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HOW SPECIAL ARE COHEN AND RANDOM FORCINGS, i.e. BOOLEAN ALGEBRAS OF THE FAMILY OF SUBSETS OF REALS MODULO MEAGRE OR NULL

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ABSTRACT

We prove that any Souslin c.c.c. forcing notion which adds a nondominated real adds a Cohen real. We also prove that any Souslin c.c.c. forcing adds a real which is not on any old "narrow" tree.

The feeling that those two forcing notions—Cohen and Random—(equivalently the corresponding Boolean algebras $\mathcal{P}(\mathbb{R})/(\text{meagre sets})$, $\mathcal{P}(\mathbb{R})/(\text{null sets})$) are special, was probably old and widespread. A reasonable interpretation is to show them unique, or "minimal" or at least characteristic in a family of "nice forcing" like Borel. We shall interpret "nice" as Souslin as suggested by Judah Shelah [JuSh 292] (discussed below). We divide the family of Souslin forcing into two, and expect that: among the first part, i.e. those adding some non-dominated real, Cohen is minimal (=is below every one), while among the rest random is quite characteristic even unique or at least minimal. Concerning the second class we have weak results, concerning the first class, our results look satisfactory.

Related is von Neumann's problem which in our language is:

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(*) is there a ${}^{\omega}\omega$-bounding c.c.c. forcing notion adding reals which is not equivalent to the measure algebra (i.e. control measure problem)?

Velickovic (and, as I have lately learnt, also Fremlin) suggests another problem (it says less on forcings which are ${}^{\omega}\omega$-bounding but it says also much on the others).

(**) is there a c.c.c. forcing notion P which adds new reals and such that for every $f \in {}^{\omega}\omega \cap V^P$ there is $h \in V$ such that $(\forall n)|h(n)| \leq 2^n$, and $f(n) \in h(n)$ for all n.

The version of it for Souslin forcing was our starting point.

We have two main results: one (1.14) says that Cohen forcing is "minimal" in the first class, the other (1.10) says that all c.c.c. Souslin forcings have a property shared by Cohen forcing and Random real forcing (this is the answer to (**) for Souslin forcing), so it gives a weak answer to the problem on how special is random forcing, but says much on all c.c.c. Souslin forcing. Earlier by Gitik Shelah [GiSh 412], any σ-centered Souslin forcing notion add a Cohen real.

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1. A Souslin forcing which adds an unbounded real adds a Cohen real

1.1 Notation:

(0) $\ell g(\eta)$ is the length of η.

(1) T denotes subtrees of ${}^{\omega>}\omega$, i.e., $T \subseteq {}^{\omega>}\omega$ is non-empty, $[\nu \triangleleft \eta \,\&\, \eta \in T \Rightarrow \nu \in T]$ and $[\nu \in T \Rightarrow (\exists \eta \in T)(\nu \triangleleft \eta)]$. For $\eta \in T$ let $T^{[\eta]} \stackrel{\text{def}}{=} \{\nu \in T\colon \nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu\}$ and let $\lim T = \{\eta \in {}^{\omega}\omega\colon \bigwedge_n \eta \restriction n \in T\}$.

(2) $\mathrm{sp}(T) = \{\eta \in T\colon (\exists^{\geq 2} k)[\eta^\frown\langle k\rangle \in T]\}$, $\ell\mathrm{sp}(T) = \{\ell g(\eta)\colon \eta \in \mathrm{sp}(T)\}$.

(3) $[A]^{\mu} = \{B \subseteq A\colon |B| = \mu\}$, $[A]^{<\mu} = \bigcup_{0 \leq \kappa < \mu}[A]^{\kappa}$.

(4) We say T is u-large if: $u \in [\omega]^{\aleph_0}$ and for some $n^* < \omega$: if $n^* < n < m < \omega, n \in u, m \in u$ then $[n, m) \cap \ell\mathrm{sp}(T) \neq \emptyset$.

(5) We say T is strongly u-large if: $u \in [\omega]^{\aleph_0}$, and for some $n^* < \omega$, if $n^* < n < m < \omega$, $n \in u$, $m \in u$ then $(\forall \eta \in T \cap {}^{n}2)(\exists \nu)[\eta \trianglelefteq \nu \in \mathrm{sp}T \,\&\, \ell g\nu < m]$.

(6) O_k is a sequence of length k of zeroes.

(7) $(\forall^{\infty} n)$ means: for every large enough $n < \omega$. $(\exists^{\infty} n)$ means for infinitely many $n < \omega$.

(8) We say T is $(u, \bar{h})$-large if : $u \in [\omega]^{\aleph_0}$, $h_k\colon \omega \to \omega \setminus \{0,1\}$, $\bar{h} = \langle h_k\colon k < \omega\rangle$ and for every $k < \omega$, T is (u, h_k)-large which means: for infinitely many $n \in u$ we have: $n \leq m \in u$ & $|u \cap m \setminus n| < h_k(n) \Rightarrow \mathrm{Min}(\ell\mathrm{sp}(T) \setminus m) < \mathrm{Min}(u \setminus (m+1))$.

Note

(*) if $h_n = n$ for every $n < \omega$ this is equivalent to: for every $k < \omega$, for some consequtive members $i_0 < i_1 < \cdots < i_k$ of u, for every $\ell < k$ we have $[i_\ell, i_{\ell+1}) \cap \ell\mathrm{sp}(T)$ is not empty.

(9) We say $\langle T_\ell\colon \ell < n\rangle$ is $(u, \bar{h})$-large if: $u \in [\omega]^{\aleph_0}, \bar{h} = \langle h_k\colon k < \omega\rangle, h_k\colon \omega \to \omega \setminus \{0,1\}$ and for every $k < \omega$ for infinitely many $n \in u$ we have $n \leq m \in u$ & $|u \cap m \setminus n| < h_k(n) \Rightarrow \bigwedge_\ell \mathrm{Min}(\ell\mathrm{sp}T_\ell \setminus m) < \mathrm{Min}(u \setminus (n+1))$.

(10) If $h_k = h$ for $k < \omega$ we write h instead of $\bar{h}$.

(11) We use forcing notions with the convention that larger means with more information.

(12) In a partial order (=forcing notion), incompatible means have no common upper bound.

1.2 Definition: A statement $\varphi(x)$ on reals is absolute if for every model M extending V with the same ordinals (mainly $M = V$ or a generic extension) and N, a model of ZFC$^-$ (which is a transitive set or class of M) with $\omega_1^M \subseteq N$ and $a \in N$, we have $N \models \varphi[a]$ iff $M \models \varphi[a]$.

1.3 Definition: (1) P is a c.c.c. Souslin forcing notion if: $P = (P, \leq)$ is such that:

(a) there is a $\sum_1^1$-definition φ^a of the set P (which is $\subseteq \mathbb{R}$),

(b) there is a $\sum_1^1$-definition φ^b of a partial order $\leq$ on P,

(c) there is a $\sum_1^1$-definition φ^c of the relation "p, q incompatible in P" (see 1.1(12)) (hence it is Δ_1^1, as by the above it is Σ_1^1, now use Definition 1.3(1) (a)+(b), it implies being compatible is $\sum_1^1$ hence being incompatible is $\prod_1^1$),

(d) $(P, \leq)$ satisfies the c.c.c.

(2) Note: we do not distinguish strictly between P and the three $\sum_1^1$-formulas $\varphi^a, \varphi^b, \varphi^c$ respectively appearing in the definition.

(3) P is a Souslin forcing notion if (a)+(b) holds.

