{L}^\alpha_{\infty,\omega}$. Since we treat all α uniformly, we write primes to enhance readability.)

8.2.2 Theorem $\alpha_0' \leq \beta_0' \leq \gamma_0'$; $\alpha_1' \leq \beta_1' \leq \gamma_1'$; $\gamma_0' < \gamma_1'$; $\alpha_0' = \alpha_1' \sim \beta_1'$, $\delta_0' \geq \beta_0'$, $\delta_1' \sim \beta_1'$.

The Infinitary Case

	Löwenheim Number	Hanf Number
$(T, \bar{L}^{\alpha}_{\infty,\omega}(Mon))$	δ_0'	δ_1'
$(Well\ order, \bar{L}^{\alpha}_{\infty,\omega}(Mon))$	α_0'	α_1'
$(Order, \bar{L}^{\alpha}_{\infty,\omega}(Mon))$	β_0'	β_1'
$\bar{L}^{\alpha}_{\infty,\omega}(2\text{nd})$	γ_0'	γ_1'

We begin by outlining the proofs of Theorems 8.2.1 and 8.2.2. Note first that $\alpha_0 \leq \beta_0 \leq \gamma_0$ and $\alpha_1 \leq \beta_1 \leq \gamma_1$ since "well order" is definable in the monadic theory of order and monadic logic is a sublogic of second-order logic. (The analogous results in the infinitary case hold for the same reason.) We will now show, in a series of separate arguments, that $\alpha_0 = \alpha_1$, $\alpha_1 \sim^* \beta_1$, and $\gamma_0 < \gamma_1$.

To make the arguments in this section more intelligible we write HL for $h^{Th(<)}_{L_{\omega,\omega}(Mon)} = \beta_1$ (the Hanf number of the finitary monadic theory of order), LW for α_0 (the Löwenheim number of the finitary monadic theory of well order), and HW for α_1 (the Hanf number of the finitary monadic theory of well order).

The following three lemmas will establish the technical tools to prove $LW = HW$ (i.e., $\alpha_0 = \alpha_1$), HW is a limit cardinal, and there is no sentence θ, which has only well-ordered models with $min(spec\ \theta) = LW$.

8.2.3 Lemma *If the ordinal $\mu = min(spec\ \theta)$ then*
(i) *There is a θ_1 with only well-ordered models which characterizes μ (as an ordinal) and therefore characterizes $|\mu|$ as a cardinal.*
(ii) *There is a θ_2 with only well-ordered models which characterizes $|\mu|^+$.*

Proof: (i) Let $\theta_1 = \theta \wedge \forall x(\neg\theta\{y : y < x\})$.

(ii) Let θ_2 be true of an ordering $(\alpha, <)$ just if α is a well-ordering with no last element and uncountable cofinality such that every closed unbounded subset of α contains a bounded subset $(D, <)$ such that $(D, <) \models \theta_1$.

It is immediate from 8.2.3(i) that $LW \leq HW$. It is easy to see that this argument applies to any logic $\mathfrak{L}$ which admits relativization to show $h^{WO}_{\mathfrak{L}} \geq ls^{WO}_{\mathfrak{L}}$. We will give a semi-abstract formulation of the argument showing $HW \leq LW$ to highlight the properties of the monadic theory of order used in the proof.

8.2.4 Lemma *Let $\mathfrak{L}$ be a logic such that for some notion $\oplus$ of sum of models*:
(a) *For each formula ϕ and structure N_0 there is a ϕ^* with $N_0 \models \phi^*$ such that for any N_0' which satisfies ϕ^*, $N_0 \oplus N_1 \models \phi$ if and only if $N_0' \oplus N_1 \models \phi$.*
(b) *For every κ and every $\lambda < \kappa$ if $|M| = \kappa$ and $|N| = \lambda$ then for some N', $M \approx N \oplus N'$.*

Then if every $\mathfrak{L}$-sentence has a model M with $|M| < \mu$, $\mu \geq h_{\mathfrak{L}}$.

