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CRITICAL CARDINALITIES AND ADDITIVITY PROPERTIES OF COMBINATORIAL NOTIONS OF SMALLNESS

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Abstract. Motivated by the minimal tower problem, an earlier work studied diagonalizations of covers where the covers are related to linear quasiorders (τ-covers). We deal with two types of combinatorial questions which arise from this study.

1. Two new cardinals introduced in the topological study are expressed in terms of well known cardinals characteristics of the continuum.
2. We study the additivity numbers of the combinatorial notions corresponding to the topological diagonalization notions.

This gives new insights on the structure of the eventual dominance ordering on the Baire space, the almost inclusion ordering on the Rothberger space, and the interactions between them.

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1. Introduction and overview

Let ω denote the set of natural numbers. We work with two spaces which carry an interesting combinatorial structure: The *Baire space* ${}^\omega\omega$ with eventual dominance $\leq^*$ ($f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n), and the *Rothberger space* $[\omega]^\omega = \{A \subseteq \omega : A \text{ is infinite}\}$ with $\subseteq^*$ ($A \subseteq^* B$ if $A \setminus B$ is finite). We write $A \subset^* B$ if $A \subseteq^* B$ and $B \not\subseteq^* A$.

A subset X of ${}^\omega\omega$ is *unbounded* if it is unbounded with respect to $\leq^*$. X is *dominating* if it is cofinal in ${}^\omega\omega$ with respect to $\leq^*$. $\mathfrak{b}$ is the minimal size of an unbounded subset of ${}^\omega\omega$, and $\mathfrak{d}$ is the minimal size of a dominating subset of ${}^\omega\omega$.

An infinite set $A \subseteq \omega$ is a *pseudo-intersection* of a family $\mathcal{F} \subseteq [\omega]^\omega$ if for each $B \in \mathcal{F}$, $A \subseteq^* B$. A family $\mathcal{F} \subseteq [\omega]^\omega$ is a *tower* if it is linearly quasiordered by $\subseteq^*$, and it has no pseudo-intersection. $\mathfrak{t}$ is the minimal size of a tower. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *centered* if the intersection of each (nonempty) finite subfamily of $\mathcal{F}$ is infinite. $\mathfrak{p}$ is the minimal size of a centered family which has no pseudo-intersection. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *splitting* if for each infinite $A \subseteq \omega$ there exists $S \in \mathcal{F}$ which *splits* A, that is, such that the sets $A \cap S$ and $A \setminus S$ are infinite. $\mathfrak{s}$ is the minimal size of a splitting family.

Let $\mathfrak{c} = 2^{\aleph_0}$. The following relations, where an arrow means $\leq$, are well-known [3]:

$$\begin{tikzcd}
 & & & \mathfrak{b} \arrow[dr] & & \\
\aleph_1 \arrow[r] & \mathfrak{p} \arrow[r] & \mathfrak{t} \arrow[ur] \arrow[dr] & & \mathfrak{d} \arrow[r] & \mathfrak{c} \\
 & & & \mathfrak{s} \arrow[ur] & &
\end{tikzcd}$$

No pair of cardinals in this diagram is provably equal, except perhaps $\mathfrak{p}$ and $\mathfrak{t}$. The *Minimal Tower problem*, which asks whether it is provable that $\mathfrak{p} = \mathfrak{t}$, is one of the most important problems in infinite combinatorics, and it goes back to Rothberger (see, e.g., [12]).

New cardinals. In [15], topological notions related to $\mathfrak{p}$ and $\mathfrak{t}$ were compared. In [17] the topological notion related to $\mathfrak{t}$ (called τ-*covers*) was studied in a wider context. This study led back to several new combinatorial questions, one of which related to the minimal tower problem.

Definition 1. For a family $\mathcal{F} \subseteq [\omega]^\omega$ and an infinite $A \subseteq \omega$, define $\mathcal{F} \restriction A = \{B \cap A : B \in \mathcal{F}\}$. If all sets in $\mathcal{F} \restriction A$ are infinite, we say that $\mathcal{F} \restriction A$ is a *large restriction* of $\mathcal{F}$. Let $\kappa_{\omega\tau}$ be the minimal cardinality of a *centered* family $\mathcal{F} \subseteq [\omega]^\omega$ such that there exists no infinite $A \subseteq \omega$ such that the restriction $\mathcal{F} \restriction A$ is large and linearly quasiordered by $\subseteq^*$.

It is not difficult to see that $\mathfrak{p} = \min\{\kappa_{\omega\mathcal{T}}, \mathfrak{t}\}$ [17]. In Section 2 we show that in fact, $\mathfrak{p} = \kappa_{\omega\mathcal{T}}$. This existence of a centered family with no large linearly quasiordered restriction shows that $\mathfrak{p}$ is combinatorially "larger" than asserted in its original definition, and suggests an additional evidence to the difficulty of separating $\mathfrak{p}$ from the combinatorially "larger" cardinal $\mathfrak{t}$: Now the consistency of $\kappa_{\omega\mathcal{T}} < \mathfrak{t}$ must be established in order to solve the Minimal Tower problem in the negative.

Definition 2. For functions $f, g \in {}^\omega\omega$, and a binary relation R on ω, define a subset $[f\,R\,g]$ of ω by:

$$[f\,R\,g] = \{n : f(n)Rg(n)\}.$$

Next, For functions $f, g, h \in {}^\omega\omega$, and binary relations R, S on ω, define $[h\,R\,g\,S\,f] \subseteq \omega$ by:

$$[f\,R\,g\,S\,h] = [f\,R\,g] \cap [g\,S\,h] = \{n : f(n)Rg(n) \text{ and } g(n)Sh(n)\}.$$

For a subset X of ${}^\omega\omega$ and $g \in {}^\omega\omega$, we say that g *avoids middles* in X with respect to $\langle R, S\rangle$ if:

1. for each $f \in X$, the set $[f\,R\,g]$ is infinite;
2. for all $f, h \in X$ at least one of the sets $[f\,R\,g\,S\,h]$ and $[h\,R\,g\,S\,f]$ is finite.

X satisfies the *excluded middle* property with respect to $\langle R, S\rangle$ if there exists $g \in {}^\omega\omega$ which avoids middles in X with respect to $\langle R, S\rangle$. $\mathfrak{x}_{R,S}$ is the minimal size of a subset X of ${}^\omega\omega$ which does not satisfy the excluded middle property with respect to $\langle R, S\rangle$.

