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# ON THE STRUCTURE OF Ext$(A, \mathbf{Z})$ IN ZFC$^+$

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**Introduction.** A fundamental problem in the theory of abelian groups is to determine the structure of Ext$(A, \mathbf{Z})$ for arbitrary abelian groups A. This problem was raised by L. Fuchs in 1958, and since then has been the center of considerable activity and progress.

We briefly summarize the present state of this problem. It is a well-known fact that

$$\mathrm{Ext}(A, \mathbf{Z}) \cong \mathrm{Ext}(tA, \mathbf{Z}) \oplus \mathrm{Ext}(A/tA, \mathbf{Z}),$$

where $tA$ denotes the torsion subgroup of $A$. Thus the structure problem for Ext$(A, \mathbf{Z})$ breaks down to the two distinct cases, torsion and torsion free groups. For a torsion group $T$,

$$\mathrm{Ext}(T, \mathbf{Z}) \cong \mathrm{Hom}(T, \mathbf{R}/\mathbf{Z}),$$

which is compact and reduced, and its structure is known explicitly [12].

For torsion free $A$, Ext$(A, \mathbf{Z})$ is divisible; hence it has a unique representation

$$\mathrm{Ext}(A, \mathbf{Z}) \cong \oplus Q \times \bigoplus_p (\oplus \mathbf{Z}(p^\infty)).$$

Thus Ext$(A, \mathbf{Z})$ is characterized by countably many cardinal numbers, which we denote as follows: $\nu_0(A)$ is the rank of the torsion free part of Ext$(A, \mathbf{Z})$, and $\nu_p(A)$ are the ranks of the $p$-primary parts of Ext$(A, \mathbf{Z})$, Ext$_p(A, \mathbf{Z})$.

If $A$ is free it is an elementary fact that Ext$(A, \mathbf{Z}) = 0$. The second named author has shown [16] that in the presence of $V = L$ the converse is also true. For countable torsion free, nonfree $A$, C. Jensen [13] has shown that $\nu_p(A)$ is either finite or $2^{\aleph_0}$ and $\nu_p(A) \le \nu_0(A)$. Therefore, the case for uncountable, nonfree, torsion free groups $A$ remains to be studied. Hulanicki [11] has shown that divisible abelian groups which admit a compact topology are characterized by the following conditions:

(i) $\nu_0(A)$ is of the form $2^\lambda$ for some infinite $\lambda$.
(ii) $\nu_p(A) \le \nu_0(A)$ for every prime $p$.
(iii) $\nu_p(A)$ is finite or of the form $2^{\lambda_p}$, $\lambda_p$ infinite.

A recent result of H. Hiller, M. Huber and S. Shelah [8] is that if $A$ is a torsion free abelian group such that Hom$(A, \mathbf{Z}) = 0$, then for every prime $p$, $\nu_p(A)$ is finite or of

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the form $2^{\lambda_p}$, $\lambda_p$ infinite. They have also shown, under the assumption of $V = L$, that for $A$ as above, $\operatorname{Ext}(A, \mathbf{Z})$ admits a compact topology, whence conditions (i), (ii) and (iii) hold. A major result of their paper [8] is the following:

(1) THEOREM ($V = L$). *If $A$ is a nonfree, torsion free abelian group, then conditions* (i) *and* (ii) *above hold.*

The question remains whether also condition (iii) holds. We shall show that this is not the case, moreover that in general there are no further restrictions other than (i) and (ii). More specifically,

(2) THEOREM (CH). *For any function $\chi$ of the primes to cardinals $\mu \le \aleph_1$ or $\mu = 2^{\aleph_1}$ there exists a nonfree torsion free abelian group $A$ of cardinality $\aleph_1$ such that $\nu_0(A) = 2^{\aleph_1}$, and $\nu_p(A) = \chi(p)$ for all primes $p$.*

We had previously conjectured [14] that in $L$, for any successor cardinal $\lambda_0$, and cardinals $\lambda_p \le \lambda_0$ $\forall_p$ prime, there exists a group $A$ such that $\nu_p(A) = \lambda_p$ and $\nu_0(A) = \lambda_0$. However we so far have only been able to establish the following theorems and their corollaries.