1.3A Remark: On (c.c.c.) Souslin forcing see Judah Shelah [JuSh 292] e.g.

1.4 Claim:

(1) *"$\varphi^a, \varphi^b, \varphi^c$ are $\sum_1^1$-formulas as in 1.3(1)" is absolute.*

(2) *For P a c.c.c. Souslin forcing notion, "$\{r_n: n < \omega\}$ is a maximal antichain of P" is a conjunction of a $\sum_1^1$ and a $\prod_1^1$ statements.*

(3) *Being a maximal antichain is absolute (even conjunction of $\prod_1^1$ and $\sum_1^1$) hence so is "being a P-name of a member of ${}^\omega 2$ (or ${}^\omega\omega$)".*

(4) *If P is a c.c.c. Souslin forcing notion , $p_0 \in P$ and $P^* =^{df} \{p \in P: P \models p_0 \leq p\}$ (with the inherited order) then P^* is a c.c.c. Souslin forcing notion too .*

Proof: E.g.

(1) For part (d) use the completeness theorem for $L_{\omega_1\omega}(a)$.

(2) The $\sum_1^1$ part is to say "$r_n \in P$", so if "$x \in P$" is also $\prod_1^1$ then this statement is $\prod_1^1$; the $\prod_1^1$ part is to say $(\forall x)\left[x \notin P \vee \bigvee_{n<\omega}(x, r_n \text{ compatible})\right]$ (by Definition 1.3(1)((a)+(c))); a third part is $\bigwedge_{n<m<\omega}$ $(r_n, r_m$ incompatible) which are $\prod_1^1$ and Δ_1^1 resp.

(3) Follows by (2). $\blacksquare_{1.4}$

1.5 Lemma: *Assume P is a c.c.c. Souslin forcing, $\underset{\sim}{r}$ a P-name of a new member of ${}^\omega 2$. Then for some infinite $u \subseteq \omega$, for every $p \in P$, the tree $T_p[\underset{\sim}{r}]$ (see Definition 1.6 below) is u-large (see 1.1(5)).*

1.6 Definition: $T_p[\underset{\sim}{r}] = \{\eta \in {}^{\omega>}2: p \nVdash_P \text{"}\eta \neq \underset{\sim}{r} \restriction \ell g\eta\text{"}\}$ (clearly it is a tree). Before we turn to proving Lemma 1.5, we prove:

1.7 Claim:

(1) *For a given c.c.c. Souslin forcing notion P (i.e. as in Definition 1.3(1)) and P-name $\underset{\sim}{r}$ of a member of ${}^\omega 2$, the conclusion of 1.5 is an absolute statement (actually $\sum_2^1$).*

(2) *The statement on u, p (and also on $\underset{\sim}{r}$) that they is as required in 1.5, is a $\sum_1^1$ statement.*

(3) *Also "$\underset{\sim}{r}$ is a P-name of a new real" is absolute in fact a $\prod_1^1$-statement.*

(4) *If P is c.c.c. Souslin forcing notion, above every $p \in P$ there are two incompatible conditions then forcing with P add a new real.*

Proof: (1) Let $\underset{\sim}{r}$ be represented by $\left\langle\langle(p_i^\eta, \mathbf{t}_i^\eta): i < \omega\rangle: \eta \in {}^{\omega>}2\right\rangle$ where

$\{p_i^\eta : i < \omega\} \subseteq P$ is a maximal antichain of P, $\mathbf{t}_i^\eta$ a truth value and $p_i^\eta \Vdash_P$ "$\eta \triangleleft \underset{\sim}{r}$ iff $\mathbf{t}_i^\eta$". For 1.5, the failure of the statement can be expressed by:

(∗) $(\forall u)(\exists p)\Big[u \subseteq \omega$ finite or $p \in P \,\&\, (\exists^\infty n \in u)(\forall \eta)[\eta \in {}^n2 \,\&\, \eta \in T_p(\underset{\sim}{r}) \Rightarrow$
$\neg(\exists \nu)[\eta \trianglelefteq \nu \,\&\, \ell g\nu < \mathrm{Min}(u \setminus (n+1)) \,\&\, \nu\char`\^\langle 0\rangle \in T_p(\underset{\sim}{r}) \,\&\, \nu\char`\^\langle 1\rangle \in T_p(\underset{\sim}{r})]\Big]$.

Now the statement "$\rho \in T_p(\underset{\sim}{r})$" is equivalent to "$p \nVdash_P [\rho \ntriangleleft \underset{\sim}{r}]$" which is equivalent to

(∗∗) $\bigvee_{i<\omega}(\mathbf{t}_i^\rho = \text{truth} \,\&\, p, p_i^\rho$ compatible).

It is enough to show that (∗) is a $\prod_2^1$-statement hence it is enough to show that inside the large parenthesis there is a $\sum_1^1$-statement. In (∗) inside the large parenthesis, ignoring quantifications over ω, we note that "$p \in P$" is $\sum_1^1$, and then we have to consider (∗∗), on which it is enough to prove that it is a Δ_1^1 statement [actually we have three instances of it — all negatives]. By Definition 1.3(1)(c) it is $\prod_1^1$ and by Definition 1.3(1)(b) (and the compatible meaning having a common upper bound) it is $\sum_1^1$.

(2) The proof is included in the proofs of parts (1) and (3).

(3) Easy. $\Big[$Why? the statement is $(\forall p)[p \notin P \vee \bigvee_{\eta \in {}^{\omega>}2} [\eta\char`\^\langle 0\rangle \in T_p(\underset{\sim}{r}) \,\&\, \eta\char`\^\langle 1\rangle \in T_p(\underset{\sim}{r})]]$. Now inside the parentheses we have $p \notin P$ which is $\prod_1^1$ and two instances of (∗∗) which, as shown above, is a $\prod_1^1$-statement.$\Big]$

(4) Easy, e.g. in $V^{Levy(\aleph_0, 2^{\aleph_0})}$ we ask: is there $p \in P$ such that $G_p =^{df} \{q : q \in P^V, q \leq p\}$ is a directed subset of P^V, generic over V, i.e. not disjoint to any maximal antichain of P^V from V? By the assumption if such p exist, necessarily $G_p \notin V$, and by the homogeneity we can find *Levy*-names $\underset{\sim}{p}$, $\underset{\sim}{G}_p$ of such objects so in V^{Levy} we can find a perfect set of such G_p's, so the p's form an antichain of size continuum but this is absolute. So there is no such p, letting $\{\{p_{i,j} : i < \omega\} : j < \omega\}$ list the maximal antichains of P^V from P^V (the list in V^{Levy}), and we define a p-name $\underset{\sim}{\eta} \in {}^\omega\omega$: (in V^{Levy}): $\underset{\sim}{\eta}(n)$ =the unique m such that $p_{n,m} \in G_P$, the generic subset of $P^{V^{Levy}}$. This is a P-name of a new real (all in $V^{Levy(\aleph_0, 2^{\aleph_0})}$ and by part (3) +1.4(2) its existence is absolute. $\blacksquare_{1.7}$

Remark: The use of Q_D below can be replaced. Q_D is called Mathias forcing. See on it [Sh-b].