Proof: Let $M \models \phi$, $|M| = \mu$. Choose any $\kappa > \mu$ and let $|N_0| = \kappa$. Choose N_0'

such that $N_0' \models \phi^*$ and $|N_0'| < |M|$. By (b), $M = N_0' \oplus N_1$ for some N_1. But then by (a), $N_0 \oplus N_1 \models \phi$ so ϕ does not characterize μ and thus $\mu \geq h_{\mathcal{L}}$.

8.2.5 Proposition *If $\mathcal{L}$ is (well order, $L_{\omega,\omega}(Mon)$) or (well order, $L_{\infty,\omega}(Mon)$) or (well order, $\bar{L}_{\infty,\omega}(Mon)$) then $\mathcal{L}$ satisfies the hypothesis of 8.2.4.*

Proof: For (a) let ϕ have quantifier rank α and let ϕ^* be $mT_0^\alpha(N_0)$ in the infinitary case (cf. 3.1.1) and the appropriate analogous finite theory in the finitary case (cf. 22). Taking $\oplus$ as ordinal sum, (b) is obvious.

8.2.6 Theorem

(i) *There is no θ so that $min(spec\ \theta) = LW$.*
(ii) *$LW = HW$* (i.e., $\alpha_0 = \alpha_1$)
(iii) *HW is a limit cardinal of cofinality $\aleph_0$.*
(iv) *If θ has arbitrarily large models less than HW, θ has arbitrarily large models.*

Proof: (i) If for some θ, $min(spec\ \theta) = LW$, then by 8.2.3(ii) there is a θ_2 characterizing $(LW)^+$ so $(LW)^+ < HW$. But by 8.2.4, $(LW)^+ \geq HW$.

(ii) We have $LW \leq HW$. By (i), we can apply 8.2.4 to LW, yielding $LW \geq HW$.

(iii) If $\kappa < LW = HW$ then by (i) there is a λ, $\kappa \leq \lambda < HW$ and a θ with $min(spec\ \theta) = \lambda$. By 8.2.3 (ii) $\lambda^+ < HW$ so HW is a limit cardinal. There are only countably many cardinals characterized by $L_{\omega,\omega}(Mon)$ sentences and HW is their limit.

(iv) Let $I \models \delta$ if and only if every proper initial segment of I contains a subsequence which satisfies θ. If θ does not have arbitrarily large models, δ characterizes HW which is impossible.

Thus we have $\alpha_0 = \alpha_1$; we now show $\alpha_1 \sim^* \beta_1$, i.e., HL is bounded in terms of HW. The idea behind the theorem is easy. Suppose M is the cardinal sum of $\langle N_i : i < \lambda\rangle$. Then for any fixed n, the theory of M for monadic sentences with fewer than n alternations of quantifiers is determined by the similar theory of the structure $\langle \lambda, Q_t\rangle$ where for each n-quantifier theory t, Q_t picks out $\{i : N_i \models t\}$. Thus, we can build large models of M by considering large models of $\langle \lambda, Q_t\rangle$. With a few minor complications, this argument goes through. We base our formalism on that in [22], since we are working in the finitary case. Note that the analogous theorem showing $\alpha_1' \leq \beta_1'$ will follow by an identical argument using the formalism in Section 3.1.

8.2.7 Lemma *For every monadic sentence $\theta(<)$ there is a sentence $\theta^*(<)$ such that for any ordinal α there is a linearly ordered model M with a subset of order type α or α^* if and only if $\alpha \models \theta^*$.*

