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FINITE DIAGRAMS STABLE IN POWER *

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In this article we define when a finite diagram of a model is stable, we investigate what is the form of the class of powers in which a finite diagram is stable, and we generalize some properties of totally transcendental theories to stable finite diagrams. Using these results we investigate several theories which have only homogeneous models in a certain power We also investigate when there exist models of a certain diagram which are λ-homogeneous and not λ^+-homogeneous in various powers. We also have new results about stable theories and the existence of maximally λ-saturated models of power μ

§ 0. Introduction

If M is a model, D(M) will be the set of complete types in the variables $x_0, \ldots, x_{n-1}$ for all $n < \omega$ which are realized in M. M is a D-model if D(M) $\subseteq$ D and M is (D, λ)-homogeneous if D(M) = D and M is λ-homogeneous. D is λ-stable if there is a (D, λ^+)-homogeneous model of power $\geq \lambda^+$ in which over every set of power $\leq \lambda$ at most λ complete types are realized. A (first-order complete) theory T is λ-stable if for every $|T|^+$-saturated model M of T D(M) is λ-stable.

Morley [6] investigated $\aleph_0$-stable theories (he called them totally transcendental theories). He proved that these theories have several properties indicating their simplicity – in every model of a $\aleph_0$-stable theory, over every infinite set of power less than the power of the

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model, there is an indiscernible set (set of indiscernibles in Morley's terminology) of fairly large power. Also, over every set included in a model of a $\aleph_0$-stable theory there is a prime model. Morley also showed that a $\aleph_0$-stable theory is stable in all powers. With the help of this last result he showed that a countable theory which is categorical in a power larger than $\aleph_0$ is categorical in every non-denumerable power.

When it was tried to generalize these results to non-countable theories problems arose principally at two points. The first was solved easily in the case of Morley: that is, if T is stable in one power then T is stable in other powers. The second is: it turns out T is not categorical exactly in those powers in which T has a non-$|T|$-saturated model. Thus Hanf numbers come into the picture here. These generalizations were treated by Rowbottom [11], Ressayre [10], and the author [13, 17].

In this article we strengthen and generalize these results to stable finite diagrams.

In § 1 we define our notation.

In § 2 we define when D is λ-stable, and we define the conditions $(*\lambda)$, $(A*\lambda)$, $(B*\lambda)$ such that: $(*\lambda) \Rightarrow (A*\lambda) \Rightarrow (B*\lambda) \Rightarrow$ D isn't stable in any power $< 2^\lambda$; and if there is a (D, $(2^{2^\lambda})^+$)-homogeneous model and D is not 2^{2^λ}-stable then D satisfies $(*\lambda)$ provided that the power of the model is $> 2^{2^\lambda}$.

In § 3 we use the results of § 2 to prove (theorem 3.1) that if D(M) is λ-stable where $\lambda < \|M\|$ and A is a set of elements of M of power $\leq \lambda$, then in M there is an indiscernible set over A of power $> \lambda$. We also prove (in 3.4) that if D is stable then for every cardinal μ there is a (D, μ)-homogeneous model of power $\geq \mu$.

In § 4 we try to characterize the class of powers in which D is stable. Our conclusion is (Theorem 4.4):

If D is stable then there are cardinals λ, κ such that D is μ-stable iff $\mu \geq \lambda$, $\mu^{(\kappa)} = \mu$. Also $\lambda < \beth[(2^{|T|})^+]$. (For stable theories $\lambda \leq 2^{|T|}$, $\kappa \leq |T|^+$; so this theorem solves the problem almost completely. For stable diagrams it is fair to assume that it is possible to improve the bound on λ).

In § 5 we define prime models over sets in a number of ways and prove several theorems about them, including their existence in certain cases.

In §6 we prove the existence of non-homogeneous models in certain cases For theories we can conclude from 6.3, 6.9 that

Theorem: *If T has a model of power $> \lambda^{|T|}$ which is $|T|^+$-saturated but not λ^+-saturated, then for every regular cardinal μ, T has models of arbitrarily high power which are μ-saturated but not μ^+-saturated.*

This theorem almost completely answers problem 4A from Keisler [3].

In §7 we solve problem D and partially solve problem C from Keisler [2]. The principle results are:

1) *If all the models of T of power $\lambda > |T|$ are homogeneous then there is μ_0 with $|T| \leq \mu_0 < \mu(T)$ such that all the models of T of power $\geq \mu_0$ are homogeneous and in every power κ with $|T| + \aleph_1 \leq \kappa < \mu_0$ T has a model which is not homogeneous* (this is Corollary 7.6).

2) (G.C.H.) *Let* SP(T, P) *be the class of cardinals λ such that every model of T of power λ which omits all the types $\{p: p \in P\}$ is homogeneous. Then if there is $\kappa \in$* SP(T, P) *with $\kappa > |T|$, then there is $\mu_0 < \beth[(2^{|T|})^+]$ such that $\lambda \geq \mu_0$ implies $\lambda \in$* SP(T, P), *and $|T| < \lambda < \mu_0$ implies $\lambda \notin$* SP(T, P) *except for perhaps one λ, when $\lambda = \beth_{\delta+1}$*; or *$\lambda = \beth_\delta$ and $\mu_0 = \lambda^{++}$ (δ a limit ordinal)*. (This is Corollary 7.8.)

3) *If T is a countable theory all of whose models of power $\aleph_1$ are homogeneous, then T is $\aleph_1$-categorical.* (This is Theorem 7 9.)

Abstracts on theorems of this paper were published in the Notices of the A.M.S. [12, 14, 15]. Keisler published an abstract [4] dealing with a theorem similar to result 2). (His hypothesis and result are stronger.)

Let D(T) *be* D(M) *for any $|T|^+$-saturated model M for T. Many of the results about* D(T), *are true for every* D *for which there exists a non-principal ultrafilter E on ω such that* D$(M) \subseteq$ D $\Rightarrow$ D$(M^\omega/E) \subseteq$ D.

Added in proof, 20 August 1970

By Shelah [18] in 1) above, $\mu_0 \leqslant (2^{|T|})^+$, and by an add theorem (7 10) (see p. 117) 2) can be improved

§ 1. Preliminaries

An ordinal is defined as the set of all smaller ordinals and a cardinal, or power, is defined as the first ordinal of its power. We shall use $\alpha, \beta, \gamma, i, j, k, l$ for ordinals, $\kappa, \lambda, \chi, \mu$ for infinite cardinals, m, n for natural numbers, and δ will denote a limit ordinal. If A is a set its power will be denoted by $|A|$. We define by induction $\beth(\lambda, 0) = \lambda$, $\beth(\lambda, \delta) = \bigcup_{i<\delta} \beth(\lambda, i)$, $\beth(\lambda, i+1) = 2^{\beth(\lambda, i)}$, $\beth_\alpha = \beth(\alpha) = \beth(\aleph_0, \alpha)$. $\alpha \cdot \beta$ will denote the product of the ordinals α, β. α divides γ if there is a β such that $\alpha \cdot \beta = \gamma$. Define $\mu^{(\lambda)} = \Sigma\{\mu^\kappa : \kappa < \lambda\}$. The empty set will be denoted by 0 or $\{\}$. $A - B = \{a : a \in A, a \notin B\}$. The domain of a function f will be denoted Dom f and its range Rang f. If f, g are functions then f extends, or is a continuation of, g if Dom $f \supseteq$ Dom g and for all $a \in$ Dom g, $f(a) = g(a)$. If f is one-to-one then f^{-1} will denote the inverse function. fg will be the composition of the two functions. $g = f|A$ if Dom $g = A \cap$ Dom f and f extends g. If $A =$ Dom $f \cap$ Dom g and $f|A = g|A$ then $f \cup g$ is a function which extends f and whose domain is Dom $f \cup$ Dom g. A sequence $\bar{t}$ is a function whose domain is an ordinal which will be called its length and will be denoted $l(\bar{t})$. If A is a set, I_A will be the function whose domain is A and $I_A(a) = a$ for all $a \in A$. If $\bar{t}$ is a sequence, then $\bar{t}_i = \bar{t}(i)$ (= the value of the function at i) The sequence $\bar{t}$ will be denoted and defined sometimes to be $\langle \bar{t}_i : i < l(\bar{t}) \rangle$. We frequently don't distinguish between t_0 and $\langle t_i : i < 1 \rangle$. The sequence $\langle a_i : i < \alpha \rangle$ is increasing with respect to the order $<$ if $i < j$ implies $a_i < a_j$. If the order relation is not specified, we assume that it is the inclusion relation. If $\bar{t}, \bar{s}$ are sequences, then $\bar{u} = \bar{t} \frown \bar{s}$ is defined to be the sequence such that $l(\bar{u}) = l(\bar{t}) + l(\bar{s})$, for $i < l(\bar{t})$, $\bar{u}_i = \bar{t}_i$, and for $l(\bar{t}) \leq i < l(\bar{u})$, $\bar{u}_i = \bar{s}(i - l(\bar{t}))$.

$(A)^n$ will designate the set of all functions from n into A. η, τ will designate sequences of ordinals, and if not specified otherwise, we shall assume that they are sequences of zeroes and ones.

T will be a first-order theory in the language $L = L(T)$ with equality sign. We always assume $|L|, |T| \geq \aleph_0$. We usually assume that T is a fixed complete theory with which we are dealing and, for simplicity, that there are no function symbols in $L(T)$ (actually this entails no loss of generality). x, y, z will designate variables, $\bar{x}, \bar{y}, \bar{z}$ finite sequences of variables, φ, ψ formulas of the language L; we write $\varphi(x_0, ..., x_{n-1})$ for

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φ if all the free variables appearing in φ are among $\{x_0, ..., x_{n-1}\}$. M, N will designate models of T or of other theories, if so specified. If M is a model, $|M|$ will be the set of its members. Thus $\|M\|$ will be its power. $M \models \varphi[a_0, ..., a_{n-1}]$ if $a_0, ..., a_{n-1} \in |M|$ and $\varphi[a_0, ..., a_{n-1}]$ is satisfied in M. If N is a model of T_1, $L(T_1) \supseteq L$ then the reduct of N to L is the model M such that $|M| = |N|$ and for every predicate R in L, $R^M = \{\langle a_0, ..., a_{n-1}\rangle : M \models R[a_0, ..., a_{n-1}]\} = R^N = \{\langle a_0, ..., a_{n-1}\rangle : N \models R[a_0, ..., a_{n-1}]\}$.

The model M is λ-saturated if for every sequence $\langle \varphi_i(x, \bar{y}) : i < i_0 < \lambda\rangle$ of formulas and sequence $\langle \bar{b}_i : i < i_0 < \lambda\rangle$ of sequences of elements of M which satisfy: for every finite subset $I \subseteq i_0$ there is a c such that $i \in I$ implies $M \models \varphi_i[c, \bar{b}_i]$; there is a c such that for all $i < i_0$, $M \models \varphi_i[c, \bar{b}_i]$.

We assume that $\bar{M}$ is a $\bar{\kappa}$ saturated model of T of power $\bar{\kappa}$ where $\bar{\kappa}$ is an inaccessible cardinal. (A proof of the existence of such a model and a general discussion of saturated models can be found in Morley and Vaught [9], where the definition is slightly different.)

M_1 is an elementary submodel of M_2 if $|M_1| \subseteq |M_2|$ and for every formula $\varphi(\bar{x})$ and sequence b of elements of M_1, $M_1 \models \varphi[\bar{b}]$ iff $M_2 \models \varphi[\bar{b}]$. We assume that all the models of T which we define are elementary sub-models of $\bar{M}$ of power $< \bar{\kappa}$. (In set theory $R(\alpha+1)$ is defined by induction to be the set of all sets included in $R(\alpha)$, $R(\delta) = \bigcup_{\beta<\delta} R(\beta)$. It is known that if κ is an inaccessible cardinal, then $R(\kappa)$ is a model of set theory. Thus we can also assume that all the elementary submodels of $\bar{M}$ of power $< \bar{\kappa}$ are in $R(\bar{\kappa})$ where $\bar{\kappa} = \|\bar{M}\|$ – and it is clear that every model of T of power $< \bar{\kappa}$ is isomorphic to such a model. These stipulations are just for convenience and it is easy to see that by a change in notation we could get by without them, with no loss of generality.) Thus it turns out that a model is determined by its set of elements, and so we sometimes don't differentiate between M and $|M|$. It is easy to see that M_1 is an elementary submodel of M_2 iff $|M_1| \subseteq |M_2|$. If M_i, $i < i_0$ is an increasing sequence of models then there is a model M with $|M| = \bigcup_{i<i_0} |M_i|$ which we sometimes denote by $\bigcup_{i<i_0} M_i$. A, B, C will denote sets included in $|\bar{M}|$ of power $< \bar{\kappa}$, $\bar{a}, \bar{b}, \bar{c}$ finite sequences of elements of $|\bar{M}|$. $\bar{a}$ is called a sequence from A (of A)

if all the elements of the sequence belong to A. Instead of $M \models \varphi[\bar{b}]$ we can write $\models \varphi[\bar{b}]$ since the particular model M makes no difference. If A is a set there is a model M with $|M| = A$ iff for every sequence $\bar{b}$ from A and formula $\varphi(x, \bar{y})$, if $\models (\exists x)\varphi(x, \bar{b})$ then there is $a \in A$ such that $\models \varphi[a, \bar{b}]$ (this is the Tarski-Vaught test).

The function F is a mapping if Rang F, Dom $F \subseteq |\bar{M}|$, $|\text{Dom } F| < \|\bar{M}\|$ and for every formula φ and sequence $\langle a_0, ..., a_{n-1}\rangle$, $\models \varphi[a_0, ..., a_{n-1}]$ iff $\models \varphi[F(a_0), ..., F(a_{n-1})]$. (Clearly, a mapping must be one-one.) F, G will denote mappings, $F(A) = \{F(a) : a \in A\}$, $F(\bar{a}) = \langle F(\bar{a}_i) : i < l(\bar{a})\rangle$. From properties of saturated models it is clear that if F is a mapping and A a set, then there is an extension G of F with domain $A \cup$ Dom F.

p is an n-type over A if p is a set of formulas of the form $\varphi(x_0, ..., x_{n-1}, \bar{b})$ where $\bar{b}$ is a sequence from A. p, q, r will denote types over a set A. p extends or continues q if $q \subseteq p$. $\bar{c}$ realizes p if for every $\varphi(\bar{x}, \bar{b}) \in p$ $(\bar{x} = \langle x_0, ..., x_{n-1}\rangle)$, $\models \varphi[\bar{c}, \bar{b}]$. For our purposes "type" will always mean a non-contradictory type, i.e. for every finite subset q of the type, there is a sequence realizing q. (From the definition of $\bar{M}$ it follows that if p is a type, there is a sequence $\bar{c}$ realizing p.) p is a complete n-type over A if for every sequence $\bar{a}$ from A and formula φ, $\varphi(x_0, ..., x_{n-1}, \bar{a}) \in p$ or $\neg\varphi(x_0, ..., x_{n-1}, \bar{a}) \in p$. If not specified otherwise, every type is a 1-type over the empty set. $S^n(A)$ will denote the set of complete n-types over A, $S(A) = S^1(A)$. Every sequence $\langle a_0, ..., a_{n-1}\rangle$ realizes a complete n-type over B which will be called "the type which $\langle a_0, ..., a_{n-1}\rangle$ realizes over B"; clearly, this type belongs to $S^n(B)$. Define $p|A = \{\psi : \psi \in p, \{\psi\}$ is a type over $A\}$, $F(p) = \{\psi(x_0, ..., x_{n-1}, F(\bar{a})) : \psi(x_0, ..., x_{n-1}, \bar{a}) \in p$, $\bar{a}$ is a sequence from Dom $F\}$. Sometimes instead of saying that p is an n-type for some $n < \omega$ we say that p is a finite type.

If A is a set, $D(A)$ is defined to be the set of finite types over 0 which are realized by finite sequences from A; $D(M) = D(|M|)$. M is a D-model (A is a D-set) if $D(M) \subseteq D$ $(D(A) \subseteq D)$. D will always denote sets of the form $D(M)$, and will be called the finite diagram (of M). A type $p \in S^n(A)$ will be called a D-n-type if for all $\langle a_0, ..., a_{n-1}\rangle$ which realize p, $A \cup \{a_0, ..., a_{n-1}\}$ is a D-set. $S^n_D(A)$ will denote the set of complete D-n-types over A, $S_D(A) = S^1_D(A)$. We usually assume that D is fixed, every set is a D-set, and every model is a D-model. In particular if we write $S_D(A)$ it is assumed that A is a D-set (if A is not a D-set, $S^n_D(A)$ is

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clearly empty). D(T) will denote D(M) where M is any $|T|^+$-saturated model of T. M is a (D, λ)-homogeneous model if for every $A \subseteq |M|$ with $|A| < \lambda$, and $p \in S_D(A)$, p is realized in M and D(M) = D. M is λ-homogeneous if it is (D(M), λ)-homogeneous. M is homogeneous if it is $\|M\|$-homogeneous. It is not difficult to show that if M is (D, λ)-homogeneous, $B \subseteq A$, A a D-set, $|B| < \lambda$, $|A| \le \lambda$, F a mapping from B into M, then there is an extension G of F which is a mapping of A into M. M is D-homogenous if it is (D, $\|M\|$)-homogeneous. It is easy to see that if $|T| < \lambda$, M is (D(T), λ)-homogeneous iff M is λ-saturated. Occasionally we shall use variables other than $x_0, \ldots, x_{n-1}$ in types, and then we shall write $p = p(y_0, \ldots, y_{n-1})$, for example, if the variables are $y_0, \ldots, y_{n-1}$. In this case we also write $p \in S^n_D(A)$ when the intention is clear.

A sequence $\langle \bar{a}_i : i < i_0 \rangle$ is indiscernible over A if every function F is a mapping if it satisfies the following conditions: Dom $F \subseteq A \cup \{\text{Rang}\, \bar{a}_i : i < i_0\}$, $F \restriction A = I_A$, $F(\bar{a}_i) \in \{\bar{a}_j : j < i_0\}$ for $i < i_0$, and if $F(\bar{a}_i) = \bar{a}_{i_1}$ and $F(\bar{a}_j) = \bar{a}_{j_1}$, then $j < i$ iff $j_1 < i_1$ (In this and the following definition we assume that $i \neq j$ implies $\bar{a}_i \neq \bar{a}_j$.)

$\{\bar{a}_i : i < i_0\}$ is an indiscernible set over A if every function F is a mapping if it satisfies the following conditions: Dom $F \subseteq A \cup \{\text{Rang}\, \bar{a}_i : i < i_0\}$, $F \restriction A = I_A$, $F(\bar{a}_i) \in \{\bar{a}_j : j < i_0\}$ for $i < i_0$, and $\bar{a}_i \neq \bar{a}_j$ implies $F(\bar{a}_i) \neq F(\bar{a}_j)$. It is easy to see that $\langle \bar{a}_i : i < i_0 \rangle$ is an indiscernible sequence over A if for all $j_0 < \ldots < j_{n-1} < i_0$, $k_0 < \ldots < k_{n-1} < i_0$, $\langle \bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}} \rangle$ and $\langle \bar{a}_{k_0}, \ldots, \bar{a}_{k_{n-1}} \rangle$ realize the same type over A. A similar condition exists for indiscernible sets. It is easy to see that if $\langle \bar{a}_i : i < i_0 \rangle$ is an indiscernible sequence over A, $\omega \le i_0 < j_0$, then it is possible to define $\bar{a}_i$ for $i_0 \le i < j_0$ such that $\langle \bar{a}_i : i < j_0 \rangle$ will be an indiscernible sequence over A. Naturally, D($A \cup \{\text{Rang}\, \bar{a}_i : i < i_0\}$) = D($A \cup \{\text{Rang}\, \bar{a}_i : i < j_0\}$). The respective claims are true for indiscernible sets. Of course, if $\bar{a}_i = \langle b_i \rangle$ we write b_i instead of $\bar{a}_i$.

§ 2. On stability of finite diagrams

In this section we define λ-stability for D and several properties ($(*\lambda)$, $(A*\lambda)$, $(B*\lambda)$) which imply the instability of D in suitable powers and which are implied by the instability of D in other powers if in addition there exist certain homogeneous D-models. These conditions will serve us later when we want to prove some theorems on stable theories.