1.8 Proof of Lemma 1.5: Assume that the conclusion fails (for $\underset{\sim}{r}$, a P-name of a new member of ${}^\omega 2$, which will be fixed until the end of the proof of Lemma 1.5). For D a filter on ω (containing the co-bounded subsets of ω) let $Q_D =$

$\{(w,A)\colon w \subseteq \omega$ finite, $A \in D$ and $\max(w) < \operatorname{Min} A$ (when $w \neq \emptyset$)$\}$ (and if $w \subseteq \omega$ is finite $A \subseteq \omega$ we identify (w,A) with $(w, A \cap (\max w, \omega))$); the order is defined by $(w_1,A_1) \le (w_2,A_2)$ iff $w_1 \subseteq w_2 \subseteq w_1 \cup A_1$, $A_1 \supseteq A_2$. Let $(w_1,A_1) \le_{\rm pr} (\omega_2, A_2)$ (pure extension) iff $w_1 = w_2$, $A_1 \supseteq A_2$. Clearly Q_D is a partial order satisfying the c.c.c. and $\{q\colon q_0 \le_{\rm pr} q\}$ is directed for each $q_0 \in Q_D$. Let $\underset{\sim}{w} = \cup\{w\colon (w,A) \in \underset{\sim}{G}_{Q_D}\}$, clearly $\underset{\sim}{w}$ is a Q_D-name and any $G \subseteq Q_D$ generic over V can be reconstructed from $\underset{\sim}{w}[G]$: $G = \{(v,A) \in Q_D\colon v \subseteq \underset{\sim}{w}[G] \subseteq v \cup A\}$. Without loss of generality CH holds (by Claim 1.7(1), e.g. force with Levy$(\aleph_1, 2^{\aleph_0})$), hence we can choose D as a Ramsey ultrafilter on ω. So as is well known that:

$\otimes_1$ if $\underset{\sim}{\ell} < 2$ is a Q_D-name and $q \in Q$ *then* for some $q', q \le_{\rm pr} q' \in Q_D$, q' forces a value to $\underset{\sim}{\ell}$.

So after forcing with Q_D, the conclusion of 1.5 still fails (by claim 1.7(1)). Hence for some $q^* \in Q_D$ and Q_D-names $\underset{\sim}{p}, \underset{\sim}{T}$ (remember that $\underset{\sim}{r}$ remains a P-name) we have $q^* \Vdash_{Q_D}$ "$\underset{\sim}{p} \in P$, $\underset{\sim}{r}$ of ${}^{\omega}2$, $\underset{\sim}{T} = T_{\underset{\sim}{p}}[\underset{\sim}{r}]$ is not $\underset{\sim}{w}$-large, such that: for arbitrarily large $n \in \underset{\sim}{w}$, ($\underset{\sim}{w}$-the Q_D-name) the interval $\Big[n, \operatorname{Min}(\underset{\sim}{w} \setminus (n+1))\Big)$ is disjoint to $\operatorname{lsp}(\underset{\sim}{T})$ " ; also we can assume that $\Vdash_{Q_D}$ "$\underset{\sim}{p} \in P$, $\underset{\sim}{r}$ remains a P-name of a new member of ${}^{\omega}2$ and $\underset{\sim}{T} = T_{\underset{\sim}{p}}[\underset{\sim}{r}]$". For $q \in Q_D$ let $S[q] =: \{\eta \in {}^{\omega>}2 :$ for some $q', q \le_{\rm pr} q'$ and $q' \Vdash_{Q_D}$ "$\eta \in \underset{\sim}{T}$"$\}$; note that $S[q]$ is also equal to $\{\eta \in {}^{\omega>}2 :$ for no q', $q \le_{\rm pr} q' \in Q_D, q' \Vdash_{Q_D}$ "$\eta \notin \underset{\sim}{T}$"$\}$ (just apply $\otimes_1$).

Now note

$\otimes_2$ $S[q]$ is a subtree (of ${}^{\omega>}2$) and if $q_1 \le_{\rm pr} q_2$ (in Q_D) *then* $S[q_1] \supseteq S[q_2]$ (in fact they are equal).

$\otimes_3$ if $q^* \le q \in Q_D$ *then* for some $q_1 \ge q$ and m we have: $S[q_1]$ has no splitting in any level $\ge m$.

Why? Let $n = \max w^q$; so q forces that: for some m, $m \in \underset{\sim}{w}$, $m \ge n$, and $\operatorname{Min}[(\ell\operatorname{sp} T_{\underset{\sim}{p}}[\underset{\sim}{r}] \setminus m] \ge \operatorname{Min}[\underset{\sim}{w} \setminus (m+1)]$. Before proving $\otimes_3$, repeatedly using $\otimes_1$ we can assume

$\otimes_4$ if $m \in A^q$, $v \subseteq m \cap A^q$, $\eta \in {}^{m}2$ then the condition $(w^q \cup v \cup \{m\}, A \setminus (m+1)) \in Q_D$ forces ($\Vdash_{Q_D}$) a truth values to the following:

(α) $\eta \in T_{\underset{\sim}{p}}(\underset{\sim}{r})$,

(β) $(\exists \nu)\Big[\eta \trianglelefteq \nu \in {}^{\omega>}2 \,\&\, \ell g \nu < \operatorname{Min}(\underset{\sim}{w} \setminus (m+1)) \,\&\, \nu \in \operatorname{sp}(T_{\underset{\sim}{p}}(\underset{\sim}{r}))\Big]$.

(Recall the definition of a Ramsey ultrafilter by game.)

By the sentence before the last, for some $m \in A^q$ and $v \subseteq A^q \cap m$ for every

$\eta^* \in {}^m2$, if we get a positive answer for (α) then we get a negative answer for (β); let $q' = (w^q \cup v \cup \{m\}, A^q \setminus (m+1))$; so q' forces those two statements. Let for $k \in A^q \setminus (m+1)$, $q^k = (w^q \cup v \cup \{m\}, A^q \setminus k)$ (so $q' \leq_{\rm pr} q^k$) and it forces ($\Vdash_{Q_D}$) "every $\eta \in {}^m2 \cap T_p[\underset{\sim}{r}]$ has a unique extension in $T_p(\underset{\sim}{r}) \cap {}^k2$", as required in $\otimes_3$.

The rest of the argument will be used again so just note that proving Claim 1.9 below is enough for finishing the proof of 1.5.

1.9 Claim: *Assume P is a c.c.c. Souslin forcing, $\underset{\sim}{r}$ a P-name of a new real, Q_D, $S[q]$ (for $q \in Q_D$) chosen as above. Then $\otimes_3$ above is impossible.*

Proof: So assume $\otimes_3$ holds and we shall get, eventually, a contradiction.

For this end we define a forcing notion $Q^* = Q^*_D$, $Q^*_D = \{(\bar{m}, \bar{q})$: for some $n = n(\bar{q}) = n(\bar{m}, \bar{q})$ we have $\bar{q} = \langle q_\eta : \eta \in {}^n2\rangle$, $q^* \leq q_\eta \in Q_D$, for each $\eta \in {}^n2$ the sequence $\langle |S[q_\eta] \cap {}^k2| : k < \omega\rangle$ is bounded, $\bar{m} = \langle m_\nu : \nu \in {}^{n>}2\rangle$, $m_\nu < \omega$ and *if* $\nu\char`\^\langle \ell\rangle \trianglelefteq \eta_\ell \in {}^n2$ for $\ell = 0, 1$ and $k < \omega$ *then* $|S[q_{\eta_0}] \cap S[q_{\eta_1}] \cap {}^k2| \leq m_\nu\}$.

The order is defined by $(\bar{m}^1, \bar{q}^1) \leq (\bar{m}^2, \bar{q}^2)$ *iff* $n(\bar{q}^1) \leq n(\bar{q}^2)$, $\bar{m}^1 = \bar{m}^2 \restriction {}^{n(\bar{q}^1)>}2$ and for $\eta \in {}^{n(\bar{q}^2)}2$ we have $q^1_{\eta\restriction n(\bar{q}^1)} \leq q^2_\eta$.