Proof: Suppose θ has n alternations of monadic quantifiers and that $\bar{k}$ indexes the number of quantifiers in each block. Then there is a function G mapping $S = \{Th_{\bar{k}}^n(I) : I \text{ a linear order}\}$ into $\{T, F\}$ such that $G(Th_{\bar{k}}^n(I)) = T$ if and only if $I \models \theta$. Choose $\bar{r}$ (by [7], 2.4) so that for any linear order J, if $J = \sum_{r\in K} I_r$ for some k and $P_t = \{r \in K\colon Th_{\bar{k}}^n(I) = t\}$ then there is a function F

mapping $S' = \{Th^n_{\bar{r}}(K, P_t) : \langle K, P_t \rangle$ a linear order with $|T|$ predicates$\}$ which computes $Th^n_{\bar{k}}(J)$ from $Th^n_{\bar{r}}(K, P_t)$. Let S'_0 be the set of $s_1 \in S'$ such that for some $s_0, s_1 \in S'$ if $\langle J, P_t \rangle = \langle I_0, P_t \rangle \oplus \langle I_1, P_t \rangle \oplus \langle I_2, P_t \rangle$ and $s_i = Th^n_{\bar{r}}(I_i, P_t)$ then $G(F(J, P_t)) = T$. Now let θ^*_{inc} (*inc* for increasing) hold of $(I, <)$ if for some subsets $\langle P_t : t \in S \rangle$ of I, $Th^n_{\bar{r}}(I, P_t) \in S'_0$. Now $\alpha \models \theta^*_{inc}$ if and only if every linear order which imbeds α satisfies θ. Construct θ^*_{dcr} (*dcr* for decreasing) by replacing $\langle I_1, P_t \rangle$ by $\langle I^*_1, P_t \rangle$ in the construction of θ^*_{inc}. Let $\theta^* = \theta^*_{inv} \vee \theta^*_{dcr}$.

8.2.8 Theorem $\sum_{\kappa<HW} (2^\kappa)^+ = HL$. *Thus* $HL \sim^* HW$.

Proof: We first show that $\kappa < HW$ and θ characterizes κ implies there is a θ^* which has a model of power 2^κ but does not have arbitrarily large models. By 8.2.6 there is a λ, $\kappa < \lambda \leq HW$ and a θ_1 which characterizes λ. Now choose θ^* so that if $(X, <) \models \theta^*$ and $(Y, <)$ is a well-ordered subset of $(X, <)$ then $(Y, <) \models \neg\theta_1$. Now 2^κ with the lexiocographic ordering satisfies θ^* (since there is no embedding of κ^+ into 2^κ) but θ^* does not have arbitrarily large models. Thus $2^\kappa < HL$ and $\sum_{\kappa<HW} (2^\kappa)^+ \leq HL$.

Now to show $HL \leq \sum_{\kappa<HW} (2^\kappa)^+$ note that if some θ characterizes $\lambda > \sum_{\alpha<HW} (2^\kappa)^+$ then by the Erdos-Rado theorem for each $\kappa < HW$ there is a model of θ imbedding either κ or κ^*. Thus θ^* has arbitrarily large models $<HW$. By 8.2.6 (iv) this is a contradiction.

Since $\sum_{\kappa<HW} (2^\kappa)^+ \leq (2^{HW})^+$ this shows $HL \leq (2^{HW})^+$ and we have $HW \leq HL$ so $HW \sim^* HL$ (i.e., $\alpha_1 \sim^* \beta_1$).

Some infinitary logics have their Hanf and Löwenheim numbers equal (e.g., $L_{\kappa,\kappa}$(2nd) where κ is $<\kappa$ compact). However, this is relatively rare. Recall that in order to prove the Hanf number exists we must restrict to logics which have a set (not a proper class) of sentences. For infinitary logics, this may require that we fix the similarity type. Thus we will consider the relation between $ls^\tau_{\mathcal{L}}$ and $h^\tau_{\mathcal{L}}$ (for some arbitrary similarity type τ).

If a logic $\mathcal{L}(\tau)$ has μ sentences it has only 2^μ theories. Thus, a sentence ψ such that if $M \models \psi$ then M is not $\mathcal{L}$-equivalent to any substructure of smaller cardinality is a bounded sentence. For, if $M_\kappa \models \psi$ with $|M_\kappa| = \kappa$ and T_κ is the $\mathcal{L}$-theory of M_κ, then for any $\lambda > \kappa$ and any model M_λ of ψ with $|M_\lambda| = \lambda$, $M_\lambda \models T_\kappa$. Thus there can be models of ψ in at most 2^μ cardinalities. In order to formulate the sentence ψ, we relied on $\mathcal{L}$ satisfying the following condition.