The cardinal $\mathfrak{x} = \mathfrak{x}_{<,\leq}$ was defined in [17]. In Section 3 we express all of the four cardinals $\mathfrak{x}_{\leq,\leq}, \mathfrak{x}_{<,\leq}, \mathfrak{x}_{\leq,<}$, and $\mathfrak{x}_{<,<}$ in terms of well-known cardinals. This solves several problems raised in [17].

Additivity of combinatorial notions of smallness. For a finite subset F of ${}^\omega\omega$, define $\max(F) \in {}^\omega\omega$ by $\max(F)(n) = \max\{f(n) : f \in F\}$ for each n. A subset Y of ${}^\omega\omega$ is *finitely-dominating* if the collection

$$\operatorname{maxfin}(Y) := \{\max(F) : F \text{ is a finite subset of } Y\}$$

is dominating.

We will use the following notations:

- $\mathfrak{B}$: the collection of all bounded subsets of ${}^\omega\omega$;
- $\mathfrak{X}$: the collection of all subsets of ${}^\omega\omega$ which satisfy the excluded middle property with respect to $\langle <, \leq\rangle$;
- $\mathfrak{D}_{\text{fin}}$: the collection of all subsets of ${}^\omega\omega$ which are not finitely dominating; and
- $\mathfrak{D}$: the collection of all subsets of ${}^\omega\omega$ which are not dominating.

Thus $\mathfrak{B} \subseteq \mathfrak{X} \subseteq \mathfrak{D}_{\text{fin}} \subseteq \mathfrak{D}$. The classes $\mathfrak{B}$, $\mathfrak{X}$, $\mathfrak{D}_{\text{fin}}$, and $\mathfrak{D}$ are used to characterize certain topological diagonalization properties [13, 16, 17].

Following [1], we define the *additivity number* for classes $\mathfrak{I} \subseteq \mathfrak{J} \subseteq P({}^{\omega}\omega)$ with $\cup\mathfrak{I} \notin \mathfrak{J}$ by

$$\mathsf{add}(\mathfrak{I},\mathfrak{J}) = \min\{|\mathfrak{F}| : \mathfrak{F} \subseteq \mathfrak{I} \text{ and } \cup\mathfrak{F} \notin \mathfrak{J}\},$$

and write $\mathsf{add}(\mathfrak{J}) = \mathsf{add}(\mathfrak{J},\mathfrak{J})$. If $\mathfrak{I}$ contains all singletons, then $\mathsf{add}(\mathfrak{I},\mathfrak{J}) \leq \mathsf{non}(\mathfrak{J})$, where $\mathsf{non}(\mathfrak{J}) = \min\{|J| : J \subseteq {}^{\omega}\omega \text{ and } J \notin \mathfrak{J}\}$ (thus $\mathsf{non}(\mathfrak{B}) = \mathfrak{b}$, $\mathsf{non}(\mathfrak{D}) = \mathsf{non}(\mathfrak{D}_{\text{fin}}) = \mathfrak{d}$, and $\mathsf{non}(\mathfrak{X}) = \mathfrak{x}$.)

For $\mathfrak{I},\mathfrak{J} \in \{\mathfrak{B},\mathfrak{X},\mathfrak{D}_{\text{fin}},\mathfrak{D}\}$, the cardinals $\mathsf{add}(\mathfrak{I},\mathfrak{J})$ bound from below the additivity numbers of the corresponding topological diagonalizations. In Section 4 we express $\mathsf{add}(\mathfrak{I},\mathfrak{J})$ for almost all $\mathfrak{I},\mathfrak{J} \in \{\mathfrak{B},\mathfrak{X},\mathfrak{D}_{\text{fin}},\mathfrak{D}\}$ in terms of well known cardinal characteristics of the continuum. In two cases for which this is not done, we give consistency results.

2. The cardinal $\kappa_{\omega\tau}$

For our purposes, a *filter* on a boolean subalgebra $\mathcal{B}$ of $P(\omega)$ is a family $\mathcal{U} \subseteq \mathcal{B}$ which is closed under taking supersets in $\mathcal{B}$ and finite intersections, and does not contain finite sets as elements.

Theorem 3. $\mathfrak{p} = \kappa_{\omega\tau}$.

Proof. Let $\mathcal{F} \subseteq [\omega]^{\omega}$ be a centered family of size $\mathfrak{p}$ which has no pseudo-intersection. Let $\mathcal{B}$ be the boolean subalgebra of $P(\omega)$ generated by $\mathcal{F}$. Then $|\mathcal{B}| = \mathfrak{p}$. Let $\mathcal{U} \subseteq \mathcal{B}$ be a filter of $\mathcal{B}$ containing $\mathcal{F}$. As $\mathcal{U}$ does not contain finite sets as elements, $\mathcal{U}$ is centered. Moreover, $|\mathcal{U}| = \mathfrak{p}$, and it has no pseudo-intersection.

Towards a contradiction, assume that $\mathfrak{p} < \kappa_{\omega\tau}$. Then there exists an infinite $A \subseteq \omega$ such that the restriction $\mathcal{U} \restriction A$ is large, and is linearly quasiordered by $\subseteq^*$. Fix any element $D_0 \cap A \in \mathcal{U} \restriction A$. As $\mathcal{U} \restriction A$ does not have a pseudo-intersection, there exist:

1. An element $D_1 \cap A \in \mathcal{U} \restriction A$ such that $D_1 \cap A \subset^* D_0 \cap A$; and
2. An element $D_2 \cap A \in \mathcal{U} \restriction A$ such that $D_2 \cap A \subset^* D_1 \cap A$.

Then the sets $(D_2 \cup (D_0 \setminus D_1)) \cap A$ and $D_1 \cap A$ (which are elements of $\mathcal{U} \restriction A$) contain the infinite sets $(D_0 \cap A) \setminus (D_1 \cap A)$ and $(D_1 \cap A) \setminus (D_2 \cap A)$, respectively, and thus are not $\subseteq^*$-comparable, a contradiction. $\square$

A closely related problem from [17] remains open.