Definition 2.1. 1) D will be called λ-*good* if there is a (D, λ)-homogeneous model of power $\geq \lambda$. D is good if it is λ-good for all λ.

2) D is λ-*stable* if D is λ^+-good and for every D-set A such that $|A| \leq \lambda$, $|S_\mathrm{D}(A)| \leq \lambda$. D is stable if it is λ-stable for some λ.

Note that 1) we say "for all λ" and in 2) "for some λ".

Claim 2.1. *If* D *is* λ^+*-good, then* D *is* λ*-stable iff for all* $n < \omega$ *and for all* A *with* $|A| \leq \lambda$, $|S^n_\mathrm{D}(A)| \leq \lambda$.

Proof: Immediate. (Note that if A is not a D-set, then $S^n_\mathrm{D}(A) = 0$.)

Definition 2.2. A type $p \in S^n(A)$ *splits over* $B \subseteq A$ if there is a formula $\psi(x, \bar{y})$ and there are two sequences $\bar{a}, \bar{b}$ from A which realize the same type over B such that $\psi(x, \bar{a}), \neg\, \psi(x, \bar{b}) \in p$.

Claim 2.2. 1) *If* $p \in S^n(A)$ *splits over* $B \subseteq A$, *then there is a mapping* F, $B \subseteq \mathrm{Dom}\, F \subseteq A$, $F|B = I_B$ *such that* $p, F(p)$ *are contradictory types.*

2) *If* $C \subseteq B \subseteq A$, $p \in S^n(A)$, *p does not split over C, and every finite type over C which is realized in A is realized in B, then* $p|B$ *has a unique continuation in* $S^n(A)$ *which doesn't split over C. If* $p|B \in S^n_\mathrm{D}(B)$ *then* $p \in S^n_\mathrm{D}(A)$.

Proof: 1) Define F such that $F|B = I_B$, $F(\bar{a}) = \bar{b}$, and $\mathrm{Dom}\, F = B \cup \mathrm{Rang}\, \bar{a}$ (in the notation of Definition 2.2). It is easy to see that F satisfies the conditions.

2) Immediate.

Theorem 2.3. 1) *If* $B \subseteq A$ *then the number of types in* $S_\mathrm{D}(A)$ *which do not split over B is* $\leq 2^{|\mathrm{D}||B|}$.

2) *If* D *is* $|B|$*-stable, the number is* $\leq |B|$.

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Proof: 1) It is easy to see that for all n the number of n-types over B which are realized in A is $\leq |D|^{|B|}$. Thus there is a set B_1, $B \subseteq B_1 \subseteq A$, $|B_1| \leq |D|^{|B|}$ such that every n-type which is realized in A is realized in B_1. By Claim 2.2 (2) we get

$$|\{p : p \in S_D(A),\ p \text{ doesn't split over } B\}| =$$

$$= |\{p|B_1 : p \in S_D(A),\ p \text{ doesn't split over } B\}| \leq$$

$$\leq |S_D(B_1)| \leq |D|^{|B_1|} \leq 2^{|D|^{|B|}} ;$$

and this proves 2.3. In addition it is clear that if $|T| \leq |B|$, then $2^{|D|^{|B|}} = 2^{2^{|B|}}$.

2) The proof is identical to the proof of (1) except that here by stability we get a B_1 with $|B_1| = |B|$ and $|S_D(B_1)| = |B|$ (here we used Claim 2.1).

Definition 2.3. 1) D *satisfies* $(*\lambda)$ if D is λ^+-good, there is an increasing sequence A_i, and a type $p \in S_D(A_\lambda)$ such that for every $i < \lambda$ $p|A_{i+1}$ splits over A_i.

2) D *satisfies* $(B*\lambda)$ if D is λ-good, there is a type $p_\eta \in S_D(A_\eta)$ for all $l(\eta) < \lambda$ such that $\eta = \tau | i$ implies $p_\eta \subseteq p_\tau$, and for every η there is a formula ψ such that $\psi \in p_{\eta\frown\langle 0\rangle}$, $\neg\psi \in p_{\eta\frown\langle 1\rangle}$ (and thus they are contradictory types), and $A_{\eta\frown\langle 0\rangle} \cup A_{\eta\frown\langle 1\rangle}$ is a D-set for every η.

3) D *satisfies* $(A*\lambda)$ if D is λ^+-good and satisfies $(B*\lambda)$.

Remark: When it is clear what the diagram D is, we shall say that one of the above conditions "holds", instead of saying that D satisfies it.

Claim 2.4. *In* Definition 2.3 *we can assume without loss of generality that*: $A_\lambda \subseteq |M|$, $A_{\tau_i} \subseteq |M|$, $|A_i| < |i|^+ + \aleph_0$, $|A_\eta| < |l(\eta)|^+ + \aleph_0$, *where M is any* (D, λ)-*homogeneous model.*

Proof: Assume that A_i is an increasing sequence, $p \in S_D(A_\lambda)$, and for all $i < \lambda$, $p|A_{i+1}$ splits over A_i; i.e., there are two sequences $\bar{a}_i$, $\bar{b}_i$ in A_{i+1} which realize the same type over A_i, and there is a formula ψ such that $\psi(x, \bar{a}_i), \neg\psi(x, \bar{b}_i) \in p$. For $i \leq \lambda$ define $B_i = (\bigcup_{j<i} \operatorname{Rang} \bar{a}_j) \cup$

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$\cup (\bigcup_{j<i} \text{Rang } \bar{b}_j)$. It is easy to see that B_i is an increasing sequence, $p_1 = p|B_\lambda \in S_D(B_\lambda)$, and for all $i < \lambda$, $p_1|B_{i+1}$ splits over B_i, $|B_i| < |i|^+ + \aleph_0$, and in a similar fashion, $|A_\eta| < |l(\eta)|^+ + \aleph_0$.

Since $|A_\lambda| \leq \lambda$ and M is (D, λ)-homogeneous, there is a mapping from A_λ into $|M|$. Thus without loss of generality $A_\lambda \subseteq |M|$.

It remains to prove that without loss of generality $A_\eta \subseteq |M|$. For every η, $l(\eta) < \lambda$ we define a mapping F_η from A_η into M such that if $\eta = \tau | i$ then $F_\eta \subseteq F_\tau$, (i.e. $F_\eta = F_\tau | \text{Dom } F_\eta$) and for every η, $F_{\eta\frown\langle 0\rangle} \cup F_{\eta\frown\langle 1\rangle}$ is a mapping. The definition is by induction on $l(\eta)$; if $l(\eta) = 0$ let $F_{\langle\rangle}$ be any mapping from $A_{\langle\rangle}$ into M. If $l(\eta) = i$, since $|A_\eta| < \lambda$, $|A_{\eta\frown\langle 0\rangle} \cup A_{\eta\frown\langle 1\rangle}| < \lambda$, there is an extension F_η of $F_{\eta|i}$ which is a mapping from $A_{\eta\frown\langle 0\rangle} \cup A_{\eta\frown\langle 1\rangle}$ into M. Define $F_{\eta\frown\langle 0\rangle} = F|A_{\eta\frown\langle 0\rangle}$, $F_{\eta\frown\langle 1\rangle} = F|A_{\eta\frown\langle 1\rangle}$. If $l(\eta) = \delta$ define $F_\eta = \bigcup_{i<\delta} F_{\eta|i}$. Define $B_\eta = F_\eta(A_\eta)$, $q_\eta = F_\eta(p_\eta)$. It is easy to see that the q_η's satisfy the conditions that we wanted for the p_η's, and so we may take $A_\eta \subseteq M$.

Theorem 2.5. *If there is an A such that* $|S_D(A)| > \mu_0 = |A|^{(\lambda)} + \sum_{\mu<\lambda} 2^{|D|^\mu}$, *then there is an increasing sequence* A_i, $i \leq \lambda$, *such that for all* $i < \lambda$ $p|A_{i+1}$ *splits over* A_i. *It follows that if there is such an A and* D *is* λ^+*-good then* D *satisfies* $(*\lambda)$.

Proof: First, we show that it is sufficient to prove the existence of $q \in S_D(A)$ such that for all $B \subseteq A$ with $|B| < \lambda$, q splits over B. For all $i<\lambda$ we define A_i by induction such that $A_i \subseteq A$, $|A_i| < |i|^+ + \aleph_0$: A_0 will be the empty set. For a limit ordinal δ, $A_\delta = \bigcup_{i<\delta} A_i$. By the hypothesis, q splits over A_i since $|A_i| < \lambda$, and thus there is an extension A_{i+1} of A_i, formed by adding a finite number of elements, such that $q|A_{i+1}$ splits over A_i, and hence $|A_{i+1}| < |i|^+ + \aleph_0$. Define $A_\lambda = \bigcup_{i<\lambda} A_i$, $p = q|A_\lambda$, and it is easy to see that the conclusion of the theorem is satisfied.

Now assume that for every type $p \in S_D(A)$ there is a set $B_p \subseteq A$, $|B_p| < \lambda$, such that p does not split over B_p. Since $|S_D(A)| > |A|^{(\lambda)}$, there is a set B such that $|\{p : B_p = B\}| > \mu_0$. Now by Theorem 2.3 the number of types in $S_D(A)$ which do not split over B is

$\leq 2^{|D|^{|B|}} \leq \mu_0$ (since $|B| < \lambda$) in contradiction to what we have just shown. Thus there is a type q as above and the theorem is proved.

Theorem 2.6. *If* D *satisfies* $(*\lambda)$ *then* D *satisfies* $(A*\lambda)$.

Proof: Let M be a (D, λ^+)-homogeneous model, A_i ($i \leq \lambda$) an increasing sequence, $A_\lambda \subseteq |M|$, $p \in S_D(A_\lambda)$, $|A_i| < \lambda$, $p|A_{i+1}$ splits over A_i. By Claim 2.2 there are mappings F_i such that $A_i \subseteq \mathrm{Dom}\, F_i \subseteq A_{i+1}$ and $p|A_{i+1}$ and $F_i(p|A_{i+1})$ are contradictory types. We want to define types p_η which will satisfy the conditions of the definition. To this end we define $p_\eta, A_\eta, G_\eta, F_\eta$ such that:

1) $p_\eta \in S_D(A_\eta)$, $A_\eta \subseteq M$, and if $\eta = \tau | i$ then $p_\eta \subseteq p_\tau$, $A_\eta \subseteq A_\tau$;

2) for every η there is a formula ψ_η such that $\psi_\eta \in p_{\eta\frown\langle 0\rangle}$, $\neg\psi_\eta \in p_{\eta\frown\langle 1\rangle}$;

3) G_η is a mapping, $\mathrm{Dom}\, G_\eta = A_{l(\eta)}$, $\mathrm{Rang}\, G_\eta = A_\eta$, and if $\eta = \tau | i$ then $G_\eta \subseteq G_\tau$, $p_\eta = G_\eta(p|A_{l(\eta)})$;

4) F_η is a mapping, $\mathrm{Dom}\, F_\eta = A_{\eta\frown\langle 0\rangle}$, $\mathrm{Rang}\, F_\eta = A_{\eta\frown\langle 1\rangle}$, $F_\eta(p_{\eta\frown\langle 0\rangle}) = p_{\eta\frown\langle 1\rangle}$, $F_\eta|A_\eta = I_{A_\eta}$, and $F_\eta \supseteq G_{\eta\frown\langle 0\rangle}\ F_{l(\eta)}\ G^{-1}_{\eta\frown\langle 0\rangle}$.

It is easy to see that if we succeed in this definition, then we have proved the theorem. For simplicity let $A_0 = 0$.

We define p_η, A_η, G_η, and F_τ by induction on k for all τ, η with $l(\eta) \leq k$, $l(\tau) + 1 \leq k$. For $k = 0$, $A_{\langle\rangle}$ will be the empty set, $p_{\langle\rangle} = p|A_{\langle\rangle}$, and $G_{\langle\rangle}$ will be the empty mapping. For $k = \delta$, a limit ordinal, $p_\eta = \bigcup_{i<\delta} B_{\eta|i}$, $A_\eta = \bigcup_{i<\delta} A_{\eta|i}$, $G_\eta = \bigcup_{i<\delta} G_{\eta|i}$. The remaining case is $k = i + 1$. Assume $l(\eta) = i$; we will define A_τ, p_τ, G_τ, and F_η for $\tau = \eta\frown\langle 0\rangle$, $\eta\frown\langle 1\rangle$. Since G_η is defined with domain A_i, $|A_i|$, $|A_k| < \lambda$, G_η has an extension $G_{\eta\frown\langle 0\rangle}$ which is a mapping from A_k into M. Define $p_{\eta\frown\langle 0\rangle} = G_{\eta\frown\langle 0\rangle}(p|A_k)$, $A_{\eta\frown\langle 0\rangle} = \mathrm{Rang}\, G_{\eta\frown\langle 0\rangle}$. $G_{\eta\frown\langle 0\rangle} F_i G^{-1}_{\eta\frown\langle 0\rangle}$ is a mapping with domain $\subseteq A_{\eta\frown\langle 0\rangle}$ which is the identity on A_η. Thus we can extend the above mapping to a mapping from $A_{\eta\frown\langle 0\rangle}$ into M. F_η will be this extension. Define $A_{\eta\frown\langle 1\rangle} = \mathrm{Rang}\, F_\eta$, $p_{\eta\frown\langle 1\rangle} = F_\eta(p_{\eta\frown\langle 0\rangle})$, $G_{\eta\frown\langle 1\rangle} = F_\eta G_{\eta\frown\langle 0\rangle}$.

All the parts of the definition follow immediately, and thus, by what was said at the beginning of the proof, we are through.

Theorem 2.7. *If* D *satisfies* $(A*\lambda)$ *and* $2^\lambda > \mu$ *then* D *is not* μ*-stable.*

(By the previous theorem it is clearly enough to assume that D satisfies $(*\lambda)$ and also $(B*\lambda)$.)

Proof: Let $\kappa = \inf\{\kappa : 2^\kappa > \mu\}$. Since $\lambda \geq \kappa$ and $(A*\lambda)$ holds, $(A*\kappa)$ also holds; i.e., there is a (D, κ^+)-homogenous model M, and there are types $p_\eta \in S_\mathrm{D}(A_\eta)$ for $l(\eta) < \kappa$ such that $A_\eta \subseteq M$ and for every η, $p_{\eta\frown\langle 0\rangle}$ and $p_{\eta\frown\langle 1\rangle}$ are contradictory types; $|A_\eta| < |l(\eta)|^+ + \aleph_0$. Define $A = \bigcup_{l(\eta)<\kappa} A_\eta$, and for $l(\eta) = \kappa$ define A_η, $A_\eta = \bigcup_{i<\kappa} A_{\eta|i}$, $p_\eta = \bigcup_{i<\kappa} p_{\eta|i}$. Clearly, $p_\eta \in S_\mathrm{D}(A_\eta)$ and since $|A_\eta| \leq \kappa$ there is an element $a_\eta \in M$ which realizes p_η. Let q_η be the type which a_η realizes over A. If $\eta \neq \tau$, $\kappa = l(\eta) = l(\tau)$ and i is the first ordinal for which $\eta_i \neq \tau_i$ (without loss of generality we can assume $\eta_i = 1, \tau_i = 0$), then $p_{(\eta|i)\frown\langle 1\rangle} \subseteq p_\eta \subseteq q_\eta$ and $p_{(\eta|i)\frown\langle 0\rangle} \subseteq p_\tau \subseteq q_\tau$. Thus q_η and q_τ are contradictory and hence not equal. From this follows $|S_\mathrm{D}(A)| \geq |\{q_\eta : l(\eta) = \kappa\}| = 2^\kappa > \mu$; $|A| = |\bigcup\{A_\eta : l(\eta) < \lambda\}| \leq \Sigma\{|A_\eta| : l(\eta) < \kappa\} \leq \Sigma\{\kappa : l(\eta) < \kappa\} = \kappa \cdot 2^{(\kappa)} \leq \mu$. Therefore D is not μ-stable.

We could now draw some conclusions about the class of powers in which a diagram is stable, but we postpone this until §4 where it shall be done in a more complete fashion.

Corollary 2.8. *If* D *is μ-stable where $\mu < 2^\lambda$, then there is no increasing sequence A_i, $i \leq \lambda$, with a complete* D*-n-type p over A_λ such that $p|A_{i+1}$ splits over A_i for all $i < \lambda$.*

Proof: As in the proofs of 2.6 and 2.7 we show that if there is such a type, then there is an A, $|A| \geq \lambda$, with $|S^n_\mathrm{D}(A)| > \lambda$, in contradiction to Claim 2.1.

Theorem 2.9. *If for every $\lambda < \beth[(2^{|T|})^+]$* D *is not λ-stable, then* D *is not stable.*

Remark: If $\mathrm{D} = \mathrm{D}(T)$, then it is enough to assume $(B*|T|^+)$ in order to get the same conclusion. If $|T| = \aleph_0$ or $= \beth_\delta$ where $cf\delta = \aleph_0$, we can take $\beth[|T|^+]$ instead of $\beth[(2^{|T|})^+]$.

Proof: If there is $\lambda < \beth[(2^{|T|})^+]$ such that D is not λ-good then D is not λ_1-good for any $\lambda_1 \geq \lambda$ and thus D is not stable in any power. It is not hard to see by 2.5 that $(*\lambda)$ holds for all $\lambda < \beth[(2^{|T|})^+]$. Thus we shall assume that for every $\lambda < \beth[(2^{|T|})^+]$ there is an increasing sequence $A_{\lambda,i}$ and there is a type $p_\lambda \in S_D(A_{\lambda,\lambda})$ such that for all $i < \lambda$, $p_\lambda | A_{\lambda,i+1}$ splits over $A_{\lambda,i}$. Let a_λ be an element realizing p_λ. Clearly $A_{\lambda,\lambda} \cup \{a_\lambda\}$ is a D set. By the definition of splitting, for every $i < \lambda$ there are sequences $\bar{a}_{\lambda,i}, \bar{b}_{\lambda,i}$ in $A_{\lambda,i+1}$ which realize the same type over $A_{\lambda,i}$, and there is a formula $\psi_{\lambda,i}$ such that $\psi_{\lambda,i}(x, \bar{a}_{\lambda,i})$, $\neg\psi_{\lambda,i}(x, \bar{b}_{\lambda,i}) \in p_\lambda$. Define $\bar{c}_{\lambda,i} = \bar{a}_{\lambda,i}{}^\frown \bar{b}_{\lambda,i}$. Then the sequence $\langle\langle a_\lambda\rangle^\frown \bar{c}_{\lambda,i} : i < \lambda\rangle$ is defined for all $\lambda < \beth[(2^{|T|})^+]$ and is length λ. As in the proof in Morley [6] by using the Erdös-Rado theorem [1] we can find an indiscernible sequence $\langle\langle a\rangle^\frown \bar{a}_i{}^\frown \bar{b}_i : i < \omega\rangle$ such that

(*) for every $n < \omega$ there are λ and $i_0 < i_1 < \ldots < i_n$ such that $^\frown\langle\langle a\rangle^\frown \bar{c}_i : i \leq n\rangle$ and $^\frown\langle\langle a_\lambda\rangle^\frown \bar{c}_{\lambda,i_j} : j \leq n\rangle$ realize the same type ($\bar{c}_i = \bar{a}_i{}^\frown \bar{b}_i$ for all i). So for all n $\bar{a}_n, \bar{b}_n$ realize the same type over $\bigcup_{i<n} \operatorname{Rang} \bar{c}_i$, and the type that a realizes over $\bigcup_{i \leq n} \operatorname{Rang} \bar{c}_i$ splits over $\bigcup_{i<n} \operatorname{Rang} \bar{c}_i$.

Assume $\mu > \aleph_0$, for all $\omega \leq i < \mu$ define $\bar{a}_i, \bar{b}_i$ such that $\langle\langle a\rangle^\frown \bar{c}_i : i < \mu\rangle$ will be an indiscernible sequence. Clearly the type which a realizes over $\bigcup_{i \leq j} \operatorname{Rang} \bar{c}_i$ is a D-type and splits over $\bigcup_{i<j} \operatorname{Rang} \bar{c}_i$. Thus either ($*\mu$) holds or D is not μ^+-good. In either case D is not μ-stable.

§3. On stable diagrams

In this section we prove two theorems on stable diagrams. One says that every stable diagram is good, and the other that if $A \subseteq |M|$, $|A| < \|M\|$, then M contains indiscernible sets over A of power $> |A|$ if M is stable in $|A|$. This theorem is a generalization of a theorem in Morley [6] on totally transcendental theories and of a theorem in [17]. We make use principally of Corollary 2.8.