Clearly Q^*_D satisfies the c.c.c. as for any $(\bar{m}^*, \bar{w}^*)$ the set $\{(\bar{m}, \bar{q}) \in Q^*_D : \bar{m} = \bar{m}^*, w^{q_\eta} = w^*_\eta$ for every $\eta \in {}^{n(\bar{q})}2\}$ is directed (in Q^*_D).

Also,

$\otimes_5$ for every n, $\{(\bar{m}, \bar{q}) \in Q^*_D : n(\bar{q}) \geq n\}$ is a dense (and open) subset of Q^*_D. [Why? It is enough to prove that for any given $(\bar{m}^0, \bar{q}^0) \in Q^*_D$ with $n(0) \overset{\rm def}{=} n(\bar{q}^0)$, there is $(\bar{m}^1, \bar{q}^1)$ such that $(\bar{m}^0, \bar{q}^0) \leq (\bar{m}^1, \bar{q}^1) \in Q^*_D$ with $n(\bar{q}^1) = n(0) + 1$; let $q^1_\eta = q^0_{\eta\restriction n(0)}$, and m^1_ν is: m^0_ν if $\nu \in {}^{n(0)>}2$, and $\max\{|S[q^0_\nu] \cap {}^k2| : k < \omega\}$ if $\nu \in {}^{n(0)}2$. Check.]

For $G^* \subseteq Q^*_D$ generic over V, let for $\eta \in {}^{\omega>}2$, $\underset{\sim}{w}_\eta[G^*]$ be $\bigcup\{w^r$: there is $(\bar{m}, \bar{q}) \in G^*$, $r = q_{\eta\restriction n}$ and $n = n(\bar{q}) \leq \ell g(\eta)\}$; it is well defined. If $\eta \in ({}^\omega2)^{V[G^*]}$ let $\underset{\sim}{w}_\eta[G^*]$ be $\bigcup_{k<\omega} w_{\eta\restriction k}[G^*]$.

Also Q^*_D adds a perfect set of generics for Q_D, moreover: in $V[G^*]$ for every $\eta \in ({}^\omega2)^{V[G^*]}$, $\underset{\sim}{w}_\eta[G^*]$ defined above is generic for Q_D over V, which means $G_\eta \overset{\rm def}{=} \{(v, A) : v \subseteq \underset{\sim}{w}_\eta[G^*] \subseteq v \cup A\}$ is a generic subset of Q_D over V; this holds by $\otimes_6$, $\otimes_7$ below.

$\otimes_6$ *if* $(\bar{m}, \bar{q}) \in Q^*_D$ and $\underset{\sim}{\tau}$ is a Q_D-name of an ordinal *then* we can find $\bar{q}^1$ such that $(\bar{m}, \bar{q}) \leq (\bar{m}, \bar{q}^1)$, $n(\bar{q}^1) = n(\bar{q})$ and for every $\eta \in {}^{n(\bar{q}^1)}2$, the condition

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q^1_η forces a value to $\underset{\sim}{\tau}$.

[Why? Let $\langle \eta_k : k < 2^{n(\bar q)} \rangle$ list ${}^{n(\bar q)}2$. We now define by induction on $k \le 2^{n(\bar q)}$, a sequence $\bar r^k = \langle r^k_\eta : \eta \in {}^{n(\bar q)}2 \rangle$ such that:

(a) $(\bar m, \bar r^k) \in Q^*_D$,

(b) $(\bar m, \bar r^k) \le (\bar m, \bar r^{k+1})$ (i.e. $Q_D \models$ "$r^k_\eta \le r^{k+1}_\eta$" for $\eta \in {}^{n(\bar q)}2$),

(c) $\bar r^0 = \bar q$,

(d) $r^{k+1}_{\eta_k}$ forces a value to $\underset{\sim}{\tau}$ (for the forcing notion Q_D).

If we succeed then $\bar q^1 \overset{\text{def}}{=} \bar r^{(2^{n(\bar q)})}$ is as required; as $\bar r^0$ is as required the only problem is to find $\bar r^{k+1}$ being given $\bar r^k$. First we can find $m_k < \omega$, such that no $S[r^k_\eta]$ (for $\eta \in {}^{n(\bar q)}2$) has a splitting node of level $\ge m_k$, and $m_k > n(\bar q)$. Second we find $r^{k,*}_{\eta_k} \in Q_D$ such that: $Q_D \models$"$r^k_{\eta_k} \le_{\text{pr}} r^{k,*}_{\eta_k}$" and $r^{k,*}_{\eta_k}$ forces a truth value to each statement of the form "$\nu \in T_{\underset{\sim}{p}}[\underset{\sim}{r}]$" for $\nu \in {}^{m_k\ge}2$. By the definition of $S[r^k_{\eta_k}]$ necessarily $r^{k,*}_{\eta_k} \Vdash_{Q_D}$ "$T_{\underset{\sim}{p}}[\underset{\sim}{r}] \cap {}^{m_k\ge} 2 \subseteq S[r^k_{\eta_k}]$". Thirdly choose $r^{k+1}_{\eta_k} \in Q_D$, such that $Q_D \models$ " $r^{k,*}_{\eta_k} \le r^{k+1}_{\eta_k}$" and $r^{k+1}_{\eta_k}$ forces a value to $\underset{\sim}{\tau}$ (possible by density) and $S[r^{k+1}_{\eta_k}]$ has no splitting above some level (use $\otimes_3$). Fourth, let $r^{k+1}_\eta = r^k_\eta$ for $\eta \in {}^{n(\bar q)}2 \setminus \{\eta_k\}$; we still have to check $|S[r^{k+1}_{\nu_0}] \cap S[r^{k+1}_{\nu_1}] \cap {}^m 2| \le m_\nu$ when $\nu^\frown\langle \ell \rangle \trianglelefteq \nu_\ell \in {}^{n(\bar q)}2$; by the induction hypothesis without loss of generality $\eta_k \in \{\nu_0, \nu_1\}$, so let $\eta_k = \nu_{\ell(*)}$. If $m \le m_k$ then $S[r^{k+1}_{\nu_{\ell(*)}}] \cap {}^m 2 \subseteq S[r^k_{\nu_{\ell(*)}}]$ and we are done by the induction hypothesis. If $m > m_k$, by the choice of m_k, $S[r^{k+1}_{\nu_{1-\ell(*)}}]$ has no splitting nodes of level $\ge m_k$ hence $|S[r^{k+1}_{\nu_{\ell(*)}}] \cap S[r^{k+1}_{\nu_{1-\ell(*)}}] \cap {}^m 2| \le |S[r^{k+1}_{\nu_{\ell(*)}}] \cap S[r^{k+1}_{\nu_{1-\ell(*)}}] \cap {}^{(m_k)}2|$, and use the previous sentence. So we can carry the induction and $\bar r^{(2^{n(\bar q)})}$ is as required in $\otimes_6$.]

$\otimes_7$ *If* $(\bar m, \bar q) \in Q^*_D$ and $k < \omega$ *then* we can find $\bar q^1$ such that $(\bar m, \bar q) \le (\bar m, \bar q^1)$, $n(\bar q^1) = n(\bar q)$ and for every $\eta \in {}^{n(\bar q^1)}2$, the condition q^1_η forces some $\underset{\sim}{m}_\eta$ to be in $w_\eta \setminus k$.