8.2.9 Definition The logic $\mathcal{L}$ is *powerful* if there is an $\mathcal{L}$-formula $\phi(U, V)$ with U and V unary relation symbols such that if $M \models \phi(U, V)$ then $U(M) \equiv_{\mathcal{L}} V(M)$.

We will describe some powerful logics in a moment. First, we continue to explore the effect of this property on Hanf and Löwenheim numbers.

We show that if $\mathcal{L}$ is powerful there is a sentence ψ^* which has a model of power $ls_{\mathcal{L}}$ but is bounded. If $ls_{\mathcal{L}}$ is attained by a model of a sentence ϕ, let $\psi^* = \phi \wedge \psi$. If not, let $\lambda = ls^\tau_{\mathcal{L}}$. For arbitrarily large $\kappa < \lambda$ there exists a sentence ϕ_κ such that ϕ_κ has a model M_κ of cardinality κ but none of smaller cardinality. Form a τ-structure M_λ by taking the disjoint union of the M_κ and defining the relation symbols of τ only within the M_κ. Now $|M_\lambda|$ is a limit cardinal and for arbitrarily large $\mu < \lambda$ there is a μ', $\mu < \mu' < \lambda$ such that there is a substructure of M_λ which has cardinality μ' and is $\mathcal{L}$-elementarily equivalent to no smaller substructure of M_λ. Since $\mathcal{L}$ is powerful this property of M_λ can be expressed by an $\mathcal{L}$-sentence ψ^*. By extending the argument for ψ, we see that ψ^* is a bounded sentence. We have proved:

8.2.10 Theorem *If $\mathcal{L}$ is a powerful logic then for any similarity type τ, $ls^\tau_{\mathcal{L}} < h^\tau_{\mathcal{L}}$.*

To apply this result we must show the relevant logics are powerful. For this we require one further definition.

8.2.11 Definition The cardinal κ is $\mathcal{L}$-definable if there is an $\mathcal{L}$-sentence ϕ whose only relation symbol is = such that all models of ϕ have cardinality κ.

8.2.12 Lemma *The logic $L_{\omega,\omega}$(2nd) is powerful; so is the logic $L^\mu_{\infty,\kappa}$(2nd) if μ and κ are definable in $L^\mu_{\infty,\kappa}$(2nd).*

Proof: Note that $U \equiv_{L^\mu_{\infty,\omega}(\text{2nd})} V$ can be defined by "back and forths" of sequences of elements and relations with length μ taken $<\kappa$ at a time (cf. [10]) and such systems are naturally described in second-order logic and thus in $\mathcal{L}$ if the cardinals μ and κ are definable in $\mathcal{L}$.

8.2.13 Theorem *The Löwenheim number of second-order logic is strictly less than the Hanf number of second-order logic. More precisely*

(i) $ls_{L_{\omega,\omega}(\text{2nd})} < h_{L_{\omega,\omega}(\text{2nd})}$.

For infinitary languages, we can say

(ii) *If μ and κ are definable in $L^\mu_{\infty,\kappa}$(2nd) then for every τ,* $ls^\tau_{L^\mu_{\infty,\kappa}(\text{2nd})} < ls^\tau_{L^\mu_{\infty,\kappa}(\text{2nd})}$.

Proof: Both claims are immediate from Theorem 8.2.10 and Lemma 8.2.12.

Finally, to complete the proof of Theorem 8.2.1 we remark that it is consistent for $\alpha_0 = \beta_0 = \gamma_0$. In fact, the consistency proofs use very weak additional axioms. For $\beta_0 = \gamma_0$ only the existence of a proper class of λ with $\lambda^{<\lambda} = \lambda$ is necessary and $V = L$ suffices for $\alpha_0 = \lambda_0$ [7].