Theorem 3.1. *Assume that* D *is* λ*-stable and* $|A| \leq \lambda < \|M\|$.

1) *If* $D(M) \subseteq D$, $A \subseteq |M|$, *then* M *contains an indiscernible set over* A *of power* $> |A|$

2) *If* E *is a set of finite sequences,* $|E| > \lambda$, *and* $\bigcup \{\mathrm{Rang}\, \bar{a} : \bar{a} \in E\} \cup A$ *is a* D*-set, then there is an* $E' \subseteq E$ *of power* $> \lambda$ which is an indiscernible set over A.

Proof: It is clearly sufficient to prove 2).

Since every sequence in E is of finite length, there is an n such that the power of $E_1 = \{\bar{a} : \bar{a} \in E,\ l(\bar{a}) = n\}$ is greater than λ; without loss of generality $|E_1| = \lambda^+$. Since D is λ^+-good, we can assume that E_1 is a set of sequences from M, where M is a D-model and $A \subseteq M$.

Lemma 3.2. *If* D *is* λ*-stable,* $D(M) \subseteq D$, $A \subseteq |M|$ $|A| \leq \lambda$, $|E_1| > \lambda$, $E_1 \subseteq |M|^n$, *then there are* B, C *with* $A \subseteq B \subseteq C \subseteq |M|$, $|B|$, $|C| \leq \lambda$ *such that: every finite type over* B *which is realized in* $|M|$ *is realized in* C, *and there is a type* $p \in S^n_D(C)$ *such that for all* C_1, $C \subseteq C_1 \subseteq |M|$, $|C_1| \leq \lambda$, p *has a continuation in* $S^n_D(C_1)$ *which is realized in* $E_1 - (C_1)^n$ *and doesn't split over* B

Proof: Assume that there are no B, C satisfying the conditions of the lemma. We shall define an increasing sequence A_i by induction for $i \leq \lambda$ such that $|A_i| \leq \lambda$, $A_i \subseteq |M|$, and every type in $S^n_D(A_{i+1})$, for which there is a sequence in $E_1 - (A_{i+1})^n$ realizing it, splits over A_i. After we define this sequence, since $|E_1| > \lambda$, $|A_\lambda| \leq \lambda$, it will follow that there exists a type $p \in S^n_D(A_\lambda)$ which is realized in $E_1 - (A_\lambda)^n$, and thus for all i, $p | A_{i+1}$ is realized in $E_1 - (A_{i+1})^n$; hence $p | A_{i+1}$ splits over A_i. But according to Corollary 2.8, this contradicts the assumption that D is λ-stable. Thus we have shown that for the proof of the lemma it is sufficient to define the sequence A_i.

Define $A_0 = 0$, $A_\delta = \bigcup_{i<\delta} A_i$ for δ a limit ordinal. Assume that A_i is defined and let $A^i \subset |M|$ be an extension of A_i such that every finite type over A_i which is realized in M is realized in A^i. By claim 2.1 we may assume $|A^i| \leq \lambda$. By assumption A_i, A^i, p for all $p \in S^n_D(A^i)$ do not satisfy the conditions of the lemma with $A_i = B$, $A^i = C$, $p = p$; so there is a set C_p, $|C_p| \leq \lambda$, $A^i \subseteq C_p \subseteq |M|$, such that every extension of p in $S^n_D(C_p)$ either splits over A_i or is not realized in $E_1 - (C_p)^n$. Define $A_{i+1} = \bigcup \{ C_p : p \in S^n_D(A^i)\}$. By the λ-stability of D and Claim 2.1 it follows that $|S^n_D(A^i)| \leq \lambda$ and thus $|A_{i+1}| \leq \lambda.\lambda = \lambda$. Also, if $p \in S^n_D(A_{i+1})$ is realized in $E_1 - (A_{i+1})^n$ then p splits over A_i. It is easy to see that all the conditions of the definition are satisfied and thus Lemma 3.2 is proved.

Let us return to the proof of the theorem. Let B, C, and $p \in S^n_D(C)$ satisfy the conditions of the lemma. By induction on $i < \lambda^+$ we define a sequence $\bar{y}_i$: if $\bar{y}_i$ is defined for all $i < j$, let $C_j = C \cup (\bigcup\{\text{Rang}\, \bar{y}_i : i < j\})$, let $p_j \in S^n_D(C_j)$ be an extension of p which is realized in $E_1 - (C_j)^n$ and does not split over B, and let $\bar{y}_j$ be a sequence in $E_1 - (C_j)^n$ which realizes p_j. We shall show that $\langle \bar{y}_i : i < \lambda^+\rangle$ is an indiscernible sequence over B. By the construction it is clear that no two $\bar{y}_i$'s in the above sequence are equal, and thus its power is $> \lambda$. Since every finite type over B which is realized in $|M|$ is realized in C, p has a unique continuation in $S^n_D(C_j)$ which doesn't split over B (by Claim 2.2.2). It follows from this that if $i < j$, then $p_j|C_i$ is a continuation of p in $S^n_D(C_i)$ which doesn't split over B, and thus $p_j|C_i = p_i$. In order to prove that $\langle \bar{y}_i : i < \lambda^+\rangle$ is an indiscernible sequence over B it is sufficient to prove for all $i_0 < \ldots < i_n$ that the sequences $\bar{y}_0{}^\frown \ldots {}^\frown \bar{y}_m$, $\bar{y}_{i_0}{}^\frown \ldots {}^\frown \bar{y}_{i_m}$ realize the same type over B. For $m = 0$ we have already proved this. Assume it for m and we shall prove the result for $m + 1$. Denote $\bar{y}^0 = \bar{y}_0{}^\frown \ldots {}^\frown \bar{y}_m$, $\bar{y}^1 = \bar{y}_{i_0}{}^\frown \ldots {}^\frown \bar{y}_{i_m}$. We must show that $\bar{y}^0{}^\frown\langle \bar{y}_{m+1}\rangle$ and $\bar{y}^1{}^\frown\langle \bar{y}_{i_{m+1}}\rangle$ realize the same type over B. Since $m + 1 \leq i_{m+1}$ we have $p_{i_{m+1}}|C_{m+1} = p_{m+1}$ and thus $\bar{y}^0{}^\frown\langle \bar{y}_{m+1}\rangle$ and $\bar{y}^0{}^\frown\langle \bar{y}_{i_{m+1}}\rangle$ realize the same type over B. It follows that if the inductive claim for $m + 1$ is not correct, there is a formula ψ and sequence $\bar{b}$ from B such that $\models \psi(\bar{y}_{i_{m+1}}, \bar{y}^0, \bar{b})$ $\models \neg\, \psi(\bar{y}_{i_{m+1}}, \bar{y}^1, \bar{b})$; hence $\psi(\bar{x}, \bar{y}^0, \bar{b})$,

$\neg\,\psi(\bar{x},\bar{y}^1,\bar{b}) \in p_{i_{m+1}}$. Since it is assumed that $\bar{y}^0, \bar{y}^1$ realize the same type over B, $p_{i_{m+1}}$ splits over B, contradiction. Thus $\bar{y}^0 \frown \langle \bar{y}_{m+1} \rangle$ and $y^1 \frown \langle \bar{y}_{i_{m+1}} \rangle$ realize the same type over B, and thus we have proved that $\langle \bar{y}_i : i < \lambda^+ \rangle$ is an indiscernible sequence over B, and hence over A.

It remains to show that this is an indiscernible set. Assume that this is not the case. Let J be an ordered set of power $> \lambda$ with a dense subset I of power $\leq \lambda$ (for example if $\mu = \inf\{\mu : 2^\mu > \lambda\}$ take J to be the set of sequences of length μ of zeroes and ones ordered by the lexicographic order and I will be the set of sequences of J which are zero from a certain point on). By the compactness theorem we can find a set $B^1 = \bigcup\{\text{Rang}\,\bar{y}_u : u \in J\} \cup A$, where $l(\bar{y}_u) = n$ for all u, such that if $u_0 < \ldots < u_m \in J$ then $\bar{y}_{u_0} \frown \ldots \frown \bar{y}_{u_m}$ satisfies the same type over A as $\bar{y}_0 \frown \ldots \frown \bar{y}_m$. It is easy to see that B^1 is a D-set and that if $u_0 \neq u_1$ then $\bar{y}_{u_0}$ and $\bar{y}_{u_1}$ realize different D-types over $B^2 = A \cup \{\text{Rang}\,\bar{y}_u : u \in I\}$. Thus $|S^n_{\mathrm{D}}(B^2)| \geq |J| > \lambda$ and $|B^2| = |A| + n$. $|I| \leq \lambda$ in contradiction to Claim 2.1. (A more detailed discussion of a similar theorem is found in Morley [6]).

Theorem 3.3. *If* D *is stable,* $B \subseteq A$, $p \in S_{\mathrm{D}}(B)$, *then* p *has an extension in* $S_{\mathrm{D}}(A)$, *when* A *is a* D-*set*

Proof: (We thank Mr. Victor Harnik for suggesting a simplification in the proof.)

Assume D is λ-stable, and thus there is a (D, λ^+)-homogeneous model M. If $|A| \leq \lambda$ there is a mapping F from A into M, and since $F(p) \in S_{\mathrm{D}}(F(B))$, $|B| \leq |A| < \lambda^+$, it is clear that $F(p)$ is realized by an element a of M. a realizes a certain type over $F(A)$, $p_1 \in S_{\mathrm{D}}(F(A))$, and clearly $F^{-1}(p_1)$ is the required extension of p.

Now assume that $A = B \cup \{a\}$ where a realizes a type $q \in S_{\mathrm{D}}(B)$. It is easy to see that there is $A_0 \subseteq B$, $|A_0| \leq \lambda$ such that p doesn't split over A_0 (otherwise $(*\lambda)$ holds in contradiction to the λ-stability of D). There is also an A_1, $A_0 \subseteq A_1 \subseteq B$, $|A_1| \leq \lambda$, such that neither p nor q splits over A_1. Define an increasing sequence B_i for $i \leq \lambda$ such that $A_1 \subseteq B_i \subseteq B$, $|B_i| \leq \lambda$, and every finite type over B_i which is realized in B is realized in B_{i+1}. (It is easy to define such a sequence since D is λ-stable.) By the first paragraph of the proof, since $|B_\lambda| \leq \lambda$, $p|B_\lambda$ has an extension to a type $p_1 \in S_{\mathrm{D}}(B_\lambda \cup \{a\})$. Let c be an element which

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realizes p_1, thus $B_\lambda \cup \{c, a\}$ is a D-set. Let r be the type which $\langle c, a\rangle$ realizes over B_λ. $r \in S^2_D(B_\lambda)$. By Corollary 2.8, since D is stable, there is $i < \lambda$ such that $r | B_{i+1}$ does not split over B_i. We define $p^1 = \{\psi(x_0, a, \bar{a}) : \bar{a}$ a sequence from B, there is a sequence $\bar{b}$ in B_{i+1} which realizes the same type as $\bar{a}$ over B_i, and $\psi(x_0, x_1, \bar{b}) \in r\}$. Is is easy to see that $p^1 \in S_D(B \cup \{a\})$.

Now we prove the theorem for the general case. Order the elements of $A = \{a_i : i < |A|\}$ and define $A^i = B \cup \{a_j : j < i\}$ for all $i \leq |A|$. By induction we define an increasing sequence of types p^i such that $p^0 = p$ and $p^i \in S_D(A^i)$: for $i = \delta$ a limit ordinal, take $p^i = \bigcup_{j<i} p^j$, and if p^i is defined we let p^{i+1} be an extension of p^i in $S_D(A^i \cup \{a_i\}) = S_D(A^{i+1})$ (as guaranteed by the previous paragraph). $p^{|A|}$ is the required type.

Theorem 3.4. *If* D *is stable then* D *is good.* More explicitly: *for every power μ and* D-*set A there is an increasing sequence of ordinals $\{i_j : j < \mu^+\}$ and a sequence of elements a_i for $i < i^0 = \bigcup_{j<\mu^+} i_j$ such that every complete* D-*type over a subset of $A_j = A \cup \{a_k : k < i_j\}$ of power $< \mu$ is realized in A_{j+1}, and $\bigcup_{j<\mu^+} A_j$ is a* (D, μ)-*homogeneous model. Also every* (D, μ)-*homogeneous model is of power $\geq \mu$.*

Remark: This partially solves a problem from Keisler and Morley [5].

Proof: We first prove that every (D, μ)-homogeneous model M is of power $\geq \mu$. Since every D-set of power $\leq \mu$ can be embedded in M, it is sufficient to prove that there is a D-set of power μ. Since D is stable, there is a λ for which D is λ-stable, and thus there is a (D, λ^+)-homogeneous model of power $\geq \lambda^+$. Hence by Theorem 3.1 there is a D-set $A = \{y_i : i < \omega\}$ which is an indiscernible set. By the compactness theorem there is an indiscernible set $\{y_i : i < \mu\} \supseteq \{y_i : i < \omega\}$ and this set is already a D-set of power μ.

Now we shall define i_j for $j < \mu^+$ and a_i for $i < i_j$ by induction on j. Define $i_0 = 0$ and $i_j = \bigcup_{k<j} i_k$ for j a limit ordinal when a_i is already defined for all $i < i_j$. Assume that the definition is completed for j and we proceed to define for $j + 1$. Let $\{p_{j,k} : i_j \leq k < i_{j+1}\}$ be the set of complete D-types over subsets of A_j. We define a_i by induction on i,

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$i_j \leq i < i_{j+1}$. If a_i is defined for all $i < k$, a_k will be an element realizing a type in $S_D(A \cup \{a_i : i < k\})$ which extends $p_{j,k}$ (such a type exists by Theorem 3.3). It is easy to see that this definition satisfies all the requirements except perhaps for the (D, λ)-homogeneity of the model M with $|M| = A \cup \{a_i : i < \bigcup_{j<\mu} + i_j\}$. Let $p \in S_D(B)$, $B \subseteq |M|$, $|B| < \mu$. Then there is a $j < \mu^+$ such that $B \subseteq A_j$, and thus there is an element in $A_{j+1} \subseteq |M|$ which realizes p. It remains to prove that M is a model of T. By the Tarski-Vaught test it is sufficient to show that if $\bar{a}$ is a sequence from $|M|$ and $\models (\exists x)\psi(x, \bar{a})$ then there is a $b \in |M|$ such that $\models \psi[b, \bar{a}]$. By the above, it is sufficient to show the existence of a type p with $\psi(x, \bar{a}) \in p \in S_D(\mathrm{Rang}\, \bar{a})$, but this follows from the existence of a (D, λ)-homogeneous model.

§4. On the class of powers in which D is stable

In this section we attempt to find the class of powers in which a finite diagram is stable. Our conclusion is that for every stable D there are cardinals λ, κ such that D is μ-stable iff $\mu \geq \lambda$ and $\mu^{(\kappa)} = \mu$.

Definition 4.1. If $p \in S^n(A)$, $B \subseteq A \subseteq C$, then *p splits strongly over B in C* if there is a sequence $\langle \bar{a}_i \ i < \omega \rangle$ of sequences in C which is an indiscernible sequence over B and there is a formula $\psi(\bar{x}, \bar{y})$ such that $\psi(\bar{x}, \bar{a}_0), \neg\, \psi(\bar{x}, \bar{a}_1) \in p$. It is clear that p splits over B.

Remark: From the proof of Theorem 3.1 it is clear that $\{\bar{a}_i : i < \omega\}$ is an indiscernible set, in the case that D is stable and C is a D-set.

Definition 4.2. D *satisfies* $(C{*}\lambda)$ if D is λ^+-good, there is an increasing sequence A_i, $i \leq \lambda$, and a type $p \in S_D(A_\lambda)$ such that for all $i < \lambda$ $p \restriction A_{i+1}$ splits strongly over A_i in A_λ.

Remark: It is clear that $(C{*}\lambda)$ implies $({*}\lambda)$, and that we can assume without loss of generality that $|A_i| < |i|^+ + \aleph_0$.

From the addition to 2.9 it is easy to conclude that if D satisfies $({*}\lambda)$ for all $\lambda < \beth[(2^{|T|})^+]$, then D satisfies $(C{*}\lambda)$ for all λ such that D is λ^+-good.

Theorem 4.1. *If* D *is* λ*-stable and there is an A such that* $|S_D(A)| > |A|^{(\kappa)} + \lambda$, *then* D *satisfies* $(C{*}\kappa)$.

Proof: Similarly to the proof of 2.5 it is sufficient to prove that there is a type $p \in S_D(A)$ such that for all $B \subseteq A$, $|B| < \kappa_1 = \min(\lambda, \kappa)$, p splits strongly over B in A. If $\kappa \leq \lambda$ it is clear that D satisfies $(C{*}\kappa)$; if $\lambda \leq \kappa$ $(C{*}\lambda)$ holds, contradiction by 2.7

This being the case, assume that for every type $p \in S_D(A)$ there is a set $B_p \subseteq A$, $|B_p| < \kappa_1$ such that p does not split strongly over B_p. Since $|S_D(A)| > |A|^{(\kappa)} + \lambda$ there is a set $B \subseteq A$ such that $|S| > |A|^{(\kappa)} + \lambda$ where $S = \{p \in S_D(A) : B_p = B\}$.

We will show that there is a sequence $\{\psi_i(x_0, \bar{a}_i) : i < \lambda^+\}$ and that there are types p_i for $i < \lambda^+$ such that $p_i \in S$ and $\{\psi_j(x_0, \bar{a}_j) : j < i\} \cup \{\neg\, \psi_i(x_0, \bar{a}_i\} \subseteq p_i$. Since $|S_D(A)| > |A| + \lambda$ it is clear that $|A| > \lambda$ and thus $|S| > \lambda^+$. We shall define S_i and an increasing sequence of $A_i \subseteq A$

for $i < \lambda^+$ such that $|A_i| \leq \lambda$. Take $A_0 = 0$ and $A_\delta = \bigcup_{i<\delta} A_i$ if δ is a limit ordinal. If A_i is defined, S_i will be the set of types $p \in S$ such that p is the unique extension of $p|A_i$ in S; A_{i+1} will be a set $A_i \subseteq A_{i+1} \subseteq A$ such that every type in $S_D(A_i)$ which has more than one extension in S has at least two extensions in $S_D(A_{i+1})$ which have continuations in S. Since $|A_i| \leq \lambda$, $|S_D(A_i)| \leq \lambda$, and thus we can choose A_{i+1} such that $|A_{i+1}| \leq \lambda$.

Now $|\bigcup_{i<\lambda^+} S_i| \leq \sum_{i<\lambda^+} |S_D(A_i)| = \lambda^+ . \lambda = \lambda^+$ and $|S| > \lambda^+$; thus there is a type $p \in S$, $p \notin \bigcup_{i<\lambda^+} S_i$. It follows that for all $i < \lambda^+$ there is a type $p^i \in S_D(A_{i+1})$, $p^i|A_i = p|A_i$, $p^i \neq p|A_{i+1}$, and p^i has an extension p_i in S. Thus there is a formula $\psi_i(x_0, \bar{a}_i)$, where $\bar{a}_i$ is a sequence from A_{i+1}, such that $\neg\, \psi_i(x_0, \bar{a}_i) \in p_i$, $\psi_i(x_0, \bar{a}_i) \in p$ It is easy to see that this sequence of ψ_i's satisfies the necessary conditions.

