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κ-fold transitive groups

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Abstract. An abelian group G of type 0 is called κ-fold transitive for some cardinal $\kappa > 0$ if for any pair of pure elements $x, y \in G$ there exist exactly κ-many $\varphi \in \operatorname{Aut} G$ such that $x\varphi = y$. We show the existence of large κ-fold transitive groups for every $\kappa \geq \aleph_0$ assuming V=L and ZFC respectively.

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1 Introduction

This paper deals with λ-free abelian groups ($\lambda \geq \aleph_1$ a given cardinal), i.e. any subgroup of cardinality $< \lambda$ is free. A central position here is occupied by free groups and $\aleph_1$-free groups, where all countable subgroups are free. All these groups share in common a very rigid group structure alongside with a plenty of pure elements (divisible only by 1 and -1); let $\mathfrak{p}G$ denote the collection of pure elements of the group G. We now call G a UT-*group* (UT for uniquely transitive) if for any pair of elements $x, y \in \mathfrak{p}G$ there exists a unique $\varphi \in \operatorname{Aut} G$ such that $x\varphi = y$. After a long period of stagnation concerning the existence of non-trivial UT-groups besides $\mathbb{Z}$ there has recently been a real rush of papers showing existence under quite different set-theoretical assumptions, see [7] for an overview and [3, 4, 5, 6, 8] for details. The methods to construct UT-groups can be summarized in the following two competing strategies: on the one hand we can try to reach the goal by purely group theoretic means resulting in groups with non-commutative free endomorphism rings and non-trivial endomorphism kernels, on the other hand we can use a shortcut through ring theory leading to a special class of principal ideal domains whose additive groups are uniquely transitive with trivial endomorphism kernels.

In this paper we want to investigate the following canonical generalizations of UT-groups.

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Definition 1.1. Let G be an $\aleph_1$-free group (or more generally: of type 0).

(a) G is *κ-fold transitive* for some cardinal $\kappa > 0$ if for any pair of elements $x, y \in \mathfrak{p}G$ there exist exactly κ-many different $\varphi \in \operatorname{Aut} M$ such that $x\varphi = y$.
(b) G is *almost uniquely transitive* if for any pair of elements $x, y \in \mathfrak{p}G$ there exist at least two but finitely many different $\varphi \in \operatorname{Aut} M$ such that $x\varphi = y$.

In Corollary 2.5 we will see that almost uniquely transitive groups are a special case of κ-fold transitive groups. Concerning the existence of κ-fold transitive groups we will prove the following result. Recall here that $\operatorname{cf}(\alpha)$ denotes the cofinality of an ordinal α and that $S^{\lambda}_{\aleph_0} := \{\alpha \in \lambda \mid \operatorname{cf}(\alpha) = \aleph_0\}$.

Theorem 1.2. *Let $\kappa \geq \aleph_0$ be a cardinal and $\lambda > \kappa$ be a regular cardinal with $\diamondsuit_S$ for some non-reflecting stationary $S \subseteq S^{\lambda}_{\aleph_0}$. Then there exists a λ-free κ-fold transitive group of cardinality λ.*

As the endomorphism rings of κ-fold transitive groups have obligatory non-trivial endomorphism kernels for $\kappa > 1$ the group theoretic approach from [6] using iterated pushouts will celebrate a fulminant comeback here. But in contrast to [6] this time the proof makes use of the Diamond Principle $\diamondsuit_S$. Remember here that assuming Gödel's universe V=L a non-reflecting stationary $S \subseteq S^{\lambda}_{\aleph_0}$ exists for every successor cardinal $\lambda > \aleph_0$, see [9]. For $\lambda = \chi^+ = 2^{\chi}$ the Diamond Principle $\diamondsuit_S$ holds for any stationary $S \subseteq \{\delta < \lambda \mid \operatorname{cf}(\delta) \neq \operatorname{cf}(\chi)\}$, see [10] for a proof and a history on earlier weaker results.

The Sections 2 to 4 of this paper engage in the proof of Theorem 1.2. In Section 5 then follows a concluding discussion of the used construction with references to UT-groups and Black Box constructions.

That we focus in this paper on the case $\kappa \geq \aleph_0$ has technical reasons: throughout our algebraic prerequisites and construction tools we will assume that for some pure element a^* in our κ-fold transitive group G the group K of automorphisms $\varphi \in \operatorname{Aut} G$ leaving a^* fixed is a non-commutative free group of cardinality κ, hence $\kappa \geq \aleph_0$. The case $\kappa < \aleph_0$ will need a much more careful and elaborate construction allowing commuting endomorphisms and a more complex endomorphism ring structure. This will be the object of a subsequent paper.

Our notations are standard (see [1, 2, 7, 9]) and homomorphisms are applied on the right. For an introduction into algebraic constructions using set theoretic tools we refer to [1, 7].

2 Algebraic preparatory work

Throughout this and the following sections let $\aleph_0 \leq \kappa < \lambda$ be cardinals with λ regular. We will emphasize the free $\diamondsuit_S$-construction. The correspondent definitions and results for the $\aleph_1$-free Black Box-construction mentioned in Section 5 are noted in brackets.

Definition 2.1.

(a) Let $\mathscr{A}$ ($\mathscr{A}^*$) be the class of all $\mathfrak{x} = (G_{\mathfrak{x}}, Y_{\mathfrak{x}}, F_{\mathfrak{x}}, \overline{G}^{\mathfrak{x}}, \overline{A}^{\mathfrak{x}}, a^*_{\mathfrak{x}}) = (G, Y, F, \overline{G}, \overline{A}, a^*)$ with:

(α) $G \neq 0$ is a free ($\aleph_1$-free) commutative group.

(β) Y is a set of non-commutative free generators with $F = \langle Y \rangle$ and $|Y| \leq |G|$.

(γ) $\overline{G} = \langle G_f : f \in F \rangle$ with $G_f \subseteq_* G$ for all $f \in F$ and G/G_{y^ε} free ($\aleph_1$-free) for all $y \in Y$, $\varepsilon \in \{-1, 1\}$.

(δ) $\overline{A} = \langle A_f : f \in F \rangle$ with $A_f : G_f \mapsto G_{f^{-1}}$ a group isomorphism. We set $G_1 := G$ and $A_1 := \mathrm{id}_G$ for $1 = 1_F$.

(ε) If $f = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n}$ ($\varepsilon_i \in \{-1, 1\}$) is the reduced representation of $f \in F$ via Y (i.e. the representation of minimal length n), then

$$A_f = A_{y_1}^{\varepsilon_1} A_{y_2}^{\varepsilon_2} \dots A_{y_n}^{\varepsilon_n}.$$

In particular

$$G_f = \mathrm{Dom}\,(A_{y_1}^{\varepsilon_1} A_{y_2}^{\varepsilon_2} \dots A_{y_n}^{\varepsilon_n}) \quad \text{and} \quad G_{f^{-1}} = \mathrm{Im}\,(A_{y_1}^{\varepsilon_1} A_{y_2}^{\varepsilon_2} \dots A_{y_n}^{\varepsilon_n}).$$

(ζ) $a^* \in \mathfrak{p}G$ and $\left|F_{\mathfrak{x}}(a^*, a^*)\right| = \kappa$, where we set $F_{\mathfrak{x}}(a, b) := \{f \in F : aA_f \in \mathbb{Z}b\}$ for all $a, b \in \mathfrak{p}G$. Here $aA_f \in \mathbb{Z}b$ includes the implications $a \in G_f$ and $b \in G_{f^{-1}}$.

(η) If $b \in \mathfrak{p}G$ with $F_{\mathfrak{x}}(a^*, b) = \emptyset$, then $F_{\mathfrak{x}}(b, b) = \{1\}$.