(2.1) THEOREM (ZFC + GCH). *For any infinite successor cardinal $\kappa < \aleph_\omega$, cardinal $\lambda \le \alpha^+$, and prime $p$, there exists an almost free abelian group $A$ of cardinality $\kappa$ such that $\nu_0(A) = \kappa^+$, $\nu_p(A) = \lambda$ and $\nu_q(A) = 0$ for all primes $q \ne p$.*

(2.2) THEOREM ($V = L$) *For any infinite regular cardinal $\kappa$ less than the first weakly compact cardinal, there exists an almost free abelian group $A$ of cardinality $\kappa$ such that $\nu_0(A) = \kappa^+$ and $\nu_p(A) = 0$, for all prime numbers $p$.*

(2.3) *Notation.* The term "group" will always mean "abelian group", and "almost free group" will mean a group such that all subgroups of cardinality less than that of the group are free.

In order to extend the above via sums we resort to the following basic fact which is also essential to our method of construction.

(3) DEFINITION. Let $H$: $\operatorname{Hom}(A, \mathbf{Z}) \to \operatorname{Hom}(A, \mathbf{Z}/p\mathbf{Z})$ be the natural homomorphism defined by:

$$[H(h)](x) = h(x)/p\mathbf{Z}, \qquad h \in \operatorname{Hom}(A, \mathbf{Z}), \quad x \in A, \quad p \text{ a prime}.$$

(4) THEOREM. *For abelian torsion free $A$*

$$\operatorname{Ext}_p(A, \mathbf{Z}) \cong \operatorname{Hom}(A, \mathbf{Z}/p\mathbf{Z})/H[\operatorname{Hom} A, \mathbf{Z}].$$

PROOF. The exact sequence

$$0 \to p\mathbf{Z} \xrightarrow{\alpha} \mathbf{Z} \xrightarrow{\beta} \mathbf{Z}/p\mathbf{Z} \to 0,$$

$\alpha$ the identity embedding, $\beta$ natural, induces the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(A, p\mathbf{Z}) &\to \operatorname{Hom}(A, \mathbf{Z}) \to \operatorname{Hom}(A, \mathbf{Z}/p\mathbf{Z}) \\ &\xrightarrow{E_*} \operatorname{Ext}(A, p\mathbf{Z}) \xrightarrow{\alpha_*} \operatorname{Ext}(A, \mathbf{Z}) \xrightarrow{\beta_*} \operatorname{Ext}(A, \mathbf{Z}/p\mathbf{Z}) \to 0 \end{aligned}$$

(see Fuchs [6]). Since the sequence is exact,

$$J = \operatorname{Hom}(A, \mathbf{Z}/p\mathbf{Z})/H[\operatorname{Hom}(A, \mathbf{Z})] \cong \operatorname{Ker}(\alpha_*) = \operatorname{Im}(E_*).$$

$\mathbf{Z}$ and $p\mathbf{Z}$ are isomorphic; hence so are $\operatorname{Ext}(A, \mathbf{Z})$ and $\operatorname{Ext}(A, p\mathbf{Z})$; in particular elements of order $p$ of $\operatorname{Ext}(A, \mathbf{Z})$ are represented by elements of order $p$ in $\operatorname{Ext}(A, p\mathbf{Z})$. All elements of $J$ are of order $p$. Hence it suffices to show that all extensions $E \in \operatorname{Ext}(A, p\mathbf{Z})$ of order $p$ are mapped to 0 by $\alpha_*$. Let $E \in \operatorname{Ext}(A, p\mathbf{Z})$, $pE = 0$, be represented by a factor set $f: A \times A \to p\mathbf{Z}$. Thus, for some function $g: A \to p\mathbf{Z}$ with $g(0) = 0$,

$$pf(x, y) = g(x) + g(y) - g(x + y) \in \operatorname{Trans}(A, \mathbf{Z}) \qquad \forall x, y \in A.$$

Since $\alpha$ is an injection, $\alpha_*(E)$ can be represented by the same $f$. Now since $A$, $\mathbf{Z}$ are torsion free, there is a unique $g': A \to \mathbf{Z}$ such that $pg'(x) = g(x)\ \forall x \in A$. Therefore $f(x, y) = g'(x) + g'(y) - g'(x + y)$; hence also $\alpha^*(E) = 0$. □