For every $i < \lambda^+$ let b_i be an element realizing p_i. Since D is λ-stable, $|B| < \kappa_1 \leq \lambda$, $|\{\langle b_i\rangle^\frown \bar{a}_i : i < \lambda^+\}| > \lambda$. By Theorem 3.1.2 $\{\langle b_i\rangle^\frown \bar{a}_i : i < \lambda^+\}$ has a subset of power λ^+ which is indiscernible over B. Without loss of generality we may take this subset to be $\{\langle b_i\rangle^\frown \bar{a}_i : i < \lambda^+\}$ itself. Since $|\bigcup_{k<\lambda} \text{Rang}\, \bar{a}_k| \leq \lambda$ we have $\left|S_D\left(\bigcup_{k<\lambda} \text{Rang}\, \bar{a}_k\right)\right| \leq \lambda$ and thus there are i, j such that $\lambda \leq j < i < \lambda^+$ and $p_i|\left(\bigcup_{k<\lambda} \text{Rang}\, \bar{a}_k\right) = p_j|\left(\bigcup_{k<\lambda} \text{Rang}\, \bar{a}_k\right)$. Also $\psi_j(x_0, \bar{a}_j) \in p_i$, $\neg\, \psi_j(x_0, \bar{a}_j) \in p_j$. If $\psi_j(x_0, \bar{a}_0) \in p_i$ then $\{\bar{a}_0, \bar{a}_j, \bar{a}_{j+1}, ..., \bar{a}_{j+n}, ...\}$ is an indiscernible set over B of sequences from A; $\psi_j(x_0, \bar{a}_0) \in p_j$ by the choice of i, j, $\neg\, \psi_j(x_0, \bar{a}_j) \in p_j$. Thus p_j spilts strongly over B in contradiction to the assumption that $p_j \in S$. If $\neg\, \psi_j(x_0, \bar{a}_0) \in p_i$ then $\{\bar{a}_j, \bar{a}_0, \bar{a}_1, ..., \bar{a}_n, ...\}$ is an indiscernible set over B of sequences from A and $\psi_j(x_0, \bar{a}_j) \in p_i$, $\neg\, \psi_j(x_0, \bar{a}_0) \in p_i$. And p_i splits strongly over B in contradiction to the assumption that $p_i \in S$. Thus there is a $p \in S_D(A)$ such that for every $B \subseteq A$, $|B| < \kappa_1$, p splits strongly over B. By what was said at the start of the proof, $(C{*}\kappa)$ holds.

Claim 4.2. *If* D *is* λ-*stable*, $\mu \leq \lambda < \lambda^\mu$, *E is an indiscernible set of sequences*, $A = \bigcup\{\text{Rang}\, \bar{y} : \bar{y} \in E\}$, $p \in S_D(A)$, *then it is not the case* *that* $|\{\bar{y}\in E : \psi(x,\bar{y})\in p\}|\geq\mu$ *and* $|\{\bar{y}\in E : \neg\,\psi(x,\bar{y})\in p\}|\geq\mu$ *for any* ψ.

Proof: Assume that E contradicts the claim. Since D is good (by 3.4) we can assume without loss of generality that $A\subseteq M$ where M is a $(\mathrm{D}, |A|^{+}+\lambda^{+})$-homogeneous model; and $a\in|M|$ realizes p. E has a subset $E_1=\{\bar{y}_i : i<\mu\}$ with the same properties, and in M we can find an $E_2=\{\bar{y}_i : i<\lambda\}\supseteq E_1$ which is an indiscernible set. Since $|\{i<\lambda : \models\psi[a,\bar{y}_i]\}|\geq\mu$, $|\{i<\lambda : \models\psi[a,\bar{y}_i]\}|\geq\mu$, we can assume without loss of generality that $|\{i<\lambda : \models\psi(a,\bar{y}_i]\}|=\lambda$. Thus we can find $E_3\subseteq E_2$, $|E_3|=\lambda$ (of course, E_3 is still indiscernibie), $E_3=\{\bar{y}^i : i<\mu+\lambda\}$ such that $\models\psi[a,\bar{y}^i]$ iff $i<\mu$. Let q be the type that a realizes over $A_3=\cup\{\mathrm{Rang}\,y^i : i<\mu+\lambda\}$, and let I be a subset of $\mu+\lambda$ of power μ whose complement is of power λ. Since E_3 is an indiscernible set, there is a mapping F_I from A_3 to A_3 such that $F_I(y^i)\in\{\bar{y}^k : k\in I\}$ iff $i<\mu$. It is easily seen that $I\neq J$ implies $F_I(q)\neq F_J(q)$, and hence $|S_{\mathrm{D}}(A_3)|\geq|\{F_I(q) : I \text{ as above}\}|\geq\lambda^{\mu}>\lambda$. But $|A_3|\leq\lambda$ in contradiction to the λ-stability of D. Thus the claim is verified.

Theorem 4.3. *If* D *is* λ*-stable,* $\kappa\geq\lambda$, $\kappa^{\kappa^1}>\kappa$, *and* D *satisfies* $(C{*}\kappa^1)$, *then* D *is not* κ*-stable.*

Proof: Let $\chi=\inf\{\chi : \kappa^{\chi}>\kappa\}$, $\mu=\inf\{u : \lambda^{\mu}>\lambda\}$. Clearly $\kappa^1\geq\chi$ and thus D satisfies $(C{*}\chi)$. Also $\chi<\kappa$; this is because D satisfies $(C{*}\chi)$ and hence also $({*}\chi)$ and thus is not (2.7) stable in any power $<2^{\chi}$; thus, $\kappa\geq\lambda\geq2^{\chi}$. Assume that A_i $(i\leq\chi)$ is an increasing sequence of sets, $|A_i|<|i|^{+}+\aleph_0$, $p\in S_{\mathrm{D}}(A_{\chi})$, such that for all $i<\chi$ $p|A_{i+1}$ splits strongly over A_i in A_{χ}, $\{\bar{a}_{i,j} : j<\omega\}$ is an indiscernible set over A_i of sequences from A_{χ}, and $\psi_i(x,\bar{a}_{i,0})$, $\neg\,\psi_i(\bar{a}_{i,1})\in p|A_{i+1}$. By Theorem 3.4 D is good and thus there is a (D,κ^{+})-homogeneous model M.

In this proof η and τ will denote sequences of ordinals $<\kappa$. For $l(\eta)<\chi$ we define p_η, A_η, and G_η by induction, such that:

1) $p_\eta\in S_{\mathrm{D}}(A_\eta)$ and $\eta=\tau|i$ implies $p_\eta\subseteq p_\tau$, $A_\eta\subseteq A_\tau$, $|A_\eta|\leq\chi$.

2) G_η is a mapping from $A_{l(\eta)}$ onto $A_\eta\subset|M|$.

3) There is no element in M which realizes $\geq\mu$ of the types $\{p_{\eta\frown\langle j\rangle} : j<\kappa\}$.

If $l(\eta) = 0$, $G_{\langle\rangle}$ will be a mapping from A_0 into M, $A_{\langle\rangle} = \text{Rang } G_{\langle\rangle}$, and $p_{\langle\rangle} = G_{\langle\rangle}(p|A_0)$.

If $l(\eta) = \delta$, $G_\eta = \bigcup_{i<\delta} G_{\eta|i}$, $A_\eta = \bigcup_{i<\delta} A_{\eta|i}$, $p_\eta = \bigcup_{i<\delta} p_{\eta|i}$.

If $l(\eta) = k$, G_η, A_η, and p_η are already defined, we proceed to define them for $\eta^\frown\langle j\rangle$ for all $j < \kappa$. Since $|A_\chi| < \chi^+$, G_η has an extension G which is a mapping from A_χ into M.

Let $\bar{b}_{\eta,i} = G(\bar{a}_{k,i})$ for all $i < \omega$. Clearly $\{\bar{b}_{\eta,i} : i < \omega\}$ is an indiscernible set over A_η and we can extend it to an indiscernible set $\{\bar{b}_{\eta,i} : i < \kappa\}$. For all $\tau = \eta^\frown\langle j\rangle$ where $j < \kappa$ we define the mapping H_τ : Dom $H_\tau = A_k \cup \text{Rang } \bar{a}_{k,0} \cup \text{Rang } \bar{a}_{k,1}$, $H_\tau | A_k = G_\eta$, and $H_\tau(\bar{a}_{k,i}) = \bar{b}_{\eta,j+i}$, $i = 0, 1$. G_τ will be a mapping from A_{k+1} into M which extends H_τ, $A_\tau = \text{Rang } G_\tau$, $p_\tau = G_\tau(p|A_{k+1})$.

Conditions 1) and 2) clearly hold. We now show 3). Since $\psi_{l(\eta)}(x, \bar{b}_{\eta,i})$, $\neg\, \psi_{l(\eta)}(x, \bar{b}_{\eta,i+1}) \in p_{\eta^\frown\langle i\rangle}$, the condition follows from Claim 4.2.

Let $A = \bigcup\{A_\eta : l(\eta) < \chi\}$. Clearly $|A| \leq \Sigma\{|A_\eta| : l(\eta) < \chi\} \leq \kappa$. $\kappa^{(\chi)} = \kappa$ ($\kappa^{(\chi)} = \kappa$ by the definition of χ). For $l(\eta) = \chi$ let $p_\eta = \bigcup_{i<\chi} p_{\eta|i}$, let a_η be an element of M which realizes p_η, and let q_η be the type which a_η realizes over A. From Condition 3) it is easily seen that for all a, $|\{\eta : a_\eta = a\}| \leq \mu^\chi$. If $\chi^\chi > \mu^\chi$ it is easy to see that $|S_D(A)| \geq |\{a_\eta : l(\eta) = \chi\}| > \kappa$. Assume now $\kappa^\chi \leq \mu^\chi$. $\lambda^\lambda > \lambda$ and thus by definition $\mu \leq \lambda$. Hence, $\lambda \leq \kappa < \kappa^\chi \leq \mu^\chi \leq \lambda^\chi$ and $\mu \leq \chi$ by the definition of μ. Thus, $\lambda < \kappa^\chi \leq \mu^\chi \leq 2^\chi$. But D satisfies $(C{*}\chi)$ and thus also $({*}\chi)$. It follows that D is not λ-stable since $\lambda < 2^\chi$ (by Theorem 2.7); contradiction. The theorem is proved.

Theorem 4.4. *Every* D *satisfies exactly one of the following:*

1) D *is not stable;*

2) *there are powers* κ, λ, $\lambda < \beth[(2^{|T|})^+]$, *such that* D *is* μ*-stable iff* $\mu \geq \lambda$ *and* $\mu^{(\kappa)} = \mu$.

Proof: If D is not stable, the conclusion of the theorem clearly is true. Assume then that D is stable and let λ be the first power such that D is λ-stable. By Theorem 2.9 $\lambda < \beth[(2^{|T|})^+]$. Define $\kappa = \inf\{\kappa :$ for all A, $|S_D(A)| \leq |A|^{(\kappa)} + \lambda\}$. We shall show that λ and κ satisfy 2).

If $\mu < \lambda$ then by the definition of λ, D is not μ-stable. Assume now that $\mu \geq \lambda$ and $\mu^{(\kappa)} = \mu$. Since D is stable, it is good, and thus μ^+-good. Also, by the definition of κ, if $|A| \leq \mu$, then $|S_D(A)| \leq |A|^{(\kappa)} + \lambda = \mu$, and thus D is μ-stable.

The last thing to show is that if $\mu \geq \lambda$ and $\mu^{(\kappa)} > \mu$, then D is not μ-stable. Let $\chi = \inf\{\chi : \mu^\chi > \mu\}$. Since $\mu^{(\kappa)} > \mu$, clearly $\chi < \kappa$. Thus by the definition of κ, there is an A such that $|S_D(A)| > |A|^{(\chi)} + \lambda$. It follows from Theorem 4.1 that D satisfies $(C{*}\chi)$. By Theorem 4.3 D is not μ-stable, and the theorem is proved.

Remark: A theory T can be found such that the values of λ and κ in 4.4 for D(T) are $\lambda = 2^{|T|}$ and κ is any power $\leq |T|^+$.

§ 5. On prime models

In this section we prove the existence of prime models over D-sets, for several definitions of primeness (especially for stable D), and several of their properties.

Definition 5.1. 1) $p \in S_D^m(A)$ is (D, 1)-*isolated*, or simply (1)-*isolated*, over $B \subseteq A$ if for all $A_1, B \subseteq A_1 \subseteq A$, $p|A_1$ is the unique extension of $p|B$ in $S_D(A_1)$. (Formally, we do have to mention the D, but it will usually be clear what D is. The same holds for the following definitions.)

2) $p \in S_D^m(A)$ is (2)-*isolated* over $B \subseteq A$ if for all A_1, $B \subseteq A_1 \subseteq A$, $p|B$ has no extension in $S_D(A_1)$ which splits over B.

3) $p \in S_D^m(A)$ is (3)-*isolated* over $B \subseteq A$ if for all A_1, $B \subseteq A_1 \subseteq A$, $p|B$ has no extension in $S_D(A_1)$ which splits strongly over B in any D-set A_2.

4) $p \in S_D^m(A)$ is (4)-*isolated* over $B \subseteq A$ if for all A_1, $B \subseteq A_1 \subseteq A$, $p|B$ has no extension in $S_D(A_1)$ which splits strongly over B in A_1.

5) p is (λ, n)-*isolated* (or (D, λ, n)-isolated) if there is a $B \subseteq A$, $|B| < \lambda$, such that p is (n)-isolated over B ($n = 1, 2, 3, 4$). (Where n is mentioned in the statement of a theorem in this section it will be assumed that $n = 1, 2, 3, 4$).

Claim 5.1. 1) *If p is (λ, n_1)-isolated, $n_1 \leq n_2$, then p is (λ, n_2)-isolated.*

2) *If $p \in S_D^m(A)$ is (n)-isolated over $B \subseteq B_1 \subseteq A_1 \subseteq A$, then $p|A_1$ is (n)-isolated over B_1. If $|B| < \lambda$, then $p|A_1$ is also (λ, n)-isolated.*

Proof: Immediate.

Definition 5.2. A D-model M is (D, λ, n)-*homogeneous* if for all $p \in S_D(A)$, $A \subseteq M$, p (λ, n)-isolated, we have p is realized in M.

Definition 5.3. 1) The D-set $A \supseteq B$ is (D, λ, n)-*prime* over B if $A = B \cup \{a_i : i < i_0\}$ where, for all $i < i_0$, a_i realizes a (D, λ, n)-isolated type over $B \cup \{a_j : j < i\}$.

2) If $|M| \supseteq B$ then M is a (D, λ, n)-*prime model* over B if $|M|$ is a (D, λ, n)-prime set over B and M is (D, λ)-homogeneous (sic).

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Claim 5.2. 1) *If A is a* (D, λ, n)*-prime set over B, M is a* (D, λ, n)*-homogeneous model, and $B \subseteq |M|$, then there is a mapping F from A onto M such that $F \upharpoonright B = I_B$.*

2) *If M is* (D, λ, n)*-homogeneous, $n_1 \leq n$, then M is* (D, λ, n_1)*-homogeneous.*

3) *If A is* (D, λ, n)*-prime over B, $n_1 \geq n$, then A is* (D, λ, n_1)*-prime over B.*

4) *If M, M_1 are* (D, λ, 1)*-prime models over A, then there is a mapping F from M into M_1 such that $F \upharpoonright A = I_A$.*

5) *If $p \in S_D(B)$, $B \subseteq A$, and there is a* (D, λ, n)*-homogeneous model $M \supseteq A$ which omits p, then every* (D, λ, n)*-prime set over A omits p. For $n = 1$, either all or none of the* (D, λ, 1)*-prime models over A omit p.*

Proof: Immediate.

Definition 5.4. D *satisfies* (P, λ, n) if for all $p \in S_D(B)$, $|B| < \lambda$, $A \supseteq B$, A a D-set, we have that p has an extension $q \in S_D(A)$ which is (λ, n)-isolated.

Remark: It apparently would seem more logical to define a (D, λ, n)-prime model by Claim 5.2.1 and say that D satisfies (P, λ, n) if for all $B \subseteq A$, $p \in S_D(B)$, p (λ, n)-isolated, p has an extension $q \in S_D(A)$ which is (λ, n)-isolated (instead of definitions 5.3.2 and 5.4). But problems would arise in Theorem 5.3 and in the theorem proving the existence of (P, λ, n) at the end of the section.

Theorem 5.3. 1) *If* D *satisfies* (P, λ, n)*, then over every* D*-set A there is a* (D, λ, n)*-prime model. If $A \subseteq |N|$ where N is* (D, λ)*-homogeneous, then there is a* (D, λ, n)*-prime model $M \subseteq N$ over A.*

2) *Moreover, if $B \supseteq A$ where B is a* D*-set, then there is a* (D, λ, n)*-prime model M over A such that $B \cup |M|$ is prime over B.*

Proof: 1) As in the proof of Theorem 3.4 it can be shown that there is a model M, $|M| \supseteq A$, $|M| = A \cup \{a_i : i < i_0\}$ such that for all $i < i_0$, a_i realizes a (λ, n)-isolated type over $A \cup \{a_j : j < i\}$ and for all $B \subseteq |M|$, $|B| < \lambda$, $p \in S_D(B)$, p is realized in M. It is clear that M is a (D, λ, n)-prime model over A.

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2) The only additional thing to be proved here is: if for all $j < i$, a_j is defined and $p \in S_D(A_1)$ where $A_1 \subseteq A \cup \{a_j : j < i\}$, $|A_1| < \lambda$, then there is a q, $p \subseteq q \in S_D(B \cup \{a_j : j < i\})$, such that q and $q \restriction (A \cup \{a_j : j < i\})$ are (λ, n)-isolated. Since D satisfies (P, λ, n) there is a q_1, $p \subseteq q_1 \in S_D(A \cup \{a_j : j < i\})$ and there is an $A_2 \subseteq A \cup \{a_j : j < i\}$, $|A_2| < \lambda$, such that q_1 is (n)-isolated over A_2. Again by (P, λ, n) $q_1 \restriction A_2$ has a continuation in $S_D(B \cup \{a_j : j < i\})$ which is (λ, n)-isolated. This continuation is the required q.

Claim 5.4. *Let M be* (D, λ, n)*-prime over A. If $C \subseteq |M|$, $|C| < \lambda$, λ regular, then there is a $B \subseteq |M|$, $B \supseteq C$, $|B| < \lambda$, such that for all $A_1 \subseteq A$, $B \cup A_1$ is* (D, λ, n)*-prime over $(A \cap B) \cup A_1$. In fact, $B = (B \cap A) \cup \{b_i : i < i_0 < \lambda\}$ and the type which b_i realizes over $A \cup \{b_j : j < i\}$ is* (D, n)*-isolated over $(B \cap A) \cup \{b_j : j < i\}$.*

Proof: Assume $|M| = A \cup \{a_j : j < i_0\}$, a_i realizes the type p_i over $A \cup \{a_j : j < i\}$, $A_i^* \subseteq A \cup \{a_j : j < i\}$, $|A_i^*| < \lambda$, and p_i is (D, n)-isolated over A_i^*. Define B_k for $k < \omega$ by $B_0 = C$, $B_{m+1} = B_m \cup \bigcup \{A_i^* : a_i \in B_m\}$, and $B = B_\omega = \bigcup_{m<\omega} B_m$. By the regularity of λ it is easy to see that $|B_k| < \lambda$ and thus $|B| < \lambda$. We now show by induction on i that $(B \cap A) \cup A_1 \cup (B \cap \{a_j : j < i\})$ is (D, λ, n)-prime over $(B \cap A) \cup A_1$. This will finish the proof. If $i = 0$, i is a limit ordinal, or $a_i \notin B$, the claim is immediate. If i is not a limit ordinal and $a_i \in B$, from Claim 5.1.2 it follows that a_i realizes a (D, λ, n)-isolated type over $(B \cap A) \cup A_1 \cup (B \cap \{a_j : j < i\})$. This completes the induction. For the last statement in the claim, we may take b_i as the ith element of $\{a_j : j < i_0\}$ which belongs to B.

Theorem 5.5. *Assume D satisfies (P, λ, n), $p \in S_D(B)$, $B \subseteq A$, A is a D-set, λ is regular, and for all $B_1 \subseteq A$ with $|B_1| < \lambda$, there is a* (D, λ, n)*-homogeneous model M such that $B \cup B_1 \subseteq |M|$ and p is not realized in M. Then every* (D, λ, n)*-prime model over A omits p.*

Proof: From Theorem 5.3.1 and Claim 5.4 it follows that for every (D, λ, n)-prime model M over A and for all $C \subseteq |M|$, $|C| < \lambda$, there is a $B \subseteq |M|$ $C \subseteq B$ such that for all $A_1 \subseteq A$, $B \cup A_1$ is (D, λ, n)-prime over $A_1 \cup (B \cap A)$.

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Assume $a \in |M|$ realizes p. Let $C = \{a\}$. By the above, there is $B_1 \subseteq |M|$, $|B_1| < \lambda$, with $a \in B_1$ such that $B_1 \cup B$ is prime over $(B_1 \cap A) \cup B$. Since $|B_1 \cap A| \leq |B_1| < \lambda$, there is a model M, $(B_1 \cap A) \cup B \subseteq |M_1|$, M_1 (D, λ, n)-homogeneous, and M_1 omits p. Thus by 5.2.5 the set $B_1 \cup B$ also omits p, in contradiction to the definition of B_1. Thus M omits p and the theorem is proved.