Definition 4. A family $Y \subseteq [\omega]^\omega$ is *linearly refinable* if for each $y \in Y$ there exists an infinite subset $\hat{y} \subseteq y$ such that the family $\hat{Y} = \{\hat{y} : y \in Y\}$ is linearly $\subseteq^*$-quasiordered. $\mathfrak{p}^*$ is the minimal size of a centered family in $[\omega]^\omega$ which is not linearly refineable.

Again it is easy to see that $\mathfrak{p} = \min\{\mathfrak{p}^*, \mathfrak{t}\}$. Thus, a solution of the following problem may shed more light on the Minimal Tower problem.

Problem 5. *Does* $\mathfrak{p} = \mathfrak{p}^*$*?*

3. The excluded middle property

Lemma 6. $\mathfrak{b} \leq \mathfrak{x}_{\leq,\leq} \leq \mathfrak{x}_{\leq,<} \leq \mathfrak{x}_{<,\leq} \leq \mathfrak{x}_{<,<} \leq \mathfrak{d}$.

Proof. The inequalities $\mathfrak{x}_{\leq,\leq} \leq \mathfrak{x}_{\leq,<}$ and $\mathfrak{x}_{<,\leq} \leq \mathfrak{x}_{<,<}$ are immediate from the definitions. We will prove the other inequalities.

Assume that Y is a bounded subset of ${}^\omega\omega$. Let $g \in {}^\omega\omega$ bound Y. Then g avoids middles in Y with respect to $\langle \leq, \leq \rangle$. This shows that $\mathfrak{b} \leq \mathfrak{x}_{\leq,\leq}$.

Next, consider a subset Y of ${}^\omega\omega$ which satisfies the excluded middle property with respect to $\langle <, < \rangle$, and let g witness that. Then g witnesses that Y is not dominating. Thus $\mathfrak{x}_{<,<} \leq \mathfrak{d}$.

It remains to show that $\mathfrak{x}_{\leq,<} \leq \mathfrak{x}_{<,\leq}$. Assume that $Y \subseteq {}^\omega\omega$ satisfies the excluded middle property with respect to $\langle \leq, < \rangle$, and let $g \in {}^\omega\omega$ avoid middles in Y with respect to $\langle \leq, < \rangle$. Define $\tilde{g} \in {}^\omega\omega$ such that $\tilde{g}(n) = g(n) + 1$ for each n. For each $f, h \in Y$ we have that $[f \leq g] = [f < \tilde{g}]$, and $[f \leq g < h] = [f < \tilde{g} \leq h]$. Therefore, $\tilde{g}$ avoids middles in Y with respect to $\langle <, \leq \rangle$. □

Theorem 7. $\mathfrak{x}_{\leq,\leq} = \mathfrak{x}_{\leq,<} = \mathfrak{b}$.

Proof. By Lemma 6, it is enough to show that $\mathfrak{x}_{\leq,<} \leq \mathfrak{b}$. Let $\langle b_\alpha : \alpha < \mathfrak{b} \rangle$ be an unbounded subset of ${}^\omega\omega$. For each $\alpha < \mathfrak{b}$ define $b^0_\alpha, b^1_\alpha \in {}^\omega\omega$ by

$$\begin{cases} b^0_\alpha(2n) & = b_\alpha(n) \\ b^0_\alpha(2n+1) & = 0 \end{cases} ; \quad \begin{cases} b^1_\alpha(2n) & = 0 \\ b^1_\alpha(2n+1) & = b_\alpha(n) \end{cases}$$

for each $n \in \omega$, and set $Y = \{b^0_\alpha, b^1_\alpha : \alpha < \mathfrak{b}\}$. Then $|Y| = \mathfrak{b}$. We will show that Y does not satisfy the excluded middle property with respect to $\langle \leq, < \rangle$. For each $g \in {}^\omega\omega$, let $\alpha < \mathfrak{b}$ be such that $\max\{g(2n), g(2n+1)\} < b_\alpha(n)$ for infinitely many n. Then

$$\begin{aligned} [b^0_\alpha \leq g < b^1_\alpha] &= \{n : b^0_\alpha(n) \leq g(n) < b^1_\alpha(n)\} \\ &\supseteq \{2n+1 : 0 \leq g(2n+1) < b_\alpha(n)\} \end{aligned}$$

is an infinite set. Similarly, $[b^1_\alpha \leq g < b^0_\alpha] \supseteq \{2n : 0 \leq g(2n) < b_\alpha(n)\}$ is also infinite. That is, g does not avoid middles in Y with respect to $\langle \leq, < \rangle$. $\square$

Lemma 8. $\mathfrak{s} \leq \mathfrak{x}_{<,\leq}$.

Proof. Assume that $Y \subseteq {}^\omega\omega$ is such that $|Y| < \mathfrak{s}$. Let $\mathcal{F} \subseteq P(\omega)$ be the family of all sets of the form $[f < h]$, where $f, h \in Y$. $|\mathcal{F}| < \mathfrak{s}$, thus there exists an infinite subset A of ω such that for each $X \in \mathcal{F}$, either $A \cap X$ is finite, or $A \setminus X$ is finite. As $|Y| < \mathfrak{s} \leq \mathfrak{d}$, there exists $g \in {}^\omega\omega$ such that for each $f \in Y$, $g \restriction A \not\leq^* f \restriction A$. (In particular, $[f < g]$ is infinite for each $f \in Y$.) We may assume that for $n \notin A$, $g(n) = 0$.

Consider any set $[f < h] \in \mathcal{F}$. If $A \cap [f < h]$ is finite, then the set

$$\begin{aligned} [f < g \leq h] &\subseteq \{n : 0 < g(n), f(n) < h(n)\} \\ &\subseteq \{n \in A : f(n) < h(n)\} = A \cap [f < h] \end{aligned}$$

is finite. Otherwise, $A \setminus [f < h]$ is finite, so we get similarly that

$$\begin{aligned} [h < g \leq f] &\subseteq \{n \in A : h(n) < f(n)\} \\ &\subseteq \{n \in A : h(n) \leq f(n)\} = A \setminus [f < h] \end{aligned}$$

is finite. Thus Y satisfies the excluded middle property with respect to $\langle <, \leq \rangle$. $\square$

Theorem 9. $\mathfrak{x}_{<,\leq} = \mathfrak{x}_{<,<} = \max\{\mathfrak{s}, \mathfrak{b}\}$.