[Why? Proof similar to that of $\otimes_6$; really a case of it.]

Now for every $\eta \in ({}^\omega 2)^{V[G^*]}$ we know that $V[G_\eta] \models$ "$\underset{\sim}{p}[G_\eta] \in P$, $\underset{\sim}{r}$ is still a P-name of a member of $(2^\omega)^{V[G_\eta][\underset{\sim}{G}_P]}$ and $T_{\underset{\sim}{p}[G_\eta]}[\underset{\sim}{r}]$ is not $\underset{\sim}{w}[G_\eta]$-large"; by 1.7 this holds in $V[G^*]$ too. A closer look shows that for $\eta \ne \nu$ (from $({}^\omega 2)^{V[G^*]}$) the tree $T_{\underset{\sim}{p}[G_\eta]}[\underset{\sim}{r}] \cap T_{\underset{\sim}{p}[G_\nu]}[\underset{\sim}{r}]$ has finitely many splittings. So the conditions $\underset{\sim}{p}[G_\eta]$, $\underset{\sim}{p}[G_\nu]$ are incompatible in $P^{V[G^*]}$. Contradiction to "P is c.c.c. Souslin and this is absolute (1.4(1))". $\blacksquare_{1.9}$

$\blacksquare_{1.5}$

Now we can answer Velickovic's question for Souslin forcings.

1.10 Conclusion: Let P be a c.c.c. Souslin forcing, adding a new real.

(1) The following is impossible: *for every* P-name of a new $\underset{\sim}{r} \in {}^{\omega}\omega$ *for some* tree $T \subseteq {}^{\omega>}\omega$, $T \in V$ and $p \in P$ we have $p \Vdash_P$ "$\underset{\sim}{r} \in \lim T$", and $\bigwedge_{n<\omega} |T \cap {}^{n}\omega| \leq 2^n$; we can replace: "for every n" by "for infinitely many n".

(2) The following is impossible: *for some* P-name of a new $\underset{\sim}{r} \in {}^{\omega}\omega$ *for every* strictly increasing $\{n_i \colon i < \omega\} \subseteq \omega$ from V *for some* tree $T \subseteq {}^{\omega>}\omega$, $T \in V$ and $p \in P$ we have $p \Vdash_P$ "$\underset{\sim}{r} \in \lim T$" and $\bigwedge_{i<\omega} |T \cap {}^{n_i}\omega| \leq 2^i$, we can replace: "for every i" by "for infinitely many i".

(3) The following is impossible: *for some* P-name $\underset{\sim}{r}$ of a new member of ${}^{\omega}2$ *for every* strictly increasing $\{n_i \colon i < \omega\} \subseteq \omega$ from V for some tree $T \subseteq {}^{\omega>}2$ and q we have $q \in P$ and $q \Vdash_P$ "$\underset{\sim}{r} \in \lim T$" and $(\exists^{\infty} i)|T \cap {}^{n_i}2| \leq 2^i$.

(4) The following is impossible: *for some* $r \in ({}^{\omega}2)^{V^P} \setminus {}^{\omega}2$ *for every* strictly increasing sequence $\langle n_i \colon i < \omega\rangle \in V$ of natural numbers, for some tree $T \subseteq {}^{\omega>}2$ from V we have: $r \in \lim T$, and $(\exists^{\infty} i)|T \cap {}^{n_i}2| \leq {}^{i}2$ or at least $(\exists^{\infty} i)|T \cap {}^{(n_i+1)}2| \leq 2^{n_i}$.

Proof: (1), (2) We shall show that they follow by part (3). Suppose that $\underset{\sim}{r}$ is a P-name of a new member of ${}^{\omega}\omega$. Let $\underset{\sim}{\eta}_n$ be (the P-name) $0_{\underset{\sim}{r}(n)+1}{}^{\frown}\langle 1\rangle$ and let $\underset{\sim}{r}^*$ be the following P-name: the concatenation of $\underset{\sim}{\eta}_0, \underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2, \ldots$. From r^* we can construct r so r^* is new too. By part (3) there is a strictly increasing sequence $\langle n_i \colon i < \omega\rangle$ of natural numbers such that for no $q \in P$ and T does $q \Vdash$ "$\underset{\sim}{r} \in \mathrm{Lim}(T)$" and for infinitely many $i < \omega$ we have $|T \cap {}^{n_i}2| \leq 2^i$. Let the tree T' be such that $s \in \lim(T')$ iff the concatenation of $O^1_{s(O)+1}\langle 1\rangle$, $O^1_{s(1)}\langle 1\rangle$, $\ldots$ is in $\lim(T)$. Clearly $|T' \cap {}^{n}2| \geq |T \cap {}^{n}2|$, so we have proved clause (2) for $\langle n'_i \colon i < \omega\rangle =: \langle n_i 2^{n_i} \colon i < \omega\rangle$. Replacing $\underset{\sim}{r}^*$ by $\langle h^*(r^* \restriction n_{(i!)!}) \colon i < \omega\rangle$, h^* is a one-to-one function from ${}^{\omega>}\omega$ into ω, will give (1) too.

(3) Follows from part (4).

(4) Let $\underset{\sim}{r}$ be a P-name of a new real.

By Lemma 1.5 for some infinite $u \subseteq \omega$ we have

$(*)$ for every $p \in P$, $T_p[\underset{\sim}{r}]$ is u-large (see 1.1(4)).

We now choose by induction on i, $n_i < \omega$, such that $n_i > \sup\{n_j \colon j < i\}$ and $|(n_i, n_{i+1}) \cap u| > 2^{n_i} + 2$. If (4) fails for $\underset{\sim}{r}$ we apply the statement to the sequence

$\langle n_i: i < \omega\rangle$, so for some $p \in P$ and subtree T of ${}^{\omega>}2$ from V, we have:

(a) $p \Vdash_P$ "$\underset{\sim}{r} \in \lim T$",

(b) for infinitely many $i < \omega$, we have $|T \cap {}^{(n_{i+1})}2| \leq 2^{n_i}$.

By the choice of u, for some $j^* < \omega$, we know that $T_p[\underset{\sim}{r}]$ has a splitting of level $\in [j_0, j_1)$ for each $j_0, j_1 \in u$, $j^* \leq j_0 < j_1$.

So if $i > j^*$, then $|T_p[\underset{\sim}{r}] \cap {}^{(n_{i+1})}2|$ is at least the number of levels $< n_{i+1}$ of splitting nodes of $T_p[\underset{\sim}{r}]$ which is $\geq |(n_i, n_{i+1}) \cap u|$ which is $> 2^{n_i}$. But $p \Vdash$ "$\underset{\sim}{r} \in \lim T$" implies $T_p[\underset{\sim}{r}] \subseteq T$ so $|T \cap {}^{(\eta_{i+1})}2|$ is $> 2^{n_i}$ (for every $i < \omega$ such that $i > j^*$), this contradicts the choice of T hence we finish. $\blacksquare_{1.10}$

1.11 Remark: This means that any c.c.c. Souslin forcing which is ${}^\omega\omega$-bounding is quite similar to the Random real forcing in some sense. More exactly every c.c.c. Souslin forcing has a property shared by the Random real forcing and the Cohen forcing.