Claim 5.6. *Assume $p_i \in S_D(B_i)$, $A_i \supseteq B_i$, A_i a D-set $(i = 1, 2)$, F is a mapping from A_1 on A_2, $F(B_1) = B_2$, $F(p_1) = p_2$. Then p_1 is realized in a (D, λ, n)-prime model over A_1 iff p_2 is realized in a (D, λ, n)-prime model over A_2*; provided that $F(A_1) = A_2$.

Proof: Immediate.

Corollary 5.7. *Assume D satisfies (P, λ, n), λ regular, $p \in S_D(A)$, $\{\bar{y}_i : i < k\}$ $(k \geq \lambda)$ an indiscernible sequence over A and $A_1 = A \cup (\bigcup \{\text{Rang}\, \bar{y}_i : i < k\})$ is a D-set. If, for all $\mu < \lambda$, there is a (D, λ, n)-homogeneous model $M \supseteq A \cup \{\bar{y}_i : i < \mu\}$ which omits p, then every (D, λ, n)-prime model over A_1 omits p.*

Proof: Immediate.

Theorem 5.8. *If D satisfies (P, λ, n), λ regular, $\{y_i : i < k\}$ an indiscernible set over A, $A_k = A \cup \{y_i : i < k\}$ a D-set, and $\lambda \leq k$, then in every (D, λ, n)-prime model over A_k, $\{y_i : i < \kappa\}$ is maximal among the indiscernible sets over A; i.e., it cannot be extended.*

Remark: A parallel theorem, with a somewhat different proof appear in [16], p. 81, theorem 6.2.

Proof: We can assume that y_i and A_i are defined for all ordinals i; $l_1 < l$ implies $A_{l_1} \subseteq A_l$. Let q_l be the type that y_l realizes over A_l. Let M be a (D, λ, n)-homogeneous model, $A_k \subseteq |M|$. In M we can find a sequence $\{y'_i : i < k_1\}$ which extends $\{y_i : i < k\}$ and is a maximal sequence among the indiscernible sets over A_k. Without loss of generality we let $y'_i = y_i$ for all $i < k_1$. Thus we have a (D, λ, n)-homogeneous model M, $A_{k_1} \subseteq |M|$, which omits q_{k_1} and hence every (D, λ, n)-prime model over A_{k_1} omits q_{k_1}. (The use of the existence of M can be easily eliminated.)

Let N be a (D, λ, n)-prime model over A_k. Since $A_k \subseteq |M|$, and M is (D, λ, n)-homogeneous, we can assume, by 5.3.1, $|N| \subseteq |M|$. Assume that $a \in N$ realizes q_k and we shall arrive at a contradiction.

By Claim 5.4 there is $B = (B \cap A_k) \cup \{ b_i : i \leq i_0 \}$, $|B| < \lambda$, $b_{i_0} = a$, such that b_i realizes a type p_i over $\{ b_j : j < i\} \cup A_k$ which is (D, n)-isolated over $(B \cap A_k) \cup \{ b_j : j < i \}$. Since $|\{ y_i : y_i \in B \}| < \lambda$ and $\{ y_i : i < k \}$ is an indiscernible set over A, we can assume without loss of generality that $A \cup (B \cap A_k) = A \cup \{ y_i : i < i_1 (< \lambda)\}$. Let $A^0 = A \cup (B \cap A_k)$ and $A^l = A^0 \cup \{ b_i : i < l \}$. We prove by induction on l that $\{ y_i : i_1 \leq i < k \}$ is an indiscernible set over A^l. For $l = 0$ this is immediate and for $l = \delta$ a limit ordinal it is clear. Assume it is true for l and we shall prove it for $l + 1$.

By way of contradiction, assume that the claim for $l + 1$ is not true. Then there are two sequences $\bar{y}_1, \bar{y}_2$ of different elements of $\{ y_i : i_1 \leq i < k \}$, a formula ψ, and a sequence $\bar{a}$ of elements of A^l such that $\models \psi[b_l, \bar{a}, y^1]$ and $\models \neg \psi[b_l, \bar{a}, \bar{y}^2]$; thus $\psi(x_0, \bar{a}, \bar{y}^1)$, $\neg \psi(x_0, \bar{a}, \bar{y}^2) \in p_l$. It follows that p_l splits strongly over A^l. (It can be assumed that Rang $\bar{y}^1$ and Rang $\bar{y}^2$ are disjoint, otherwise we take a third sequence, with range disjoint from both of these, in place of $\bar{y}^1$ or $\bar{y}^2$, and then we can find a sequence $\bar{y}^n$ for $n < \omega$ such that Rang $\bar{y}^n$ are disjoint in pairs and contained in $\{ y_i : i_1 \leq i < k\}$. This shows the strong splitting.) By Claim 5.1 we get a contradiction to the definition of B in all cases of (D, λ, n). Thus the induction works for $l + 1$. It follows that $\{ y_i : i_1 \leq i < k \}$ is an indiscernible set over A^{i_0+1} $(= A \cup B)$. Now we show by induction on l that $A_{k_1} \cup \{ b_i : i < l\}$ is a (D, λ, n)-prime set over A_{k_1} and that $\{ y_i : i_1 \leq i < k_1 \}$ is an indiscernible set over $A \cup \{ y_i : i < i_1\} \cup \{ b_i : i < l \}$.

For $l = 0$ or a limit ordinal, immediate. Assume it for l. To show it for $l + 1$ it is sufficient to prove that b_l realizes over $A_{k_1} \cup \{ b_i : i < l\}$ a (D, λ, n)-isolated type p^l. (The indiscernibility of $\{ y_i : i_1 \leq i < k_1 \}$ follows as in the previous paragraph). Further we shall show that the type p^l is (D, n)-isolated over $(B \cap A_k) \cup \{ b_i : i < l \}$. First, it is clear that this type is a D-type since $B \cup A_{k_1} \subseteq M$. We take the case $n = 1$ since the others have a similar proof. If p^l is not (D, n)-isolated over $(B \cap A_k) \cup \{ b_i : i < l\}$ then $p^l \upharpoonright [(B \cap A_k) \cup \{ b_i : i < l \}]$ has two distinct continuations in $S_D(A_{k_1} \cup \{ b_i . i < l\})$, say q^1, q^2. It is clear that

$q^1|(A_k \cup \{b_i : i < l\}) = q^2|(A_k \cup \{b_i : i < l\})$ Thus there is a sequence $\bar{c}$ from A^l and a sequence $\bar{y}$ from $\{y_i : i_1 \leq i < k_1\}$ and a formula ψ such that $\psi(x, \bar{y}, \bar{c}) \in q^1$, $\neg\, \psi(x, \bar{y}, \bar{c}) \in q^2$. By induction hypothesis there is a mapping F from $A_{k_1} \cup \{b_i : i < l\}$ onto itself such that $F|A^l = I_{A^l}$ and $F(\bar{y})$ is a sequence from $\{y_i : i_1 \leq i < k\}$. It is easy to show that $F(q^1)|(A_l \cup \{y_i : i_1 \leq i < k\})$ and $F(q^2)|(A_l \cup$ $\cup \{y_i : i_1 \leq i < k\})$ belong to $S_D(A_k \cup \{y_i : i_1 \leq i < k\})$ and extend $p^l|[(B \cap A_k) \cup \{b_i : i < l\}]$, in contradiction to the assumption that b_l realizes over $A_l \cup \{y_i : i_1 \leq i < k\}$ a (D, n)-isolated type over $(B \cap A_k) \cup \{b_i : i < l\}$.

It follows that $B \cup A_{k_1} \subseteq M$ is indeed a (D, λ, n)-prime set over A_{k_1}. It was proved that $\{y_i : i_1 \leq i < k_1\}$ is an indiscernible set over $A \cup \{y_i : i < i_1\} \cup \{b_i : i \leq i_0\}$ and thus over $A \cup \{y_i : i < i_1\} \cup \{a\}$. It is known that $\{y_i : i < k\} \cup \{a\}$ is an indiscernible set over A. Since $i_1 < \lambda \leq k$, we get that $\{y_i : i < k_1\} \cup \{a\}$ is an indiscernible set over A. Thus in M, $\{y_i : i < k_1\}$ is not a maximal indiscernible set over A: contradiction. This proves the theorem.

Now we shall check when the conditions (P, λ, n) are satisfied.

Theorem 5.9. *If* D *is* λ*-good and* $(B{*}\lambda)$ *is not satisfied, then* D *has* $(P, \lambda, 1)$.

Proof: Assume $p \in S_D(B)$, $|B| < \lambda$, $B \subseteq A$, A a D-set. Since D is λ-good, it is easy to see that every $q \in S_D(B_1)$ has an extension in $S_D(B_2)$ where $B_1 \subseteq B_2 \subseteq A$ and $|B_2| < \lambda$. Also, if $q \in S_D(B_1)$, $|B_1| < \lambda$, and for all $B_2 \subseteq A$, $|B_2| < \aleph_0$, q has a unique extension in $S_D(B_1 \cup B_2)$, then q has a unique extension in $S_D(C)$ for all $C \subseteq A$, and it is clear that the extension of q in $S_D(A)$ is a $(\lambda, 1)$-isolated type.

We must show that p has an extension $p_1 \in S_D(A)$ which is $(\lambda, 1)$-isolated. Assume there is none, and we shall produce a contradiction. Since there is no such extension, for all B_1, $B \subseteq B_1 \subseteq A$, $|B_1| < \lambda$ and for all $q \in S_D(B_1)$, $q \supseteq p$, there is a finite set $C_q \subseteq A$ such that q has at least two extensions in $S_D(B_1 \cup C_q)$. We want to define by induction on $l(\eta) < \lambda$ p_η and A_η such that $p_\eta \in S_D(A_\eta)$, $\eta = \tau | i$ implies $p_\eta \subseteq p_\tau$, $p_{\langle\rangle} = p$, $A_\eta \subseteq A$, $|A_\eta| < \aleph_0 + |l(\eta)|^+$, and there is a formula φ_η such that $\varphi_\eta \in p_{\eta^\frown\langle 0\rangle}$, $\neg\, \varphi_\eta \in p_{\eta^\frown\langle 1\rangle}$. From the completed induction will follow the existence of $(B{*}\lambda)$, in contradiction to the hypothesis of the theorem. For $l(\eta) = 0$ define $A_{\langle\rangle} = B$, $p_{\langle\rangle} = p$. If $l(\eta) = \delta$ let $A_\eta = \bigcup_{i<\delta} A_{\eta|i}$, $p_\eta = \bigcup_{i<\delta} p_{\eta|i}$; $|A_\eta| \leq \sum_{i<\delta} |A_{\eta|i}| \leq \sum_{i<\delta} (\aleph_0 + |i|) = |\delta| < \aleph_0 + |l(\eta)|^+$. Now assume $l(\eta) = k$ and A_η, p_η, are defined. Since $B \subseteq A_\eta$, $p \subseteq p_\eta$, $|A_\eta| < \lambda$, there is a finite set $C_{p_\eta} \subseteq A$ such that p_η has two extensions $p_{\eta\frown\langle 0\rangle}$, $p_{\eta\frown\langle 1\rangle}$ in $S_D(A_\eta \cup C_{p_\eta})$. We define $A_{\eta\frown\langle 0\rangle} = A_{\eta\frown\langle 1\rangle} = A_\eta \cup C_q$. It is easily seen that this definition satisfies the requirements, and thus the theorem is proved.

Theorem 5.10. *If* D *is good and does not satisfy* $(*\lambda)$, *then* $(P, \lambda, 2)$ *holds. If* D *is good and does not satisfy* $(C*\lambda)$ *then* $(P, \lambda, 3)$ *and* $(P, \lambda, 4)$ *hold.*

Proof: Immediate.

Theorem 5.11. 1) *If* D *is* λ*-stable,* $2^\mu > \lambda$, *then* D *satisfies* $(P, \mu, 1)$, $(P, \mu, 2)$, $(P, \mu, 3)$, $(P, \mu, 4)$. *If* D *is* λ*-stable,* $\lambda^\kappa > \lambda$, *then* D *satisfies* $(P, \kappa, 3)$, $(P, \kappa, 4)$.

2) *If* D *is* λ*-stable,* $\mu > \lambda$, *then* p *is* (D, μ, n)*-isolated iff* p *is* (D, μ, m)*-isolated, for all* $1 \leq m, n \leq 4$.

Proof: 1) If D is λ-stable, $2^\mu > \lambda$, then by 2.7, 2.6, 4.3, D doesn't satisfy $(A*\lambda)$, $(*\lambda)$, or $(C*\lambda)$, and thus the theorem follows from 5.10, 5.9. (If D is stable, there is no difference between $(A*\lambda)$, $(B*\lambda)$.)

If D is λ-stable, $\lambda^\kappa > \lambda$, then by 4.3 D doesn't satisfy $(C*\lambda)$ Thus the theorem follows from 5.10.

2) The proof is similar to the proof of Theorem 4.1.

§6. On the existence of maximally (D, λ)-homogeneous models

Definition 6.1. A model is *maximally* (D, λ)*-homogeneous* (λ*-homo.*) if it is (D, λ)-homogeneous (resp. λ-homo.) but not (D, λ^+)-homogeneous (resp. λ^+-homo.).

Theorem 6.1. *If* $|D| \leq \lambda$ $(|T| \leq \lambda)$, $\lambda^{(\kappa)} = \lambda$, *κ a regular cardinal and* D *satisfies* $(B*(2^\lambda)^+)$, *then there is a maximally* (D, κ)*-homogeneous (resp. κ-homo.)* D*-model* M_0 *of power* λ. *Furthermore, if N is a* D*-model of power* $\leq \lambda$ *we can choose* M_0 *so that* $|N| \subseteq |M_0|$. *Instead of demanding that* D *satisfy* $(B*(2^\lambda)^+)$ *we can take* D *to be good and satisfying* $(C*\kappa)$.

Proof: We just prove the case where $(B*(2^\lambda)^+)$ holds. By Claim 2.4 we can assume that there is a (D, $(2^\lambda)^+$)-homogeneous model M and there are p_η, A_η for all $l(\eta) < (2^\lambda)^+$ such that $p_\eta \in S_D(A_\eta)$, $A_\eta \subseteq |M|$, $\eta = \tau | i$ implies $A_\eta \subseteq A_\tau$, $p_\eta \subseteq p_\tau$, and there is $\varphi_\eta \in p_{\eta\frown\langle 0\rangle}$, $\neg\varphi_\eta \in p_{\eta\frown\langle 1\rangle}$, where $\varphi_\eta = \psi_\eta(x_0, \bar{a}_\eta)$.

Since M is (D, $(2^\lambda)^+$)-homogeneous we can assume without loss of generality that $|N| \subseteq |M|$.

We define N_i, C_i, and η_i for $i \leq \kappa$ such that:

1) $|N_i| \subseteq |M|$, $\|N_i\| = \lambda$, every complete D(N_i)-type over a subset of $|N_i|$ which is of power $< \kappa$ is realized in N_{i+1} : $N_\delta = \bigcup_{i<\delta} N_i$, and thus N_i is an increasing sequence. If $|D| > \lambda$, $N_0 = N$ and if $|D| \leq \lambda$, N_0 is an extension in M of N which realizes every type from D.

2) $C_i \subseteq |N_{i+1}|$, $C_i \subseteq A_{\eta_{i+1}}$, $|C_i| < \aleph_0$, $p_{\eta_{i+1}} | C_i$ is not realized by any element of N_i but $p_{\eta_{i+1}} | (\bigcup_{j \leq i} C_j)$ is realized in N_{i+1}. Also if $l(\eta_i) = l_i$, $i < j$, then $\eta_j | l_i = \eta_i$.

(Let D_i denote D(N_i).)

It is easy to see that when the definition is completed, N_κ will be the required model. This follows because if $A \subseteq |N_\kappa|$, $|A| < \kappa$, $p \in S_{D_\kappa}(A)$ then there is an $i < \kappa$ such that $p \in S_{D_i}(A)$ and $A \subseteq |N_i|$, and thus p is realized in N_{i+1}, and a fortiori in N_κ; i.e., N_κ is (D_κ, κ)-homogeneous. If in addition $|D| \leq \lambda$, then $D = D(M) \supseteq D(N_\kappa) \supseteq D(N_0) = D$, and thus N_κ is (D, κ)-homogeneous.

On the other hand, $p_{\eta_\kappa} | (\bigcup_{i<\kappa} C_i)$ is a type such that its restriction to every finite subset of $\bigcup_{i<\kappa} C_i$ is realized, but itself is not realized. Thus N_κ is not κ^+-homogeneous. (Actually the proof shows that the above type has a sub-type of power κ which is not realized.)

We now proceed to carry out the definition. N_0 was already defined; take $\eta_0 = \langle\rangle$. It is clear what are N_δ, η_δ for limit ordinals δ. Assume that N_i and η_i are defined; we define C_i, N_{i+1} and η_{i+1} as follows: Since $\lambda^{(\kappa)} = \lambda$, the number of subsets of $|N_i|$ of power $< \kappa$ is $\leq \lambda$, and if $|A| < \kappa$, $A \subseteq |N_i|$, then $|S_{D(N_i)}(A)| \leq |D(N_i)|^{|A|} \leq \|N_i\|^{|A|} \leq \lambda^{(\kappa)} = \lambda$. Thus there is a set $B \supseteq |N_i|$, $|B| \leq \lambda$, $B \subseteq |M|$, in which every type $p \in S_{D(N_i)}(A)$, for all $A \subseteq |N_i|$, $|A| < \kappa$, is realized.

For all $k < (2^\lambda)^+$ define $\tau_k = \eta_i{}^\frown 0_k$, where 0_k is a sequence of zeroes of order type k. For all $a \in |N_i|$ define $W_a = \{ k : k < (2^\lambda)^+,$ $\models \psi_{\tau_k}(a, \bar{a}_{\tau_k}) \}$. Since there are $\leq \lambda$ sets W_a, and $W_a \subseteq (2^\lambda)^+$, there are ordinals l, j $(j < l)$ such that for all $a \in |N_i|$, $l \in W_a$ iff $j \in W_a$. Let $\eta_{i+1} = \tau_{l^\frown\langle 1\rangle}$ and $C_i = \text{Rang}\, \bar{a}_{\tau_l} \cup \text{Rang}\, \bar{a}_{\tau_j}$. Since every $a \in |N_i|$ satisfies $\models \psi_{\tau_l}[a, \bar{a}_{\tau_l}] \leftrightarrow \psi_{\tau_j}[a, \bar{a}_{\tau_j}]$ and $\psi_{\tau_j}(x, \bar{a}_{\tau_j}) \wedge \neg\, \psi_{\tau_l}(x, \bar{a}_{\tau_l}) \in p_{\eta_{i+1}} | C_i$, it is clear that no element of N_i realizes $p_{\eta_{i+1}} | C_i$. Now it is easy to find a model N_{i+1} such that $B \cup C_i \subseteq |N_{i+1}| \subseteq |M|$ and such that $p_{\eta_{i+1}}$ is realized in N_{i+1}. Thus we have finished the inductive definition and proved the theorem.

Claim 6.2. *If there is a* (D, λ^+)*-homogeneous model M such that* $\|M\| = 2^{2^\lambda}$, $S_D(|M|)| > \|M\|$, $N \subseteq M$, $\|N\| \leq \lambda$, *κ a regular cardinal,* $\lambda = \lambda^{(\kappa)}$, $|D| \leq \lambda$ $(|T| \leq \lambda)$, *then there is a* D*-model* $M_1 \supseteq N$ *of power* λ *which is maximally* (D, κ)*-homogeneous* (*resp.* κ*-homo.*).

Proof: The proof is similar to 6.1. We shall prove the case $|D| \leq \lambda$. As in the proof of 2.5, there is a $p \in S_D(M)$ which splits over every subset of $|M|$ of power $\leq \lambda$. Let us define an increasing sequence N_i, $i \leq \kappa$, such that $|N_0| \supseteq |M|$; $|N_0| \subseteq |M|$; $\|N_i\| = \lambda$; every D-type (over the empty set) is realized in N_0; for all $\delta \leq \kappa$, $|N_\delta| = \bigcup_{i<\delta} |N_i|$; for all $i < \kappa$, every complete D-type over a subset of $|N_i|$ of power $< \kappa$ is realized in N_{i+1}; and there is a finite set $C_i \subseteq |N_{i+1}|$ such that $p | C_i$ is not realized in N_i, $p\restriction \bigcup_{j\le i} C_j$ is realized in N_{i+1}. The only problem is to define C_i.