Proof. By Lemmas 6 and 8, we have that $\max\{\mathfrak{s}, \mathfrak{b}\} \leq \mathfrak{x}_{<,\leq} \leq \mathfrak{x}_{<,<}$. We will prove that $\mathfrak{x}_{<,<} \leq \max\{\mathfrak{s}, \mathfrak{b}\}$. The argument is an extension of the proof of Theorem 7.

Let $\mathfrak{b}^*$ be the minimal size of a subset B of ${}^\omega\omega$ such that B is unbounded on each infinite subset of ω. According to [3], $\mathfrak{b} = \mathfrak{b}^*$. Thus there exists a subset $B = \langle b_\alpha : \alpha < \mathfrak{b} \rangle$ of ${}^\omega\omega$ such that B is increasing with respect to $\leq^*$ and unbounded on each infinite subset of ω. Let $\mathcal{S} = \langle \mathcal{S}_\alpha : \alpha < \mathfrak{s} \rangle \subseteq [\omega]^\omega$ be a splitting family. For each $\alpha < \mathfrak{s}$ and $\beta < \mathfrak{b}$ define $b^0_{\alpha,\beta}, b^1_{\alpha,\beta} \in {}^\omega\omega$ by

$$b^0_{\alpha,\beta}(n) = \begin{cases} b_\beta(n) & n \in S_\alpha \\ 0 & n \notin S_\alpha \end{cases}; \quad b^1_{\alpha,\beta}(n) = \begin{cases} 0 & n \in S_\alpha \\ b_\beta(n) & n \notin S_\alpha \end{cases}$$

and set $Y = \{b^i_{\alpha,\beta} : i < 2, \alpha < \mathfrak{s}, \beta < \mathfrak{b}\}$. Then $|Y| = 2 \cdot \mathfrak{s} \cdot \mathfrak{b} = \max\{\mathfrak{s}, \mathfrak{b}\}$. We will show that Y does not satisfy the excluded middle property with respect to $\langle <, < \rangle$. Assume that $g \in {}^\omega\omega$ avoids middles in Y with respect to $\langle <, < \rangle$. Then the set $A = [0 < g]$ is infinite; thus there exists $\alpha < \mathfrak{s}$ such that the sets $A \cap S_\alpha$ and $A \setminus S_\alpha$ are infinite. Pick $\gamma < \mathfrak{b}$ such that

$b_\gamma \restriction A \cap S_\alpha \not\leq^* g \restriction A \cap S_\alpha$, and $\beta > \gamma$ such that $b_\beta \restriction A \setminus S_\alpha \not\leq^* g \restriction A \setminus S_\alpha$. Then

$$\begin{aligned}[b^0_{\alpha,\beta} < g < b^1_{\alpha,\beta}] &\supseteq \{n \in A \setminus S_\alpha : b^0_{\alpha,\beta}(n) < g(n) < b^1_{\alpha,\beta}(n)\} \\ &= \{n \in A \setminus S_\alpha : 0 < g(n) < b_\beta(n)\} \\ &= \{n \in A \setminus S_\alpha : g(n) < b_\beta(n)\}\end{aligned}$$

is an infinite set. Similarly, the set

$$\begin{aligned}[b^1_{\alpha,\beta} < g < b^0_{\alpha,\beta}] &\supseteq \{n \in A \cap S_\alpha : b^1_{\alpha,\beta}(n) < g(n) < b^0_{\alpha,\beta}(n)\} \\ &= \{n \in A \cap S_\alpha : 0 < g(n) < b_\beta(n)\} \\ &= \{n \in A \cap S_\alpha : g(n) < b_\beta(n)\}\end{aligned}$$

is also infinite, because $b_\gamma \leq^* b_\beta$; a contradiction. □

Remark 10. The cardinal $\max\{\mathfrak{s}, \mathfrak{b}\}$ is also equal to the *finitely splitting number* $\mathfrak{fs}$ studied in [8].

Several variations of the excluded middle property are studied in the appendix to the online version of this paper [14].

4. Additivity of combinatorial properties

The additivity number $\mathsf{add}(\mathfrak{I}, \mathfrak{J})$ is monotone decreasing in the first coordinate and increasing in the second. Our task in this section is to determine, when possible, the cardinals in the following diagram in terms of the usual cardinal characteristics $\mathfrak{b}$, $\mathfrak{d}$, etc. (In this diagram, an arrow means $\leq$.)

$$\begin{array}{ccccccc}
\mathsf{add}(\mathfrak{D},\mathfrak{D}) & \to & \mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D}) & \to & \mathsf{add}(\mathfrak{X},\mathfrak{D}) & \to & \mathsf{add}(\mathfrak{B},\mathfrak{D}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D}_{\text{fin}}) & \to & \mathsf{add}(\mathfrak{X},\mathfrak{D}_{\text{fin}}) & \to & \mathsf{add}(\mathfrak{B},\mathfrak{D}_{\text{fin}}) \\
 & & & & \uparrow & & \uparrow \\
 & & & & \mathsf{add}(\mathfrak{X},\mathfrak{X}) & \to & \mathsf{add}(\mathfrak{B},\mathfrak{X}) \\
 & & & & & & \uparrow \\
 & & & & & & \mathsf{add}(\mathfrak{B},\mathfrak{B})
\end{array}$$

4.1. Results in ZFC.

Theorem 11. *The following equalities hold:*

1. $\mathsf{add}(\mathfrak{B}, \mathfrak{D}_{\text{fin}}) = \mathsf{add}(\mathfrak{B}, \mathfrak{D}) = \mathfrak{d}$*;*
2. $\mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D}_{\text{fin}}) = \mathsf{add}(\mathfrak{X}, \mathfrak{X}) = \mathsf{add}(\mathfrak{X}, \mathfrak{D}_{\text{fin}}) = 2$*; and*
3. $\mathsf{add}(\mathfrak{D}, \mathfrak{D}) = \mathsf{add}(\mathfrak{B}, \mathfrak{B}) = \mathsf{add}(\mathfrak{B}, \mathfrak{X}) = \mathfrak{b}$*.*

Proof. (1) As $\mathsf{non}(\mathfrak{D}) = \mathfrak{d}$, it is enough to show that $\mathsf{add}(\mathfrak{B}, \mathfrak{D}_{\text{fin}}) \geq \mathfrak{d}$. Assume that $|I| < \mathfrak{d}$, and that $Y = \bigcup_{i\in I} Y_i$ where each Y_i is bounded. For each $i \in I$ let g_i bound Y_i. As $|I| < \mathfrak{d}$, the family $\operatorname{maxfin}(\{g_i : i \in I\})$ is not dominating; let h be a witness for that. For each finite $F \subseteq Y$, let $\tilde{I}$ be a finite subset of I such that $F \subseteq \bigcup_{i\in\tilde{I}} Y_i$. Then $\max(F) \leq^* \max(\{g_i : i \in \tilde{I}\}) \not\geq^* h$. Thus $\max(F) \not\geq^* h$, so $Y \in \mathfrak{D}_{\text{fin}}$.