1.12 CLAIM:

(1) *Assume*

(a) *P is a forcing notion,*

(b) *$\underset{\sim}{r}$ is a P-name of a member of ${}^\omega 2$,*

(c) *$\bar{h} = \langle h_n: n < \omega\rangle, h_n = n$ (i.e. $h_n(i) = n$ for every $i < \omega$) and $u \subseteq \omega$ is infinite,*

(d) *for every $p \in P$, for some $\eta \in T_p[\underset{\sim}{r}]$ the set $\{k: \eta^\frown O_{k-\ell g\eta}{}^\frown\langle 1\rangle \in T_p[\underset{\sim}{r}]\}$ is $(u, \bar{h})$-large (see (*) of 1.1(8)).*

Then forcing with P adds a Cohen real.

(2) *We can weaken (d) to*

(d)$^-$ *for every $p \in P$ for some $n < \omega$, and $\eta_0, \dots, \eta_{n-1} \in T_p[\underset{\sim}{r}]$ the set $\{k$: for some $\ell < n$, $\eta_\ell{}^\frown O_{k-\ell g\eta_\ell}{}^\frown\langle 1\rangle \in T_p[\underset{\sim}{r}]\}$ is $(u, \bar{h})$-large.*

Proof: (1), (2) Let $u \setminus \{0\} = \{n_i: 1 \leq i < \omega\}$, $n_0 \stackrel{\text{def}}{=} 0 < n_1 < n_2 < \cdots$, let $\langle k(i,\ell): \ell < \omega\rangle$ be such that $i = \sum_\ell k(i,\ell)2^\ell$, $k(i,\ell) \in \{0,1\}$, so $k(i,\ell) = 0$ when $2^\ell > i$. Let $\rho^*_m = \langle k(i,\ell): \ell \leq [\log_2(i+1)]\rangle$ where $i = i_u(m)$ is the unique i such that $n_i \leq m < n_{i+1}$. We define a P-name $\underset{\sim}{s}$ (of a member of $({}^\omega 2)^{V^P}$): let $\{\underset{\sim}{k}_i: i < \omega\}$ list in increasing order $\{k < \omega: \underset{\sim}{r}(k) = 1\}$ and $\underset{\sim}{s}$ be $\rho^*_{k_0}{}^\frown\rho^*_{k_1}{}^\frown\rho^*_{k_2}{}^\frown\dots$.

Clearly by condition (d)$^-$, for every $p \in P$ and $n < \omega$ we have $p \nVdash$ "$\underset{\sim}{r}(k) = 0$ for every $k \geq n$". Hence $\Vdash_P$ "$\{k < \omega: \underset{\sim}{r}(k) = 1\}$ is infinite, hence $\Vdash_P$ "$\underset{\sim}{s} \in {}^\omega 2$".

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It is enough to prove that $\Vdash_P$"$\underset{\sim}{s}$ is a Cohen real over V". So let $T \in V$ be a given subtree of ${}^{\omega>}2$ which is nowhere dense, i.e. $(\forall \eta \in T)(\exists \nu)[\eta \triangleleft \nu \in {}^{\omega>}2 \setminus T]$, and we should prove $\Vdash_P$"$\underset{\sim}{s} \notin \lim T$". So assume $p \in P$, $p \Vdash_P$ "$\underset{\sim}{s} \in \lim T$" and we shall get a contradiction. Having our $p \in P$ we can apply $(d)^-$ (or (d), which is stronger), so we can find $n < \omega$ and $\eta_0, \ldots, \eta_{n-1} \in T_p[\underset{\sim}{r}]$ as there such that $A = \{k < \omega$: for some $\ell < n$, $\eta_\ell {}^\frown O_{k-\ell g \eta_\ell} {}^\frown \langle 1 \rangle \in T_p[\underset{\sim}{r}]\}$ is $(u, \bar{h})$-large.

Let for each $\ell < n$, $\{k^\ell_j : j < j_\ell\}$ list in increasing order $\{k < \ell g(\eta_\ell) : \eta_\ell(k) = 1\}$ and let $\rho^\ell = \rho^*_{k^\ell_0} {}^\frown \rho^*_{k^\ell_1} {}^\frown \ldots \rho^*_{k^\ell_{j_\ell - 1}}$. Now we can choose by induction on $\ell \leq n$, a sequence $\nu_\ell \in {}^{\omega>}2$ such that: $\nu_0 = \langle\rangle$, $\nu_\ell \trianglelefteq \nu_{\ell+1}$ and $\rho^\ell {}^\frown \nu_{\ell+1} \notin T$ (each time use "T is nowhere dense").

Next we choose $m(*) \in A$ such that $\nu_n \trianglelefteq \rho^*_{m(*)}$; possible as A is $(n, \bar{h})$-large (check Definition 1.1(8): the set $\{i_u(m) : m \in A\}$ contains an interval of length $> 2^{\ell g(\nu_n)}$, so by the definition of ρ^*_m, some $m(*)$ in this interval is as required). Now we can find $p_1 \in P$ such that $p \leq p_1$ and $p_1 \Vdash_P$ "for some $\ell < n$, $\eta_\ell {}^\frown O_{m(*)-\ell g \eta_\ell} {}^\frown \langle 1 \rangle \trianglelefteq \underset{\sim}{r}$" hence $p_1 \Vdash_P$ "for some $\ell < n$, $\rho^\ell {}^\frown \rho^*_{m(*)} \trianglelefteq \underset{\sim}{s}$", so by the choice of $\nu_{\ell+1}$, and as $\nu_{\ell+1} \trianglelefteq \nu_n \trianglelefteq \rho^*_{m(*)}$ we get $p_1 \Vdash_P$ "$\underset{\sim}{s} \notin \lim T$" hence we get contradiction to: $p \Vdash_P$"$\underset{\sim}{s} \in \lim T$", hence we finish proving $\Vdash_P$ "$\underset{\sim}{s}$ is a Cohen real over V." $\blacksquare_{1.12}$

1.13 Claim: *Let P be a c.c.c. Souslin forcing*

(1) *"P adds a Cohen real" is absolute (as well as "x is a P-name of a Cohen real").*

(2) *"x is a P-name of a dominating real" is absolute.*

(3) *"P add a non dominated real" is absolute (as well as "x is a P-name of a non dominated real").*

(4) *For a given $\bar{h}$, "there is $u \in [\omega]^{\aleph_0}$ such that (d) of Claim 1.12(1) holds" is absolute; similarly $(d)^-$ of 1.12(2).*

(5) *"x is a P-name of a member of ${}^\omega\omega$, dominating $\eta_1 \in {}^\omega\omega$ and not dominating $\eta_2 \notin {}^\omega\omega$" is absolute (in fact, a conjunction of $\prod^1_1$ and $\sum^1_1$ statements).*

Proof: (1) Let $\varphi(x)$ say:

(a) $x = \langle\langle p^\eta_i, \mathbf{t}^\eta_i : i < \omega\rangle : \eta \in {}^{\omega>}\omega\rangle$, $p^\eta_i \in P$, $\mathbf{t}^\eta_i$ a truth value, $\langle p^\eta_i : i < \omega\rangle$ a maximal antichain (for each $\eta \in {}^{\omega\geq}\omega$),

(b) if $\eta, \nu \in {}^{\omega>}\omega$ and p^η_i, p^ν_j are compatible *then*: $[\eta \trianglelefteq \nu \wedge \mathbf{t}^\nu_j$ truth $\Rightarrow \mathbf{t}^\eta_i$=truth$]$ and $[\eta, \nu$ are $\triangleleft$-incomparable $\bigwedge \& \mathbf{t}^\eta_i$=truth$\Rightarrow \mathbf{t}^\nu_j$=false$]$,

(c) for every $p \in P$ for some $\eta \in {}^{\omega>}\omega$ for every ν, $\eta \trianglelefteq \nu \in {}^{\omega>}\omega$, we have $\bigvee_{i<\omega}(p, p_i^\nu \text{ compatible} \wedge \mathbf{t}_i^\nu=\text{truth})$.