(b) For every $\mathfrak{x} \in \mathscr{A}$ ($\mathscr{A}^*$) set $K := K_{\mathfrak{x}} := F_{\mathfrak{x}}(a^*, a^*) \subseteq F_{\mathfrak{x}}$ as subgroup.

The maps A_y ($y \in Y$) were called "partial automorphisms" in [6] as their main purpose is to grow up by algebraic manipulation to automorphisms of the whole group G. Recall here that the composition $\varphi\mu$ of two partial automorphisms φ, μ was defined canonically as having domain $\mathrm{Dom}\,(\varphi\mu) = (\mathrm{Dom}\,\mu \cap \mathrm{Im}\,\varphi)\varphi^{-1}$ and image $\mathrm{Im}\,(\varphi\mu) = (\mathrm{Dom}\,\mu \cap \mathrm{Im}\,\varphi)\mu$.

Corollary 2.2.
(1) *If $f = f_1 f_2 \dots f_n$ for elements $f, f_1, \dots, f_n \in F$, then $A_f \supseteq A_{f_1} A_{f_2} \dots A_{f_n}$ holds.*
(2) *For every $f \in F, g \in G_f$ holds $g \in \mathfrak{p}G \iff gA_f \in \mathfrak{p}G$.*
(3) *For every $f \in F$ the group G/G_f is free ($\aleph_1$-free).*

Proof. Easy. Clause (3) is proven by induction on the length of the reduced representation $f = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n}$ of $f \in F$. To demonstrate the keynote for A_y, A_z ($y, z \in Y$) observe that

$$\mathrm{Im}\,A_y/(\mathrm{Dom}\,A_z \cap \mathrm{Im}\,A_y) \cong (\mathrm{Dom}\,A_z + \mathrm{Im}\,A_y)/\mathrm{Dom}\,A_z \subseteq G/\mathrm{Dom}\,A_z$$

is free. Multiplication by A_y^{-1} shows that

$$\mathrm{Dom}\,A_y/(\mathrm{Dom}\,A_z \cap \mathrm{Im}\,A_y)A_y^{-1} = \mathrm{Dom}\,A_y/\mathrm{Dom}\,(A_y A_z)$$

and $G/\mathrm{Dom}\,(A_y A_z)$ are free. □

Thus in particular $aA_f \in \mathbb{Z}b$ for $a, b \in \mathfrak{p}G$ means $aA_f \in \{-b, b\}$ by Corollary 2.2(2).

Definition 2.3. Let be $\mathfrak{x} \in \mathscr{A}$ ($\mathscr{A}^*$).

(a) We define the relation $\mathscr{E}_{\mathfrak{x}} := \{(a, b) \mid a, b \in \mathfrak{p}G, \exists f \in F : aA_f \in \mathbb{Z}b\}$.
(b) We call $\mathfrak{x}$ *full* if $G_f = G$ for all $f \in F$.

(c) We call $\mathfrak{x}$ *very full* if $\mathfrak{x}$ is full with $\mathscr{E}_{\mathfrak{x}} = \mathfrak{p}G \times \mathfrak{p}G$.
(d) Let $\mathcal{U} \subseteq \mathscr{A}$ ($\mathcal{U}^* \subseteq \mathscr{A}^*$) be the class of all very full $\mathfrak{x}$.

Plain consequences of this definition include the following.

Corollary 2.4.
(1) *$\mathscr{E}_{\mathfrak{x}}$ is an equivalence relation.*
(2) *For every $(a,b) \in \mathfrak{p}G \times \mathfrak{p}G$ holds $(a,b) \in \mathscr{E}_{\mathfrak{x}} \iff F_{\mathfrak{x}}(a,b) \neq \emptyset$.*
(3) *If $a, b \in \mathfrak{p}G$ with $a/\mathscr{E}_{\mathfrak{x}} = b/\mathscr{E}_{\mathfrak{x}} = a^*/\mathscr{E}_{\mathfrak{x}}$, then $F_{\mathfrak{x}}(a,b) = sK_{\mathfrak{x}}t$ for suitable $s, t \in F_{\mathfrak{x}}$, in particular $F_{\mathfrak{x}}(a,a) \cong K_{\mathfrak{x}}$ as groups. Otherwise $|F_{\mathfrak{x}}(a,b)| \leq 1$ holds.*

Proof. Easy.

From Corollary 2.4(3) the following link between κ-fold transitive, uniquely transitive and almost uniquely transitive groups is immediate.

Corollary 2.5.
(1) *A group G is uniquely transitive iff it is 1-fold transitive.*
(2) *A group G is almost uniquely transitive iff it is κ-fold transitive for some $1 < \kappa < \aleph_0$.*

Proof. Easy.

We should emphasize that the class $\mathscr{A}$ ($\mathscr{A}^*$) is non-trivial. For the more complicated class $\mathcal{U}$ ($\mathcal{U}^*$) this will follow from Theorem 3.5.

Lemma 2.6. $\mathscr{A} \neq \emptyset$ ($\mathscr{A}^* \neq \emptyset$).

Proof. Let G be a free group of cardinality $\kappa \leq |G| < \lambda$. (We need a bounded cardinality to have an appropriate starting point for the recursive construction later on.) For some $a^* \in \mathfrak{p}G$ we then define $Y := \langle y_\alpha : \alpha < \kappa \rangle$ and let $\overline{G}$ and $\overline{A}$ be induced by

$$G_{y_\alpha} := G_{y_\alpha^{-1}} := \mathbb{Z}a^*, \quad A_{y_\alpha} := \mathrm{id}_{\mathbb{Z}a^*}.$$

Now check the definitions. □

Definition 2.7.
(a) We define a relation $\subseteq_{\mathscr{A}}$ on $\mathscr{A}$ ($\subseteq_{\mathscr{A}^*}$ on $\mathscr{A}^*$) where $\mathfrak{x} \subseteq_{\mathscr{A}} \mathfrak{y}$ ($\mathfrak{x} \subseteq_{\mathscr{A}^*} \mathfrak{y}$) means that:
  (α) $G_{\mathfrak{x}} \subseteq G_{\mathfrak{y}}$, $Y_{\mathfrak{x}} \subseteq Y_{\mathfrak{y}}$ and $F_{\mathfrak{x}} \subseteq F_{\mathfrak{y}}$.
  (β) $G_f^{\mathfrak{x}} \subseteq G_f^{\mathfrak{y}}$ and $A_f^{\mathfrak{x}} \subseteq A_f^{\mathfrak{y}}$ for all $f \in F_{\mathfrak{x}}$.
  (γ) If $y \in Y_{\mathfrak{x}}$ and $G_y^{\mathfrak{x}} \neq G_y^{\mathfrak{y}}$, then $G_{\mathfrak{x}} \subseteq G_y^{\mathfrak{y}}, G_{y^{-1}}^{\mathfrak{y}}$.
  (δ) $a_{\mathfrak{x}}^* = a_{\mathfrak{y}}^*$ and $K_{\mathfrak{x}} = K_{\mathfrak{y}}$.
(b) We define a relation $\leq_{\mathscr{A}}$ on $\mathscr{A}$ ($\leq_{\mathscr{A}^*}$ on $\mathscr{A}^*$) where $\mathfrak{x} \leq_{\mathscr{A}} \mathfrak{y}$ ($\mathfrak{x} \leq_{\mathscr{A}^*} \mathfrak{y}$) means that in addition to (a), (α) – (δ) also
  (ε) $G_{\mathfrak{y}}/G_{\mathfrak{x}}$ is free ($\aleph_1$-free).

Corollary 2.8. *The relations $\subseteq_{\mathscr{A}}$ and $\leq_{\mathscr{A}}$ ($\subseteq_{\mathscr{A}^*}$ and $\leq_{\mathscr{A}^*}$) are partial orders on $\mathscr{A}$ ($\mathscr{A}^*$).*

Proof. Easy.

