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POSITIONAL STRATEGIES IN LONG EHRENFEUCHT–FRAÏSSÉ GAMES

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Abstract. We prove that it is relatively consistent with ZF + CH that there exist two models of cardinality $\aleph_2$ such that the second player has a winning strategy in the Ehrenfeucht–Fraïssé-game of length ω_1 but there is no σ-closed back-and-forth set for the two models. If CH fails, no such pairs of models exist.

§1. Introduction. Suppose $\mathcal{A} = (A, \dots)$ and $\mathcal{B} = (B, \dots)$ are structures for the same vocabulary $\mathcal{L}$ of cardinality $< \kappa$. We say that a set $\mathcal{I}$ of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ has the κ*-back-and-forth property* if for every $p \in \mathcal{I}$, and every $A_0 \subseteq A$ and $B_0 \subseteq B$ of size $< \kappa$ there is $q \in \mathcal{I}$ extending p such that $A_0 \subseteq \mathrm{dom}(q)$ and $B_0 \subseteq \mathrm{ran}(q)$. We say that $\mathcal{A}$ and $\mathcal{B}$ κ*-partially isomorphic* and write $\mathcal{A} \simeq^p_\kappa \mathcal{B}$ if there is a κ-back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. The relation $\mathcal{A} \simeq^p_\kappa \mathcal{B}$ has a metamathematical interpretation. Namely, for regular κ it coincides with elementary equivalence relative to the infinitary language $L_{\infty\kappa}$. In particular, $\simeq^p_\kappa$ is an equivalence relation on the class of all $\mathcal{L}$-structures. If κ is uncountable then even for models of cardinality κ the relation $\simeq^p_\kappa$ is strictly weaker than isomorphism. This was first proved by Morley (1968, unpublished, see [7]). For instance, for $\kappa = \aleph_1$, one can take a pair of $\aleph_1$-like dense linear orders one of which contains a closed copy of ω_1 while the other doesn't.

In this paper, we investigate a strengthening of the relation $\simeq^p_\kappa$. Namely, given two cardinals κ and λ and two structures $\mathcal{A}$ and $\mathcal{B}$ in a vocabulary of size $< \kappa$, we say that $\mathcal{A}$ and $\mathcal{B}$ are (κ, λ)*-partially isomorphic* and write $\mathcal{A} \simeq^p_{\kappa,\lambda} \mathcal{B}$ if there is a κ-back-and-forth set $\mathcal{I}$ between $\mathcal{A}$ and $\mathcal{B}$ such that any increasing chain of length $< \lambda$ in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. The point is that the relation $\simeq^p_{\kappa,\kappa}$, unlike the weaker version $\simeq^p_\kappa$, implies isomorphism in the case that the models are of cardinality at most κ, and many classical isomorphism-proofs can be interpreted as results about the relation $\simeq^p_{\kappa,\lambda}$. Indeed, suppose κ is regular. Then any two η_κ-sets are in the relation $\simeq^p_{\kappa,\kappa}$. If they are of cardinality κ, they are isomorphic. Also, it is well known that any two real closed fields whose underlying orders are of type η_{ω_1} and are of cardinality ω_1 are isomorphic, see [3]. In fact, if κ is regular then any

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two real closed fields whose underlying orders are of type η_κ are in the relation $\simeq^p_{\kappa,\kappa}$, see [2]. Another example concerns saturated models. Any two κ-saturated elementary equivalent structures of cardinality κ are isomorphic, and the proof shows that any two κ-saturated elementary equivalent structures are in the relation $\simeq^p_{\kappa,\kappa}$. Finally, consider two κ-homogeneous structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \simeq^p_\kappa \mathcal{B}$. If they happen to be of cardinality κ they are isomorphic and the proof goes by showing that $\mathcal{A} \simeq^p_{\kappa,\kappa} \mathcal{B}$.

Thus, the relation $\simeq^p_{\kappa,\kappa}$ seems like an attractive weaker version of isomorphism. However, there are some simple questions concerning it that are still open. The most important one was raised by Dickmann [1] and Kueker [6], and asks if $\simeq^p_{\kappa,\kappa}$ is equivalent to elementary equivalence in some logic. In fact, it is not clear if $\simeq^p_{\kappa,\kappa}$ is even transitive. This was a serious obstacle to generalizing first order logic. In order to overcome this Karttunen [5] defined *tree-like partial isomorphisms*. This leads to a transitive relation which coincides with elementary equivalence in a certain logic called $\mathcal{N}_{\infty\kappa}$ and implies isomorphism for models of size κ. One can translate Karttunen's concept in terms of the existence of a winning strategy in a certain Ehrenfeucht–Fraïssé game which we now describe. To begin, we fix two regular cardinals κ and λ and two structures $\mathcal{A}$ and $\mathcal{B}$ in the same vocabulary $\mathcal{L}$ of size $< \kappa$.

Definition 1.1 ($\mathrm{EF}^\lambda_\kappa(\mathcal{A},\mathcal{B})$). There are two players $\forall$ and $\exists$. The game runs in λ rounds and proceeds as follows.

$$\begin{array}{c|cc} \forall & A_0, B_0 \quad \dots A_\alpha, B_\alpha & \dots \\ \hline \exists & p_0 \dots & p_\alpha \dots \end{array} \qquad (\alpha < \lambda)$$

At stage $\alpha < \lambda$, player $\forall$ picks $A_\alpha \subseteq A$ and $B_\alpha \subseteq B$, both of size $< \kappa$. Player $\exists$ responds by a partial isomorphism p_α between a substructure of $\mathcal{A}$ of size $< \kappa$ containing A_α and a substructure of $\mathcal{B}$ containing B_α. We require that p_α extends the p_ξ, for $\xi < \alpha$. Player $\exists$ wins the game if she plays λ rounds while obeying the rules. Otherwise player $\forall$ wins.

We write $\mathcal{A} \equiv_{\kappa,\lambda} \mathcal{B}$ if $\exists$ has a winning strategy in $\mathrm{EF}^\lambda_\kappa(\mathcal{A},\mathcal{B})$. This is clearly transitive. This concept has allowed the study of infinitary languages to take off and has been very fruitful (see e.g. [9]). One of the first new results was obtained by Hyttinen [4] who proved the Craig Interpolation Theorem and other classical results for this new logic. Still the following question remains.

Question 1.2. *What is the relation between* $\simeq^p_{\kappa,\lambda}$ *and* $\equiv_{\kappa,\lambda}$?

Clearly, if $\mathcal{A} \simeq^p_{\kappa,\lambda} \mathcal{B}$ then $\mathcal{A} \equiv_{\kappa,\lambda} \mathcal{B}$. Indeed, if $\mathcal{A} \simeq^p_{\kappa,\lambda} \mathcal{B}$ then there is a *positional* winning strategy for $\exists$ in $\mathrm{EF}^\lambda_\kappa(\mathcal{A},\mathcal{B})$, in the sense that $\exists$ only needs to know the current position in order to know how to play and win. Thus, Question 1.2 simply asks if the converse is true. Note that the positive answer implies that $\simeq^p_{\kappa,\lambda}$ is transitive. We concentrate on the first nontrivial case, namely the relation between $\simeq^p_{\aleph_1,\aleph_1}$ and $\equiv_{\aleph_1,\aleph_1}$. Let us first note the well-known fact that $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ can be expressed as the existence of *potential isomorphism*[1] an isomorphism in a forcing extension obtained by σ-closed forcing.

[1]Recall that for purely relational structures $\mathcal{A} \equiv_{\omega,\omega} \mathcal{B}$ is equivalent to the existence of an isomorphism of $\mathcal{A}$ and $\mathcal{B}$ in *some* forcing extension.

PROPOSITION 1.3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures in the same vocabulary $\mathcal{L}$. Then $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ if and only if there is a σ-closed forcing notion $\mathcal{P}$ such that $\Vdash_{\mathcal{P}} \mathcal{A} \cong \mathcal{B}$.*

We recall the following results from [8] where the equivalence of $\simeq^p_{\aleph_1,\aleph_1}$ and $\equiv_{\aleph_1,\aleph_1}$ has been established in some special cases.

THEOREM 1.4. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two structures in the same vocabulary $\mathcal{L}$. Then $\mathcal{A} \simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$ and $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ are equivalent in any of the following cases:*

(1) $|\mathcal{A}|, |\mathcal{B}| \leq 2^{\aleph_0}$.
(2) *$\mathcal{A}$ and $\mathcal{B}$ have different cardinality.*
(3) *$\mathcal{A}$ and $\mathcal{B}$ are trees of height $\aleph_1$.*

On the basis of these results it seems interesting to investigate the case when $\mathcal{A}$ and $\mathcal{B}$ are of cardinality $\aleph_2$ and CH holds. Even in this case we can have a positive result if we look at partial isomorphisms of size $\aleph_1$ rather than of size $\aleph_0$. The following result was proved in [8].

THEOREM 1.5. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two structures of cardinality $\aleph_2$ in the same vocabulary $\mathcal{L}$. Then $\mathcal{A} \simeq^p_{\aleph_2,\aleph_1} \mathcal{B}$ if and only if $\mathcal{A} \equiv_{\aleph_2,\aleph_1} \mathcal{B}$.*

The main result of this paper is that the relations $\simeq^p_{\aleph_1,\aleph_1}$ and $\equiv_{\aleph_1,\aleph_1}$ may not be equivalent for structures of size $\aleph_2$.