(2) It is enough to show that $\mathsf{add}(\mathfrak{X}, \mathfrak{D}_{\text{fin}}) = 2$. Thus, let

$$\begin{aligned} Y_0 &= \{f \in {}^\omega\omega : (\forall n) f(2n) = 0 \text{ and } f(2n+1) \geq 1\} \\ Y_1 &= \{f \in {}^\omega\omega : (\forall n) f(2n) \geq 1 \text{ and } f(2n+1) = 0\}. \end{aligned}$$

Then the constant function $g \equiv 1$ witnesses that $Y_0, Y_1 \in \mathfrak{X}$, but $Y_0 \cup Y_1$ is 2-dominating, and in particular finitely dominating.

(3) It is folklore that $\mathsf{add}(\mathfrak{D}, \mathfrak{D}) = \mathsf{add}(\mathfrak{B}, \mathfrak{B}) = \mathfrak{b}$ – see, e.g., [2, full version] for a proof. It remains to show that $\mathsf{add}(\mathfrak{B}, \mathfrak{X}) \leq \mathfrak{b}$. Let B be a subset of ${}^\omega\omega$ which is unbounded on each infinite subset of ω, and such that $|B| = \mathfrak{b}$. For each $f \in B$ let $Y_f = \{g \in {}^\omega\omega : g \leq^* f\}$. (Thus each Y_f is bounded.) We claim that $Y = \bigcup_{f\in B} Y_f \notin \mathfrak{X}$. To this end, consider any function $g \in {}^\omega\omega$ which claims to witness that $Y \in \mathfrak{X}$. In particular, $[0 < g]$ must be infinite. Choose $f \in B$ such that $f \restriction [0 < g] \not\leq^* g \restriction [0 < g]$, that is, $[0 < g < f]$ is infinite. Let A_0, A_1 be a partition of $[0 < g < f]$ into two infinite sets, and define $f_0 \in Y_f$ by $f_0(n) = g(n)$ when $n \in A_0$ and 0 otherwise; similarly define $f_1 \in Y_f$ by $f_1(n) = g(n)$ when $n \in A_1$ and 0 otherwise. Then $f_0, f_1 \in Y$, but both of the sets $[f_0 < g \leq f_1]$ and $[f_1 < g \leq f_0]$ are infinite. $\square$

4.2. Consistency results.

The only cases which we have not solved yet are $\mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})$ and $\mathsf{add}(\mathfrak{X}, \mathfrak{D})$. In [2, full version] it was proved that $\mathfrak{b} \leq \mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})$. In Theorem 2.2 of [10] it is (implicitly) proved that $\mathfrak{g} \leq \mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})$. Thus

$$\max\{\mathfrak{b}, \mathfrak{g}\} \leq \mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D}) \leq \mathsf{add}(\mathfrak{X}, \mathfrak{D}) \leq \mathfrak{d}.$$

Moreover, for any $\mathfrak{I} \subseteq \mathfrak{J}$, $\mathsf{cf}(\mathsf{add}(\mathfrak{I}, \mathfrak{J})) \geq \mathsf{add}(\mathfrak{J})$, and therefore

$$\mathsf{cf}(\mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})), \mathsf{cf}(\mathsf{add}(\mathfrak{X}, \mathfrak{D})) \geq \mathsf{add}(\mathfrak{D}, \mathfrak{D}) = \mathfrak{b}.$$

The notion of ultrafilter will be used to obtain upper bounds on $\mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})$ and $\mathsf{add}(\mathfrak{X}, \mathfrak{D})$. A family $\mathcal{U} \subseteq [\omega]^\omega$ is a *nonprincipal ultrafilter* if it is closed under taking supersets and finite intersections, and cannot be extended, that is, for each infinite $A \subseteq \omega$, either $A \in \mathcal{U}$ or $\omega \setminus A \in \mathcal{U}$. Consequently, a linear quasiorder $\leq_{\mathcal{U}}$ can be defined on ${}^\omega\omega$ by

$$f \leq_{\mathcal{U}} g \qquad \text{if} \qquad [f \leq g] \in \mathcal{U}.$$

The *cofinality* of the reduced product ${}^\omega\omega/\mathcal{U}$ is the minimal size of a subset C of ${}^\omega\omega$ which is cofinal in ${}^\omega\omega$ with respect to $\leq_\mathcal{U}$.

Theorem 12. *For each cardinal number κ, the following are equivalent:*

1. $\kappa < \mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})$;
2. *for each κ-sequence $\langle(g_\alpha, \mathcal{U}_\alpha) : \alpha < \kappa\rangle$ with each $\mathcal{U}_\alpha$ an ultrafilter on ω and each $g_\alpha \in {}^\omega\omega$ there exists $g \in {}^\omega\omega$ such that for each $\alpha < \kappa$, $[g_\alpha \leq g] \in \mathcal{U}_\alpha$.*

Proof. $1 \Rightarrow 2$: For each $\alpha < \kappa$ let $Y_\alpha = \{f \in {}^\omega\omega : [f < g_\alpha] \in \mathcal{U}_\alpha\}$. Then each $Y_\alpha \in \mathfrak{D}_{\text{fin}}$, thus by (1) $Y = \bigcup_{\alpha<\kappa} Y_\alpha$ is not dominating. Let $g \in {}^\omega\omega$ be a witness for that. In particular, for each α $g \notin Y_\alpha$, that is, $[g < g_\alpha] \notin \mathcal{U}_\alpha$. As $\mathcal{U}_\alpha$ is an ultrafilter, we have that $[g_\alpha \leq g] = \omega \setminus [g < g_\alpha] \in \mathcal{U}_\alpha$.