For our recursive construction the notion of limits in $\mathscr{A}$ is of central importance.

Definition 2.9.
(a) For elements $\mathfrak{x}, \mathfrak{x}_\alpha$ $(\alpha < \delta)$ in $\mathscr{A}$ ($\mathscr{A}^*$) we define $\mathfrak{x} = \bigcup\{\mathfrak{x}_\alpha : \alpha < \delta\}$ to mean that:

(α) $\langle \mathfrak{x}_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathscr{A}}$-increasing ($\leq_{\mathscr{A}^*}$-increasing) sequence with δ a limit ordinal.

(β) $G_{\mathfrak{x}_\delta} = \bigcup_{\alpha<\delta} G_{\mathfrak{x}_\alpha}$, $Y_{\mathfrak{x}_\delta} = \bigcup_{\alpha<\delta} Y_{\mathfrak{x}_\alpha}$ and $F_{\mathfrak{x}_\delta} = \bigcup_{\alpha<\delta} F_{\mathfrak{x}_\alpha}$.

(γ) $G_f^{\mathfrak{x}_\delta} = \bigcup_{\alpha<\delta} G_f^{\mathfrak{x}_\alpha}$ and $A_f^{\mathfrak{x}_\delta} = \bigcup_{\alpha<\delta} A_f^{\mathfrak{x}_\alpha}$ for all $f \in F_{\mathfrak{x}_\delta}$.

(b) $\langle \mathfrak{x}_\alpha : \alpha < \alpha_* \rangle$ is continuously $\leq_{\mathscr{A}}$-increasing ($\leq_{\mathscr{A}^*}$-increasing), if:

(α) $\mathfrak{x}_\alpha \leq_{\mathscr{A}} \mathfrak{x}_\beta$ ($\mathfrak{x}_\alpha \leq_{\mathscr{A}^*} \mathfrak{x}_\beta$) for all $\alpha \leq \beta < \delta$.

(β) $\mathfrak{x}_\delta = \bigcup\{\mathfrak{x}_\alpha : \alpha < \delta\}$ for every limit ordinal $\delta < \alpha_*$.

Corollary 2.10. *Let $\langle \mathfrak{x}_\alpha : \alpha < \delta \rangle$ be continuously $\leq_{\mathscr{A}}$-increasing ($\leq_{\mathscr{A}^*}$-increasing). Then for a unique $x_\delta \in \mathscr{A}$ ($\mathscr{A}^*$) holds*

$$\mathfrak{x}_\delta = \bigcup\{\mathfrak{x}_\alpha : \alpha < \delta\}.$$

Proof. Easy. Observe that all properties are of finite character. In the $\aleph_1$-free case make use of Pontryagins Theorem.

3 Construction tools

Next we describe the construction tools for reaching our main goal. We start with a lemma that will be useful in growing up partial automorphisms A_f to full automorphisms.

Lemma 3.1. *Let be $\mathfrak{x} \in \mathscr{A}$ ($\mathscr{A}^*$) and $y_* \in Y_{\mathfrak{x}}$. Then there exists some $\mathfrak{x} \neq \mathfrak{y} \in \mathscr{A}$ ($\mathscr{A}^*$) with:*
(i) $\mathfrak{x} \leq_{\mathscr{A}} \mathfrak{y}$ ($\mathfrak{x} \leq_{\mathscr{A}^*} \mathfrak{y}$) *and* $|G_{\mathfrak{y}}| = |G_{\mathfrak{y}} \setminus G_{\mathfrak{x}}| = |G_{\mathfrak{x}}|$.
(ii) $Y_{\mathfrak{y}} = Y_{\mathfrak{x}}$ *and* $G_y^{\mathfrak{y}} = G_y^{\mathfrak{x}}$ ($y_* \neq y \in Y_{\mathfrak{x}}$).
(iii) $G_{\mathfrak{x}} \subseteq G_{y_*}^{\mathfrak{y}}, G_{y_*^{-1}}^{\mathfrak{y}}$.

Proof. We follow a two-step construction.

Step 1 (free case): According to Definition 2.1(a)(γ) holds $G_{y_*}^{\mathfrak{x}} \sqsubseteq G_{\mathfrak{x}}$, thus $G_{\mathfrak{x}} = G_{y_*}^{\mathfrak{x}} \oplus C$ for a suitable free summand C. Thus setting

$$G'_{\mathfrak{x}} := G_{\mathfrak{x}} \oplus C'$$

with $C' \cong C$ we can continue $A_{y_*}^{\mathfrak{x}}$ to a partial automorphism $A'_{y_*} : G'_{y_*} \to G'_{y_*^{-1}}$ of $G'_{\mathfrak{x}}$ by setting $G'_{y_*} := G_{\mathfrak{x}}$, $G'_{y_*^{-1}} := G_{y_*^{-1}}^{\mathfrak{x}} \oplus C'$ and $A'_{y_*} \restriction C = \varphi$ for an arbitrarily chosen isomorphism $\varphi : C \to C'$.

Step 1 ($\aleph_1$-case): In this case we proceed by using a pushout construction similar to [6]. Set $G'_{\mathfrak{x}} := G_{\mathfrak{x}} \times G_{\mathfrak{x}}/H$ with $H := \{(gA^{\mathfrak{x}}_{y_*}, -g) : g \in G^{\mathfrak{x}}_{y_*}\}$. Furthermore let for $U \subseteq G_{\mathfrak{x}}$ be $U_0 := (U \times 0 + H)/H$, $U_1 := (0 \times U + H)/H$. Identify $G_{\mathfrak{x}}$ with $(G_{\mathfrak{x}})_0 \subseteq G'_{\mathfrak{x}}$ and continue $A^{\mathfrak{x}}_{y_*}$ to a partial automorphism $A'_{y_*} : G'_{y_*} \to G'_{y_*^{-1}}$ of $G'_{\mathfrak{x}}$ via $G'_{y_*} := (G_{\mathfrak{x}})_0$, $G'_{y_*^{-1}} := (G_{\mathfrak{x}})_1$ and $((g, 0) + H)A'_{y_*} = (0, g) + H$.

Now in both cases it can be verified that:

- $G_{\mathfrak{x}} \subseteq G'_{\mathfrak{x}}$ and $G'_{\mathfrak{x}}/G_{\mathfrak{x}}$ are free ($\aleph_1$-free).
- $A^{\mathfrak{x}}_{y_*} \subseteq A'_{y_*}$ with $G'_{y_*} = G_{\mathfrak{x}}$, and $G'_{\mathfrak{x}}/G'_{y_*}$, $G'_{\mathfrak{x}}/G'_{y_*^{-1}}$ are free ($\aleph_1$-free).

Step 2: Repeat Step 1 with y_*^{-1} instead of y_* to result in $G_{\mathfrak{y}}$ and $A^{\mathfrak{y}}_{y_*}$ for the desired $\mathfrak{x} \leq_{\mathscr{A}} \mathfrak{y}$. To verify Definitions 2.1(a) and 2.7 now is a merely straightforward calculation (see also Corollary 3.2). Concerning clause (i) observe that in case of A'_{y_*} being a full automorphism $G_{\mathfrak{y}} \neq G_{\mathfrak{x}}$ can always be achieved replacing $G_{\mathfrak{y}}$ by $G_{\mathfrak{y}} \oplus \mathbb{Z}$. □

We emphasize some special features of the proof separately for later use.