Now by 1.4(1)+(2) part (a) is a conjunction of $\prod_1^1$ and $\sum_1^1$ statements, part (b) is both $\prod_1^1$ and $\sum_1^1$ and part (c) is $\prod_1^1$ (we use: compatibility is both $\prod_1^1$ and $\sum_1^1$ and $(\forall p \in P)\,[\ldots]$ means $(\forall p)[p \notin P \vee \ldots])$. So $\varphi(x)$ is a conjunction of $\prod_1^1$ and $\sum_1^1$ statements. Now $\varphi(x)$ says "x represents a P-name of a Cohen real" so $(\exists x)\varphi(x)$ which is a $\sum_2^1$ statement, express the statement "forcing with P adds a Cohen real."

(2) We repeat the proof of part (1) but clause (c) is replaced by:

(c)$'$ *for every* $p \in P$ and $f \in ({}^\omega\omega)^V$ *there are* $q \in P$ and n^* such that $p \le q$ and:

(c)$_{q,f,n^*}$ *if* q, p_i^η are compatible, $\mathbf{t}_i^\eta$ truth and $n^* \le n < \ell g(\eta)$ *then* $f(n) \le \eta(n)$.

Now (c)$_{q,f,n^*}$ is a $\prod_1^1$ and $\sum_1^1$, so (c)$'$ has the form $(\forall p, f)[p \notin P \vee (\exists q, n^*)[q \in P \& p \le q \& (c)_{q,f,n^*}]$ which is $\prod_2^1$ hence "x is a P-name of a dominating real" is an absolute statement.

(3) Use the proof of Part (1) but clause (c) is replaced by:

(c)$''$ for $p \in P$ for infinitely many n the set $\{\eta(n)\colon \eta \in {}^{\omega>}\omega, \ell g\eta > n, i < \omega, \mathbf{t}_i^n = \text{truth}, p_i^\eta, p \text{ compatible}\}$ is infinite.

Now (c)$''$ is $\prod_1^1$ and we can finish as there.

(4) The statements (d) and (d)$^-$ from 1.12 for given $p, \underset{\sim}{r}, \bar{h}, u$ is a $\prod_1^1$ statement (as by the proof of 1.7(1) $\nu \in T_p[\underset{\sim}{r}]$ is a $\prod_1^1$-statement and a $\sum_1^1$-statement).

(5) Easier than the proof of (3). $\blacksquare_{1.14}$

1.14 Conclusion: If P is a c.c.c. Souslin forcing notion adding $\underset{\sim}{g} \in {}^\omega\omega$ not dominated by any $f \in ({}^\omega\omega)^V$ *then* forcing with P adds a Cohen real.

Proof: Without loss of generality, $\underset{\sim}{g}$ is strictly increasing. Let $\underset{\sim}{r} = \{\underset{\sim}{g}(i)\colon i < \omega\}$; it is a subset of ω identified with its characteristic function. We imitate the proof of Lemma 1.5 (using here 1.13(3) instead of 1.7 there) so as there, without loss of generality, there is a Ramsey ultrafilter D on ω and let the forcing nótion Q_D be as there. Let $\bar{h}$ be as in Claim 1.12 condition (c); we ask:

$\otimes^1$ is there an infinite $u \subseteq \omega$ such that condition $(d)^-$ of Claim 1.12 holds?

If yes we are done by Claim 1.12 . So from now on we asssume not.

Let $G \subseteq Q_D$ be generic over V, condition (d)$^-$ fails also in $V[G]$ (using

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absoluteness which holds by Claim 1.13(4)), in particular for $u =^{df} \underset{\sim}{w}[G]$. For $p \in P^{V[G]}$ and $\eta \in {}^{\omega>}2$ we let

$C[\eta, p] =^{df} \{k\colon \eta^\frown 0_{k-\ell g(\eta)}{}^\frown\langle 1\rangle \in T_p[\underset{\sim}{r}]\}$.

Hence for some $p^* \in P^{V[G]}$ (remember (*) of 1.1(8)) :

$(*)_1$ $V[G] \models$ " for every $j < \omega$ and $\eta^*_\ell \in T_{p^*}[\underset{\sim}{r}]$ for $\ell < j$ for some n^* letting $C =^{df} \bigcup\{C[\eta^*_\ell, p^*]\colon \ell < j\}$, for no $n^* + 1$ consequtive members $i_0 < i_1 < \cdots < i_{n^*}$ of u do we have: for every $m < n^*$ the sets $[i_m, i_{m+1})$, C are not disjoint".

Let for $n < \omega$, $h_0(n)$ be like n^* in $(*)_1$ for $\{\eta^*_\ell\colon \ell < j\} =^{df} T_{p^*}[\underset{\sim}{r}] \cap {}^{n\geq}2$. Let for $n < \omega$, $h(n) =^{df} \sum_{i\leq n}(h_0(i)+n)$. Note that we have: h is strictly increasing.

So for some $q^* \in G$ (which is a subset of Q_D), and Q_D-name $\underset{\sim}{p}^*$ of a member of P, we have: q^* forces that $\underset{\sim}{p}^*, \underset{\sim}{h}, \underset{\sim}{C}[\eta, \underset{\sim}{p}^*]$ (for $\eta \in {}^{\omega>}2$,) are as above (for our fixed $\underset{\sim}{r}$).

Wlog $0 \in w^{q^*}$.

Let $\underset{\sim}{w} = \underset{\sim}{w} \cup \{0\} = \{\underset{\sim}{n}_i\colon i < \omega\}$ be strictly increasing, note $\underset{\sim}{w}$, $\underset{\sim}{n}_i$ are Q_D-names.

Wlog

$\otimes^2$ for every $k \in A^{q^*}$ and subset v of $A^{q^*} \cap k$ the condition $(w^{q^*} \cup v \cup \{k\}, A^{q^*} \setminus (k+1))$ forces a value to the following:

(A) $T_{\underset{\sim}{p}^*}[\underset{\sim}{r}] \cap {}^{k\geq}2$, say t,

(B) truth value to $[\underset{\sim}{n}_i, \underset{\sim}{n}_{i+1}) \cap C[\eta, \underset{\sim}{p}^*] = \emptyset$ for $\eta \in t$ and $i \leq k+1$.

So

$\otimes^3$ Assume $q_1 \geq q^*$, $n < \omega$ and q_1 forces $\underset{\sim}{h}(n) = n^*$ and $T_{\underset{\sim}{p}^*}[\underset{\sim}{r}] \cap {}^{n\geq}2 = t$. If $i_0 < \cdots < i_{n^*}$ are the first $n^* + 1$ members of A^{q_1}, then for some $m < n^*$ the condition $q_2 =^{df} (w^{q_1} \cup \{i_0, \ldots, i_{m-1}\}, A^{q_1} \setminus i_m)$ forces that:

for no $\eta \in t$ and j do we have: $i_{m-1} \leq j < \mathrm{Min}(\underset{\sim}{w} \setminus (i_{m-1}+1))$ and $j \in C[\eta, \underset{\sim}{p}^*]$.