8.2.14 Theorem *It is consistent with ZFC that the Löwenheim numbers of second-order logic, the monadic theory of well order and the monadic theory of order are the same.*

Proof: By [7] it is consistent that there is a cardinal preserving interpretation of second-order logic into each of the other two theories. By Lemma 1.4.3, this establishes one inequality and the other is easy.

We include the previous result for completeness but we do not need it for Corollary 8.2.16, which is crucial to establishing claim (ii). To prove the corollary we need one more lemma.

8.2.15 Lemma *If $\mathcal{L}$ is $L_{\omega,\omega}$(2nd), $L_{\infty,\omega}$(2nd) or $\bar{L}_{\infty,\omega}$(2nd) then for any τ both $h^{\tau}_{\mathcal{L}}$ and $ls^{\tau}_{\mathcal{L}}$ are strong limit cardinals.*

Proof: Note first that for any sentence $\psi \in \mathcal{L}$ there is a sentence ϕ^* such that $Spec(\phi^*) = \{2^{\kappa} : \kappa \in Spec(\phi)\}$. ($\lambda \in Spec\ \phi^*$ just if there is an $N \subseteq M$ with $|M| = \lambda$, $N \models \phi$, and M "is" the power set of N.) Now if $\lambda = inf(spec(\phi))$, $inf(spec(\phi^*)) = 2^{\lambda}$ and if $\lambda = sup(spec\ \phi)$, $2^{\lambda} = sup(spec(\phi^*))$. Thus both $h^{\tau}_{\mathcal{L}}$ and $ls^{\tau}_{\mathcal{L}}$ are strong limit cardinals.

8.2.16 Corollary *Let the similarity type τ contain only the equality symbol.*

(i) $h^{Th(<)}_{L_{\omega,\omega}(Mon)} < h^{\tau}_{L_{\omega,\omega}(Mon)}$.

(ii) $h^{Th(<)}_{L_{\omega_1,\omega}(Mon)} < h^{\tau}_{L_{\omega_1,\omega}(Mon)}$.

(iii) $h^{Th(<)}_{\bar{L}_{\omega_1,\omega}(Mon)} < h^{\tau}_{\bar{L}_{\omega_1,\omega}(Mon)}$.

Proof: Consider first the case, $L_{\omega,\omega}(Mon)$. We want to show $\beta_1 < \gamma_1$. By 8.2.8, $\beta_1 \leq (2^{\alpha_1})^+$; but $\alpha_1 = \alpha_0$ by 8.2.6 and $\alpha_0 \leq \gamma_0$ so $\beta_1(2^{\gamma_0})^+$. Since $\gamma_0 < \gamma_1$ (8.2.8) and γ_1 is a strong limit (8.2.15), this yields $\beta_1 \leq (2^{\gamma_0})^+ < \gamma_1$.

All of the coding in Lemmas 8.2.3 through 8.2.8 readily extends to the infinitary case and the Feferman-Vaught Theorem (3.1.13) was proved for infinitary logic. Thus a similar argument can be carried out in the infinitary case.

Suppose that a monadic theory T_1 can be interpreted into a monadic theory T_2 via sentences in a logic $\mathcal{L}$ (for various reasonable logics) possibly containing additional unary predicates but with no monadic quantification. We show that in an infinitary logic with infinitely many unary parameters we can guarantee that the interpretation preserves Hanf number. We state and prove the most interesting case of the theorem. Afterwards, we note the extension to uncountably infinitary logics.

8.2.17 Theorem *If $(T_1, L_{\omega,\omega}(Mon)) \leq_{\bar{L}_{\omega,\omega}} (T_2, L_{\kappa,\omega})$ then via $\bar{L}_{\omega_1,\omega}$ there is an almost decreasing interpretation of $(T_1, L_{\omega_1,\omega}(Mon))$ into $(T_2, L_{\omega_1,\omega}(Mon))$.*

Proof: By Definition 3.1.3 we have formulas π, θ, and χ_i such that if $M_2 \models \pi(\bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})$ then $(\theta(M_2), \chi_i(M_2, \bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1}), \chi_{m-1}(M_2(\bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1}))$ is a model of T_1 and for each $M_1 \models T$, there exists an $M_1^* \models T_2$ such that $(M_1, \bar{R}) \approx \theta(M_1^*, \bar{X})$. Our task is to extend π to a π^* so that if $M_1^* \models \pi^*(\bar{a}, \bar{S}_0, \ldots, \bar{S}_{n-1})$ then $|M_1^*| \leq \beth_{\omega}(|M_1|)$. We do this by altering the choice of M_1^* and noting that the new choice can be specified in $\bar{L}_{\infty,\omega}(Mon)$.