Corollary 3.2. *With $\mathfrak{x}' = (G'_{\mathfrak{x}}, Y_{\mathfrak{x}}, F_{\mathfrak{x}}, \overline{G}'^{\mathfrak{x}}, \overline{A}'^{\mathfrak{x}})$ defined as in Step 1 of the last lemma holds:*
(1) $\mathfrak{x}' \in \mathscr{A}$ $(\mathscr{A}^*)$.
(2) $\operatorname{Im} A'_{y_*} \cap G_{\mathfrak{x}} = \operatorname{Im} A^{\mathfrak{x}}_{y_*}$.
(3) $F_{\mathfrak{x}'}(g, g) = F_{\mathfrak{x}}(g, g)$ *and* $F_{\mathfrak{x}'}(gA'_{y_*}, gA'_{y_*}) = A'^{-1}_{y_*} F_{\mathfrak{x}}(g, g) A'_{y_*}$ *for all* $g \in G_{\mathfrak{x}}$.
(4) $a^*/\mathscr{E}_{\mathfrak{x}'} = a^*/\mathscr{E}_{\mathfrak{x}} \cup A'_{y_*}(a^*/\mathscr{E}_{\mathfrak{x}})$ *and* $F_{\mathfrak{x}'}(g, g) = \{1\}$ *for all* $g \in G'_{\mathfrak{x}} \setminus (G_{\mathfrak{x}} \cup \operatorname{Im} A'_{y_*})$.

Proof. Easy. Concerning clause (3) let be $g \in G_{\mathfrak{x}}$ and $f = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n} \in F_{\mathfrak{x}'} = F_{\mathfrak{x}}$ with $gA^{\mathfrak{x}'}_f = g(A^{\mathfrak{x}'}_{y_1})^{\varepsilon_1}(A^{\mathfrak{x}'}_{y_2})^{\varepsilon_2} \ldots (A^{\mathfrak{x}'}_{y_n})^{\varepsilon_n} = g$. For $y_i \neq y_*$ we can replace in the last equation $\mathfrak{x}'$ directly by $\mathfrak{x}$, while for $y_i = y_*$ by clause (2) either $\mathfrak{x}'$ can be replaced by $\mathfrak{x}$ or otherwise y_* in f is directly followed by $(y_*)^{-1}$ and therefore can be reduced. □

Next we actually demonstrate how recursive application of Lemma 3.1 grows partial automorphisms to full automorphisms.

Lemma 3.3. *Let be $\mathfrak{x} \in \mathscr{A}$ $(\mathscr{A}^*)$. Then there exists some $\mathfrak{x} \neq \mathfrak{y} \in \mathscr{A}$ $(\mathscr{A}^*)$ with:*
(i) $\mathfrak{x} \leq_{\mathscr{A}} \mathfrak{y}$ $(\mathfrak{x} \leq_{\mathscr{A}^*} \mathfrak{y})$ *and* $\left|G_{\mathfrak{y}}\right| = \left|G_{\mathfrak{y}} \setminus G_{\mathfrak{x}}\right| = \left|G_{\mathfrak{x}}\right|$.
(ii) $Y_{\mathfrak{y}} = Y_{\mathfrak{x}}$ *and* $\mathfrak{y}$ *is full.*

Proof. Set $\mu := \left|Y_{\mathfrak{x}}\right|$ and let $Y_{\mathfrak{x}} = \{y_\alpha : \alpha < \mu\}$ be a listing of the set $Y_{\mathfrak{x}}$. Then $Y'_{\mathfrak{x}} := \{y'_\alpha : \alpha < \mu\omega\}$ with $y'_{\mu n+\alpha} := y_\alpha$ $(n \in \omega, \alpha \in \mu)$ is a listing with ω-repetition. Define $\langle \mathfrak{x}_\alpha : \alpha \leq \mu\omega \rangle$ as continuously $\leq_{\mathscr{A}}$-increasing ($\leq_{\mathscr{A}^*}$-increasing) sequence:

- $\mathfrak{x}_0 := \mathfrak{x}$.
- $\mathfrak{x}_\alpha := \bigcup\{\mathfrak{x}_\beta : \beta < \alpha\} \in \mathscr{A}$ $(\mathscr{A}^*)$ for limit ordinals α using Corollary 2.10.
- $\mathfrak{x}_\alpha$ for $\alpha = \beta + 1$ is derived from $\mathfrak{x}_\beta$ using Lemma 3.1 and $y_* := y'_\alpha$.

Setting $\mathfrak{y} := \mathfrak{x}_{\mu\omega}$ claims (i) and (ii) are obvious. □

Next a construction tool that will be useful to achieve transitivity.

Lemma 3.4. *Let be $\mathfrak{x} \in \mathcal{A}$ ($\mathcal{A}^*$) and $b^* \in \mathfrak{p}G_\mathfrak{x}$ with $(a^*, b^*) \notin \mathcal{E}_\mathfrak{x}$. Then there exists some $\mathfrak{x} \neq \mathfrak{y} \in \mathcal{A}$ ($\mathcal{A}^*$) with:*

(i) $\mathfrak{x} \leq_\mathcal{A} \mathfrak{y}$ ($\mathfrak{x} \leq_{\mathcal{A}^*} \mathfrak{y}$) *and* $\left|G_\mathfrak{y}\right| = \left|G_\mathfrak{y} \setminus G_\mathfrak{x}\right| = \left|G_\mathfrak{x}\right|$.
(ii) $a^* \mathcal{E}_\mathfrak{y} b^*$.

Proof. Set

- $G_\mathfrak{y} := G_\mathfrak{x} \oplus \mathbb{Z}$,
- $Y_\mathfrak{y} := Y_\mathfrak{x} \cup \{y_*\}$ for some new free generator y_*,
- $G_y^\mathfrak{y} := G_y^\mathfrak{x}$, $A_y^\mathfrak{y} := A_y^\mathfrak{x}$ for $y \in Y_\mathfrak{x}$,
- $G_{y_*}^\mathfrak{y} := \mathbb{Z}a^*$, $G_{y_*^{-1}}^\mathfrak{y} := \mathbb{Z}b^*$ and $A_{y_*}^\mathfrak{y} : \mathbb{Z}a^* \to \mathbb{Z}b^*$, $a^* \mapsto b^*$.

Now verify the definitions. We want to give details on Definition 2.1(a)(ζ),(η) and Definition 2.7(δ) where the most interesting arguments take place:

Let be $a, b \in \mathfrak{p}G_\mathfrak{y}$ and $f = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n} \in F_\mathfrak{y}(a, b)$ reduced, thus $aA_f^\mathfrak{y} = b$. We define $c_0 := a$, $c_i := a(A_{y_1}^\mathfrak{y})^{\varepsilon_1}(A_{y_2}^\mathfrak{y})^{\varepsilon_2} \dots (A_{y_i}^\mathfrak{y})^{\varepsilon_i}$ for $1 \leq i \leq n$, $u := \{1 \leq i \leq n |\, y_i = y_*\}$ and the partition $u := u^+ \cup u^-$, where $u^+ := \{i \in u |\, \varepsilon_i = 1\}$, $u^- := \{i \in u |\, \varepsilon_i = -1\}$. So $(c_{i-1}, c_i) \in \{(a^*, b^*), (-a^*, -b^*)\}$ for $i \in u^+$ and $(c_{i-1}, c_i) \in \{(b^*, a^*), (-b^*, -a^*)\}$ for $i \in u^-$. This gives cause to the following observations:

1.) For $a \notin (a^*/\mathcal{E}_\mathfrak{x}) \cup (b^*/\mathcal{E}_\mathfrak{x})$ holds either $F_\mathfrak{y}(a, b) = F_\mathfrak{x}(a, b)$ or $F_\mathfrak{y}(a, b) \subseteq \{1\}$.
For $f \neq 1$ we can prove by induction $i \notin u$ and $c_i \notin (a^*/\mathcal{E}_\mathfrak{x}) \cup (b^*/\mathcal{E}_\mathfrak{x})$, thus $f \in F_\mathfrak{x}$.