THEOREM 1.6. *It is relatively consistent with* ZFC + CH *that there exist two relational structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ in a countable vocabulary such that $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$ and $\mathcal{A} \not\simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$.*

The remainder of the paper is organized as follows. In Section2 we introduce the persistency game played on a given family of countable partial functions from ω_2 to ω_1. Given an $(\omega_1, 1)$-simplified morass $\mathfrak{M}$ we define a family $\mathcal{F} = \mathcal{F}(\mathfrak{M})$ which is strategically persistent. If $\mathfrak{M}$ is a generic morass we show that $\mathcal{F}$ does not have a σ-closed persistent subfamily. In Section 3 we use the family $\mathcal{F}$ from the previous section to define two structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$. If $\mathcal{F}$ is derived from a generic morass we show that $\mathcal{A} \not\simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$. Finally, in Section 4 we state some open questions and directions for further research.

§2. Persistent families of functions. In this section we change the original problem and instead of considering the Ehrenfeucht–Fraïssé game on a pair of structures, we consider a certain game on a given family of countable partial functions from ω_2 to ω_1.

Let $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ be the collection of all countable partial functions from ω_2 to ω_1. We say that a subfamily $\mathcal{F}$ of $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ is *persistent* if for every $p \in \mathcal{F}$ and $\alpha < \omega_2$ there is $q \in \mathcal{F}$ extending p such that $\alpha \in \mathrm{dom}(q)$. We will also consider the following *persistency game* on $\mathcal{F}$.

DEFINITION 2.1 ($\mathcal{G}_{\omega_1}(\mathcal{F})$). Suppose $\mathcal{F}$ is a subfamily of $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$. The game $\mathcal{G}_{\omega_1}(\mathcal{F})$ is played by players $\forall$ and $\exists$ and runs as follows:

$$\begin{array}{c|ccccc} \forall & \alpha_0 & \alpha_1 & \ldots\alpha_\xi & \ldots \\ \hline \exists & p_0 & p_1\ldots & p_\xi\ldots & \end{array} \quad (\xi < \omega_1)$$

At stage ξ player $\forall$ plays an ordinal $\alpha_\xi < \omega_2$ and $\exists$ plays $p_\xi \in \mathcal{F}$ extending p_η, for $\eta < \xi$, such that $\alpha_\xi \in \mathrm{dom}(p_\xi)$. $\exists$ wins the game if she is able to play ω_1 moves. Otherwise, $\forall$ wins.

We say that $\mathcal{F}$ is *strategically persistent* if $\exists$ has a winning strategy in $\mathcal{G}_{\omega_1}(\mathcal{F})$. One way to guarantee the existence of a winning strategy for $\exists$ is that there exist a persistent subfamily $\mathcal{D}$ of $\mathcal{F}$ which is σ*-closed*, i.e., for every sequence $(p_n)_n$ which is increasing under inclusion there is $q \in \mathcal{D}$ such that $p_n \subseteq q$, for all n. Indeed, given such a family $\mathcal{D}$, $\exists$ has a trivial winning strategy in $\mathcal{G}_{\omega_1}(\mathcal{F})$: at stage ξ she plays any $p_\xi \in \mathcal{D}$ which extends $\bigcup_{\eta<\xi} p_\eta$ and such that $\alpha_\xi \in \mathrm{dom}(p_\xi)$. The main goal of this section is to show that it is relatively consistent with ZFC that there exist a downward closed family $\mathcal{F}$ which is strategically persistent but does not have a σ-closed persistent subfamily. Indeed, given a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$ we can read off a certain family $\mathcal{F} = \mathcal{F}(\mathfrak{M})$ which is strategically persistent. If $\mathfrak{M}$ is obtained by the standard forcing construction we show that $\mathcal{F}$ does not have a σ-closed persistent subfamily.

We start by recalling the relevant definitions from Velleman [10].

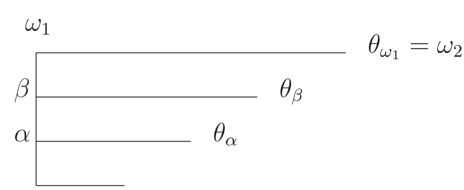
Definition 2.2 ([10]). A simplified $(\omega_1, 1)$-morass is a pair

$$\mathfrak{M} = \langle\langle \theta_\alpha : \alpha \leq \omega_1\rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \omega_1\rangle\rangle,$$

where $\langle \theta_\alpha : \alpha \leq \omega_1\rangle$ is a sequence of countable ordinals, $\mathcal{F}_{\alpha,\beta}$ is a family of order preserving embeddings from θ_α to θ_β, for $\alpha < \beta \leq \omega_1$, and the following conditions are satisfied (Figure 1):

(1) **(Successor)** For every α there are $\gamma_\alpha, \eta_\alpha \leq \theta_\alpha$ such that $\theta_\alpha = \gamma_\alpha + \eta_\alpha$, $\theta_{\alpha+1} = \theta_\alpha + \eta_\alpha$ and $\mathcal{F}_{\alpha,\alpha+1} = \{\mathrm{id}_{\theta_\alpha}, s_\alpha\}$, where $\mathrm{id}_{\theta_\alpha}$ is the identity on θ_α and $s_\alpha \restriction \gamma_\alpha = \mathrm{id}_{\gamma_\alpha}$ and $s_\alpha(\gamma_\alpha + \xi) = \theta_\alpha + \xi$, for all $\xi < \eta_\alpha$. We call s_α the *shift* at α. (Figure 2).

(2) **(Composition)** If $\alpha < \beta < \gamma$ then $\mathcal{F}_{\alpha\gamma} = \{g \circ f : f \in \mathcal{F}_{\alpha\beta}, g \in \mathcal{F}_{\beta\gamma}\}$.



θ_α is the α-th approximation of ω_2

Figure 1. A simplified morass.

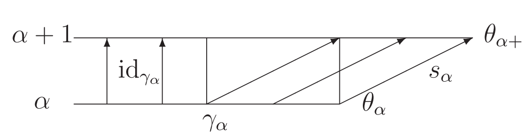


Figure 2. A shift.

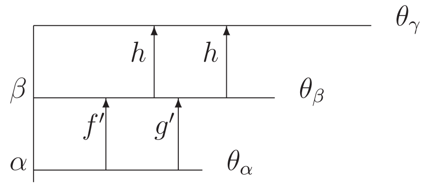


FIGURE 3. Factoring.

(3) **(Factoring)** Suppose γ is limit, $\alpha < \gamma$ and $f, g \in \mathcal{F}_{\alpha\gamma}$. Then there exists β such that $\alpha < \beta < \gamma$, and $f', g' \in \mathcal{F}_{\alpha,\beta}$ and $h \in \mathcal{F}_{\beta\gamma}$ such that $f = h \circ f'$ and $g = h \circ g'$. (Figure 3).
(4) **(Fullness)** If $\alpha < \beta$ then $\theta_\beta = \bigcup\{f[\theta_\alpha] : f \in \mathcal{F}_{\alpha\beta}\}$. Moreover, $\theta_{\omega_1} = \omega_2$.

We then have (see [10]) that if $\alpha < \beta$ and $\xi < \theta_\beta$, then there is a *unique* predecessor of ξ on level α, i.e., there is a unique $\eta < \theta_\alpha$ such that $f(\eta) = \xi$, for some $f \in \mathcal{F}_{\alpha\beta}$. Moreover, any such f is uniquely determined on $\eta+1$. We call η the α-th *predecessor* of ξ and write

$$\pi_\alpha^\beta(\xi) = \eta.$$

DEFINITION 2.3. Given a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$ we define the ordering $\preceq^{\mathfrak{M}}$ on ω_2 as follows:

$$\xi \preceq^{\mathfrak{M}} \eta \quad \text{iff} \quad \pi_\alpha^{\omega_1}(\xi) \leq \pi_\alpha^{\omega_1}(\eta), \text{ for all } \alpha < \omega_1.$$

We also define the ordering $\preceq_\alpha^{\mathfrak{M}}$ by:

$$\xi \preceq_\alpha^{\mathfrak{M}} \eta \quad \text{iff} \quad \xi \preceq^{\mathfrak{M}} \eta \quad \& \quad \pi_\alpha^{\omega_1}(\xi) = \pi_\alpha^{\omega_1}(\eta).$$

If $\mathfrak{M}$ is clear from the context we write $\preceq$ for $\preceq^{\mathfrak{M}}$ and $\preceq_\alpha$ for $\preceq_\alpha^{\mathfrak{M}}$ (Figure 4).

Given a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$, we define a certain subfamily $\mathcal{F}(\mathfrak{M})$ of $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ and show that it is strategically persistent.

DEFINITION 2.4. Suppose $\mathfrak{M}$ is a simplified $(\omega_1, 1)$-morass. Let $\mathcal{F}(\mathfrak{M})$ be the set of all $f \in \mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ such that:

(1) if $\xi, \eta \in \mathrm{dom}(f)$, $f(\eta) = \alpha$ and $\xi \preceq_\alpha \eta$, then $f(\xi) = \alpha$.
(2) $f^{-1}\{\alpha\}$ is $\preceq$-bounded, for all $\alpha \in \mathrm{ran}(f)$.