Since by the definition of p, p splits over $|N_i|$, there are sequences $\bar a$, $\bar a_1 \in |M|$ which realize the same type over $|N_i|$ and $\varphi(x,\bar a) \wedge \wedge \neg\varphi(x,\bar a_1) \in p$. Thus $C_i = \operatorname{Rang}\bar a \cup \operatorname{Rang}\bar a_1$ is the required set.

Corollary 6.3. *If* D *is good but not st le, and* $|D| \le \lambda$, $\lambda^{(\kappa)} = \lambda$, κ *regular.* N *a* D*-model,* $\|N\| \le \lambda$, *then there is a* D*-model* $M \supseteq N$ *of power* λ *which is maximally* (D, κ)*-homogeneous.*

Proof: Since D is not 2^{2^λ}-stable, there is a D-set A, $|A| = 2^{2^\lambda}$, such that $|S_D(A)| > |A|$. Since D is good, there is a (D, λ)-homogeneous model M, $A \subseteq |M|$, $|A| = \|M\|$, $M \supseteq N$. Since D is good $|S_D(|M|)| \ge |S_D(A)| > 2^{2^\lambda}$. Thus the corollary follows from Claim 6.2. It can also be derived from Theorem 6.1.

Theorem 6.4. *Let* $\lambda = \beth_\delta$, $\kappa = cf\,\lambda < \lambda$. *If* D *satisfies* $(B*\lambda)$, *then there exists a* D*-model* N, $\|N\| \le \lambda$, *such that there is no* (D, κ^+)*-homogeneous model* $M \supseteq N$ *with* $\|M\| = \lambda$.

Remark: As in Theorem 6.1, in place of the assumption $(B*\lambda)$, we can take D to be λ-good and satisfying $(C*\kappa)$.

Proof: Since D satisfies $(B*\lambda)$, by Claim 2.4 there is a model N, $\|N\| = \lambda$, for all $l(\eta) < \lambda$ there are A_η, p_η such that $A_\eta \subseteq |N|$, $p_\eta \in S_D(A_\eta)$, $\tau = \eta\restriction i$ implies $p_\tau \subseteq p_\eta$ and $A_\tau \subseteq A_\eta$, and there are formulas $\psi_\eta(x,\bar a_\eta)$ such that $\psi_\eta(x,\bar a_\eta) \in p_{\eta\frown\langle 0\rangle}$, $\neg\psi_\eta(x,\bar a_\eta) \in p_{\eta\frown\langle 1\rangle}$. Since cf $\lambda = \kappa$, there is an increasing sequence of cardinals $\lambda_i < \lambda$ such that $\lambda = \sum_{i<\kappa} \lambda_i$.

Let M be any D-model of power λ, $|M| \supseteq |N|$. We shall prove that M is not (D, κ^+)-homogeneous. Since $\|M\| = \lambda$, there is an increasing sequence of sets A_i, $|A_i| = \lambda_i$, such that $|M| = \bigcup_{i<\kappa} A_i$. Now we shall define an increasing sequence η_i for $i < \kappa$ (i.e., $\eta_j\restriction l(\eta_i) = \eta_i \Longleftrightarrow i < j$) and a sequence of finite sets $C_i \subseteq |M|$ such that: $l(\eta_i) < (2^{\lambda_i})^+$ and $p_{\eta_i}\restriction C_i$ is not realized by any element of A_i. If η_i are defined for all $i < j$ then $l(\eta_i) < (2^{\lambda_j})^+$ and there is an η with $l(\eta) < (2^{\lambda_j})^+$ such that $\eta_i = \eta\restriction l(\eta_i)$ for all $i < j$. Take $\tau_k = \eta\frown\langle 0 : i < k\rangle$. As in the proof of 6.1 we can find

$l < k < (2^{\lambda_i})^+$ such that all elements a of A_j satisfy $\models \psi_{\tau_k}[a, \bar{a}_{\tau_k}] \leftrightarrow \psi_{\tau_l}[a, \bar{a}_{\tau_l}]$. Choose $\eta_i = \tau_k{}^\frown\langle 1\rangle$ and $C_j = \text{Rang}\,\bar{a}_{\tau_k} \cup \text{Rang}\,\bar{a}_{\tau_l}$. It is easy to see that all the conditions are satisfied. If $C = \bigcup_{i<\kappa} C_i$, $p = \bigcup_{i<\kappa} p_{\eta_i}$ then $p|C \in S_D(C)$, $|C| \leq \kappa$, and $p|C$ is omitted by M. Thus M is not (D, κ^+)-homogeneous, as was to be proved.

Claim 6.5. *Let* D *satisfy* $(B*\lambda)$. 1) *If* M *is* (D, λ^+)*-homogeneous then* $\|M\| \geq 2^\lambda$.

2) *There is a* (D, λ)*-homogeneous model* M *of power* λ *iff* $\lambda^{(\lambda)} = \lambda$ (D *as above*).

Proof: 1) Assume M is (D, λ^+)-homogeneous. As in Claim 2.4 we can find A_η, p_η, ψ_η such that $A_\eta \subseteq |M|$, $p_\eta \in S_D(A_\eta)$, $\tau = \eta|i$ implies $A_\tau \subseteq A_\eta$ and $p_\tau \subseteq p_\eta$, $\psi_\eta \in p_{\eta^\frown\langle 0\rangle}$, $\neg\,\psi_\eta \in p_{\eta^\frown\langle 1\rangle}$, $|A_\eta| < |l(\eta)|^+ + \aleph_0$ For all η, $l(\eta) = \lambda$ define $A_\eta = \bigcup_{i<\lambda} A_{\eta|i}$, $p_\eta = \bigcup_{i<\lambda} p_{\eta|i}$. It is easy to see that $p_\eta \in S_D(A_\eta)$, $|A_\eta| \leq \lambda$. Thus for all η, $l(\eta) = \lambda$ there is $a_\eta \in |M|$ which realizes p_η. It is easy to see that if $\eta \neq \tau$, $l(\eta) = l(\tau)$, then $a_\eta \neq a_\tau$. Thus $\|M\| \geq |\{a_\eta : l(\eta) = \lambda\}| = |\{\eta : l(\eta) = \lambda\}| = 2^\lambda$.

2) Follows from 6.5.1 and 6.4.

Claim 6.6. *If* D *is stable and does not satisfy* $(C*\lambda)$ *with* λ *regular, then there is a maximally* (D, λ)*-homogeneous model* M (*of power* $\geq |D|$, *of course*).

Proof: Since D is stable, with the help of Theorem 3.1 we can find a D-set $A = \{y_i : i < \lambda\}$ which is an indiscernible set. Let M be a $(D, \lambda, 4)$-prime model over A. By definition M is (D, λ)-homogeneous. Since it realizes every type from D, its power is $\geq |D|$. On the other hand, by Theorem 5.8 A has no extension in $|M|$ which is also an indiscernible set, and thus M is not (D, λ^+)-homogeneous.

Theorem 6.7. *Assume* $D = D(M)$ *and let* λ *and* μ *be cardinals such that* $A \subseteq |M|$, $|A| \leq \lambda$ *implies* $|S_D(A)| \leq \mu$ *where* $|T| \leq \lambda \leq \mu < \|M\|$. *Then at least one of the following possibilities holds:*

1) *There is a* D*-set* $A = \{y_i : i < \omega\}$ *which is an indiscernible sequence.*

2) *There is a submodel of M of power λ which is not $\aleph_1$-homogeneous, and if* $|\mathrm{D}| \leq \lambda$, *its finite diagram is* D.
(*If* $cf(\lambda) = \omega$, *we demand only* $|A| < \lambda$ *implies* $|S_{\mathrm{D}}(A)| \leq \mu$.

Proof: First, assume that

(a) there is $N \subseteq M$, $\|N\| = \lambda$, and there is a $p \in S_{\mathrm{D}}(|N|)$ which is realized $> \mu$ times in M, such that for all N_1, $|N| \subseteq |N_1| \subseteq |M|$, $\|N_1\| \leq \lambda$, and for all $p_1 \in S_{\mathrm{D}}(|N_1|)$, $p_1 \supseteq p$, p_1 either is realized $\leq \mu$ times in M or p_1 does not split over $|N|$ (or both).

From this we shall prove that 1) holds. By induction on n define N_n, p_n, y_n such that $p_n \in S_{\mathrm{D}}(|N_n|)$, $|N_n| \subseteq |N_{n+1}|$, $\|N_n\| \leq \lambda$, $|N_n| \subseteq |M|$, $p_n \subseteq p_{n+1}$, p_n is realized $> \mu$ times in M, $y_n \in |N_{n+1}|$, y_n realizes p_n, p_n does not split over N_0. As in the proof of Theorem 3.1 we can prove here that $\{ y_n : n < \omega \}$ is an indiscernible sequence, and thus 1) holds.

We take $N_0 = N$, $p_0 = p$ (in (a)), and we let y_0 be any element of M which realizes p. Now let $m > 0$ and assume N_n, p_n, y_n are defined for all $n < m$. N_m will then be an elementary submodel of M of power λ, $\{ y_{m-1} \} \cup |N_{m-1}| \subseteq |N_m|$ (such a model exists by the Downward Löwenheim-Skolem theorem). By induction hypothesis p_{m-1} is realized $> \mu$ times in M and thus there is a $p_m \in S_{\mathrm{D}}(|N_m|)$ which is realized $> \mu$ times in M, $p_m \supseteq p_{m-1}$ (this because $|S_{\mathrm{D}}(|N_m|)| \leq \mu$). By (a) p_m does not split over N_0. y_m will be any element of M which realizes p_m. Thus we have finished the inductive definition and 1) holds.

Now assume that (a) does not hold. Then we have:

(b) For every model N, $|N| \subseteq |M|$, $\|N\| = \lambda$, and for every $p \in S_{\mathrm{D}}^{*}(|N|)$ which is realized $> \mu$ times in M, there is N_1, $|N| \subseteq |N_1| \subseteq |M|$, $\|N_1\| \leq \lambda$, and there is $p_1 \in S_{\mathrm{D}}(|N_1|)$, $p_1 \supseteq p$, which is realized $> \mu$ times in M and splits over $|N|$.

We shall show that M has a submodel of power λ which is not $\aleph_1$-homogeneous. Let N_0 be any elementary submodel of M of power λ, and let p_0 be any type in $S_{\mathrm{D}}(|N_0|)$ which is realized $> \mu$ times in M (since $|S_{\mathrm{D}}(|N_0|)| \leq \mu < \|M\|$). Define increasing sequences of models N_n and types p_n such that $|N_n| \subseteq |M|$, $\|N_n\| = \lambda$, $p_n \in S_{\mathrm{D}}(|N_n|)$, p_n is realized in N_n and p_{n+1} splits over $|N_n|$. As in the proof of 6.2 we get that $N^1 = \bigcup_{n<\omega} N_n$ is not $\aleph_1$-homogeneous and clearly $\|N^1\| = \lambda$, $N^1 \subseteq M$. Thus it will follow that 2) holds and the theorem will be proved.

The sequences are easily defined with the help of (b): If p_n, N_n are defined then by assumption there are p_{n+1}, N_{n+1} such that $|N_n| \subseteq |N_{n+1}| \subseteq |M|$, $p_n \subseteq p_{n+1} \in S_D(|N_{n-1}|)$, $\|N_{n+1}\| = \lambda$, p_{n+1} is realized $> \mu$ times in M, and p_{n+1} splits over N_n.

Theorem 6.8. *Assume that M is a model with $D(M) = D$, $|D| \leq \kappa^{(\kappa)} = \kappa$, $2^\kappa < \|M\|$, and D is not stable. Then there is a D-model of power κ^+ which is not homogeneous.*

Remark: If in addition D satisfies $(B*\kappa)$, then it is sufficient to require $|T| \leq \kappa$ instead of $|D| \leq \kappa$.

Proof (of the theorem): If M is not (D, κ^+)-homogeneous, then the theorem is immediate. Thus assume that M is (D, κ^+)-homogeneous. Since D is not κ-stable there is a D-set $A \subset |M|$ (in fact $D(A) = D$) such that $|A| = \kappa$, $|S_D(A)| > \kappa$. Let N be a submodel of M, $|N| \supseteq A$, $\|N\| = 2^\kappa$ (and so $M \neq N$), and N (D, κ^+)-homogeneous. We inductively define an increasing sequence M_i for $i \leq \kappa$ such that:

1) $|M_i| \subseteq |M|$, $\|M_i\| = \kappa$, $|M_i| \supseteq A$, $|M_i| \cap (|M| - |N|) \neq 0$;

2) If F is a mapping, $A \subseteq \mathrm{Dom}\, F \subseteq |M_i|$, $\mathrm{Rang}\, F \subseteq |M_i|$, $F|A = I_A$, $|\mathrm{Dom}\, F - A| < \kappa$, and $a \in |M_i|$, then there is an extension G of F, $\mathrm{Dom}\, G = \mathrm{Dom}\, F \cup [a]$, $\mathrm{Rang}\, G \subseteq |M_{i+1}|$, and if $\mathrm{Rang}\, F \subseteq |N|$, then $G(a) \in |N|$;

3) $M_\delta = \bigcup_{i<\delta} M_i$.

It is easy to see that such a sequence can be defined. It is also clear that if we add the elements of A as distinguished elements to the models $M^2 = M_\kappa$, M^1, $|M^1| = |M^2| \cap |N|$, then M^1 and M_2 are homogeneous, $D(M^1) = D(M^2)$. Since $|M^1| \subseteq |M^2|$, it is easy to see that there is an increasing sequence of models M^i for $0 < i < \kappa^+$ such that each one can be embedded in M^2 (as in Morley, Vaught [9] Theorem 6.2). Let M^{κ^+} be such that $|M^{\kappa^+}| = \bigcup_{i<\kappa^+} |M^i|$ and let N^1 be its reduct to the language $L(T)$. Clearly $D = D(M) \supseteq D(N^1) \supseteq D(A) = D$, or $D(N^1) = D$. Also every type in $S_D(A)$ which is realized in N^1 already was realized in M^2, and since $\|M^2\| \leq \kappa$, $|S_D(A)| > \kappa$, there is $p \in S_D(A)$ which is not realized in M^2 and thus not in N^1. Hence N^1 is not homogeneous.

Theorem 6.9. *If* D *is* λ_1-*stable and* λ_3-*stable and there is a maximally* (D, λ_2)-*homogeneous model* M *of power* $> \lambda_3$ *where* $\lambda_1 < \lambda_2 \leq \lambda_3$, *then for all regular cardinals* μ, *there are maximally* (D, μ)-*homogeneous models of arbitrarily large powers.*

By a slight refinement in the proof we can conclude:

Corollary 6.10. *If a theory* T *has a maximally* λ-*saturated model of power* $> \lambda^{|T|}$ *where* $|T| < \lambda$, *then for all regular cardinals* μ, T *has models of arbitrarily large power which are* μ-*saturated but not* μ^+-*saturated. (More exactly, maximally* (D(T), μ)-*homo.)*

Remark: The proof may be skipped in the first reading of the article.

Proof of 6.9: Let λ_0 be the first cardinal for which $(C*\lambda_0)$ does not hold. Clearly $\lambda_0 \leq \lambda_1$.

Claim 6.11. *Under the conditions of* 6.9 *there are sets* $A \supseteq A_1 \supseteq A_2$, $|A| \leq \lambda_3$, $|A_1| < \lambda_1$, $|A_2| < \lambda_0$, *and a type* $p \in S_D(A)$ *such that for all* B, $A \subseteq B \subseteq M$, $|B| \leq \lambda_3$, p *has an extension in* $S_D(B)$ *which does not split over* A_1, *is realized in* M, *and does not split strongly over* A_2. *Also every finite* D-*type over* A_1 *which is realized in* M *is realized in* A.

Proof: Assume the contrary and we shall derive a contradiction. By induction on $i \leq \lambda_1$ define A^i and A_p, B_p for all $p \in S_D(A^i)$ such that:

1. If $p \in S_D(A^i)$ then $B_p \subseteq A_p \subseteq A^i \subseteq M$, $|B_p| < \lambda_0$, $|A_p| < \lambda_1$;
2. If $i < j$ then $A^i \subseteq A^j$; for all i $|A^i| \leq \lambda_3$; $A^\delta = \bigcup_{i<\delta} A^i$.
3. If $p \in S_D(A^j)$, $q = p|A^i$, $i < j$, then $A_q \subseteq A_p$, $B_q \subseteq B_p$;
4. p does not split over A_p and does not split strongly over B_p;
5. Every finite D-type over A^i which is realized in M is realized in A^{i+1};
6. If $p \in S_D(A^i)$ then every continuation of p in $S_D(A^{i+1})$ which is realized in M either splits over A_p or splits strongly over B_p. If $p \subseteq q \in S_D(A^{i+1})$ and q does not split strongly over B_p, then $B_p = B_q$.

It is not difficult to see that the definition may be carried out and if $p \in S_D(A^{\lambda_1})$ is realized in M (and there certainly is such a p) then it follows that either $(C*\lambda_0)$ or $(*\lambda_1)$ holds; contradiction.

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Claim 6.12. *Under the conditions of* 6.9 *there are sets* $\{y_i : i < \lambda_2\}$ *and* $\{y_j^1 : j < \lambda_3^+\}$, *in M such that:*

1. $\{y_i : i < \lambda_2\}$ *is an indiscernible set over* $A_2 \cup \{y_i^1 : i < \lambda_3^+\}$;
2. $\{y_i^1 : i < \lambda_3^+\}$ *is an indiscernible set over* $A_2 \cup \{y_i : i < \lambda_2\}$;
3. $\{y_i : i < \lambda_2)$ *is a maximal indiscernible set over* A_2.

Proof: As M is maximally (D, λ_2)-homogeneous, it is not λ_2^+-homogeneous. So M omits a type q, $q \in S_{\mathrm{D}}(B)$, $|B| = \lambda_2$, $B \subseteq M$. Without loss of generality, there exists $B_2 \subseteq B_1 \subseteq B$, $|B_1| < \lambda_1$, $|B_2| < \lambda_0$, such that q does not split over B_1, and does not split strongly over B_2, and for every $C \subseteq M$, $|C| \leq \lambda_2$ q has an extension q^0 in $S_{\mathrm{D}}(B \cup C)$ which does not split over B_1, and does not split strongly over B_2, and every finite type on B_1 which is realized in M is realized in B. (This is true since D satisfies neither $(C{*}\lambda_0)$ nor $({*}\lambda_1)$; every continuation of q is also omitted, and so we can easily find an extension of q which satisfies all the above conditions).

Let us define by induction y_i for $i < \lambda_2$: Let $B = \{b_j : j < \lambda_2\}$. If for every $j < i$ y_j is defined, then, by the above, q has a continuation q_i, $q_i \in S_{\mathrm{D}}(B \cup A_2 \cup \{y_j : j < i\})$ (A_2 is defined in Claim 6.11) such that q_i does not split over B_1, and does not split strongly over B_2. As $|B_1 \cup \{b_j : j < i\} \cup A_2 \cup \{y_j : j < i\}| < \lambda_2$, there is an element y_i of M which realizes the type $q_i | (B_1 \cup \{b_j . j < i\} \cup A_2 \cup \{y_j : j < i\})$. As every finite type over B_1 which is realized in M is realized in B, $i > j$ implies $q_i \supseteq q_j$; so, as in the proof of 3.1, $\{y_i : i < \lambda_2\}$ is an indiscernible set over $B_1 \cup A_2$, and so also over A_2. Let $\{y_i : i < \alpha\}$ be a maximal indiscernible set (over A_2) which extends $\{y_i : i < \lambda_2\}$. We shall show that $\alpha < \lambda_2^+$. Suppose $\alpha \geq \lambda_2^+$. As q is omitted in M, for every y_i, $i < \alpha$, there is a formula $\psi(\lambda, \bar{a}) \in p$, such that $\models \neg \psi[y_i, \bar{a}]$. As $p \in S_{\mathrm{D}}(B)$, and $|\mathrm{D}| \leq \lambda_1 \leq |B|$ it is clear that there exists a formula $\psi(x, \bar{a}) \in p$ such that for λ_2^+ y_i's $\models \neg \psi[y_i, \bar{a}]$. On the other hand it is clear that there is $i_0 < \lambda_2$ such that Rang $a \subseteq \{b_j : j < i_0 < \lambda_2\}$. So, by the definition of the y_i's, for every i, $i_0 \leq i < \lambda_2$, $\models \psi[y_i, \bar{a}]$. So $|\{y_i : \models \psi[y_i, \bar{a}]\}| \geq \lambda_2 > \lambda_0$ and $|\{y_i : \models \neg \psi[y_i, a]\}| \geq \lambda_2 \geq \lambda_0$, but $(C{*}\lambda_0)$ does not hold, a contradiction by 4.2.