2.) $K_\mathfrak{y} = K_\mathfrak{x}$.
Assume that $f \in K_\mathfrak{y} \setminus K_\mathfrak{x}$, thus $u \neq \emptyset$. For $i_0 := \min u$ holds $c_{i_0 - 1} \in a^*/\mathcal{E}_\mathfrak{x} \cap \{\pm a^*, \pm b^*\} = \{a^*, -a^*\}$, in particular $i_0 \in u^+$ and $c_{i_0} \in \{b^*, -b^*\}$. Similarly holds $i_k \in u^-$ for $i_k := \max u$, thus $|u| > 1$. We denote by $i_0 < i_1$ the second member of u and conclude again $c_{i_1 - 1} \in \{\pm a^*, \pm b^*\}$. Take a look at

$$f' := y_{i_0+1}^{\varepsilon_{i_0+1}} y_{i_0+2}^{\varepsilon_{i_0+2}} \dots y_{i_1-1}^{\varepsilon_{i_1-1}} \in F_\mathfrak{x}.$$

Thus $c_{i_1-1} = c_{i_0} A_{f'}^\mathfrak{x}$. For $c_{i_1-1} \in \{a^*, -a^*\}$ follows $(f')^{-1} \in F_\mathfrak{x}(a^*, b^*)$ contradicting $(a^*, b^*) \notin \mathcal{E}_\mathfrak{x}$. Thus $c_{i_1-1} \in \{b^*, -b^*\}$, $f' \in F_\mathfrak{x}(b^*, b^*) = \{1\}$ (see Definition 2.1(a)(η)), $f' = 1$ and $i_1 = i_0 + 1$ as f is reduced. But now $y_{i_0}^{\varepsilon_{i_0}} = y_*$, $y_{i_0+1}^{\varepsilon_{i_0+1}} = y_*^{-1}$ finally contradicts f reduced and our assumption $K_\mathfrak{y} \neq K_\mathfrak{x}$.

Very similar to 2.) is the proof of
3.) $a^*/\mathcal{E}_\mathfrak{y} = (a^*/\mathcal{E}_\mathfrak{x}) \cup (b^*/\mathcal{E}_\mathfrak{x})$. □

We summarize our efforts to

Theorem 3.5.
(1) *For every $\mathfrak{x} \in \mathcal{A}$ ($\mathcal{A}^*$) exists a $\mathfrak{x} \neq \mathfrak{y} \in \mathcal{A}$ ($\mathcal{A}^*$) with:*
  (i) $\mathfrak{x} \leq_\mathcal{A} \mathfrak{y}$ ($\mathfrak{x} \leq_{\mathcal{A}^*} \mathfrak{y}$) *and* $|G_\mathfrak{y}| = |G_\mathfrak{y} \setminus G_\mathfrak{x}| = |G_\mathfrak{x}|$.
  (ii) $\mathfrak{y}$ *is full with* $\mathfrak{p}G_\mathfrak{x} \times \mathfrak{p}G_\mathfrak{x} \subseteq \mathcal{E}_\mathfrak{y}$.
(2) *In* (1) *we can tighten $\mathfrak{x} \neq \mathfrak{y} \in \mathcal{A}$ ($\mathcal{A}^*$) to $\mathfrak{x} \neq \mathfrak{y} \in \mathcal{U}$ ($\mathcal{U}^*$).*

Proof.
Clause (1): Let $\langle b_\varepsilon | \varepsilon < \mu\rangle$ be a list of $\mathfrak{p}G_{\mathfrak{x}}$. Define $\langle \mathfrak{x}_\varepsilon | \varepsilon < 2\mu\rangle$ as continuously $\leq_{\mathcal{A}}$-increasing ($\leq_{\mathcal{A}^*}$-increasing) sequence:

- $\mathfrak{x}_0 := \mathfrak{x}$.
- $\mathfrak{x}_\varepsilon := \bigcup\{\mathfrak{x}_\alpha : \alpha < \varepsilon\} \in \mathcal{A}$ ($\mathcal{A}^*$) for limit ordinals ε using Corollary 2.10.
- $\mathfrak{x}_\varepsilon$ for odd $\varepsilon = \alpha + 1$ is derived from $\mathfrak{x}_\alpha$ using Lemma 3.3.
- $\mathfrak{x}_\varepsilon$ for even $\varepsilon = 2(\alpha + 1)$ is derived from $\mathfrak{x}_{2\alpha+1}$ using Lemma 3.4 and $b^* := b_\alpha$.

Now check!

For clause (2) repeat the construction in (1). □

Thus we are able to upgrade every element of $\mathcal{A}$ ($\mathcal{A}^*$) to an element of $\mathcal{U}$ ($\mathcal{U}^*$) and therefore to work in the highly appreciated class $\mathcal{U}$ ($\mathcal{U}^*$) entirely.

In the main construction we will make use of the following possibility to code F (and therefore in the end the desired automorphism group itself) into our groups G.

Lemma 3.6. *Let $\mathfrak{x} \in \mathcal{A}$ ($\mathcal{A}^*$) be full. Then some full $\mathfrak{x} \leq_{\mathcal{A}} \mathfrak{y} \in \mathcal{A}$ ($\mathfrak{x} \leq_{\mathcal{A}^*} \mathfrak{y} \in \mathcal{A}^*$) with $|G_{\mathfrak{y}}| = |G_{\mathfrak{y}} \setminus G_{\mathfrak{x}}| = |G_{\mathfrak{x}}|$ is defined by setting:*

(i) $G_{\mathfrak{y}} := G_{\mathfrak{x}} \oplus_{f \in F_{\mathfrak{x}}} \mathbb{Z}e_f$.
(ii) $Y_{\mathfrak{y}} := Y_{\mathfrak{x}}$ *and* $F_{\mathfrak{y}} := F_{\mathfrak{x}}$.
(iii) $A_y^{\mathfrak{y}} \restriction G_{\mathfrak{x}} = A_y^{\mathfrak{x}}$ *and* $e_f A_y^{\mathfrak{y}} = e_{fy}$ ($y \in Y_{\mathfrak{x}}, f \in F_{\mathfrak{x}}$).

Proof. Easy.

Our list of useful construction tools is completed by a standardized method for killing undesired endomorphisms. This comes in two parts: we start with killing totally savage candidates.

Step Lemma 3.7. *Let $\langle \mathfrak{x}_n | n \leq \omega\rangle$, φ, $\langle e_n | n < \omega\rangle$ and a prime element p be given such that:*

(a) $\mathfrak{x}_n \in \mathcal{A}$ ($\mathcal{A}^*$) *full for all* $n \leq \omega$.
(b) $\langle \mathfrak{x}_n | n \leq \omega\rangle$ *is continuously* $\leq_{\mathcal{A}}$*-increasing* ($\leq_{\mathcal{A}^*}$*-increasing*).
(c) $\varphi \in \operatorname{Aut} G_{\mathfrak{x}_\omega}$ *and* $\varphi \restriction G_{\mathfrak{x}_n} \in \operatorname{Aut} G_{\mathfrak{x}_n}$ *for all* $n < \omega$.
(d) $G_{\mathfrak{x}_n} \oplus \bigoplus_{f \in F_{\mathfrak{x}_n}} \mathbb{Z}e_{nf} \subseteq_* G_{\mathfrak{x}_{n+1}}$ *as p-pure subgroup with* $e_{n1} := e_n$ *and* $e_{nf} A_y^{\mathfrak{x}_{n+1}} = e_{nfy}$ *for all* $n < \omega$, $y \in Y_{\mathfrak{x}_n}$, $f \in F_{\mathfrak{x}_n}$.
(e) $e_n\varphi \notin G_{\mathfrak{x}_n} \oplus \bigoplus_{f \in F_{\mathfrak{x}_n}} \mathbb{Z}e_{nf}$ *for all* $n < \omega$.