Note that the family $\mathcal{F}(\mathfrak{M})$ is closed under subfunctions. If $\mathfrak{M}$ is clear from the context, we will write $\mathcal{F}$ for $\mathcal{F}(\mathfrak{M})$. We first show the following.

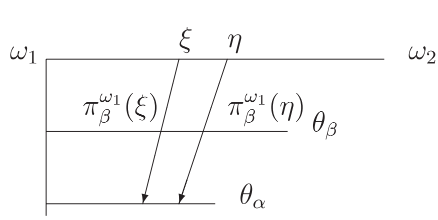


FIGURE 4. The ordering $\preceq$.

LEMMA 2.5. *Suppose $\mathfrak{M}$ is a simplified $(\omega_1, 1)$-morass. Then $\mathcal{F}(\mathfrak{M})$ is strategically persistent.*

PROOF. Given $\xi, \eta < \omega_2$, by (4) and (3) of Definition 2.2 there exists $\alpha < \omega_1$ and $f \in \mathcal{F}_{\alpha,\omega_1}$ such that $\xi, \eta \in \operatorname{ran}(f)$. Let $\mu(\xi, \eta)$ be the least such α. If $\xi < \eta$ it follows that $\pi_\beta^{\omega_1}(\xi) < \pi_\beta^{\omega_1}(\eta)$, for every β such that $\mu(\xi, \eta) \leq \beta < \omega_1$. We now describe a strategy for $\exists$ in the persistency game on $\mathcal{F}(\mathfrak{M})$. At every stage j if player $\forall$ plays some $\xi_j < \omega_2$ then player $\exists$ picks an ordinal $\alpha_j < \omega_1$ and plays $p_j = \{\langle \xi_i, \alpha_i\rangle : i \leq j\}$. Thus, we only need to describe how to choose the ordinals α_j and check that the corresponding function p_j belongs to $\mathcal{F}(\mathfrak{M})$. Suppose we are at stage j and player $\forall$ plays ξ_j. Player $\exists$ first asks if there is an ordinal $i < j$ such that $\xi_j \preceq_{\alpha_i} \xi_i$. If so, then $\exists$ picks the least such i and sets $\alpha_j = \alpha_i$. Otherwise, $\exists$ picks any ordinal α_j strictly bigger than the α_i, for $i < j$, and $\mu(\alpha_i, \alpha_j)$, for $i < j$. In order to check that the corresponding functions p_j are in $\mathcal{F}(\mathfrak{M})$ we need the following.

CLAIM. *At every stage j there is at most one α for which there is $i < j$ such that $\xi_j \preceq_\alpha \xi_i$ and $\alpha_i = \alpha$.*

PROOF. Suppose there were two distinct such ordinals, say α and β. Let k be the least such that $\alpha_k = \alpha$ and $\xi_j \preceq_\alpha \xi_k$ and, similarly, let l be the least such that $\alpha_l = \beta$ and $\xi_j \preceq_\beta \xi_l$. Suppose that $k < l$. Notice that, by the minimality of l, there is no $i < l$ such that $\alpha_i = \beta$ and $\xi_l \preceq_\beta \xi_i$. Therefore, by the definition of α_l, it follows that β is bigger than α and $\mu(\xi_k, \xi_l)$. We consider two cases.

Case 1. $\xi_k < \xi_l$. Since $\beta > \mu(\xi_k, \xi_l)$ we have that $\pi_\beta^{\omega_1}(\xi_k) < \pi_\beta^{\omega_1}(\xi_l)$. Since $\xi_j \preceq_\alpha \xi_k$ and $\alpha < \beta$ we have that $\pi_\beta^{\omega_1}(\xi_j) \leq \pi_\beta^{\omega_1}(\xi_k)$. Therefore, we have that $\pi_\beta^{\omega_1}(\xi_j) < \pi_\beta^{\omega_1}(\xi_l)$. On the other hand, we have that $\xi_j \preceq_\beta \xi_l$, which means that, in particular, $\pi_\beta^{\omega_1}(\xi_j) = \pi_\beta^{\omega_1}(\xi_l)$, a contradiction.

Case 2. $\xi_l < \xi_k$. Since $\beta > \mu(\xi_k, \xi_l)$ we have that $\pi_\gamma^{\omega_1}(\xi_l) < \pi_\gamma^{\omega_1}(\xi_k)$, for all $\gamma \geq \beta$. We also have that $\pi_\beta^{\omega_1}(\xi_j) = \pi_\beta^{\omega_1}(\xi_l)$. Since $\xi_j \preceq_\alpha \xi_k$ and $\alpha < \beta$ it follows that $\xi_l \preceq_\alpha \xi_k$. Therefore, at stage l we should have let $\alpha_l = \alpha$, a contradiction. ⊣

Now, we check that the functions p_j belong to $\mathcal{F}(\mathfrak{M})$, for all j. Condition (1) of Definition 2.4 is satisfied by the construction. To verify (2), suppose $\alpha \in \operatorname{ran}(p_j)$ and notice that if i is the least such that $\alpha_i = \alpha$ then ξ_i is the $\preceq_\alpha$-largest element of $p_j^{-1}\{\alpha\}$. Therefore, $p_j^{-1}\{\alpha\}$ is $\preceq$-bounded. This completes the proof of Lemma 2.5. ⊣

In order to show that $\mathcal{F}(\mathfrak{M})$ does not have a σ-closed persistent family we will need to assume certain properties of $\mathfrak{M}$.

DEFINITION 2.6. Let $\mathfrak{M}$ be a simplified $(\omega_1, 1)$-morass.

(1) We say that $\mathfrak{M}$ is *stationary* if $\mathcal{S}(\mathfrak{M}) = \{f[\theta_\alpha] : \alpha < \omega_1 \text{ and } f \in \mathcal{F}_{\alpha,\omega_1}\}$ is a stationary subset of $[\omega_2]^\omega$.
(2) We say that $\mathfrak{M}$ satisfies the $\aleph_2$-*antichain condition* if for every $X \subseteq (\omega_2)^\omega$ of size ω_2 there are distinct $s, t \in X$ such that $s(n) \preceq t(n)$, for all n, i.e., there is no antichain of size $\aleph_2$ in $(\omega_2, \preceq)^\omega$ under the product ordering.

We first show that if $\mathfrak{M}$ has the above properties then $\mathcal{F}(\mathfrak{M})$ does not have a σ-closed persistency subfamily. Then we show that if $\mathfrak{M}$ is obtained by the standard forcing for adding a simplified $(\omega_1, 1)$-morass then $\mathfrak{M}$ has the above properties.

LEMMA 2.7. *Suppose* CH *holds and* $\mathfrak{M}$ *is a simplified* $(\omega_1, 1)$*-morass which satisfies the* $\aleph_2$*-antichain condition. Let* $\mathcal{A}$ *be a subset of* $\mathcal{F}(\mathfrak{M})^\omega$ *of size* $\aleph_2$*. Then there is* $\vec{g} \in \mathcal{A}$ *and* $\mathcal{B} \subseteq \mathcal{A}$ *of size* $\aleph_2$ *such that for every* $\vec{h} \in \mathcal{B}$*, every* n*, and every* $f \in \mathcal{F}(\mathfrak{M})$*, if* f *extends* h_n *and* $\operatorname{dom}(g_n) \subseteq \operatorname{dom}(f)$ *then* f *extends* g_n*.*

PROOF. First, observe that if X is a subset of $(\omega_2)^\omega$ of size $\aleph_2$ then there is $s \in X$ and $Y \subseteq X$ of size $\aleph_2$ such that $s(n) \preceq t(n)$, for all $t \in Y$ and all n. To see this, let Z be a maximal antichain in X. Then every element of X is comparable with an element of Z. Since $\preceq$ refines the usual ordering on ω_2, by CH, for every $s \in Z$ the set of $t \in X$ such that $t(n) \preceq s(n)$, for all n, has size at most $\aleph_1$. Therefore, there is $s \in Z$ such that the set

$$Y = \{t \in X : s(n) \preceq t(n), \text{ for all } n\}$$

is of size $\aleph_2$. Then s and Y are as required.