Let the additional unary predicates be $\langle P_i : i < \omega \rangle$. Let π' be an infinitary sentence which asserts that $P_0(x) \leftrightarrow \theta(x, \bar{y}, s_0, \ldots, s_{n-1})$ and that each first-order type in the original language union $\{S_0, \ldots, S_{n-1}\}$ over $\bigcup_{n<\omega} P_n$ is realized exactly once in P_{n+1}. Then for any M_i^*, $Q = \bigcup_{i<\omega} P_i(M_i^*) \prec M_i^*$ and $|Q| < \beth_{\omega}(P_0)$. Thus, if π^* is $\pi \wedge \pi'$, we have required interpretation.

The preceding argument extends to infinitary logic to show that if $(T_1, L_{\kappa,\omega}(Mon)) \leq_{\bar{L}_{\kappa,\omega}} (T_2, L_{\kappa,\omega}(Mon))$ then $(T_1, L_{\kappa,\omega}(Mon)) \leq_{\bar{L}_{\kappa^+,\omega}} (T_2, L_{\kappa,\omega}(Mon))$.

We need to add the additional unary predicates and describe the process in infinitary logic because we are claiming not merely that each $M \models T_1$ can be interpreted in a model of T_2 which is not much larger than M, but that any model into which it is interpreted is not much bigger. The sentence π' guarantees this fact, which is crucial as we now establish claim (ii) in a precise way.

8.2.18 Theorem $(T_\infty, \text{2nd}) \not\leq (Th(<), Mon)$.

Proof: If $(T_\infty, \text{2nd}) \leq (Th(<), Mon)$ then by 8.2.15 and 2.3.6 $h_{\bar{\mathfrak{L}}_{\omega_1,\omega}(\text{2nd})}$ is bounded in terms of $h_{\bar{L}_{\omega_1,\omega}(Mon)}$. But in 8.2.16 we showed $h^{Th(<)}_{\bar{L}_{\omega_1,\omega}(Mon)} < h_{\bar{L}_{\omega_1,\omega}(\text{2nd})}$ and since the power set function is definable in second-order logic this implies $\beth_\omega(h^{Th(<)}_{L_{\omega_1,\omega}(Mon)}) < h_{\bar{L}_{\omega_1,\omega}(Mon)}$.

A. H. Lachlan has given the following direct argument for 8.2.18 which avoids the appeal to $\bar{L}_{\omega_1,\omega}(Mon)$. He argues first via 3.1.13 that $h^{Th(<)}_{\bar{L}_{\omega,\omega}(Mon)} = h^{Th(<)}_{\bar{L}_{\omega,\omega}(Mon)}$. Then he points out that $(T_\infty, \text{2nd}) \leq (Th(<), Mon)$ implies that for every $\lambda < \gamma_1$ there is a sentence ϕ_λ in the language of order with an additional predicate U_1 such that for some M, $|U_1(M)| \leq \lambda$ but ϕ_λ is a bounded sentence. Clearly any such $\lambda < h^{Th(<)}_{\bar{L}_{\omega,\omega}(Mon)} = h^{Th(<)}_{L_{\omega,\omega}(Mon)} = \beta_1$. But by 8.2.16, $\beta_1 < \gamma_1$ so we have $\beta_1 < \beta_1$ yielding the result.

This method by which we proved Theorem 8.2.16 provides some evidence for claim (iii).