Then there exists some full $\mathfrak{y} \in \mathcal{A}$ ($\mathcal{A}^*$) such that:

(i) $\mathfrak{x}_\omega \subseteq_{\mathcal{A}} \mathfrak{y}$ ($\mathfrak{x}_\omega \subseteq_{\mathcal{A}^*} \mathfrak{y}$) *and* $\mathfrak{x}_n \leq_{\mathcal{A}} \mathfrak{y}$ ($\mathfrak{x}_n \leq_{\mathcal{A}^*} \mathfrak{y}$) *for all* $n < \omega$.
(ii) $G_{\mathfrak{y}}/G_{\mathfrak{x}_\omega}$ *is p-divisible.*
(iii) $Y_{\mathfrak{y}} := Y_{\mathfrak{x}_\omega}$ *and* $F_{\mathfrak{y}} := F_{\mathfrak{x}_\omega}$.
(iv) φ *does not extend to an endomorphism of* $G_{\mathfrak{y}}$.

Proof. As usual we work in the p-adic closure $\widehat{G_{\mathfrak{x}_\omega}}$ of $G_{\mathfrak{x}_\omega}$. More precisely: Let $\widehat{A_f^{\mathfrak{x}_\omega}}$ be the continuous extension of $A_f^{\mathfrak{x}_\omega}$ to $\widehat{G_{\mathfrak{x}_\omega}}$ and set

$$G_{\mathfrak{y}} := \langle G_{\mathfrak{x}_\omega}, y\widehat{A_f^{\mathfrak{x}_\omega}} | f \in F_{\mathfrak{x}_\omega}\rangle_* \subseteq_* \widehat{G_{\mathfrak{x}_\omega}}$$

as p-purification with $y := \sum_{n<\omega} p^n e_n$ and $A_f^{\mathfrak{y}} := \widehat{A_f^{\mathfrak{x}_\omega}} \restriction G_{\mathfrak{y}}$. Observe that

$$yA_f^{\mathfrak{y}} = \Big(\sum_{n<k} p^n e_n\Big) A_f^{\mathfrak{x}_k} + \sum_{k\le n<\omega} p^n e_{nf}$$

for $f \in F_{\mathfrak{x}_k}$ and $y\varphi \notin G_{\mathfrak{y}}$. Now check! □

We then kill those half-way tame candidates that survived the first trial.

Step Lemma 3.8. *Let $\langle \mathfrak{x}_n | n \le \omega\rangle$, φ, $\langle e_n | n < \omega\rangle$ and a prime element p be given such that:*

(a) $\mathfrak{x}_n \in \mathscr{A}$ $(\mathscr{A}^*)$ *full for all* $n \le \omega$.
(b) $\langle \mathfrak{x}_n | n \le \omega\rangle$ *is continuously* $\le_{\mathscr{A}}$*-increasing* ($\le_{\mathscr{A}^*}$*-increasing*).
(c) $\varphi \in \operatorname{Aut} G_{\mathfrak{x}_\omega}$ *and* $\varphi \restriction G_{\mathfrak{x}_n} \in \operatorname{Aut} G_{\mathfrak{x}_n}$ *for all* $n < \omega$.
(d) $G_{\mathfrak{x}_n} \oplus \bigoplus_{f\in F_{\mathfrak{x}_n}} \mathbb{Z}e_{nf} \subseteq_* G_{\mathfrak{x}_{n+1}}$ *as p-pure subgroup with* $e_{n1} := e_n$ *and* $e_{nf} A_y^{\mathfrak{x}_{n+1}} = e_{nfy}$ *for all* $n < \omega$, $y \in Y_{\mathfrak{x}_n}$, $f \in F_{\mathfrak{x}_n}$.
(e) $e_n\varphi \in G_{\mathfrak{x}_n} \oplus \bigoplus_{f\in F_{\mathfrak{x}_n}} \mathbb{Z}e_{nf}$ *for all* $n < \omega$.
(f) $\varphi \notin \mathbb{Z}\overline{A}^{\mathfrak{x}_\omega}$, *where* $\mathbb{Z}\overline{A}^{\mathfrak{x}_\omega}$ *is the group ring induced by* $\overline{A}^{\mathfrak{x}_\omega}$.

Then there exists some full $\mathfrak{y} \in \mathscr{A}$ $(\mathscr{A}^*)$ *such that:*

(i) $\mathfrak{x}_\omega \subseteq_{\mathscr{A}} \mathfrak{y}$ $(\mathfrak{x}_\omega \subseteq_{\mathscr{A}^*} \mathfrak{y})$ *and* $\mathfrak{x}_n \le_{\mathscr{A}} \mathfrak{y}$ $(\mathfrak{x}_n \le_{\mathscr{A}^*} \mathfrak{y})$ *for all* $n < \omega$.
(ii) $G_{\mathfrak{y}}/G_{\mathfrak{x}_\omega}$ *is p-divisible.*
(iii) $Y_{\mathfrak{y}} := Y_{\mathfrak{x}_\omega}$ *and* $F_{\mathfrak{y}} := F_{\mathfrak{x}_\omega}$.
(iv) φ *does not extend to an endomorphism of* $G_{\mathfrak{y}}$.

Proof. Let $\widehat{A_f^{\mathfrak{x}_\omega}}$ again be the continuous extension of $A_f^{\mathfrak{x}_\omega}$ to $\widehat{G_{\mathfrak{x}_\omega}}$ and set

$$G_{\mathfrak{y}^i} := \langle G_{\mathfrak{x}_\omega}, y^i\widehat{A_f^{\mathfrak{x}_\omega}} | f \in F_{\mathfrak{x}_\omega}\rangle_* \subseteq_* \widehat{G_{\mathfrak{x}_\omega}}$$

and $A_f^{\mathfrak{y}^i} := \widehat{A_f^{\mathfrak{x}_\omega}} \restriction G_{\mathfrak{y}^i}$ for suitable elements $y^i \in \widehat{G_{\mathfrak{x}_\omega}}$. This leads to full elements $\mathfrak{y}^1, \mathfrak{y}^2 \in \mathscr{A}$ $(\mathscr{A}^*)$, but the correct choice of y^1, y^2 demands skill.

We start with the guess $y^1 := \sum_{n<\omega} p^n e_n$. For $y^1\varphi \notin G_{\mathfrak{y}^1}$ the proof is finished. Thus assume $y^1\varphi \in G_{\mathfrak{y}^1}$. Therefore

(1) $\quad p^k y^1\varphi = g + y^1\psi$

holds for some $k \in \omega$ and $g \in G_{\mathfrak{x}_k}, \psi \in \mathbb{Z}\langle A_f^{\mathfrak{x}_\omega} | f \in F_{\mathfrak{x}_k}\rangle$. For $p^k\varphi = \psi$ evaluation at e_k leads to $\varphi \in \mathbb{Z}\langle A_f^{\mathfrak{x}_\omega} | f \in F_{\mathfrak{x}_k}\rangle$, contradiction! Thus $p^k\varphi - \psi \neq 0$ and there exist some $a \in G_{\mathfrak{x}_\omega}$ and a suitable p-adic number $\pi \in \widehat{\mathbb{Z}_p}$ with

$$\pi a(p^k\varphi - \psi) \notin G_{\mathfrak{x}_\omega}. \tag{2}$$

Without loss of generality let be $a \in G_{\mathfrak{x}_k}$ and $\pi a(p^k\varphi - \psi) \in \widehat{G_{\mathfrak{x}_k}} \setminus G_{\mathfrak{x}_k}$. We now set $y^2 := \pi a + \sum_{n<\omega} p^n e_n = \pi a + y^1$ as our second guess. For $y^2\varphi \notin G_{\mathfrak{y}^2}$ the proof is finished. Thus assume $y^2\varphi \in G_{\mathfrak{y}^2}$ and without loss of generality

$$p^k y^2 \varphi = g' + y^2 \psi' \tag{3}$$

holds for suitable $g' \in G_{\mathfrak{x}_k}, \psi' \in \mathbb{Z}\langle A_f^{\mathfrak{x}_\omega} | f \in F_{\mathfrak{x}_k}\rangle$. Subtracting (1) from (3) gives

$$\pi a(p^k\varphi - \psi') = (g' - g) + y^1(\psi' - \psi). \tag{4}$$

From a support argument follows $\psi' = \psi$ and thus $\pi a(p^k\varphi - \psi) = g' - g \in G_{\mathfrak{x}_k}$, a final contradiction to (2). □