We now turn to the proof of the lemma. First of all, we may assume that there is a fixed ordinal $\alpha < \omega_1$ such that $\alpha = \sup(\bigcup_n \operatorname{ran}(g_n))$, for all $\vec{g} \in \mathcal{A}$. By CH, we may assume that there is a fixed ordinal $\mu > \alpha$ and, for each n a subset E_n of θ_μ such that, for every $\vec{g} \in A$, there is $f_{\vec{g}} \in \mathcal{F}_{\mu,\omega_1}$ such that $f_{\vec{g}}[E_n] = \operatorname{dom}(g_n)$. Consider now the functions $e_{n,\vec{g}} = g_n \circ f_{\vec{g}}$, for $\vec{g} \in \mathcal{A}$ and $n < \omega$. By CH again, we may assume that there are fixed functions e_n, such that $e_{n,\vec{g}} = e_n$, for all $\vec{g} \in \mathcal{A}$ and n. By the first paragraph of this proof, there is $\vec{g} \in \mathcal{A}$ and a subset $\mathcal{B}$ of $\mathcal{A}$ of size $\aleph_2$ such that $f_{\vec{g}}(\xi) \preceq f_{\vec{h}}(\xi)$, for all $\vec{h} \in \mathcal{B}$ and $\xi < \theta_\mu$. We claim that $\vec{g}$ and $\mathcal{B}$ are as required. To see this, consider some $\vec{h} \in \mathcal{B}$ and some integer n. Let u be any extension of h_n which belongs to $\mathcal{F}(\mathfrak{M})$ and is defined on $\operatorname{dom}(g_n)$. We check that u extends g_n. Let $\rho \in \operatorname{dom}(g_n)$. Then there is $\xi \in E_n$ such that $f_{\vec{g}}(\xi) = \rho$. Let $\rho' = f_{\vec{h}}(\xi)$. Then $\rho \preceq_\mu \rho'$. Since u extends h_n and $h_n(\rho') \leq \mu$, by (1) of Definition 2.4 it follows that $u(\rho) = h_n(\rho')$. On the other hand, $g_n(\rho) = h_n(\rho') = e_n(\xi)$. Therefore, $u(\rho) = g_n(\rho)$. Since ρ was arbitrary it follows that u extends g_n. ⊣

LEMMA 2.8. *Assume* CH *and let* $\mathfrak{M}$ *be a simplified* $(\omega_1, 1)$*-morass which is stationary and satisfies the* $\aleph_2$*-antichain condition. Then there is no* σ*-closed persistent subfamily of* $\mathcal{F}(\mathfrak{M})$*.*

PROOF. Fix a persistent subfamily $\mathcal{G}$ of $\mathcal{F}(\mathfrak{M})$. We need to show that $\mathcal{G}$ is not σ-closed. Let τ be a sufficiently large regular cardinal. Since $\mathcal{S}(\mathfrak{M})$ is stationary in $[\omega_2]^\omega$, we can find a countable elementary submodel M of H_τ containing all the relevant objects such that $M \cap \omega_2 \in \mathcal{S}(\mathfrak{M})$. Let $\zeta = \sup(M \cap \omega_2)$ and fix an increasing sequence $\{\zeta_n\}_n$ of ordinals in M which is cofinal in ζ.

We now work in M. For each $\delta < \omega_2$ fix $g^0_\delta \in \mathcal{G}$ such that $\delta \in \operatorname{dom}(g^0_\delta)$. We can find $\alpha < \omega_1$ and $X_0 \subseteq \omega_2 \setminus \zeta_0$ of size $\aleph_2$ such that $g^0_\delta(\delta) = \alpha$, for all $\delta \in X_0$. Since $\mathfrak{M}$ satisfies the $\aleph_2$-antichain condition, by Lemma 2.7 we can fix $\delta_0 \in X_0$ and $X_1 \subseteq X_0 \setminus \zeta_1$ of size $\aleph_2$ such that, for all $\delta \in X_1$, any extension of g^0_δ to a function in $\mathcal{F}(\mathfrak{M})$ which is defined on $\operatorname{dom}(g^0_{\delta_0})$ must extend $g^0_{\delta_0}$. For each $\delta \in X_1$ fix some $g^1_\delta \in \mathcal{G}$ which extends g^0_δ and is defined on $\operatorname{dom}(g^0_{\delta_0})$. It follows that $g^0_{\delta_0} \cup g^0_\delta \subseteq g^1_\delta$. By Lemma 2.7 again, we can fix $\delta_1 \in X_1$ and $X_2 \subseteq X_1 \setminus \zeta_2$ of size $\aleph_2$ such that, for all $\delta \in X_2$ and all $h \in \mathcal{F}(\mathfrak{M})$, if h extends g^1_δ and is defined on $\operatorname{dom}(g^1_{\delta_1})$ then

h extends $g^1_{\delta_1}$. We continue like this and get an increasing sequence $(\delta_n)_n$ of ordinals from M, a decreasing sequence $(X_n)_n$ of subsets of ω_2 of size $\aleph_2$, and, for each n and $\delta \in X_n$, a function $g^n_\delta \in \mathcal{G}$ such that:

(1) $\delta_n \geq \zeta_n$, for all n,
(2) $g^n_{\delta_n} \cup g^n_\delta \subseteq g^{n+1}_\delta$, for all $\delta \in X_{n+1}$.

While the sequence $(\zeta_n)_n$ does not belong to M, at each stage we need to know only finitely many of the ζ_n. Therefore, we can perform each step of the construction inside M. It follows that $(g^n_{\delta_n})_n$ is an increasing sequence of functions from $\mathcal{G}$ and $g^n_{\delta_n}(\delta_n) = \alpha$, for all n. The sequence $(\delta_n)_n$ is cofinal in ζ and, since $M \cap \omega_2 \in \mathcal{S}(\mathfrak{M})$, it follows that it is unbounded in the sense of $\preceq$. Therefore, any functions which extends $\bigcup_n g^n_{\delta_n}$ violates (2) of Definition 2.4 and cannot be in $\mathcal{F}(\mathfrak{M})$. It follows that $\mathcal{G}$ is not σ-closed. ⊣

We now consider the standard forcing notion for adding a simplified $(\omega_1, 1)$-morass and show that the generic morass is stationary and satisfies the $\aleph_2$-antichain condition. Before we start, it will be convenient to make the following definition.

DEFINITION 2.9. For $\beta < \omega_2$ let I_β be the interval $[\omega_1 \cdot \beta, \omega_1 \cdot (\beta + 1))$. We say that a subset A of ω_2 is ω_1-*full* if $A \cap I_\beta$ is an initial segment of I_β, for all $\beta < \omega_2$.

We now state a slight variation of the standard forcing for adding a simplified $(\omega_1, 1)$-morass from [10].

DEFINITION 2.10 ([10]). The forcing notion $\mathcal{P}$ consists of tuples

$$p = \langle \langle \theta^p_\alpha : \alpha \leq \delta_p \rangle, \langle \mathcal{F}^p_{\alpha,\beta} : \alpha < \beta \leq \delta_p \rangle, A_p, i_p \rangle,$$

where $\delta_p < \omega_1$, $\langle \theta^p_\alpha : \alpha \leq \delta_p \rangle$ is a sequence of limit ordinals $< \omega_1$, $\mathcal{F}^p_{\alpha,\beta}$ is a collection of order-preserving mappings from θ^p_α to θ^p_β, A_p is an ω_1-full subset of ω_2, i_p is an order preserving bijection between $\theta^p_{\delta_p}$ and A_p, and the following conditions hold:

(1) $\mathcal{F}^p_{\alpha,\alpha+1} = \{\mathrm{id}_{\theta_\alpha}, s_\alpha\}$, where s_α is a shift as in Definition 2.2 (1).
(2) If $\alpha < \beta < \gamma \leq \delta_p$ then $\mathcal{F}^p_{\alpha,\gamma} = \{g \circ f : f \in \mathcal{F}^p_{\alpha,\beta}, g \in \mathcal{F}^p_{\beta,\gamma}\}$.
(3) Suppose $\alpha < \gamma \leq \delta_p$, γ limit and $f, g \in \mathcal{F}^p_{\alpha,\gamma}$. Then there is β such that $\alpha < \beta < \gamma$ and there are $f', g' \in \mathcal{F}^p_{\alpha,\beta}$ and $h \in \mathcal{F}^p_{\beta,\gamma}$ such that $f = h \circ f'$ and $g = h \circ g'$.
(4) If $\alpha < \beta \leq \delta_p$ then $\theta^p_\beta = \bigcup\{f[\theta^p_\alpha] : f \in \mathcal{F}^p_{\alpha\beta}\}$.

The ordering of $\mathcal{P}$ is defined as follows. We say that $q \leq p$ if $\delta_p \leq \delta_q$, $\theta^p_\alpha = \theta^q_\alpha$ for $\alpha \leq \delta_p$, $\mathcal{F}^p_{\alpha,\beta} = \mathcal{F}^q_{\alpha,\beta}$ if $\alpha < \beta \leq \delta_p$, and $i_p = i_q \circ h$, for some $h \in \mathcal{F}^q_{\delta_p,\delta_q}$. Note that, in particular, this means that $A_p \subseteq A_q$.