$2 \Rightarrow 1$: Assume that $Y = \bigcup_{\alpha<\kappa} Y_\alpha$ where each $Y_\alpha \in \mathfrak{D}_{\text{fin}}$. For each α, let $\mathcal{U}_\alpha$ be an ultrafilter such that $Y_\alpha/\mathcal{U}_\alpha$ is bounded, say by $g_\alpha \in {}^\omega\omega$ [13]. By (2) let $g \in {}^\omega\omega$ be such that for each $\alpha < \kappa$, $[g_\alpha \leq g] \in \mathcal{U}_\alpha$. Then g witnesses that Y is not dominating: For each $f \in Y$, let α be such that $f \in Y_\alpha$. Then $[f \leq g_\alpha] \in \mathcal{U}_\alpha$, thus $[f < g] \supseteq [f < g_\alpha] \cap [g_\alpha \leq g] \in \mathcal{U}_\alpha$; therefore $[f < g]$ is infinite. □

Corollary 13. *Assume that $\mathcal{U}$ is a nonprincipal ultrafilter on ω. Then $\mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D}) \leq \mathsf{cof}({}^\omega\omega/\mathcal{U})$.*

Proof. Assume that $\kappa < \mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D})$ and let $\langle g_\alpha : \alpha < \kappa\rangle$ be any κ-sequence of elements of ${}^\omega\omega$. For each α set $\mathcal{U}_\alpha = \mathcal{U}$. Then by Theorem 12 there exists $g \in {}^\omega\omega$ such that for each α, $[g_\alpha \leq g] \in \mathcal{U}_\alpha = \mathcal{U}$. Thus $\langle g_\alpha : \alpha < \kappa\rangle$ is not cofinal in ${}^\omega\omega/\mathcal{U}$. □

Corollary 14. $\mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D}) \leq \mathsf{cf}(\mathfrak{d})$.

Proof. Canjar [7] proved that there exists a nonprincipal ultrafilter $\mathcal{U}$ with $\mathsf{cof}({}^\omega\omega/\mathcal{U}) = \mathsf{cf}(\mathfrak{d})$. Now use Corollary 13. □

Lemma 15. *$g \in {}^\omega\omega$ avoids middles in Y if, and only if, for each $f \in Y$ $[f < g]$ is infinite, and the family $\{[f < g] : f \in Y\}$ is linearly quasiordered by $\subseteq^*$.*

Theorem 16. *For any cardinal κ, the following are equivalent:*

1. $\kappa < \mathsf{add}(\mathfrak{X}, \mathfrak{D})$;

2. *for each κ-sequence $\langle(g_\alpha,\mathcal{F}_\alpha):\alpha<\kappa\rangle$, such that each $g_\alpha\in{}^\omega\omega$, and for each α the restriction $\mathcal{F}_\alpha\restriction[0<g_\alpha]$ is large and linearly quasiordered by $\subseteq^*$, there exists $h\in{}^\omega\omega$ such that for each $\alpha<\kappa$, the restriction $\mathcal{F}_\alpha\restriction[g_\alpha\le h]$ is large.*

Proof. $2\Rightarrow 1$: Assume that $Y=\bigcup_{\alpha<\kappa}Y_\alpha$ where each $Y_\alpha\in\mathfrak{X}$. For each α let $g_\alpha\in{}^\omega\omega$ be a function avoiding middles in Y_α, and set $\mathcal{F}_\alpha=\{[f<g_\alpha]:f\in Y_\alpha\}$. By Lemma 15, $\mathcal{F}_\alpha\subseteq[\omega]^\omega$ is linearly quasiordered by $\subseteq^*$. As $\mathcal{F}_\alpha\restriction[0<g_\alpha]=\mathcal{F}_\alpha$, the restriction is large and linearly quasiordered by $\subseteq^*$. By the assumption (2), there exists $h\in{}^\omega\omega$ such that for each $\alpha<\kappa$ and each $f\in Y_\alpha$, $[f<g_\alpha]\cap[g_\alpha\le h]$ is infinite; therefore $h\not\le^* f$. Thus h witnesses that $Y\in\mathfrak{D}$.

$1\Rightarrow 2$: Replacing each $\mathcal{F}_\alpha$ with $\mathcal{F}_\alpha\restriction[0<g_\alpha]$, we may assume that each $A\in\mathcal{F}_\alpha$ is an infinite subset of $[0<g_\alpha]$.

For each $\alpha<\kappa$ let

$$Y_\alpha=\{f\in{}^\omega\omega:[f<g_\alpha]\in\mathcal{F}_\alpha\}.$$

For each $A\in\mathcal{F}_\alpha$ and each $h\in{}^\omega\omega$, define

$$\tilde{h}(n)=\begin{cases}g_\alpha(n)-1 & n\in A\\ \max\{g_\alpha(n),h(n)\} & \text{otherwise.}\end{cases}\tag{4.1}$$

Then $[\tilde{h}<g_\alpha]=A$, and $[\tilde{h}<h]\subseteq A$. Thus, for each α,

$$\mathcal{F}_\alpha=\{[h<g_\alpha]:h\in Y_\alpha\}\subseteq[\omega]^\omega.$$

As $\mathcal{F}_\alpha$ is linearly quasiordered by $\subseteq^*$, we have by Lemma 15 that g_α avoids middles in Y_α. By (1), $Y=\bigcup_{\alpha<\kappa}Y_\alpha$ is not dominating; let $h\in{}^\omega\omega$ be a witness for that.

For each $\alpha<\kappa$ and $A\in\mathcal{F}_\alpha$, let $\tilde{h}\in Y_\alpha$ be the function defined in Equation (4.1). Then $\tilde{h}\in Y$, therefore $[\tilde{h}<h]$ is infinite. By the definition of $\tilde{h}$, $[\tilde{h}<h]\subseteq A\cap[g_\alpha\le h]$; therefore the restriction $\mathcal{F}_\alpha\restriction[g_\alpha\le h]$ is large. $\square$

A nonprincipal ultrafilter $\mathcal{U}$ is a *simple P_κ point* if it is generated by a κ-sequence $\langle A_\alpha:\alpha<\kappa\rangle\subseteq[\omega]^\omega$ which is decreasing with respect to $\subseteq^*$. $\mathcal{U}$ is a *pseudo-P_κ point* if every family $\mathcal{F}\subseteq\mathcal{U}$ with $|\mathcal{F}|<\kappa$ has a pseudo-intersection. Clearly every simple P_κ point is a pseudo-P_κ point.