4 Constructing κ-fold transitive groups

In this section we provide the construction and the proof needed for Theorem 1.2. For this let λ be a regular cardinal with $\diamondsuit_S$ for some non-reflecting stationary $S \subseteq S^\lambda_{\aleph_0}$. Choose a set G of cardinality $|G| = \lambda$. Also choose a λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ with $|G_0| = |G_1 \setminus G_0| = \kappa$ and $|G_\alpha| = |G_{\alpha+1} \setminus G_\alpha| = \kappa \cdot |\alpha|$ for all $0 < \alpha < \lambda$. Let $\{\Phi_\alpha : G_\alpha \to G_\alpha | \alpha \in S\}$ be a system of predicting Jensen-functions for the λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$.

We want to assign inductively a group structure to the sets G_α defining a $\subseteq_{\mathcal{A}}$-ascending and sufficiently continuously $\leq_{\mathcal{A}}$-ascending chain $\langle \mathfrak{x}_\alpha | \alpha < \lambda\rangle$ in $\mathfrak{U}$ with $\mathfrak{x}_\alpha = (G_\alpha, Y_\alpha, F_\alpha, \overline{G}^\alpha, \overline{A}^\alpha, a^*_\alpha)$. For the canonical union

$$\mathfrak{x} := \bigcup_{\alpha<\lambda} \mathfrak{x}_\alpha := (\bigcup_{\alpha<\lambda} G_\alpha, \bigcup_{\alpha<\lambda} Y_\alpha, \bigcup_{\alpha<\lambda} F_\alpha, \bigcup_{\alpha<\kappa} \overline{G}^\alpha, \bigcup_{\alpha<\kappa} \overline{A}^\alpha, a^*_\alpha)$$

of this chain the group $G = \bigcup_{\alpha<\lambda} G_\alpha$ will be κ-fold transitive satisfying Theorem 1.2. We will carry out the following steps inductively.

Choose $\mathfrak{x}_0 \in \mathfrak{U}$ with Lemma 2.6 and Theorem 3.5. Suppose that $\mathfrak{x}_\beta$ $(\beta < \alpha)$ is already defined.

Case A: $\alpha = \beta + 1,\ \beta \notin S$.
Construct $\mathfrak{x}_\alpha$ using Lemma 3.6 first (giving e_β) and then Theorem 3.5.

Case B: $\alpha = \beta + 1,\ \beta \in S$.
To construct $\mathfrak{x}_\alpha$ work your way through the following graded flowchart. Have a look at the Jensen-function $\Phi_\beta : G_\beta \to G_\beta$ and then decide.

B1: If Φ_β satisfies the conditions of Step Lemma 3.7 for some suitable subchain $\langle \mathfrak{x}'_n | n \le \omega\rangle \subseteq \langle \mathfrak{x}_\gamma | \gamma < \alpha\rangle$ with $\mathfrak{x}'_\omega := \mathfrak{x}_\beta$ kill it and proceed to B5. Otherwise proceed to B2.
B2: If Φ_β^{-1} satisfies the conditions of Step Lemma 3.7 for some suitable subchain $\langle \mathfrak{x}'_n | n \le \omega\rangle \subseteq \langle \mathfrak{x}_\gamma | \gamma < \alpha\rangle$ with $\mathfrak{x}'_\omega := \mathfrak{x}_\beta$ kill it and proceed to B5. Otherwise proceed to B3.
B3: If Φ_β satisfies the conditions of Step Lemma 3.8 for some suitable subchain $\langle \mathfrak{x}'_n | n \le \omega\rangle \subseteq \langle \mathfrak{x}_\gamma | \gamma < \alpha\rangle$ with $\mathfrak{x}'_\omega := \mathfrak{x}_\beta$ kill it and proceed to B5. Otherwise proceed to B4.
B4: If Φ_β^{-1} satisfies the conditions of Step Lemma 3.8 for some suitable subchain $\langle \mathfrak{x}'_n | n \le \omega\rangle \subseteq \langle \mathfrak{x}_\gamma | \gamma < \alpha\rangle$ with $\mathfrak{x}'_\omega := \mathfrak{x}_\beta$ kill it and proceed to B5. Otherwise proceed to B5 directly.
B5: Construct $\mathfrak{x}_\alpha$ using Lemma 3.6 first (giving e_β) and then Theorem 3.5.

Case C: α is a limit ordinal.
Set $\mathfrak{x}_\alpha := \bigcup_{\gamma<\mu} \mathfrak{x}'_\gamma$ for some unbounded continuously $\le_{\mathscr{A}}$-ascending subchain $\langle \mathfrak{x}'_\gamma | \gamma < \mu\rangle \subseteq \langle \mathfrak{x}_\beta | \beta < \alpha\rangle$. Here we use that S is non-reflecting.

We list some easy facts about the constructed chain $\langle \mathfrak{x}_\alpha | \alpha < \lambda\rangle$.

Lemma 4.1.

(1) $\langle \mathfrak{x}_\alpha | \alpha < \lambda\rangle$ *is a well-defined* $\subseteq_{\mathscr{A}}$*-ascending chain in* $\mathfrak{U}$.
(2) *If* $\alpha \le \beta < \lambda$ *with* $\alpha \notin S$*, then* $\mathfrak{x}_\alpha \le_{\mathscr{A}} \mathfrak{x}_\beta$.
(3) G_α *is a* p*-pure subgroup of its* p*-adic completion* $\widehat{G_\alpha}$ *for all* $\alpha \in \lambda$.

Proof. Easy.

Now Theorem 1.2 is part of the following list of properties of $\mathfrak{x} = \bigcup_{\alpha\in\lambda} \mathfrak{x}_\alpha$.

Theorem 4.2.

(1) $\mathfrak{x} \in \mathfrak{U}$.
(2) G *is a* λ*-free group of cardinality* λ.
(3) $\operatorname{Aut} G = \pm\overline{A}^{\mathfrak{x}} \cong \pm F_{\mathfrak{x}}$.
(4) G *is a* κ*-fold transitive group.*

Proof. Clauses (1) and (2) are immediate consequences of Lemma 4.1 while clause (4) follows easily from clause (3). Clause (3) now is where the interesting combinatorics takes place: to start with choose an arbitrary $\varphi \in \operatorname{Aut} G$ and let $S' \subseteq S \subseteq S^\lambda_{\aleph_0}$ be the stationary set where $\varphi \restriction G_\alpha = \Phi_\alpha$ ($\alpha \in S'$) is predicted by $\diamondsuit_S$. We first want to prove

$$(1) \qquad e_\alpha\varphi \in G_{\mathfrak{x}_\alpha} \oplus \bigoplus_{f\in F_{\mathfrak{x}_\delta}} \mathbb{Z}(e_\alpha A_f^{\mathfrak{x}_{\alpha+1}}) = G_{\mathfrak{x}_\alpha} \oplus e_\alpha \left(\mathbb{Z}\langle A_f^{\mathfrak{x}_{\alpha+1}} | f \in F_{\mathfrak{x}_\delta}\rangle\right)$$

for all $\delta < \alpha < \lambda$,

where δ is some fixed ordinal.