LEMMA 2.11. *Let $(p_n)_n$ be a decreasing sequence of conditions in $\mathcal{P}$. Then there is $q \in \mathcal{P}$ such that $A_q = \bigcup_n A_{p_n}$ and $q \leq p_n$, for all n. In particular, $\mathcal{P}$ is σ-closed.*

PROOF. Suppose $(p_n)_n$ is a decreasing sequence of conditions in $\mathcal{P}$. We define the required condition q. We let $A_q = \bigcup_n A_{p_n}$ and $\delta_q = \sup_n \delta_{p_n}$. Note that, since the sequence of the A_{p_n} is increasing and each of them is ω_1-full, then so is A_q. Let $\theta^q_{\delta_q}$ be the order type of A_q and i_q the order preserving bijection between $\theta^q_{\delta_q}$ and A_q. For $\alpha < \delta_q$ we let θ^q_α be equal to $\theta^{p_n}_\alpha$, for any sufficiently large n. Also, for

$\alpha < \beta < \delta^q$ we let $\mathcal{F}^q_{\alpha,\beta}$ be equal to $\mathcal{F}^{p_n}_{\alpha,\beta}$, for any sufficiently large n. It remains to define the collections $\mathcal{F}^q_{\alpha,\delta_q}$, for $\alpha < \delta_q$. Fix some $\alpha < \delta_q$ and let n be sufficiently large such that $\alpha < \delta_{p_n}$. We let

$$\mathcal{F}^q_{\alpha,\delta_q} = \{i_q^{-1} \circ i_{p_n} \circ f : f \in \mathcal{F}^{p_n}_{\alpha,\delta_{p_n}}\}.$$

It is straightforward to check that $q = \langle\langle \theta^q_\alpha : \alpha \leq \delta_q\rangle, \langle \mathcal{F}^q_{\alpha,\beta} : \alpha < \beta \leq \delta_q\rangle, A_q, i_q\rangle$ is a condition and $q \leq p_n$, for all n. ⊣

It follows that $\mathcal{P}$ preserves ω_1. We now need a lemma on the compatibility of conditions in $\mathcal{P}$. First, let us say that two condtions p and q are *isomorphic* if $\delta_p = \delta_q$, $\theta^p_\alpha = \theta^q_\alpha$, for all $\alpha \leq \delta_p$, and $\mathcal{F}^p_{\alpha,\beta} = \mathcal{F}^q_{\alpha,\beta}$, for all $\alpha < \beta \leq \theta^p_{\delta_p}$. If p and q are isomorphic, we say that they are *directly compatible* if there is $r \leq p, q$ such that $\delta_r = \delta_p + 1$. We call such r the *amalgamation* of p and q.

LEMMA 2.12. *Suppose p and q are two isomorphic conditions in $\mathcal{P}$ such that $A_p \cap A_q$ is an initial segment of both A_p and A_q, and $\sup(A_p \setminus A_q) < \inf(A_q \setminus A_p)$. Then p and q are directly compatible.*

PROOF. We define a condition r which is the amalgamation of p and q. For simplicity, set $\delta = \delta_p = \delta_p$ and $\theta_\alpha = \theta^p_\alpha = \theta^q_\alpha$, for all $\alpha \leq \delta$. Let $\delta_r = \delta + 1$ and $A_r = A_p \cup A_q$. Note that, since A_p and A_q are ω_1-full, then so is A_r. Let $\theta^r_{\delta_r}$ be the order type of A_r and i_r the order preserving bijection between $\theta^r_{\delta_r}$ and A_r. For $\alpha < \beta \leq \delta$ let $\mathcal{F}^r_{\alpha,\beta} = \mathcal{F}^p_{\alpha,\beta}$. Let $R = A_p \cap A_q$, let γ be the order type of R and η the order type of $A_p \setminus A_q$ and $A_q \setminus A_p$. Since $\sup(A_p \setminus A_q) < \inf(A_q \setminus A_p)$ it follows that $\theta^r_{\delta_r} = \theta_\delta + \eta$. Let $s : \theta_\delta \to \theta^r_{\delta_r}$ be the shift of θ_δ at γ, i.e., it is the identity on γ and $s(\gamma + \xi) = \theta_\delta + \xi$, for all $\xi < \eta$. We let $\mathcal{F}^r_{\delta,\delta_r} = \{\mathrm{id}_{\theta_\delta}, s\}$. Finally, for $\alpha < \delta$ let

$$\mathcal{F}^r_{\alpha,\delta_r} = \{g \circ f : f \in \mathcal{F}^p_{\alpha,\delta}, g \in \mathcal{F}^r_{\delta,\delta_r}\}.$$

Then r is as required. ⊣

REMARK Let p and q be as in Lemma 2.12 and let r be the amalgamation of p and q. Let i be the order preserving bijection between A_p and A_q. What is important for our purposes is that r forces that $\xi \preceq_{\delta_p} i(\xi)$, for all $\xi \in A_p$.

LEMMA 2.13. *Let $\alpha < \omega_2$. Then, for every $p \in \mathcal{P}$ there is $r \leq p$ such that $\alpha \in A_r$.*

PROOF. Let β be such that $\alpha \in I_\beta$. We show that every condition p has an extension r such that $A_r \cap I_\beta$ is a proper extension of $A_p \cap I_\beta$. Since $A_r \cap I_\beta$ is an initial segment of I_β, for every r, the order type of I_β is ω_1 and $\mathcal{P}$ is σ-closed, by iterating this operation countably many times we can find a condition $s \leq p$ such that $\alpha \in A_s$. So, fix some $p \in \mathcal{P}$. Assume first that $A_p \setminus \omega_1 \cdot (\beta + 1)$ is nonempty and let η be its order type. Note that η is a countable ordinal. Let $\mu = \min(I_\beta \setminus A_p)$. Since A_p is ω_1-full we have that $A_p \cap [\mu, \omega_1 \cdot (\beta + 1)) = \emptyset$. Let $\nu = \mu + \eta$ and let $A_q = (A_p \cap \omega_1 \cdot \beta) \cup [\omega_1 \cdot \beta, \nu)$. Then A_p and A_q have the same order type, $A_p \cap A_q$ is an initial segment of both of them, and $\sup(A_q \setminus A_p) < \inf(A_p \setminus A_q)$. Also note that $A_p \cup A_q$ is ω_1-full. Let i_q be the isomorphism between $\theta^p_{\delta_p}$ and A_q. Let $\delta_p = \delta_q$, $\theta^p_\alpha = \theta^q_\alpha$, for all $\alpha \leq \delta_p$, $\mathcal{F}^p_{\alpha,\beta} = \mathcal{F}^q_{\alpha,\beta}$, for all $\alpha < \beta \leq \theta^p_{\delta_p}$. Then p and q satisfy the assumptions of Lemma 2.12. Let r be their amalgamation. Then $r \leq p$ and $A_r \cap I_\beta$ is a proper extension of $A_p \cap I_\beta$, as required.

Assume now that $A_p \subseteq \omega_1 \cdot (\beta + 1)$. For simplicity, let $\delta = \delta_p$ and $\theta_\alpha = \theta^p_\alpha$, for $\alpha \leq \delta$. Recall that this implies that θ_δ is the order type of A_p. Let $\mu = \min(I_\beta \setminus A_p)$.

We are going to define the condition r directly. We let $A_r = A_p \cup [\mu, \mu + \theta_\delta)$. We let $\delta_r = \delta + 1$. We let $\theta^r_\alpha = \theta_\alpha$, for all $\alpha \leq \delta$ and $\theta^r_{\delta+1} = \theta_\delta \cdot 2$. We let $\mathcal{F}^r_{\alpha,\beta} = \mathcal{F}^p_{\alpha,\beta}$, for all $\alpha < \beta \leq \delta$. We let $\mathcal{F}^r_{\delta,\delta+1} = \{\mathrm{id}_{\theta_\delta}, s\}$, where $\mathrm{id}_{\theta_\delta}$ is the identity on θ_δ and s is the shift of θ_δ at 0, i.e., $s(\rho) = \theta_\delta + \rho$, for all $\rho < \theta_\delta$. For $\alpha < \delta$ we let $\mathcal{F}^r_{\alpha,\delta+1}$ consist of all functions of the form $g \circ f$, where $f \in \mathcal{F}^p_{\alpha,\delta}$ and $g \in \mathcal{F}^r_{\delta,\delta+1}$. Finally, let i_r be the order preserving bijection between $\theta_\delta \cdot 2$ and A_r. Then r is an extension of p and $A_r \cap I_\beta$ is a proper extension of $A_p \cap I_\beta$. ⊣

LEMMA 2.14. *Assume* CH. *Then $\mathcal{P}$ satisfies the $\aleph_2$-chain condition.*

PROOF. Let $\mathcal{A}$ be a subset of $\mathcal{P}$ of size $\aleph_2$. By CH we may assume that all the conditions in $\mathcal{A}$ are compatible. Therefore, we can fix an ordinal δ, a sequence $\langle \theta_\alpha : \alpha \leq \delta \rangle$ and a sequence $\langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \delta$ such that every condition p in A is of the form $p = \langle \langle \theta_\alpha : \alpha \leq \delta \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha \leq \beta \leq \delta \rangle, A_p, i_p \rangle$, for some A_p of order type θ_δ, where i_p is the order preserving bijection between θ_δ and A_p. By CH again and the Δ-system lemma, we may find distinct $p, q \in \mathcal{A}$ such that $A_p \cap A_q$ is an initial segment of both A_p and A_q and such that $\sup(A_p \setminus A_q) < \inf(A_q \setminus A_p)$. By Lemma 2.12 p and q are compatible, as required. ⊣

Assume CH. By Lemmas 2.11 and 2.14 $\mathcal{P}$ preserves cardinals. Let G be a $\mathcal{P}$-generic filter over V. For $\alpha < \omega_1$, we let θ^G_α be equal to θ^p_α, for any $p \in G$ such that $\alpha \leq \delta_p$. We also let $\theta^G_{\omega_1} = \omega_2$. For $\alpha < \beta < \omega_1$ we let $\mathcal{F}^G_{\alpha,\beta}$ be equal to $F^p_{\alpha,\beta}$, for any $p \in G$ such that $\beta \leq \delta_p$. For $\alpha < \omega_1$ we define:

$$\mathcal{F}^G_{\alpha,\omega_1} = \{i_p \circ f : f \in \mathcal{F}^p_{\alpha,\delta_p}, p \in G \text{ and } \alpha \leq \delta_p\}.$$

It follows that

$$\mathfrak{M}_G = \langle \langle \theta^G_\alpha : \alpha \leq \omega_1 \rangle, \langle \mathcal{F}^G_{\alpha,\beta} : \alpha < \beta \leq \omega_1 \rangle \rangle$$

is a simplified $(\omega_1, 1)$-morass. Let $\dot{\mathfrak{M}}$ be the canonical $\mathcal{P}$-name for $\mathfrak{M}_G$.