Corollary 17. *If $\mathcal{U}$ is a simple P_κ point, then $\mathsf{add}(\mathfrak{X},\mathfrak{D})\le\mathsf{cof}({}^\omega\omega/\mathcal{U})$.*

Proof. Assume that $\lambda<\mathsf{add}(\mathfrak{X},\mathfrak{D})$. Let $\langle A_\beta:\beta<\kappa\rangle\subseteq[\omega]^\omega$ be a κ-sequence which generates $\mathcal{U}$ and is linearly quasiordered by $\subseteq^*$, and set $\mathcal{F}_\alpha=\mathcal{F}=\{A_\beta:\beta<\kappa\}$ for all $\alpha<\lambda$. Assume that $g_\alpha\in{}^\omega\omega$, $\alpha<\lambda$, are given. We will show that these functions g_α are not cofinal in ${}^\omega\omega/\mathcal{U}$.

We may assume that for each $\alpha < \lambda$, $[0 < g_\alpha] = \omega$. Use Theorem 16 to obtain a function $h \in {}^\omega\omega$ such that for each $\alpha < \lambda$, the restriction $\mathcal{F} \restriction [g_\alpha \leq h]$ is large. Assume that for some $\alpha < \lambda$, $[g_\alpha \leq h] \notin \mathcal{U}$. Then $[h < g_\alpha] \in \mathcal{U}$, thus there exists $\beta < \kappa$ such that $A_\beta \subseteq^* [h < g_\alpha]$, therefore $A_\beta \cap [g_\alpha \leq h]$ is finite, a contradiction. Thus $h+1$ witnesses that the functions g_α are not cofinal in ${}^\omega\omega/\mathcal{U}$, therefore $\lambda < \mathsf{cof}({}^\omega\omega/\mathcal{U})$. □

In the remaining part of the paper we will consider the remaining standard cardinal characteristics of the continuum (see [3]). Let $\mathfrak{u}$ denote the minimal size of an ultrafilter base.

Theorem 18. *It is consistent (relative to ZFC) that the following holds:*

$$\mathfrak{u} = \mathsf{add}(\mathfrak{D}_{\text{fin}}, \mathfrak{D}) = \mathsf{add}(\mathfrak{X}, \mathfrak{D}) = \aleph_1 < \aleph_2 = \mathfrak{s} = \mathfrak{c}.$$

Thus, it is not provable that $\mathfrak{s} \leq \mathsf{add}(\mathfrak{X}, \mathfrak{D})$.

Proof. In [5] a model of set theory is constructed where $\mathfrak{c} = \aleph_2$ and there exist a simple $P_{\aleph_1}$ point and a simple $P_{\aleph_2}$ point. The simple $P_{\aleph_1}$ point is generated by $\aleph_1$ many sets, thus $\mathfrak{u} = \aleph_1$. As $\mathfrak{b} \leq \mathfrak{u}$, $\mathfrak{b} = \aleph_1$ as well.

Nyikos proved that if there exists a pseudo P_κ point $\mathcal{U}$ and $\kappa > \mathfrak{b}$, then $\mathsf{cof}({}^\omega\omega/\mathcal{U}) = \mathfrak{b}$ (see [4]). Thus by Corollary 17, $\mathsf{add}(\mathfrak{X}, \mathfrak{D}) \leq \mathfrak{b} = \aleph_1$ in this model. In [4] it is proved that if there exists a pseudo P_κ point $\mathcal{U}$, then $\mathfrak{s} \geq \kappa$. Therefore $\mathfrak{s} \geq \aleph_2$ in this model. □

$\text{Depth}^+([\omega]^\omega)$ is defined as the minimal cardinal κ such that there exists no $\subset^*$-decreasing κ-sequence in $[\omega]^\omega$. (Thus, e.g., $\mathfrak{t} < \text{Depth}^+([\omega]^\omega)$.) Each linearly quasiordered family $\mathcal{F} \subseteq [\omega]^\omega$ has a cofinal subfamily which forms a $\subset^*$-decreasing sequence of length $< \text{Depth}^+([\omega]^\omega)$.

Theorem 19.

1. *If* $\text{Depth}^+([\omega]^\omega) < \mathfrak{d}$, *then* $\mathsf{add}(\mathfrak{X}, \mathfrak{D}) = \mathfrak{d}$.
2. *If* $\text{Depth}^+([\omega]^\omega) = \mathfrak{d}$, *then* $\mathsf{cf}(\mathfrak{d}) \leq \mathsf{add}(\mathfrak{X}, \mathfrak{D})$.

Proof. To prove (1) it is enough to show that for each κ satisfying $\text{Depth}^+([\omega]^\omega) \leq \kappa < \mathfrak{d}$, we have that $\kappa < \mathsf{add}(\mathfrak{X}, \mathfrak{D})$. To prove (2) we will show that for each $\kappa < \mathsf{cf}(\mathfrak{d})$, $\kappa < \mathsf{add}(\mathfrak{X}, \mathfrak{D})$. We will use Theorem 16, and prove both cases simultaneously.

Assume that $\text{Depth}^+([\omega]^\omega) \leq \kappa < \mathfrak{d}$ (respectively, $\kappa < \mathsf{cf}(\mathfrak{d})$). Consider any κ-sequence $\langle (g_\alpha, \mathcal{F}_\alpha) : \alpha < \kappa \rangle$ where each $g_\alpha \in {}^\omega\omega$, each $\mathcal{F}_\alpha \subseteq [\omega]^\omega$ is linearly quasiordered by $\subseteq^*$, and the restriction $\mathcal{F}_\alpha \restriction [0 < g_\alpha]$ is large. We must show that there exists $h \in {}^\omega\omega$ such that for each $\alpha < \kappa$, the restriction $\mathcal{F}_\alpha \restriction [g_\alpha < h]$ is large.

Use the fact that $\mathrm{Depth}^+([\omega]^\omega) \le \kappa$ (respectively, $\mathrm{Depth}^+([\omega]^\omega) = \mathfrak{d}$) to choose for each $\alpha < \kappa$ a cofinal subfamily $\tilde{\mathcal{F}}_\alpha$ of $\mathcal{F}_\alpha$ such that $|\tilde{\mathcal{F}}_\alpha| < \kappa$ (respectively, $|\tilde{\mathcal{F}}_\alpha| < \mathfrak{d}$).