So $\alpha < \lambda_2^+$, but as $\{y_i : i < \alpha\}$ is an indiscernible set (over A_2), we can, by changing notation, get $\alpha = \lambda_2$.

Now, we shall define y_i^1 for $i < \lambda_3^+$. If y_j is defined for every $j < i$,

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let p_i be the continuation of p in $S_D(A \cup \{y_j : j < \lambda_2\} \cup \{y_j^1 : j < i\})$ which does not split over A_1 (and does not split strongly over A_2). (A, A_1, A_2 were defined in 6.11) and p_i is realized in M. Let y_i be an element in M which realizes p_i.

As in the proof of Theorem 3.1 it is clear that $\{y_i^1 ; i < \lambda_3^+\}$ is an indiscernible set over $A_2 \cup \{y_i : i < \lambda_1\}$.

It is also clear that $\{y_i : i < \lambda_2\}$ is a maximal indiscernible set over A_2.

It remains to be proved only that $\{y_i : i < \lambda_2\}$ is an indiscernible set over $A_2 \cup \{y_i^1 : i < \lambda_3^+\}$.

We shall prove by induction on j that $\{y_i : i < \lambda_2\}$ is an indiscernible set over $A_2 \cup \{y_i^1 : i < j\}$. For $j = 0$ and j a limit ordinal it is clear. Suppose it is true for j and we shall prove for $j + 1$. If it is not true for $j + 1$, there exist sequences, of different elements, $\bar{y}^1$, $\bar{y}^2$ from $\{y_i : i < \lambda_2\}$ and a sequence $\bar{c}$ from $A_2 \cup \{y_i^1 : i < j\}$ and a formula ψ, such that $\models \psi[y_j^1, \bar{c}, \bar{y}^1]$, $\models \neg \psi[y_j^1, \bar{c}, \bar{y}^2]$. As we have shown in the proof of Theorem 5.8, it follows that the type which y_j^1 realizes over $A_2 \cup \{y_i : i < \lambda_2\} \cup \{y_i^1 : i < j\}$ splits strongly over A_2, in contradiction to the definition of y_j^1. This proves 6.12.

Continuation of proof of 6.9: Suppose $\mu \le \lambda_2$, $\mu \ge \lambda_0$ and μ is regular. In any (D, μ, 4)-prime model over $A \cup \{y_i : i < \lambda_2\} \cup \{y_i^1 : i < \lambda_3^+\}$ $\{y_i : i < \lambda_2\}$ is a maximal indiscernible set over A_2 (M is (D, λ_2)-homogeneous, $\lambda_2 > \lambda_1$, and D is stable in λ_1. So, as was remarked in 5.11.2, M is (D, λ_2, 4)-homogeneous. As $M \supseteq A_2 \cup \{y_i : i < \lambda_2\} \cup$ $\cup \{y_i^1 : i < \lambda_3\}$ and $\{y_i : i < \lambda_2\}$ is maximal indiscernible in M it follows by 5.2.5).

Let $\mu_1 \ge \lambda_3$. We define y_i^1 for $\lambda_3^+ \le i \le \mu_1$ such that $\{y_i : i < \mu_1\}$ will be an indiscernible set over $A_2 \cup \{y_i : i < \lambda_2\}$. By Theorem 6.7 it is clear that in any (D, μ, 4)-prime model over $A_2 \cup \{y_i : i < \lambda_2\} \cup$ $\cup \{y_i^1 : i < \mu_1\}$ $\{y_i : i < \lambda_2\}$ is a maximal indiscernible set over A_2. Since $\mu \ge \lambda_0$, as in the proof of 5.8, it follows that in any (D, λ, 4)-prime model over $A_2 \cup \{y_i : i < \mu\} \cup \{y_i^1 : i < \mu_1\}$, $\{y_i : i < \mu\}$ is a maximal indiscernible set over A_2. So there is a maximally (D, μ)-homogeneous set of power $\ge \mu_1$, (there is a (D, μ, 4)-prime model since $\mu \ge \lambda_0$, and so ($C{*}\mu$) does not hold, and so by 5.10 (P, μ, 4) holds).

If $\mu < \lambda_0$, μ is regular then ($C{*}\mu$) holds, and so by Theorem 6.1

there exist maximally (D, μ)-homogeneous models of arbitrarily large power.

So in order to prove the theorem there remains the case $\mu > \lambda_2$. Suppose $\mu > \lambda_2$ is a regular cardinal, and let $\mu_1 \geq \lambda_3^+$ be any cardinal. We define y_i for $\lambda_2 \leq i \leq \mu$ and y_i^1 for $\lambda_3^+ \leq i < \mu_1$ such that $\{ y_i : i \leq \mu \}$ will be an indiscernible set over $A_2 \cup \{ y_i^1 : i < \mu_1 \}$, and $\{ y_i^1 : i < \mu_1 \}$ will be an indiscernible set over $A_2 \cup \{ y_i : i \leq \mu \}$. Let N be a (D, μ, 1)-prime model over $A_2 \cup \{ y_i : i < \mu \} \cup \{ y_i^1 : i < \mu_1 \}$ which is clearly a D-set. Let p^0 be the type which y_μ realizes over $A_2 \cup \{ y_i : i < \mu \}$. If we prove that p^0 is omitted in N (i.e. $\{ y_i : i < \mu \}$ is a maximal indiscernible set over A_2 in N) it will follow that N is not (D, μ^+)-homogeneous. As it is clear that N is (D, μ)-homogeneous, this will finish the proof.

Suppose there is an element $a \in N$ which realizes p^0. By Theorem 5.4 there is a sequence $\{ c_i : i < i_0 < \mu \}$ of elements of N, and a set $B \subseteq A_2 \cup \{ y_i : i < \mu \} \cup \{ y_i^1 : i < \mu_1 \}$ $|B| < \mu$, such that $a = b_{i_0}$, and for every $i \leq i_0$ c_i realizes over $A_2 \cup \{ y_j : j < \mu \} \cup \{ y_j^1 : j < \mu_1 \} \cup \cup \{ c_j : j < i \}$ a type q^i which is (D, 1)-isolated over $B \cup \{ c_j : j < i \}$. Without loss of generality let $B = A_2 \cup \{ y_i : i < i_1 < \mu \} \cup \cup \{ y_i^1 : i < i_2 < \mu \}$. As in 5.8 it follows that $\{ y_i : i_1 \leq i < \mu \}$ is an indiscernible set over $A_2 \cup \{ c_i : i \leq i_0 \} \cup \{ y_i^1 : i < \mu_1 \} \cup \{ y_i : i < i_1 \}$, and similarly for $\{ y_i^1 : i_2 \leq i < \mu_1 \}$. As D is stable in λ_1, there exists $B_i^1 \subseteq B_i \subseteq B \cup \{ c_j : j < i \}$, $|B_i| \leq \lambda_1$ and q_i is the only extension of $q_i | B_i$ in $S_D(A_2 \cup \{ y_i : i < \mu \} \cup \{ y_i^1 : i < \mu_1 \} \cup \{ c_j : j < i \})$ which does not split over B_i^1.

Let us define by induction $C_n \subseteq A_2 \cup \{ y_i, y_j^1 : i < i_1, j < i_2 \} \cup \cup \{ c_j : j \leq i_0 \}$, $|C_n| \leq \lambda_1$, $C_0 = \{ a \} \cup A_2 = \{ a_{i_0} \} \cup A_2$.

Suppose C_n is defined. If $c_k \in C_n$ let $A[n, k, l]$, $l \leq \lambda_1$ be an increasing sequence of sets included in

$$A^k = A_2 \cup \{ y_j : j < \mu \} \cup \{ y_j^1 : j < \mu_1 \} \cup \{ c_i : i < k \}$$

such that:

1) $A[n, k, 0] \supseteq C_n \cap (A_2 \cup \{ y_i, y_j^1 : i < i_1, j < i_2 \} \cup \{ c_i : i < k \})$;

2) $A[n, k, 0] \supseteq B_k$;

3) Every finite type over $A[n, k, l]$ which is realized in A^k is realized in $A[n, k, l+1]$;

4) $|A[n, k, l]| \leq \lambda_1$.

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We define $C_{n+1} = C_n \cup \bigcup \{ A[n, k, \lambda_1] : c_k \in C_n \}$.

It is easily seen that $|C_n| \leq \lambda_1$. We define $C_\omega = \bigcup_{n<\omega} C_n$; clearly $|C_\omega| \leq \lambda_1$.

Let $B^1 = C_\omega \cup \{ y_i : i_1 \leq i < \mu\} \cup \{ y_i^1 : i_2 \leq i < \mu_1 \}$.

$$B^0 = B^1 - \{c_i : i \leq i_0\} = A_2 \cup \{y_i : i_1 \leq i < \mu\} \cup$$
$$\cup \{ y_i^1 : i_2 \leq i < \mu_1 \} \cup (C_\omega - \{ c_i : i \leq i_0\}) .$$

We shall prove that B^1 is $(D, \lambda_1^+, 1)$-prime over B^0. Thus in a $(D, \lambda_1^+, 1)$-prime model over $A_2 \cup \{ y_i : i < \lambda_2 \} \cup \{y_i^1 : i < \lambda_3^+ \}$, $\{y_i : i < \lambda_2 \}$ is not a maximally indiscernible set over A_2, a contradiction, by 5.6.

So we prove by induction on k that $B^0 \cup (B^1 \cap \{c_i : i \leq k\})$ is $(D, \lambda_1^+, 1)$-prime over B^0. For this it suffices to prove that if $c_k \in B^1$, i.e. $c_k \in C_\omega$, then c_k realizes over $B^0 \cup (B^1 \cap \{c_i : i < k\})$ a $(D, \lambda_1^+, 1)$-isolated type r_k. We prove further, that r_k is $(D, 1)$-isolated over $C^k = (C_\omega - \{c_i : i \geq k\}) \cup \{ y_i : i_1 \leq i < i_1 + \lambda_1 + \omega\} \cup \{y_i^1 : i_2 \leq i \leq i_2 + \lambda_1 + \omega\}$ (as obviously $|C_k| \leq \lambda_1 < \lambda_1^+$ this implies the $(D, \lambda_1^+, 1)$-isolation of r_k).

Suppose $r \in S_D(B^0 \cup (B^1 \cap \{c_i : i < k\}))$, $r|C^k = r_k|C^k$, $r \neq r_k$. Let us denote

$$C^{k,1} = (C_\omega - \{c_i : i \geq k\}) \cup \{ y_i : i_1 \leq i < i_1 + \lambda_1\} \cup$$
$$\cup \{ y_i^1 : i_2 \leq i < i_2 + \lambda_1 \} .$$

It is clear that r_k does not split over $B_k^1 \subseteq B_k \subseteq C^{k,1} \subseteq C^k$ so, as $r|C^k = r_k|C^k$, r splits over $C^{k,1}$. This says that there are sequences of different elements $\bar{y}_1, \bar{y}_2$ (from $\{ y_i : i_1 + \lambda_1 \leq i < \mu\}$) and $\bar{y}_1^1, \bar{y}_2^1$ (from $\{y_i^1 : i_2 + \lambda \leq i < \mu_1\}$) a formula ψ^0, and a sequence $\bar{c}$ from $C^{k,1} \cap C_\omega$ such that

$$\models \psi^0[c, \bar{y}_2, \bar{y}_1, \bar{c}] , \qquad \models \neg \psi^0[c, \bar{y}_2^1, \bar{y}_1^1, \bar{c}]$$

where c realizes r.

Let $\bar{r} = \bar{r}(x, \bar{x}_1, \bar{x}_2, \bar{x}_1^1, \bar{x}_2^1)$ be the type which $\langle c\rangle^\frown \bar{y}_1{}^\frown \bar{y}_2{}^\frown \bar{y}_1^1{}^\frown \bar{y}_2^1$ (of length m) realizes over $C^{k,1}$. As $C_\omega = \bigcup_{n<\omega} C_n$, there exists $n < \omega$ such

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that $c_k \in C_n$, Rang $\bar{c} \subseteq C_n$. From this it is clear that the sequence $A[n,k,i]$, $i \leq \lambda_1$ is defined, $A[n,k,i] \subseteq C^{k,1}$. So as $(*\lambda_1)$ does not hold, by Claim 2.8, there is $l < \lambda_1$ such that $\bar{r} | A[n,k,l+1]$ does not split over $A[n,l,k]$. By the definition of $A[n,k,l]$ it follows that $\bar{r} | A[n,k,l+1]$ has a (unique) extension r^1 in $S_D^m(A_3)$, which does not split over $A[n,l,k]$, where $A_3 = A_2 \cup \{c_i : i < k\} \cup$ $\cup \{y_i : i < i_2 + \lambda_1\} \cup \{y_i^1 : i < i_2 + \lambda_1\}$. We define r^2, r^3 such that $r^3(\bar{x}_1, \bar{x}_2, \bar{x}_1^1, \bar{x}_2^1)$, $r^2(x) \subseteq \bar{r}(\bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_1^1, \bar{x}_2^1)$ and $r^2 \in S_D(A_3)$, $r^3 \in S_D^{m-1}(A_3)$. It is clear that $r^2 | A[n,k,l+1] = r_k | A[n,k,l+1] = q_k | A[n,k,l+1]$, r^2 does not split over $A[n,k\ l]$ and q^k does not split over $B_k^1 \subseteq A[n,k,0] \subseteq A[n,k,l]$. As every type over $A[n,k,l]$ which is realized in A^k is realized in $A[n,k,l+1]$, and $B \cup [c_i : i < k] \subseteq A^k$ it follows that $q^k | (B \cup \{c_i : i < k\}) = r^2 | (B \cup \{c_i : i < k\})$. Also it is not difficult to see that we can find sequences $\bar{z}_1, \bar{z}_2$ (from $\{y_i : i_1 + \lambda_1 \leq i < \mu\}$) and $\bar{z}_1^1, \bar{z}_2^1$ (from $\{y_i : i_2 + \lambda \leq i < \mu_1\}$) which realizes r^3. We define r^4

$$r^4 = \{\psi(x, \bar{z}_1, \bar{z}_2, \bar{z}_1^1, \bar{z}_2^1, \bar{a}) : \psi(x, \bar{x}_1, \bar{x}_2, \bar{x}_1^1, \bar{x}_2^1, \bar{a}) \in r^1\} .$$

It is easily seen that $r^4 \in S_D^4(A_3 \cup \text{Rang}\, \bar{z}_1 \cup \text{Rang}\, \bar{z}_2 \cup \text{Rang}\, \bar{z}_1^1 \cup$ $\cup\, \text{Rang}\, \bar{z}_2^1)$, and also that $r^4 \supseteq r^2$, and r^4 splits over A_3, and so also over $B \cup \{c_i : i < k\}$. As $q^k | (B \cup \{c_i : i < k\}) = r^4 | (B \cup \{c_i : i < k\})$; q^k is (D, 1)-isolated and so (D, 2)-isolated over $B \cup \{c_i : i < k\}$, we get a contradiction, and so, the theorem is proved.

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§ 7. On SP(T, P)

Definition 7.1. 1) If P is a set of finite types (over the empty set) in the language L(T_1) then EC(T_1, P) will be the class of models of T_1 which omit all the types in P.

2) SP(T_1, P) will be the class of powers λ such that every model $M \in$ EC(T_1, P) of power λ is λ-homogeneous, and $\lambda \geq |T| + \aleph_1$.

Remark: If D(M) $\subseteq$ D(N) and $N \in$ EC(T, P), then $M \in$ EC(T, P).

The following theorems will not be proved since a similar discussion appears in Keisler [2], Shelah [17] and similar proofs appear in Morley [7], Vaught [19] and Chang [20]. Theorem 7.1, in essence, appears in Keisler [2].

Theorem 7.1. *For every theory T and set of finite types P in* L(T) *there is a theory $T_1 \supseteq T$, where* L(T_1) *contains an additional predicate* Q ($|T_1| = |T|$), *and there is a type p in* L(T_1) *such that:*
There is a model $M \in$ EC(T, P), $\|M\| = \lambda$, which is not κ^+-homogeneous iff there is a model $M_1 \in$ EC(T_1, $P \cup \{p\}$), $\|M_1\| = \lambda$, such that $|Q^{M_1}| \leq \kappa$.

Theorem 7.2. *If for all $\delta < (2^{|T|})^+$ there is a model $M \in$ EC(T, P) of power $\geq \beth(|Q^M|, \delta)$ ($|Q^M| \geq \beth_\delta$) then for all $\lambda \geq |T|$ (and μ) ($|T| \leq \mu \leq \lambda$) there is a model $N \in$ EC(T, P), $\|N\| = \lambda$, $|Q^N| = |T|$ ($|Q^N| = \mu$) and for all $A \subseteq |N|$ at most $|A| + |T|$ complete types over A in* L(T) *are realized in N*

Corollary 7.3. *If for all $\lambda < \beth[(2^{|D|})^+]$ there is a model M_λ of power $\geq \lambda$,* D(M_λ) = D, *then for all $\mu \geq |D|$ there is a model N_μ of power μ,* D(N_μ) = D *such that for all $A \in |N_\mu|$ not more than $|A| + |D|$ types in* S(A) *are realized in N_μ.*

Hint for the proof: Adjoin |D| constants to each model M_λ such that each type in D is realized by one of them and then use 7.1, 7.2 with Q^{M_1} the empty set.

Corollary 7.4. *If there is a $\lambda > |T|$ in* SP(T, P) *then*

1) *There is* $\delta_0 < (2^{|T|})^+$ *such that if* $M \in \mathrm{EC}(T,P)$ *is of power* $\geq \beth(\mu, \delta_0)$ *then M is μ-homogeneous. If* $\delta_0 \cdot \omega$ *divides* δ *then* $\beth_\delta \in \mathrm{SP}(T,P)$.

2) *If there is an M of power* $\geq \beth[(2^{|\mathrm{D}|})^+]$ *with* $\mathrm{D}(M) = \mathrm{D}$ *then* D *is* κ*-stable for all* $\kappa \geq |\mathrm{D}| + |T|$. *If* $|T| < 2^\lambda$ *then* D *does not satisfy* $(B{*}\lambda)$. *It follows that* D *also does not satisfy* $({*}\lambda)$, $(C{*}\lambda)$. All this assuming $M \in \mathrm{EC}(T,P)$.

Proof: 1) The first assertion follows from 7.1, 7.2. If $\delta_0 \cdot \omega$ divides δ then $\mu < \beth_\delta$ implies $\beth(\mu, \delta_0) < \beth_\delta$. Thus every $M \in \mathrm{EC}(T,P)$ of power $\beth_\delta$ is μ-homogeneous for all $\mu < \beth_\delta$, and hence homogeneous.

2) The first statement is proved by choosing $\mu = \beth_{\delta} > \kappa$ in $\mathrm{SP}(T,P)$ (by 1) and using 1) and 7.3 since we then get a κ^+-homogeneous model and over every set A of power $\leq \kappa$ there are κ types realized.