LEMMA 2.15. $\Vdash_{\mathcal{P}} \dot{\mathfrak{M}}$ *is stationary.*

PROOF. Suppose $p \Vdash_{\mathcal{P}} \dot{C}$ is a club in $[\omega_2]^\omega$. Set $p_0 = p$. By using Lemmma 2.13 and 2.11 repeatedly and the fact that p forces $\dot{C}$ to be unbounded in $[\omega_2]^\omega$, we can build a decreasing sequence $(p_n)_n$ of conditions in $\mathcal{P}$ and an increasing sequence $(B_n)_n$ of countable subsets of ω_2 such that $A_{p_n} \subseteq B_n \subseteq A_{p_{n+1}}$ and $p_{n+1} \Vdash_{\mathcal{P}} B_n \in \dot{C}$, for all n. Let q be the limit of the sequence $(p_n)_n$ as in Lemma 2.11. Then $A_q = \bigcup_n A_{p_n} = \bigcup_n B_n$. Since $\dot{C}$ is forced by p to be closed and $q \leq p$ it follows that $q \Vdash_{\mathcal{P}} A_q \in \dot{C}$. Since $q \Vdash_{\mathcal{P}} A_q \in \mathcal{S}(\dot{\mathfrak{M}})$ and $\dot{C}$ was arbitrary, it follows that $\dot{\mathfrak{M}}$ is forced to be stationary. ⊣

LEMMA 2.16. *Assume* CH *holds in V. Then $\Vdash_{\mathcal{P}}$ $\dot{\mathfrak{M}}$ satisfies the $\aleph_2$-antichain condition.*

PROOF. Suppose $p \in \mathcal{P}$ forces that $\dot{X}$ is a subset of $(\omega_2)^\omega$ of size $\aleph_2$. We can find a subset S of $(\omega_2)^\omega$ of size $\aleph_2$ and, for each $s \in S$, a condition $p_s \leq p$ such that $p_s \Vdash_{\mathcal{P}} s \in \dot{X}$. By Lemma 2.13 we may assume that $\mathrm{ran}(s) \subseteq A_{p_s}$, for all s. By CH we may assume that the conditions p_s, for $s \in S$, are all isomorphic. Let us fix an ordinal δ, a sequence $\langle \theta_\alpha : \alpha \leq \delta \rangle$ and a sequence $\langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \delta \rangle$ such that every condition p_s, for $s \in S$, is of the form $p_s = \langle \langle \theta_\alpha : \alpha \leq \delta \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha \leq \beta \leq \delta \rangle, A_{p_s}, i_{p_s} \rangle$, for some A_{p_s} of order type θ_δ, where i_{p_s} is the order preserving bijection between θ_δ and A_{p_s}. Further, again by CH, we may assume that there are fixed ordinals $\xi_n < \theta_\delta$, for $n < \omega$, such that $s(n) = i_{p_s}(\xi_n)$, for all $s \in S$ and all n. By the Δ-system lemma, we may find distinct $s, t \in S$ such that $A_{p_s} \cap A_{p_t}$ is an initial segment of both A_{p_s} and A_{p_t} and such that $\sup(A_{p_s} \setminus A_{p_t}) < \inf(A_{p_t} \setminus A_{p_s})$. Let r be the amalgamation of p_s and p_t. Then $r \Vdash_{\mathcal{P}} s, t \in \dot{X}$. By the remark following Lemma 2.12 it follows that that $r \Vdash_{\mathcal{P}} s(n) \preceq_\delta t(n)$, for all n. Therefore, r forces that $\dot{X}$ is not an antichain in $(\omega_2, \preceq)^\omega$, as required. ⊣

By putting together the results of this section we obtain the following.

THEOREM 2.17. *It is relatively consistent with* ZFC + CH *that there exist a downward closed subfamily $\mathcal{F}$ of* $\mathrm{Fn}(\omega_2, \omega_1, \omega_1)$ *which is strategically persistent but does not have a σ-closed persistent subfamily.*

§3. The main theorem. The goal of this section is to prove Theorem 1.6. Before we do that we show that if $\mathcal{A} \simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$ then we can find an ω_1-back and forth family $\mathcal{I}$ of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ with additional closure properties. Recall that we defined $\mathcal{I}$ to be σ-closed if every increasing sequence $(p_n)_n$ of members of $\mathcal{I}$ has an upper bound in $\mathcal{I}$. We will say that $\mathcal{I}$ is *strongly σ-closed* if $\bigcup_n p_n \in \mathcal{I}$, for every such sequence $(p_n)_n$. We will need the following.

LEMMA 3.1. *Assume* CH *and let $\mathcal{A}$ and $\mathcal{B}$ be two structures of size $\aleph_2$ in the same vocabulary such that $\mathcal{A} \simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$. Then there is an ω_1-back and forth set $\mathcal{J}$ for $\mathcal{A}$ and $\mathcal{B}$ which is strongly σ-closed.*

PROOF. Let $\mathcal{I}$ be a σ-closed ω_1-back and forth set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. We build another ω_1-back and forth set $\mathcal{J}$ which is strongly σ-closed. We may assume that the base set of both $\mathcal{A}$ and $\mathcal{B}$ is ω_2. Since $\mathcal{I}$ consists of countable partial functions from ω_2 to ω_2, by CH it follows that it is of cardinality ω_2. Let us fix an enumeration $\{p_\alpha : \alpha < \omega_2\}$ of $\mathcal{I}$. We may assume that the empty function belongs to $\mathcal{I}$ and is enumerated as p_0. We let $q \in \mathcal{J}$ if q is a permutation of a countable subset D_q of ω_2 containing 0 and the following hold:

(1) if $\alpha \in D_q$ then $\mathrm{dom}(p_\alpha) \cup \mathrm{ran}(p_\alpha) \subseteq D_q$,
(2) if $\alpha \in D_q$ and $p_\alpha \subseteq q$ then for every $\xi \in D_q$ there is $\beta \in D_q$ such that $p_\alpha \subseteq p_\beta \subseteq q$, and $\xi \in \mathrm{dom}(p_\beta) \cap \mathrm{ran}(p_\beta)$.

Note that if $q \in \mathcal{J}$ then, by (2) and the fact that $0 \in D_q$, we can find a sequence $(\alpha_n)_n$ of elements of D_q such that $p_{\alpha_0} \subseteq p_{\alpha_1} \subseteq \ldots \subseteq p_{\alpha_n} \subseteq \ldots$, and $q = \bigcup_n p_{\alpha_n}$. Since each p_{α_n} is a countable partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then so is q. Moreover, since $\mathcal{I}$ is σ-closed there is $p \in \mathcal{I}$ such that $q \subseteq p$. Note also that $\mathcal{J}$ is strongly σ-closed. In order to show that $\mathcal{J}$ has the ω_1-back and forth property it suffices to show the following.

CLAIM. For every $p \in \mathcal{I}$ there is $q \in \mathcal{J}$ such that $p \subseteq q$.

PROOF. Fix a sufficiently large regular cardinal τ and a countable elementary submodel M of H_τ containing p and the enumeration of $\mathcal{I}$. Fix an enumeration $\{\alpha_n : n < \omega\}$ of $M \cap \omega_2$. We define an increasing sequence $(r_n)_n$ of elements of $\mathcal{I} \cap M$ as follows. Let $r_0 = p$. Suppose we have defined r_n. By the fact that $\mathcal{I}$ is an ω_1-back and forth set and M is elementary, we can find $r_{n+1} \in M \cap \mathcal{I}$ extending r_n such that:

(a) r_{n+1} either extends p_{α_n} or is incompatible with it,
(b) $\alpha_n \in \mathrm{dom}(r_{n+1}) \cap \mathrm{ran}(r_{n+1})$.