Assume that (1) is wrong. Then

$$C_1 := \{\alpha < \lambda |\, \varphi \restriction G_\alpha \text{ satisfies the conditions of Step Lemma 3.7 for some suitable subchain } \langle \mathfrak{x}'_n |\, n \le \omega\rangle \subseteq \langle \mathfrak{x}_\gamma |\, \gamma < \alpha\rangle\}$$

is an ω-cub (unbounded and ω-closed) and we can choose $\alpha \in C_1 \cap S' \neq \emptyset$. In particular $\mathfrak{x}_{\alpha+1}$ is constructed from $\mathfrak{x}_\alpha$ via Step Lemma 3.7, thus $G_{\alpha+1} \subseteq_* \widehat{G_\alpha}$ is constructed as subgroup of the p-adic closure and some $y \in G_{\alpha+1}$ exists with $y\varphi \notin G_{\alpha+1}$. We now can write any $g \in G_{\alpha+1}$ as a p-adic limit of a sequence $\langle g_i |\, i \in \omega\rangle \subseteq G_\alpha$, and $\langle g_i\varphi |\, i \in \omega\rangle \subseteq G_\alpha$ converges to $g\varphi$ by continuity. By Lemma 4.1 $G/G_{\alpha+1}$ is κ-free, $G_{\alpha+1}$ is p-adically closed in G and $g\varphi \in G_{\alpha+1}$. But this explicitly includes $y\varphi \in G_{\alpha+1}$, a contradiction to the step lemma.

Next we sharpen (1) to

$$\text{(2)} \qquad e_\alpha\varphi, e_\alpha\varphi^{-1} \in G_{\mathfrak{x}_\alpha} \oplus e_\alpha \left(\mathbb{Z}\langle A_f^{\mathfrak{x}_{\alpha+1}} |\, f \in F_{\mathfrak{x}_\delta}\rangle\right) \quad \text{for all} \quad \delta < \alpha < \lambda$$

and some fixed ordinal δ. Otherwise make use of the then existing ω-cub

$$C_2 := \{\alpha < \lambda |\, \varphi \restriction G_\alpha \text{ fails but } \varphi^{-1} \restriction G_\alpha \text{ satisfies the conditions of Step Lemma 3.7 for some suitable subchain } \langle \mathfrak{x}'_n |\, n \le \omega\rangle \subseteq \langle \mathfrak{x}_\gamma |\, \gamma < \alpha\rangle\}$$

for a contradiction similar to above. Using (2)

$$C := \{\alpha < \lambda |\, \varphi \restriction G_\alpha \text{ and } \varphi^{-1} \restriction G_\alpha \text{ fail Step Lemma 3.7 and } \varphi \restriction G_\alpha \in \operatorname{Aut} G_\alpha\}$$

is a cub now. With Step Lemma 3.8 we proceed to

$$\text{(3)} \qquad \varphi \restriction G_\alpha,\ \varphi^{-1} \restriction G_\alpha \in \mathbb{Z}\overline{A}^{\mathfrak{x}_\alpha} \quad \text{for all} \quad \alpha \in C \cap S',$$

where $C \cap S' \subseteq S^\lambda_{\aleph_0}$ is stationary.

Assume that (3) is wrong. If $\varphi \restriction G_\alpha \notin \mathbb{Z}\overline{A}^{\mathfrak{x}_\alpha}$ for some $\alpha \in C \cap S'$ it gets killed by Step Lemma 3.8 during procedure B3, contradiction. Thus $\varphi \restriction G_\alpha \in \mathbb{Z}\overline{A}^{\mathfrak{x}_\alpha}$ is obligatory. This again clears the way for procedure B4 and also $\varphi^{-1} \restriction G_\alpha \in \mathbb{Z}\overline{A}^{\mathfrak{x}_\alpha}$ follows.

Next we fix some $\alpha \in C \cap S'$ and take a closer look at (3): as $\alpha \in S^\lambda_{\aleph_0}$ there exists some $\beta < \alpha$ with $\varphi \restriction G_\alpha,\ \varphi^{-1} \restriction G_\alpha \in \mathbb{Z}\langle A_f^{\mathfrak{x}_\alpha} |\, f \in F_{\mathfrak{x}_\beta}\rangle$. Evaluating $\varphi\varphi^{-1} = 1$ at e_β we can jump from $\mathbb{Z}\langle A_f^{\mathfrak{x}_\alpha} |\, f \in F_{\mathfrak{x}_\beta}\rangle$ to the freely generated group ring $\mathbb{Z}F_{\mathfrak{x}_\beta}$ resulting in

$$\text{(4)} \qquad \varphi \restriction G_\alpha,\ \varphi^{-1} \restriction G_\alpha \in \pm\overline{A}^{\mathfrak{x}_\alpha} \quad \text{for all} \quad \alpha \in C \cap S'.$$

Recalling (2) we can sharpen (4) directly to

$$\text{(5)} \qquad \varphi \restriction G_\alpha,\ \varphi^{-1} \restriction G_\alpha \in \langle \pm A_f^{\mathfrak{x}_\alpha} |\, f \in F_{\mathfrak{x}_\delta}\rangle \quad \text{for all} \quad \alpha \in C \cap S'$$

and some fixed ordinal δ. Otherwise for $\delta < \alpha \in C \cap S' \subseteq S^\lambda_{\aleph_0}$ we can choose $\delta < \beta < \alpha$ with $\varphi \restriction G_\alpha, \varphi^{-1} \restriction G_\alpha \in \langle \pm A_f^{\mathfrak{x}_\alpha} |\, f \in F_{\mathfrak{x}_\beta}\rangle$. Now evaluate φ (respectively φ^{-1}) at e_β for a contradiction.

For $\alpha \in C \cap S'$ choose $f_\alpha \in F_{\mathfrak{x}_\delta}$ with $\varphi \restriction G_\alpha \in \{-A^{\mathfrak{x}_\alpha}_{f_\alpha}, A^{\mathfrak{x}_\alpha}_{f_\alpha}\}$. For $\delta < \alpha_1, \alpha_2 \in C \cap S'$ evaluation of φ at e_δ then leads to $f_{\alpha_1} = f_{\alpha_2}$. Thus the sequence $\langle f_\alpha | \alpha \in C \cap S' \rangle$ becomes constant and there exists a unique $f \in F_{\mathfrak{x}_\delta}$ with either $\varphi = A^{\mathfrak{x}}_f$ or $\varphi = -A^{\mathfrak{x}}_f$. □

5 Further Discussion

The presented construction can be manipulated canonically to offer some further results.

To start with: from Definition 2.1 onwards we assumed that for every $\mathfrak{x} \in \mathscr{A}$ the group $K_\mathfrak{x} = |F_\mathfrak{x}(a^*, a^*)|$ is freely generated of cardinality κ. This is possible only for $\kappa \geq \aleph_0$ or else $\kappa = 1$, and indeed the whole construction works fine also for 1-fold transitive groups with trivial $K_\mathfrak{x} = \{1_{F_\mathfrak{x}}\}$. Therefore we formulate

Corollary 5.1. *Let λ be a regular cardinal with $\diamondsuit_S$ for some non-reflecting stationary $S \subseteq S^\lambda_{\aleph_0}$. Then there exists a λ-free uniquely transitive group of cardinality λ.*

Furthermore it is easy to replace the Diamond argumentation by a Black Box argumentation and we formulate as dual result to Theorem 1.2.

Theorem 5.2. *Let $\kappa \geq \aleph_0$ be a cardinal and $\lambda > \kappa$ be a regular cardinal with $\lambda^{\aleph_0} = \lambda$. Then there exists an $\aleph_1$-free κ-fold transitive group of cardinality λ.*

Proof. Replace $\mathscr{A}$ and $\mathscr{U}$ by $\mathscr{A}^*$ and $\mathscr{U}^*$ respectively. Concerning the main construction progress similar to [6]. □

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