8.2.19 Corollary *Let T be a countable unstable theory. Then* $h^T_{\bar{L}_{\omega_1,\omega}(Mon)} \leq h^{Th(<)}_{\bar{L}_{\omega_1,\omega}(Mon)}$.

Proof: By 8.1.6 we have $(T, Mon) \leq (Th(<), Mon)$. Now applying 8.2.17 and 2.3.6 (ii) we have the result.

9 Further problems

There are several directions for further research connected with this area. The first direction is to refine or amplify the specific results discussed here.

1. Can this analysis be extended to unstable theories? The simple class should be those which do not have the independence property even allowing expansion by unary predicates. A test problem for this situation is to compute the Hanf number of such a theory but it is more important to find a structure theory.

Conjecture: If no expansion of T by a finite number of unary predicates admits the independence property then each model with power λ can be decomposed into a tree of models, each with cardinality $\leq 2^{|T|}$ indexed by the structure $\langle \lambda^{\leq\lambda}, \lhd, \leq \rangle$ where $\lhd$ denotes the initial segment order on sequences and $\leq$ the lexiocographic order.

A less important problem here is to compute the exact Hanf number of (T, Mon). There is some information relevant to a possible structure theory in [19]. Shelah shows the Hanf number of (T, Mon) is between w.o., the Hanf number of the monadic theory of well-orderings, and, $\beth_\omega$(w.o.). This bound on the Hanf number shows the property provides a significant cutting point.

2. We remarked in Section 8.2 that for deep superstable theories the "nice" concept does not suffice to finding a cutting point between theories with high and low Löwenheim numbers. Problem: Find such a cutting point.

We found in Section 7 exact conditions for computing the Hanf number of $(T, L_{\omega_1;\omega}(Mon))$. Can this be improved to $(T, L_{\omega,\omega}(Mon))$?

3. Suppose T is shallow but not nice. Then one can define a depth of T by counting only those decompositions which are not nice. Call this the *cruel depth* of T. Problem: Suppose T is a shallow superstable theory. Show the cruel depth of T is α if and only if the Löwenheim number of $(T, L_{\omega,\omega}(Mon))$ is approximately $\beth_\alpha$. For $\alpha = 0$ or 1 this follows from the treatment in Section 7.

A second direction for research is more set theoretic in nature. The problem is to investigate the Hanf and Löwenheim numbers of our prototype classes.

4. What can be said about the Hanf number of the monadic theory of well orderings? Gurevich, Magider and Shelah [6] show it is consistent for it to be large. Is it consistent for it to be small?

5. Consider the class, κ, of structures $\{\lambda^{\leq\omega} : \lambda \in Card\}$. In [20], Shelah shows it is consistent for the Löwenheim number of $(K, L_{\omega,\omega}(Mon))$ to be large. Is it consistently small?

6. Consider the class, K_0, of structures $\{\lambda^{<\omega} : \lambda \in Card\}$. Let Q_{pd} denote quantification over functions which press down on $\lambda^{<\omega}$. It is easy to see that Q_{pd} is definable in $L_{\omega_1,\omega}(Mon)$ if we add unary predicates for the levels. (It is possible to make this definition with only one additional unary predicate.) But even $((\omega, <), Q_{pd})$ is stronger than $L_{\omega,\omega}(Mon)$.

In [20], Shelah shows that if $V = L$, the Löwenheim number of (K_0, Q_{PD}) is large (i.e., same as Q_{II}). Is it consistently small?

A third direction of research is to try to derive a similar theory for larger classes of quantifiers. The most obvious possibility is to expand the notion of quantifier by permitting some algebraic information about the object one quantifies over. Thus one might quantify over automorphisms, endomorphisms, subalgebras, etc. For example, in [20] Shelah shows that quantification over the endomorphisms of a free algebra is complicated by interpreting set theory into the resulting logic.

Can this result be strengthened to an interpretation of second-order logic?

A fourth and vaguer query is to find a more general rubric which encompasses not only the theory here, but the other avatars of classification theory.

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REFERENCES

[1] Baldwin, J. T., *Basic Stability Theory*, Springer-Verlag, New York, 1985.