We may assume that each g_α is increasing. For each α and each $A \in \mathcal{F}_\alpha$, let $\vec{A} \in {}^\omega\omega$ be the increasing enumeration of A. The collection $\{g_\alpha \circ \vec{A} : \alpha < \kappa,\ A \in \mathcal{F}_\alpha\}$ has less than $\mathfrak{d}$ many elements and therefore cannot be dominating. Let $h \in {}^\omega\omega$ be a witness for that. Fix $\alpha < \kappa$. For all $A \in \mathcal{F}_\alpha$, there exist infinitely many n such that

$$g_\alpha(\vec{A}(n)) = g_\alpha \circ \vec{A}(n) < h(n) \le h(\vec{A}(n)),$$

that is, $A \cap [g_\alpha < h]$ is infinite. □

Theorem 20. *Assume that V is a model of CH and $\aleph_1 < \kappa = \kappa^{\aleph_0}$. Let $\mathbb{C}_\kappa$ be the forcing notion which adjoins κ many Cohen reals to V. Then in the Cohen model $V^{\mathbb{C}_\kappa}$, the following holds:*

$$\mathsf{add}(\mathfrak{D}_{\mathrm{fin}}, \mathfrak{D}) = \mathfrak{s} = \mathfrak{a} = \mathsf{non}(\mathcal{M}) = \aleph_1 < \mathsf{cov}(\mathcal{M}) = \mathsf{add}(\mathfrak{X}, \mathfrak{D}) = \mathfrak{c}.$$

Proof. The assertions $\mathfrak{s} = \mathfrak{a} = \mathsf{non}(\mathcal{M}) = \aleph_1 < \mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ are well-known to hold in $V^{\mathbb{C}_\kappa}$, see [3]. It was proved by Kunen [9] that $V^{\mathbb{C}_\kappa} \models \mathrm{Depth}^+([\omega]^\omega) = \aleph_2$. As $\mathsf{cov}(\mathcal{M}) \le \mathfrak{d}$, we have that $\mathfrak{d} = \mathfrak{c} = \kappa$ in this model. If $\kappa = \aleph_2$, use Theorem 19(1) and the fact that $\mathfrak{d}$ is regular in this model to obtain $\mathfrak{d} \le \mathsf{add}(\mathfrak{X}, \mathfrak{D})$. Otherwise use Theorem 19(2) and the fact that $\mathrm{Depth}^+([\omega]^\omega) = \aleph_2 < \kappa = \mathfrak{d}$ to obtain this.

In [6, 11] it is proved that there exists a nonprincipal ultrafilter $\mathcal{U}$ in $V^{\mathbb{C}_\kappa}$ such that $\mathsf{cof}({}^\omega\omega/\mathcal{U}) = \aleph_1$. By Corollary 13, we have that $\mathsf{add}(\mathfrak{D}_{\mathrm{fin}}, \mathfrak{D}) = \aleph_1$ in $V^{\mathbb{C}_\kappa}$. □

In particular, the cardinals $\mathsf{add}(\mathfrak{D}_{\mathrm{fin}}, \mathfrak{D})$ and $\mathsf{add}(\mathfrak{X}, \mathfrak{D})$ are not provably equal.

Corollary 21. *It is not provable that* $\mathsf{add}(\mathfrak{X}, \mathfrak{D}) \le \mathsf{cf}(\mathfrak{d})$.

Proof. Use Theorem 20 with $\kappa = \aleph_{\aleph_1}$. In $V^{\mathbb{C}_\kappa}$, $\mathfrak{d} = \mathfrak{c} = \aleph_{\aleph_1}$, therefore $\mathsf{cf}(\mathfrak{d}) = \aleph_1 < \mathsf{add}(\mathfrak{X}, \mathfrak{D})$ in this model. □

Remark 22. In the remaining canonical models of set theory which are used to distinguish between the various cardinal characteristics of the continuum (see [3]), $\max\{\mathfrak{b}, \mathfrak{g}\} = \mathfrak{d}$ holds, and therefore $\mathsf{add}(\mathfrak{D}_{\mathrm{fin}}, \mathfrak{D}) = \mathsf{add}(\mathfrak{X}, \mathfrak{D}) =$

$\mathfrak{d}$ too. These models show that none of the following is provable:

$$\min\{\mathsf{cov}(\mathcal{N}),\mathfrak{r}\} \le \mathsf{add}(\mathfrak{X},\mathfrak{D}) \text{ (\textit{Random reals} model)},$$
$$\mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D}) \le \max\{\mathsf{cov}(\mathcal{N}),\mathfrak{s}\} \text{ (\textit{Hechler reals} model)},$$
$$\mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D}) \le \max\{\mathsf{non}(\mathcal{N}),\mathsf{cov}(\mathcal{N})\} \text{ (\textit{Laver reals} model), and}$$
$$\mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D}) \le \max\{\mathfrak{u},\ \mathfrak{a},\mathsf{non}(\mathcal{N}),\mathsf{non}(\mathcal{M})\} \text{ (\textit{Miller reals} model)}.$$

Collecting all of the consistency results, we get that the only possible additional lower bounds on $\mathsf{add}(\mathfrak{X},\mathfrak{D})$ are $\mathsf{cov}(\mathcal{M})$ and $\mathfrak{e}$ (observe that $\mathfrak{e} \le \mathsf{cov}(\mathcal{M})$ [3].)

Problem 23. *Is* $\mathsf{cov}(\mathcal{M}) \le \mathsf{add}(\mathfrak{X},\mathfrak{D})$*? And if not, is* $\mathfrak{e} \le \mathsf{add}(\mathfrak{X},\mathfrak{D})$*?*

No additional cardinal characteristic can serve as an upper bound on $\mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D})$.

Another question of interest is whether $\mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D})$ or $\mathsf{add}(\mathfrak{X},\mathfrak{D})$ appear in the lattice generated by the cardinal characteristics with the operations of maximum and minimum. In particular, we have the following.

Problem 24. *Is it provable that* $\mathsf{add}(\mathfrak{D}_{\text{fin}},\mathfrak{D}) = \max\{\mathfrak{b},\mathfrak{g}\}$*?*

We have an indication that the answer to Problem 24 is negative, but this is a delicate matter which will be treated in a future work.

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