This completes the definition of the sequence $(r_n)_n$. Let $q = \bigcup_n r_n$. Clearly, q is a permutation of $M \cap \omega_2$, i.e., $D_q = M \cap \omega_2$. We check that $q \in \mathcal{J}$. Condition (1) is satisfied by elementary of M. To see that condition (2) is satisfied consider some $\alpha, \xi \in D_q$ such that $p_\alpha \subseteq q$. Let k and l be such that $\alpha = \alpha_k$ and $\xi = \alpha_l$. Choose some $n > k, l$. Then $p_\alpha \subseteq r_n$ and $\xi \in \mathrm{dom}(r_n) \cap \mathrm{ran}(r_n)$. By elementary of M there is $\beta \in D_q$ such that $r_n = p_\beta$. Then β witnesses condition (2) for α and ξ. $\dashv$

This completes the proof that $\mathcal{J}$ is a strongly σ-closed ω_1-back and forth set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. $\dashv$

We now turn to the proof of Theorem 1.6. We work in a model of ZFC + CH in which there is a simplified $(\omega_1, 1)$-morass $\mathfrak{M}$ which is stationary and satisfies the $\aleph_2$-antichain condition. Let $\mathcal{F} = \mathcal{F}(\mathfrak{M})$ be the family defined in Definition 2.4. Our plan is to define one structure $\mathcal{C}$ and two distinct elements a and b of $\mathcal{C}$ and let $\mathcal{A} = (\mathcal{C}, a)$ and $\mathcal{B} = (\mathcal{C}, b)$. $\mathcal{C}$ will consist of two parts, one is ω_2 with the usual ordering. Its only role is to ensure certain amount of rigidity of $\mathcal{C}$. The second part of $\mathcal{C}$ consists of layers indexed by countable subsets of ω_2. Given $u \in [\omega_2]^\omega$ let

$$\mathcal{F}_u = \{f \in \mathcal{F} : \mathrm{dom}(f) = u\}.$$

We let $\mathcal{G}_u$ be $[\mathcal{F}_u]^{<\omega}$. Since we wish these structures to be disjoint and $\emptyset$ belongs to all them, we will replace $\emptyset$ in $\mathcal{G}_u$ by another object, which we denote by $\emptyset_u$, such that the $\emptyset_u$ are all distinct. We still denote the modified structure by $\mathcal{G}_u$. Let $\mathcal{G} = \bigcup\{\mathcal{G}_u : u \in [\omega_2]^\omega\}$. For $a \in \mathcal{G}$ we let $u(a)$ be the unique u such that $a \in \mathcal{G}_u$. The base set of $\mathcal{C}$ will be

$$C = \omega_2 \cup \mathcal{G}.$$

We now describe the language of $\mathcal{C}$. First, we will have two binary relation symbols, $\leq$ and E. The interpretation $\leq^{\mathcal{C}}$ of $\leq$ will be the usual ordering on ω_2. The interpretation of E is as follows:

$$(\alpha, a) \in E^{\mathcal{C}} \text{ iff } \alpha < \omega_2, a \in \mathcal{G} \text{ and } \alpha \in u(a).$$

This guarantees that any isomorphism of $\mathcal{C}$ is the identity on ω_2 and maps each $\mathcal{G}_u$ to itself. We now put some structure on the $\mathcal{G}_u$. Note that $(\mathcal{G}_u, \Delta)$ is a Boolean group, where Δ denotes the symmetric difference. We will keep only the affine structure of this group, i.e., we want the automorphisms of $\mathcal{G}_u$ to be precisely the shifts by some member of $\mathcal{G}_u$, i.e., maps of the form:

$$x \mapsto x\Delta a,$$

for some fixed element a of $\mathcal{G}_u$. In order to achieve this, we will add countably many binary relation symbols $R_{n,i}$, for $i = 0, 1$ and $n < \omega$. In each $\mathcal{G}_u$ we will interpret these relation symbols as follows. First, we index the members of $\mathcal{F}_u$ by elements of 2^ω, say $\mathcal{F}_u = \{f^u_x : x \in 2^\omega\}$. If $a, b \in \mathcal{G}_u$ and $a\Delta b$ is a singleton, say $\{f^u_x\}$, for each n and i, we let

$$R^{\mathcal{C}}_{n,i}(a, b) \text{ if and only if } x(n) = i.$$

Otherwise, no relation between a and b holds. Also, if $u \neq v$ then no relation $R^{\mathcal{C}}_{n,i}$ holds between elements of $\mathcal{G}_u$ and $\mathcal{G}_v$. We also need to connect the different layers of our structure. Suppose $u, v \in [\omega_2]^\omega$ and $u \subseteq v$. We define a homomorphism $\pi_{u,v} : \mathcal{G}_v \to \mathcal{G}_u$ as follows. First, for $f \in \mathcal{F}_u$ we let $\pi_{u,v}(\{f\}) = \{f \restriction u\}$. Then we extend $\pi_{u,v}$ to a homomorphism of $\mathcal{G}_v$ to $\mathcal{G}_u$. Note that, in general, $\pi_{u,v}(a)$ may be different from $\{f \restriction u : f \in a\}$, since there may be cancelation, i.e., there could exist $f, f' \in a$ with $f \neq f'$ but $f \restriction u = f' \restriction u$. Now we add a binary relation symbol S and we let:

$$S^{\mathcal{C}}(a,b) \text{ iff } [a, b \in \mathcal{G}, u(a) \subseteq u(b) \text{ and } \pi_{u(a),u(b)}(b) = a].$$

This guarantees the following: if τ is an automorphism of our structure $\mathcal{C}$ then, for each layer u, τ is the shift by some $a_u \in \mathcal{G}_u$ and if $u \subseteq v$ then $\pi_{u,v}(a_v) = a_u$. This completes the definition of the structure $\mathcal{C}$.

Now, we turn to the definition of $\mathcal{A}$ and $\mathcal{B}$. Recall that $\exists$ has a winning strategy, say σ, in the persistency game on $\mathcal{F}$. Consider the play of length ω in which, at stage n, player $\forall$ plays n and player $\exists$ responds by following σ. Let p^* be the resulting position after ω moves and let f^* be the corresponding function. So, $f^* \in \mathcal{F}_\omega$. Now, we introduce a new constant symbol, c. Then we let $\mathcal{A}$ be the expansion of $\mathcal{C}$ obtained by interpreting c as $\emptyset_\omega$ and $\mathcal{B}$ the expansion of $\mathcal{C}$ in which we interpret c as $\{f^*\}$.

Lemma 3.2. $\mathcal{A} \equiv_{\aleph_1,\aleph_1} \mathcal{B}$.

Proof. We describe informally a winning strategy for player $\exists$ in $\mathrm{EF}_{\aleph_1}^{\aleph_1}(\mathcal{A},\mathcal{B})$. Suppose player $\forall$ starts by playing A_0 and B_0, where A_0 is a countable subset of $\mathcal{A}$ and B_0 is a countable subset of $\mathcal{B}$. Since the base sets of $\mathcal{A}$ and $\mathcal{B}$ are the same, we may assume $A_0 = B_0$. Let's call this set C_0. Let $C_0' = C_0 \cap \omega_2$ and $C_0'' = C_0 \cap \mathcal{G}$. Now, let $U_0 = \{u(a) : a \in C_0''\}$. Then, U_0 is a countable collection of countable subsets of ω_2. Let $u_0 = \bigcup U_0$. Then player $\exists$ simulates a play in the persistency game on $\mathcal{F}$ continuing the play p^* in which player $\forall$ enumerates the elements of $u_0 \setminus \omega$ in some order after the ω-th move and $\exists$ uses her winning strategy σ. Let p_0 be the resulting position and f_0 the corresponding function. Then $f_0 \in \mathcal{F}_{u_0}$. Let φ_0 be the function on C_0'' defined by:

$$\varphi_0(a) = a\Delta\{f_0 \restriction u(a)\},$$

and let $\psi_0 = \varphi_0 \cup \mathrm{id}_{C_0'}$. Note that ψ_0 is an involution and $\psi_0(\emptyset_\omega) = \{f^*\}$, since f_0 extends f^*. Thus, we can consider ψ_0 as a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ such that $A_0 \subseteq \mathrm{dom}(\psi_0)$ and $B_0 \subseteq \mathrm{ran}(\psi_0)$. Player $\exists$ then plays ψ_0 as her first move in $\mathrm{EF}_{\aleph_1}^{\aleph_1}(\mathcal{A},\mathcal{B})$.

In general, in the ξ-th move of $\mathrm{EF}_{\aleph_1}^{\aleph_1}(\mathcal{A},\mathcal{B})$ player $\forall$ plays a countable subset A_ξ of $\mathcal{A}$ and a countable subset B_ξ of $\mathcal{B}$. We may assume that $A_\xi = B_\xi$ and we call this set C_ξ. We let $C_\xi' = C_\xi \cap \omega_2$ and $C_\xi'' = C_\xi \cap \mathcal{G}$. We let $U_\xi = \{u(a) : a \in C_\xi''\}$ and

$$u_\xi = \bigcup\{u_\eta : \eta < \xi\} \cup \bigcup U_\xi.$$

Player $\exists$ simulates a play p_ξ in the persistency game on $\mathcal{F}$ which extends the p_η, for $\eta < \xi$, such that after $\bigcup_{\eta<\xi} p_\eta$ player $\forall$ continues by enumerating in some order the elements of $u_\xi \setminus \bigcup_{\eta<\xi} u_\eta$ and player $\exists$ plays by following her strategy σ. Let f_ξ be the function corresponding to p_ξ. Notice that f_ξ extends f_η, for $\eta < \xi$. Now, let φ_ξ be the function defined on C_ξ'' by

$$\varphi_\xi(a) = a\Delta\{f_\xi \restriction u(a)\}.$$

Finally, let

$$\psi_\xi = \bigcup_{\eta<\xi} \psi_\eta \cup \varphi_\xi \cup \mathrm{id}_{C'_\xi}.$$

It is easy to see that ψ_ξ extends ψ_η, for $\eta < \xi$. Since σ is a winning strategy for player $\exists$ in the persistency game on $\mathcal{F}$, player $\exists$ can continue playing like this for ω_1 moves. Therefore, she has a winning strategy in $\mathrm{EF}^{\aleph_1}_{\aleph_1}(\mathcal{A},\mathcal{B})$, as required. $\dashv$

Lemma 3.3. $\mathcal{A} \not\simeq^p_{\aleph_1,\aleph_1} \mathcal{B}$.