[2] Baldwin, J. T., "Definable second order quantifiers," in *Model Theoretic Logics* ed. by J. Barwise and S. Feferman, Springer-Verlag, Heidelberg, 1985.

[3] Baldwin, J. T. and Leo Harrington, "Trivial pursuit: Remarks on the main gap," *Proceedings of the Trento Conference*, 1984.

[4] Baldwin, J. T. and R. McKenzie, ''Counting models in universal Horn classes,'' *Algebra Universalis*, vol. 15 (1982), pp. 359–384.

[5] Buechler, S., "Theories with very few types," preprint.

[6] Gurevich, Y., M. Magidor, and S. Shelah, "The monadic theory of w_2," *The Journal of Symbolic Logic*, vol. 48 (1983), pp. 387–398.

[7] Gurevich, Y. and S. Shelah, "Monadic theory of the next world," to appear in *Israel Journal of Mathematics.*

[8] Harnik, V. and L. Harrington, "Fundamentals of forkings," *Annals of Pure and Applied Logic*, vol. 26 (1984), pp. 245–286.

[9] Harrington, L. and M. Makkai, "An exposition of Shelah's 'Main Gap': Counting uncountable models of ω-stable and super-stable theories," *Notre Dame Journal of Formal Logic*, vol. 26, no. 2 (1985), pp. 139–177.

[10] Kueker, D. W., "Back and forth arguments and infinitary logics," in *Infinitary Logic*: In Memoriam Carol Karp, Springer-Verlag Lecture Notes 492, (1975), ed. by D. W. Kueker, pp. 17-73.

[11] Kuratowski, K. and A. Mostowski, *Set Theory*, North Holland, Amsterdam, 1976.

[12] Lascar, D. and B. Poizat, "An introduction to forking," *The Journal of Symbolic Logic*, vol. 44 (1979), pp. 330–351.

[13] Morley, M., "Omitting classes of elements," pp. 265–273 in *Symposium on the Theory of Models*, Berkeley 1963, North Holland Publishing Co., 1965.

[14] Rubin, M. and S. Shelah, "On the elementary equivalence of automorphism groups of Boolean algebras, downward Löwenheim Skolem theorems and compactness of related quantifiers," *The Journal of Symbolic Logic*, vol. 45 (1980), pp. 265–283.

[15] Shelah, S., *Classification Theory and the Number of Nonisomorphic Models*, North-Holland, Amsterdam 1978.

[16] Shelah, S., "Errata to: First order theory of permutation groups," *Israel Journal of Mathematics*, vol. 15 (1973), pp. 437–441.

[17] Shelah, S., "First order theory of permutation groups," *Israel Journal of Mathematics*, vol. 14 (1973), pp. 149–162.

[18] Shelah, S., "Interpreting set theory in the endomorphism group of a free algebra in a category," Proceedings of a Symposium in Clermont-Firrenel, July 1975, *Annales Scientifiques de l'Université de Clermont: Mathématiques*, vol. 13 (1976), pp. 1–29.

[19] Shelah, S., "Monadic logic: Hanf numbers," to appear in *Annals of Pure and Applied Logic.*

[20] Shelah, S., "Monadic logic: Löwenheim numbers," to appear in *Annals of Pure and Applied Logic.*

[21] Shelah, S., "There are just four possible second order quantifiers," *Israel Journal of Mathematics*, vol. 15 (1973), pp. 282–300.

[22] Shelah, S., "The monadic theory of order," *Annals of Mathematics*, vol. 102 (1975), pp. 379–419.

[23] Shelah, S., "The spectrum problem. $\aleph_0$-saturated models, the main gap," *Israel Journal of Mathematics*, vol. 43 (1982). pp. 324–356.

J. Baldwin
Department of Mathematics
University of Illinois Chicago Circle
Chicago, Illinois

S. Shelah
Institute of Advanced Studies
Jerusalem, Israel
and
Institute of Mathematics
The Hebrew University
Jerusalem, Israel