Assume $\lambda = \inf\{\lambda : 2^\lambda > |T|\}$ and $(B{*}\lambda)$ holds. By 7.4.1 M is $(\mathrm{D}, |T|^+)$-homogeneous and thus we can find $A_\eta \subseteq |M|$, $p_\eta \in S_\mathrm{D}(A_\eta)$, a_η realizing p_η, $\eta = \tau | i$ implies $p_\eta \subseteq p_\tau$, $\varphi_\eta \in p_{\eta^\frown\langle 0\rangle}$, $\neg\varphi_\eta \in p_{\eta^\frown\langle 1\rangle}$, $|A_\eta| < |l(\eta)|^+ + \aleph_0 \leq |T|$ for $l(\eta) < \lambda$. Let $A = \bigcup\{A_\eta : l(\eta) < \lambda\} \cup \cup\{a_\eta : l(\eta) < \lambda\}$. It is easy to see that $|A| \leq 2^{(\lambda)} \leq |T|$. Adjoin the elements of A to the model as distinguished constants. As in 7.7 we find a model (of the extended language) of power μ which omits every $p \in P$ and over every set A at most $|A| + |T|$ complete types are realized. The reduct of this model also satisfies this property, and thus there is η, $l(\eta) = \lambda$ such that $p = \bigcup_{i<\lambda} p_{\eta|i}$ is omitted but for all $i < \lambda$, $p_{\eta|i}$ is realized. Thus $p \in S_{\mathrm{D}(M)}(\bigcup_{i<\lambda} A_{\eta|i})$, $|\bigcup_{i<\lambda} A_{\eta|i}| \leq |T|$ and M is not homogeneous, contradiction.

Theorem 7.5. *If* D *is good or stable, and not in every power* $\mu > |T|$ *is there a non-homogeneous* D*-model, then there is a cardinal* μ_0, $|\mathrm{D}| < \mu_0 < \beth[(2^{|T|})^+]$ *such that: every* D*-model of power* $> \mu_0$ *is homogeneous, for every* μ, $|T| + \aleph_1 \leq \mu < \mu_0$, *there is a* D*-model of power* μ *which is not homogeneous*

Proof: It is easy to see that there is a P such that $\mathrm{D}(M) \subseteq \mathrm{D}$ iff $M \in \mathrm{EC}(T,P)$. By hypothesis there is a $\lambda \in \mathrm{SP}(T,P)$, $\lambda > |T|$. Since everything stable is good, D is good, and thus there is a model M of power $\geq \beth[(2^{|\mathrm{D}|})^+]$ with $\mathrm{D}(M) = \mathrm{D}$. Then by 7.4 D is κ-stable for all $\kappa \geq |\mathrm{D}| + |T|$ and by 4.3 D does not satisfy $(C{*}\aleph_0)$.

Since D is stable there is a D-set $A = [y_i : i < \omega]$ which is indiscernible. Since D is stable, and does not satisfy $(C*\aleph_0)$ by 5.10, D satisfies $(P, \aleph_0, 3)$. Let M be a $(D, \aleph_0, 3)$-prime model over A. By 5.8 $\|M\| \geq |D|$ and M is not $\aleph_1$-homogeneous. Thus for all μ, $|T| + \aleph_1 \leq \mu \leq |D|$, there is a D-model of power μ which is not homogeneous.

Now assume that M is a D-model of power $> |D| + |T|$ which is not homogeneous. If M is not $|T|$-homogeneous, then for all μ, $|T| \leq \mu \leq \|M\|$, there is a D-model which is not μ-homogeneous, and hence $\mu \notin SP(T, P)$ Assume that M is $|T|$-homogeneous. Since M is not homogeneous, there is a $p \in S_{D(M)}(A)$, where $A \subseteq |M|$ is of power $< \|M\|$, which is omitted by M. Since by 7.4 D does not satisfy $(B*|T|)$ and D is x-stable where $x = |D| + |T| + |A| < \|M\|$, we get by 5.7, 3.1 and 7.4 that there is a non-$|A|^+$-homogeneous D-model of power $\beth(|A|, (2^{|T|})^+)$ contradiction.

It follows that every non-homo D-model of power $> |D| + |T|$ is not $|T|$-homogeneous. Define $\mu_0 = \inf\{\mu : |T| + \aleph_1 \leq \mu \in SP(T, P)\}$. Since there is a $\lambda \in SP(T, P)$ of power $> |T|$, μ_0 exists. Earlier we proved that $|T| + \aleph_1 \leq \mu \leq |D|$ implies $\mu \notin SP(T, P)$ and thus $\mu_0 > |D|$. If $\aleph_1 + |T| \leq \mu < \mu_0$, then by definition $\mu \notin SP(T, P)$. If $\mu \geq \mu_0$, then since every D-model of power μ is μ_0-homogeneous, by the above it is homogeneous.

It remains only to show that $\mu_0 < \beth[(2^{|T|})^+]$. We have shown that for all $\mu < \mu_0$ there is a D-model of power μ which is not $|T|^+$-homogeneous. $|D| \geq |T|$ and thus the assertion follows from 7.4.1.

Corollary 7.6. *If all the models of T of power $\lambda > |T|$ are homogeneous, then there is a cardinal μ_0, $|D(T)| < \mu_0 < \mu(|T|) < \beth[(2^{|T|})^+]$, such that for all $\mu \geq \aleph_1 + |T|$, T has a non-homogeneous model of power μ iff $\mu < \mu_0$*

Remark: 1) Assume $\aleph_0 < |T| \in SP(T, 0)$. In the proof of 7.5 we showed that there is a $(D(T), \aleph_0, 3)$-homogeneous model M of power $\geq |D(T)|$ which is not $\aleph_1$-homogeneous. Thus $|T| > \|M\| \geq |D(T)|$. From here we see that T is a definitional extension of a theory of power $< |T|$. Thus the restriction that $\lambda > |T|$ can be replaced by $\lambda \geq |T| + \aleph_1$. See [16].

2) Added in proof: in fact $\mu_0 = |D(T)|^+$.

3) $\mu_{|T|} = \mu(|T|)$ is defined in Vaught [19].

Proof of 7.6: The only part which does not follow immediately from 7.5 is $\mu_0 < \mu(|T|)$. We show this as follows: We shall find $T_1 \supseteq T$, $|T_1| = |T|$, and a type p in $L(T_1)$ such that T has a model of power μ which is not $|T|^+$-homogeneous iff T_1 has a model of power μ omitting p. By the definition of $\mu(|T|)$ this will suffice. The language $L(T_1)$ will be $L(T)$ with new constant symbols $\{c_i, d_j : i < |T|, j \leq |T|\}$. Define

$$T_1 = T \cup \{ \psi(c_{i_1}, ..., c_{i_m}) \leftrightarrow \psi(d_{i_1}, ..., d_{i_m}) : \psi \text{ a formula of } L(T), i_1, ..., i_m < |T|\},$$

$$p = \{ \psi(x, c_{i_1}, ..., c_{i_n}) \to \psi(d_{|T|}, d_{i_1}, ..., d_{i_n}) : \psi \text{ a formula of } L(T), i_1, ..., i_n < |T|\}.$$

It is easy to see that T_1 and p satisfy our requirements. This proves the corollary.

Theorem 7.7. *Assume $\lambda \in SP(T,P)$, $M \in EC(T,P)$ is a non-homogeneous model of power μ, $D(M) = D$, $|D| \leq \lambda < \|M\|$. Then*

A. *$\kappa \geq |T|$, $\lambda > 2^{2^\kappa}$ implies $\kappa \notin SP(T,P)$ and there is a D-model of power κ which is not $\aleph_1$-homogeneous.*

B. *At least one of the following holds:*

1) *$\mu \leq 2^\lambda$ and there is no $\kappa \in SP(T,P)$ such that $|T| \leq \kappa < \lambda$.*

2) *$\lambda^{(\lambda)} = \lambda$ and if $|T| \leq \kappa < \lambda$, $\kappa \in SP(T,P)$, then $\lambda = 2^\kappa$, $\kappa^{(\kappa)} = \kappa$. (Thus there is no more than one such κ). Also there is no D-model which is $(2^\lambda)^+$-homogeneous of power $\geq (2^\lambda)^+$.*

C. *There is $\mu_0 < \beth[(2^{|T|})^+]$ such that $\mu \geq \mu_0$ implies $\mu \in SP(T,P)$, and $|T| \leq \mu < \mu_0$ implies $\mu \notin SP(T,P)$, except for possibly two powers.*

Proof: A) By Theorem 7.5 D is not stable and by 7.4.2. there is a power such that there are no models M with $D(M) = D$ of cardinality greater than this power. If $\lambda > 2^{2^\kappa}$ then D is not 2^{2^κ}-stable, and thus there is a D-set A of power $2^{2^\kappa} < |S_D(A)|$. Since there is a (D, λ)-homogeneous model, there is a D-model N of power $|A|$ containing A, and $\|N\| = |A| < |S_D(A)| \leq |S_D(N)|$. By 6.2 it follows that there is a D-model of power κ which is not $\aleph_1$-homogeneous.

B) First assume that there is no D-set which is an indiscernible sequence. We shall show that 1) holds.

If $\mu > 2^{\lambda}$, then since $|A| \leq \lambda$ implies $|S_D(A)| \leq 2^{|A|} \leq 2^{\lambda}$, by 6.7 we get a contradiction. Thus $\mu \leq 2^{\lambda}$. If $|T| \leq \kappa < \lambda$, $\kappa \in SP(T, P)$, then $|A| \leq \kappa$ implies $|S_D(A)| \leq \lambda < \mu$, and thus we again get a contradiction by 6.7. Thus 1) holds.

Assume from now on that there is a D-set $\{y_i : i < \omega\}$ which is an indiscernible sequence. It follows that there are D-sets of arbitrarily large power. If D were to satisfy $(P, \kappa, 1)$ for some κ, then we would get arbitrarily large models M with $D(M) = D$, contrary to what was said at the start of the proof of A). In particular, since there is a (D, λ)-homogeneous model, $(B*\lambda)$ holds, by 5.9. By Theorem 6.1 we are able to conclude that if $\lambda > 2^{\kappa}$, $\kappa \geq |T|$, then there is a D-model of power κ which is not $\aleph_1$-homogeneous, and thus $\kappa \notin SP(T, P)$. It is also clear that $(B*\kappa)$ holds for all $\kappa \leq \lambda$, and thus by Claim 6.5, if N is a (D, κ^+)-homogeneous model, $\lambda \geq \kappa^+$, then $\|N\| \geq 2^{\kappa}$. Therefore, if $\lambda \geq \kappa \geq |T|$, $\kappa \in SP(T, P)$ then $2^{(\kappa)} = \kappa$; in particular, $2^{(\lambda)} = \lambda$. If $\kappa = \beth_{\delta}$, $cf(\delta) < \kappa$, $\lambda \geq \kappa \geq |T|$, by 6.4 $\kappa \notin SP(T, P)$. Thus $\kappa^{(\kappa)} = \kappa$; in particular, $\lambda^{(\lambda)} = \lambda$. If in addition $\lambda > \kappa$, then we have proved that $2^{\kappa} \geq \lambda$ and $2^{\kappa} \leq 2^{(\lambda)} = \lambda$, and hence $2^{\kappa} = \lambda$. If $\|M\| \geq (2^{\lambda})^+$, M is a $(2^{\lambda})^+$-homogeneous model, it follows as before that $(B*(2^{\lambda})^+)$ holds, in contradiction to $\lambda \in SP(T, P)$.

C) By 7.4.2 SP(T, P) is infinite. If μ^0 is the third element of SP(T, P) which is greater than $2^{|T|}$, then by the previous theorem $\mu^0 \leq \mu$ implies $\mu \in SP(T, P)$. Let μ_0 be the first cardinal such that $\mu_0 \leq \mu$ implies $\mu \in SP(T, P)$. By 7.4.1 it is not the case that for all $\mu < \beth[(2^{|T|})^+]$ there is a model in EC(T, P) of power μ which is not $(2^{|T|})^+$-homogeneous. Let μ_1 be the first cardinal such that $M \in EC(T, P)$ is of power $\geq \mu_1$ implies M is $\aleph_1$-homogeneous. Clearly $\mu_1 < \beth[(2^{|T|})^+]$. Assume $M \in EC(T, P)$ is a $\mu_2 = (2^{2^{\mu_1}})^+$-homogeneous, non-homogeneous model (of power $> \mu_2$). If $D(M)$ is not μ_2-stable, we get, as in the proof of A), that there is a model in EC(T, P) of power μ_1 which is not $\aleph_1$-homogeneous; contradiction. Thus, $D(M)$ is stable. Now we get that there are arbitrarily large models which are not homogeneous (as in the proof of 7.5); contradiction. Thus, if $M \in EC(T, P)$ is of power $\geq \mu_2$ and is not homogeneous, then M is not μ_2-homogeneous. If $\mu_0 \geq \beth[(2^{|T|})^+]$ then it follows that for all $\mu < \beth[(2^{|T|})^+]$ there is a model of power $\geq \mu$

which is not μ_2-homogeneous. This is a contradiction to 7.4.1. Thus $\mu_0 < \beth[(2^{|T|})^+]$. The rest follows by B).

Remark: Theorem 6.8 completes the picture.

Corollary 7.8. (G.C.H.) *If there is a* $\lambda \in \mathrm{SP}(T, P)$ *with* $|T| < \lambda$ *then there is a* $\mu_0 < \beth[(2^{|T|})^+]$ *such that:*

$\mu \geq \mu_0$ *implies* $\mu \in \mathrm{SP}(T, P)$;

$|T| < \mu < \mu_0$ *implies* $\mu \notin \mathrm{SP}(T, P)$ *except for, perhaps, one* μ *when* $\mu = \beth_{\delta+1}$, *or when* $\mu = \beth_\delta$, $\mu_0 = \mu^{++}$.

Proof: 7.7 and 6.8.

Theorem 7.9. *If T is a countable theory with only homogeneous models of power* $\aleph_1$, *then T is* $\aleph_1$*-categorical.*

Remark: This solves problem D in Keisler [2].

Proof: By 7.6 there is a cardinal $\mu_0 > |D(T)|$ such that $|T| + \aleph_1 \leq \mu < \mu_0$ implies $\mu \notin \mathrm{SP}(T,P)$. Since $\aleph_1 \in \mathrm{SP}(T, P)$, $\mu_0 \leq \aleph_1$. Thus $|D(T)| < \aleph_1$, i.e. $|D(T)| \leq \aleph_0$. By 7.4.2 $D(T)$ is $\aleph_0$-stable and thus T is $\aleph_0$-stable, or in the terminology of Morley [6], T is totally transcendental. Assume T is not $\aleph_1$-categorical. By Morley [8] T has a model M of power $\aleph_1$, and there is a formula $\psi(x, \bar{y})$ and a sequence $\bar{a}$ from M such that $|\{b \in |M| : \models \psi[b, \bar{a}]\}| = \aleph_0$. Define $A = \mathrm{Rang}\,\bar{a} \cup \{b : \models \psi[b, \bar{a}]\}$, $p = \{\psi(x, \bar{a}) \wedge x \neq b : \models \psi[b, \bar{a}], b \in |M|\}$. Since M is a $D(T)$-model and $D(T)$ is $\aleph_0$-stable, by 3.1 there is an indiscernible set $\{y_i : i < \aleph_1\}$ over A in M. Let $D = D(M) \subseteq D(T)$. Clearly D is $\aleph_0$-stable, and there is a $(D, \aleph_1)$-homogeneous model (M). Thus by 5.11, D satisfies $(P, \aleph_0, 1)$. Define $\{y_i : i < \beth_\omega\} \supseteq \{y_i : i < \omega\}$ such that it too is an indiscernible set over A. Clearly $A_1 = A \cup \{y_i : i < \beth_\omega\}$ is a D-set. Let M_1 be a $(D, \aleph_0, 1)$-prime model over A_1. By 5.7 M_1 omits p. Thus $|\{b : \models \psi[b, a], b \in |M_1|\}| = \aleph_0$. It is also clear that $\|M_1\| \geq |\{y_i : i < \beth_\omega\}| = \beth_\omega$. Let $M_2 = M_1^\omega/E$ where E is any non-principal ultrafilter over ω. By known properties of ultrapowers (see e.g. Keisler [2]) M_2 is a model of T of power $\geq \beth_\omega$, $|\{b : \models \psi[b, \bar{a}], b \in M\}| = 2^{\aleph_0}$, and $D(M) = D(T)$. Define $B = \mathrm{Rang}\,\bar{a} \cup \{b : \models \psi[b, \bar{a}], b \in |M_2|\}$, $q = \{\psi(x, \bar{a}) \wedge x \neq b; \models \psi[b, \bar{a}], b \in |M_2|\}$. It is easily seen that $B \subseteq |M_2|$, $|B| = 2^{\aleph_0}$, q is not realized in M_2, q is a type over B. Let $q \subseteq q_1 \in S_{D(T)}(B)$. By 7.6 it follows that M_2 is $(D(T), \beth_\omega)$-homogeneous, and thus q_1 must be realized, contradiction. Thus, the theorem is proved.

Added in proof 20 August 1970: We can improve 7.7, 7.8 to

Theorem 7.10. If $|T| < \lambda_1 \in SP(T, P)$ then there is μ_0 such that: (1) $\mu \geq \mu_0$ implies $\mu \in SP(T, P)$, and $\mu_0 < \beth[(2^{|T|})^+]$. (2) $|T| < \lambda \in SP(T, P)$ implies $(2^\lambda)^T \geq \mu_0$. (3) If there are two λ, $\mu_0 > \lambda \in SP(T, P)$ then $\mu_0 \leq (2^{|T|})^+$.

Proof. Let μ_0 be the first satisfying (1) (by 7.7 it exists). Suppose (2) fails. Then there are $\lambda \in SP(T, P)$ $|T| < \lambda$, $M \in EC(T, P)$, $2^\lambda < \|M\|$, M is not homo and $D = D(M)$. As is 7.7, $\lambda^{(\lambda)} = \lambda$, hence $\lambda > |T|$ implies $\lambda \geq 2^{|T|} \geq |D|$. Define: $p \in S_D(A)$ *suitably splits* over $B \subset A \subset |M|$ if there are a (D, λ)-homo. N, $B \subset |N| \subset |M|$, and sequences $\bar{a}, \bar{b}$ from A, which realize the same type over $|N|$, and $\varphi(x, \bar{a}) \neg \varphi(x, \bar{b}) \in p$. In the definitions of §5 we can replace $n = 2$ by $n = 5$, and splitting by suitably splitting. Now if there are $A_n \subset |M|$, $A_n \subset A_{n+1}$, and $p \in S_D(\bigcup_{n>\omega} A_n)$ where $p | A_{n+1}$ suitably splits over A_n, then there is a D-model N, $\|N\| = \lambda$, which is not $\aleph_1$-homogeneous. (Note that w.l.o.g. $|A_n| \leq \aleph_0$). A contradiction, so $(P, \aleph_0, 5)$ holds. As by 7.7, $\lambda = \lambda^{(\lambda)}$, and $|T| < \lambda < \mu_0 < \beth[(2^{|T|})^T]$, implies λ is not strongly inaccessible, and as in 7.7 λ is not $\beth_\delta$ clearly $\lambda = \chi^+ = 2^\chi$. As D is not χ-stable there is $A \subset |M|$, $|A| = \chi$, $|S_D(A)| = \chi^+$. Now in the proof of 6.7, (a) should hold, and we use its notation. W.l.o.g. $A \subset N_0$, and clearly N_n, $\bigcup N_n$ are D-homo. Now, using a property slightly stronger than $(P, \aleph_0, 5)$ which clearly hold we can find N^0, $|N^0| = A \cup \{y_i: i < \omega\} \cup \{b_i: i < \lambda\} \subset \bigcup N_n$ (which should be D-homo) and for $i < \alpha$ there is a finite $B^i \subset \{y_i: i < \omega\} \cup \{b_j: j < i\} = B_i$, such that the type b_i realize over $A \cup B_i$ is $(D, 5)$-isolated over $A \cup B^i$. If $\langle y_i: i < \omega \rangle$ is a maximal indis. seq. over A, we get contradiction, and prove (2). Suppose $\langle y_i: i < \omega + 1 \rangle$ is indis. seq. over A, $y_\omega \in N^0$. By changing notations $y_{\omega+1} = b_m$. Let n be such that $b_0, \ldots, b_m \in N_n$, $B^0, \ldots, B^m \subset \{y_i: i < n\} \cup \{b_i: i \leq m\}$. By the proof of 6.7, $\{y_i: n \leq i < \omega\}$ is indis. seq. over $|N_n|$. We can prove by induction on $k \leq m + 1$ that $\{y_i: n \leq i \leq \omega + 1\}$ is indis. seq. over $A \cup \{y_i: i < n\} \cup \{b_i: i \leq m\}$. As y_ω satisfies $x = b_m$, it follows $y_n = b_m = y_{n+1}$, a contradiction. So we prove (2), (3) can be proved easily.

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