Proof. This is similar to the proof of Lemma 2.8. Suppose Ω is a σ-closed family of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ with the back-and-forth property. By Lemma 3.1, we may assume that Ω is strongly σ-closed. Let ψ be a member of Ω. Then, the domain of ψ is a countable subset A_ψ of $\mathcal{A}$ and the range is a countable subset B_ψ of $\mathcal{B}$. Let $A'_\psi = A_\psi \cap \omega_2$ and let $A''_\psi = A_\psi \cap \mathcal{G}$. Since Ω has the back and forth property, it is easy to see that ψ has to be the identity on A'_ψ and preserve the layers of $\mathcal{G}$. Let $U_\psi = \{u(a) : a \in A''_\psi\}$. Since Ω is also strongly σ-closed, the set of $\psi \in \Omega$ such that U_ψ is directed under inclusion is dense in Ω. By replacing Ω by this set we may assume that U_ψ is directed, for all $\psi \in \Omega$. Let $u(\psi) = \bigcup U_\psi$, for $\psi \in \Omega$. For $u \in U_\psi$ let $A_{\psi,u} = A''_\psi \cap \mathcal{G}_u$. It follows that $\psi \restriction A_{\psi,u}$ has to be the shift by some element of $\mathcal{G}_u$, say $a_{\psi,u}$. Moreover, if $u, v \in U_\psi$ and $u \subseteq v$ then $\pi_{u,v}(a_{\psi,v}) = a_{\psi,u}$. Each $a_{\psi,u}$ is finite and since U_ψ is directed under inclusion and ψ can be extended to a function ρ in Ω which is defined on some point of $\mathcal{G}_{u(\psi)}$, it follows that there exists $a_\psi \in \mathcal{G}_{u(\psi)}$ such that $\psi \restriction A_{\psi,u}$ is the shift by $\pi_{u,u(\psi)}(a_\psi)$, for every $u \in U_\psi$. Let n_ψ be the cardinality of a_ψ. Note that $n_\psi > 0$, since $\psi(\emptyset_\omega) = \{f^*\}$, so ψ cannot be the identity on its domain. Moreover, since Ω is σ-closed and $n_\psi \le n_\rho$, for every $\psi, \rho \in \Omega$ such that $\psi \subseteq \rho$, there is $\psi_0 \in \Omega$ and an integer n such that $n_\psi = n$, for all $\psi \in \Omega$ such that $\psi_0 \subseteq \psi$. We can replace Ω by $\{\psi \in \Omega : \psi_0 \subseteq \psi\}$, so without loss of generality we may assume that $n_\psi = n$, for all $\psi \in \Omega$.

Now, we proceed as in the proof of Lemma 2.8. We fix a sufficiently large regular cardinal τ. Since $\mathcal{S}(\mathfrak{M})$ is stationary in $[\omega_2]^\omega$, we can find a countable elementary submodel M of H_τ containing all the relevant objects such that $M \cap \omega_2 \in \mathcal{S}(\mathfrak{M})$. Let $\zeta = \sup(M \cap \omega_2)$ and fix an increasing sequence $\{\zeta_n\}_n$ of ordinals in M which is cofinal in ζ. We now work in M. For each $\delta < \omega_2$, fix $\psi_{\delta,0} \in \Omega$ such that $\delta \in u(\psi_{\delta,0})$. Let us enumerate $a_{\psi_{\delta,0}}$ as, say $\{f^0_{\delta,0}, \dots f^{n-1}_{\delta,0}\}$. We can find $\alpha < \omega_1$ and $X_0 \subseteq \omega_2 \setminus \zeta_0$ of size $\aleph_2$ such that $f^0_{\delta,0}(\delta) = \alpha$, for all $\delta \in X_0$. Since $\mathfrak{M}$ satisfies the $\aleph_2$-antichain condition, by Lemma 2.7 we can fix $\delta(0) \in X_0$ and $X_1 \subseteq X_0 \setminus \zeta_1$ of size $\aleph_2$ such that, for all $\delta \in X_1$, and all $i < n$, any extension of $f^i_{\delta,0}$ to a function in $\mathcal{F}$ which is defined on $\mathrm{dom}(f^i_{\delta(0),0})$ must extend $f^i_{\delta(0),0}$. For each $\delta \in X_1$ fix some $\psi_{\delta,1} \in \Omega$ which extends $\psi_{\delta,0}$ and is defined on $A_{\psi_{\delta(0),0}}$. Then $\psi_{\delta,1}$ must be the identity on $A'_{\psi_{\delta(0),0}}$ and

$$\pi_{u(\psi_{\delta,0}),u(\psi_{\delta,1})}(a_{\psi_{\delta,1}}) = a_{\psi_{\delta,0}}.$$

Since $a_{\psi_{\delta,1}}$ has the same size as $a_{\psi_{\delta,0}}$, we can enumerate it as $\{f^0_{\delta,1}, \dots f^{n-1}_{\delta,1}\}$ such that $f^i_{\delta,1}$ extends $f^i_{\delta,0}$, for all $i < n$. Moreover, $f^i_{\delta,1}$ is defined on $\mathrm{dom}(f^i_{\delta(0),0})$ and so it must extend $f^i_{\delta(0),0}$. In other words, $f^i_{\delta(0),0} \cup f^i_{\delta,0} \subseteq f^i_{\delta,1}$, for all $i < n$. It follows that $\psi_{\delta,1}$ extends $\psi_{\delta(0),0}$, for all $\delta \in X_1$. By Lemma 2.7 again, we can fix $\delta(1) \in X_1$ and $X_2 \subseteq X_1 \setminus \zeta_2$ of size $\aleph_2$ such that, for all $\delta \in X_2$ and all $i < n$, any extension of

$f^i_{\delta,1}$ to a function in $\mathcal{F}$ which is defined on $\mathrm{dom}(f^i_{\delta(1),1})$ must extend $f^i_{\delta(1),1}$. For each $\delta \in X_2$ fix some $\psi_{\delta,2} \in \Omega$ which extends $\psi_{\delta,1}$ and is defined on $A_{\psi_{\delta(1),1}}$. As before, $\psi_{\delta,2}$ must be the identity on $A'_{\psi_{\delta,2}}$ so it must agree with $\psi_{\delta(1),1}$ on $A'_{\psi_{\delta(1),1}}$ Also, we can enumerate $a_{\psi_{\delta,2}}$ as $\{f^0_{\delta,2}, \dots, f^{n-1}_{\delta,2}\}$ such that $f^i_{\delta(1),1} \cup f^i_{\delta,1} \subseteq f^i_{\delta,2}$. We conclude that $\psi_{\delta,2}$ extends $\psi_{\delta(1),1} \cup \psi_{\delta,1}$, for all $\delta \in X_2$. Continuing in this way we get an increasing sequence $(\delta(k))_k$ of ordinals from M, a decreasing sequence $(X_k)_k$ of subsets of ω_2 of size $\aleph_2$, and, for each k and $\delta \in X_k$, $\psi_{\delta,k} \in \Omega$ and an enumeration $\{f^0_{\delta,k}, \dots, f^{n-1}_{\delta,k}\}$ of $a_{\psi_{\delta,k}}$ such that:

(1) $\delta(k) \geq \zeta_k$, for all k,
(2) $\psi_{\delta(k),k} \cup \psi_{\delta,k} \subseteq \psi_{\delta,k+1}$, for all $\delta \in X_{k+1}$,
(3) $f^i_{\delta(k),k} \cup f^i_{\delta,k} \subseteq f^i_{\delta,k+1}$, for all $i < n$ and all $\delta \in X_{k+1}$.

Now, $(\psi_{\delta(k),k})_k$ is an increasing sequence of members of Ω and since Ω is σ-closed there is $\rho \in \Omega$ extending all the $\psi_{\delta(k),k}$. It follows that there is an enumeration $\{f^0, \dots, f^{n-1}\}$ of a_ρ such that $f^i_{\delta(k),k} \subseteq f^i$, for each $i < n$ and $k < \omega$. Recall that $f^0_{\delta,0}(\delta) = \alpha$, for all $\delta \in X_0$. Moreover, $f^i_{\delta,0} \subseteq \ldots \subseteq f^i_{\delta,k}$, for all $i < n$ and $\delta \in X_k$. It follows that $f^0(\delta(k)) = \alpha$, for all k. However, all the $\delta(k)$ belong to $M \cap \omega_2$ and the sequence $(\delta(k))_k$ is cofinal in ζ. Since $M \cap \omega_2$ belongs $\mathcal{S}(\mathfrak{M})$ it follows that this sequence is $\preceq$-unbounded. Therefore, f^0 violates condition (2) of Definition 2.4 and so it cannot belong to $\mathcal{F}$, a contradiction. $\dashv$

This completes the proof of Theorem 1.6.

§4. Open questions.

We mention a couple of questions which remain open.

QUESTION 4.1. *Is it consistent that* $\equiv_{\aleph_1,\aleph_1}$ *and* $\simeq^p_{\aleph_1,\aleph_1}$ *are equivalent for structures of size* $\aleph_2$ *in the context of* CH ?

QUESTION 4.2. *Is it consistent that* $\simeq^p_{\aleph_1,\aleph_1}$ *is not transitive? This would show that* $\simeq^p_{\aleph_1,\aleph_1}$ *is not the right concept, i.e., it does not represent equivalence in some logic.*

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