Now we want to imitate the proof of 1.9. We define a forcing notion $Q^{**} = Q^{**}_D$ as follows:

a member of Q^{**}_D has the form $(\bar{m}, \bar{q})$ such that:

(1) $\bar{q} = \langle q_\nu\colon \nu \in {}^n2\rangle$, $q^* \leq q_\nu \in Q_D$, $n = n(\bar{q})$,

(2) for each $\nu \in {}^n2$ the number $i[q_\nu] =^{df} \mathrm{Max}(w^{q_\nu})$ is well defined,

(3) q_ν forces (in Q_D) a value t_ν to $T_{\underset{\sim}{p}^*}[\underset{\sim}{r}] \cap {}^{n\geq}2$ and a value $h_\nu(j)$ to $\underset{\sim}{h}(j)$ for $j \leq n$ and a value $c_\nu(\eta, \underset{\sim}{p}^*)$ to $\underset{\sim}{C}(\eta, \underset{\sim}{p}^*) \cap i[q_\nu]$,

(4) $q_\nu \Vdash_{Q_D}$ "if $\eta \in t_\nu$ then for no k do we have: $i[q_\nu] \le k < \mathrm{Min}(\underset{\sim}{w} \setminus (i[q_\nu]+1))$ and $k \in C[\eta, \underset{\sim}{p}^*]$",

(5) $\bar m = \langle m_i : i \le n \rangle$, $m_i < \omega$,

(6) if $\nu^\frown\langle j\rangle \trianglelefteq \nu_j \in {}^n 2$ for $j = 0, 1$ and $\eta \in t_{\nu_0} \cap t_{\nu_1}$ then $c_{\nu_0}[\eta, p^*] \cap c_{\nu_1}[\eta, p^*]$ is bounded by $\max\{m_{\ell g(\eta)}, m_{\ell g(\nu)}\}$.

The order is defined by: $(\bar m^1, \bar q^1) \le (\bar m^2, \bar q^2)$ *iff* $n(\bar q^1) \le n(\bar q^2)$, $\bar m^1 = \bar m^2 \restriction (n(\bar q^1) + 1)$ and for $\eta \in {}^{n(\bar q^2)}2$ we have $q^1_{\eta \restriction n(\bar q^1)} \le q^2_\eta$.

We now continue as in the proof of 1.9 .

Why are the $\underset{\sim}{p}^*[G_\nu]$ for $\nu \in {}^\omega 2$ pairwise incompatible? If ν_1, ν_2 are not equal, and $\underset{\sim}{p}^*[G_{\nu_1}], \underset{\sim}{p}^*[G_{\nu_2}]$ are compatible in P, let p be a common upper bound. We know that for some $\eta \in T_p[\underset{\sim}{r}]$ we have $C[\eta, p]$ is infinite, as otherwise from $T_p[\underset{\sim}{r}]$ we can define a function $f \in {}^\omega\omega$ such that $q \Vdash_P$ "$\underset{\sim}{g} \le f$", contradicting the assumption. $\blacksquare_{1.14}$

* * *

The question is whether a forcing adding half Cohen real (see below) adds a Cohen real is due to Bartoszyński and Fremlin, appears in [B].

1.15 Definition: If $V \subseteq V^1$, $r \in ({}^\omega\omega)^{V^1}$ we say that r is a half Cohen real over V *if* for every $\eta \in ({}^\omega\omega)^V$, for infinitely many $n < \omega$, $r(n) = \eta(n)$.

1.16 Conclusion: If a c.c.c. Souslin forcing adds a half Cohen real then it adds a Cohen real.

Proof: If $\underset{\sim}{r}$ is a P-name a member of ${}^\omega\omega$ which is (forced to be) half Cohen over V, then $\Vdash_P$ "$\underset{\sim}{r} \in {}^\omega\omega$ is not dominated by any "old" $h \in {}^\omega\omega$ (i.e. $h \in ({}^\omega\omega)^V$") trivially—you can use the definition on $h+1$ to get strict inequality. Now use 1.14. $\blacksquare_{1.16}$

We remark on another claim in the same line.

1.17 CLAIM: *Assume Q is a ccc Souslin forcing and $\underset{\sim}{r}$ is a Q-name of a member of ${}^\omega\omega$. Assume further that for some $\eta \in {}^\omega\omega$, a Cohen real over V and G , a subset of $Q^{V[\eta]}$ generic over $V[\eta]$ we have $\underset{\sim}{r}[G]$ dominate η, i.e. for every $n < \omega$ large enough $\eta(n) \le \underset{\sim}{r}[G](n)$. Then (in V) $\underset{\sim}{r}$ is forced by some $q \in Q^V$ to be a dominating real, i.e. for every G, a subset of Q^V generic over V to which p belongs and $\rho \in ({}^\omega\omega)^{V[G]}$, for every $n < \omega$ large enough $\rho(n) \le \underset{\sim}{r}[G](n)$.*

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Proof: Assume that not; then some $p \in Q^{V[\eta]}$ force the negation, i.e. $\underset{\sim}{r}$ is as above but the conclusion fails. By the homogeneity of Cohen forcing there is a Cohen name $\underset{\sim}{p}$ of such a condition.

Also in V there is a maximal antichain J of Q and sequence $\langle \rho_q \colon q \in J \rangle$ such that for each $q \in J$ either q forces that $\underset{\sim}{r}$ does not dominate ρ_q and $\rho_q \in ({}^{\omega}(\omega \setminus \{0\})$ OR q forces that $\underset{\sim}{r}$ dominate every $\rho \in ({}^{\omega}\omega)^V$ and $\rho_q(n) = 0$ for every $n < \omega$ (all in V). Now we can find $\rho^* \in {}^{\omega}\omega$ dominating all ρ_q for $q \in J$.

Also wlog $\underset{\sim}{p}$ is above some $q^* \in J$ so necessarily $\rho_{q^*}(n) \neq 0$.

We define a forcing notion R as follows: members have the from (f, g) where for some $n = n(f) < \omega$, f is a function from ${}^{n>}2$ to ω and g is a function from ω to ω. The order is: $(f_1, g_2) \leq (f_2, g_2)$ iff $n(f_1) \leq n(f_2)$, f_2 extend f_1, for every $n < \omega$ we have $g_1(n) \leq g_2(n)$ and for every k satisfying $n(f_1) \leq k < n(f_2)$, for all except at most one $\nu \in {}^{k}2$ we have $g_1(k) \leq f_2(\nu)$.

SUBCLAIM: *Let G be a subset of R generic over V.*

(1) *In $V[G]$, for every $\nu \in ({}^{\omega}2)^V$, $\eta_\nu =^{df} f \circ \nu = \langle f(\nu \restriction \ell) \colon \ell < \omega \rangle$ is a Cohen real over V, so*

(2) *$p_{\eta_\nu} =^{df} \underset{\sim}{p}[\eta_\nu]$ is a member of $Q^{V[\eta]}$, which by absoluteness is a subset of $Q^{V[G]}$.*

Now in $V[\eta_\nu]$ clearly p_{η_ν} forces that $\underset{\sim}{r}$ dominate η_ν but not ρ^* . By absoluteness this holds also in $V[G]$. Now if ν_1, ν_2 are distinct members of $({}^{\omega}2)^{V[G]}$ and $p_{\eta_{\nu_1}}, p_{\eta_{\nu_2}}$ are compatible in $Q^{V[G]}$, let $p^* \in Q^{V[G]}$ be a common upper bound; then it forces that $\underset{\sim}{r}$ dominates ρ^*, but it is also above a member q^* of J such that ρ_{q^*} is not a sequence of zeroes.

By absoluteness we get a contradiction to "Q is c.c.c. Souslin forcing". $\blacksquare_{1.17}$

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