Just using the elementary fact that

$$\text{(4.01)} \qquad \operatorname{Ext}_p\left(\bigoplus A_i, \mathbf{Z}\right) = \prod \operatorname{Ext}_p(A_i, \mathbf{Z})$$

and (2.1) we obtain

(4.1) COROLLARY (ZFC + GCH). *For any cardinal $\aleph_0 < \kappa < \aleph_\omega$, and cardinals $\lambda_p \leq \kappa^+$, there exists an almost free group $A$ of cardinality $\kappa$ such that $\nu_0(A) = \kappa^+$ and $\nu_p(A) = \lambda_p$ for all prime numbers $p$.*

Using (4), (4.01), (2.1) and (2.2), we also easily obtain

(4.2) COROLLARY ($V = L$). (i) *For any successor cardinal $\kappa > \aleph_0$ less than the first weakly compact and regular cardinals $\lambda_p \leq \kappa^+$, there exists a group $A$ of cardinality $\kappa$ such that $\nu_0(A) = \kappa^+$ and $\nu_p(A) = \lambda_p$, for all primes $p$.*

(ii) *For any regular cardinal $\kappa > \aleph_0$ less than the first weakly compact and regular cardinals $\lambda_p$, $\lambda_p = \kappa^+$ or $\lambda_p < \kappa$, there exists a group $A$ of cardinality $\kappa$ such that $\nu_0(A) = \kappa^+$ and $\nu_p(A) = \lambda_p$.*

(4.21) Note that such an $A$ will generally *not be almost free.*

Eklof and Huber [5], in their proof of Theorem 2, observed that the only properties required of $\mathbf{Z}$ in treating the $p$-ranks of $\operatorname{Ext}(A, \mathbf{Z})$ are just those of a reduced rational group $G$. Since we do not resort to any further properties of $\mathbf{Z}$, all our constructions can also be modified to yield the parallel results concerning the $p$-ranks of $\operatorname{Ext}(A, G)$ for reduced rational groups $G$.

We do not know if $\nu_p(A)$ can be singular! We also do not know the situation for groups of singular cardinality or the state of affairs for groups of cardinality larger than the first weakly compact. Nor do we know whether there is a group $G$ of inaccessible non-weakly-compact cardinality $\kappa$ for which $\nu_p(G) = \kappa$. This question is particularly interesting since weakly compact cardinals display the following phenomenon.

(4.3) THEOREM (ZFC). If $G$ is a group of weakly compact cardinality $\kappa$ for which $\nu_p(G) \geq \kappa$, then $\nu_p(G) = 2^\kappa$.

For the proof of (4.3) see [15].

In the course of this paper we will expound a method of "$\operatorname{Ext}_p$ forging", i.e. a way of constructing abelian groups $G$ for which the ranks of the $p$-parts of $\operatorname{Ext}(G, \mathbf{Z})$ have prescribed cardinalities. The keystone of this method is the following notion of "*coiling*".

(4.4) DEFINITION. Let $G \subseteq G^*$ be abelian groups.

i) A sequence of elements $y^n \in G^* - G$, $n < \omega$ is said to be *coiled* (*over* $G$) or a coil (over $G$) iff there exist integers $k_n$ and elements $a^n \in G$, $n < \omega$, such that the following *coiling* relations hold in $G^*$:

$$\begin{aligned} k_0 y^1 &= y^0 + a^0, \\ k_1 y^2 &= y^1 + a^1, \\ k_2 y^3 &= y^2 + a^2, \\ &\vdots \\ k_n y^{n+1} &= y^n + a^n, \quad n < \omega. \end{aligned}$$

ii) $G^*$ is said to be *coiled over G* or *obtained from G by coiling*, iff it is generated from $G$ by a union of disjoint coils, i.e., by a set $\bigcup_{i<\lambda}\{y_i^n : n < \omega\}$, where $y_i^n$, $n < \omega$, are coiled over $G$.

We will generally restrict out attention to groups $G^*$ coiled over a free group $G$, with coiling relations of the form $py^{n+1} = y^n + a^n$ for a fixed prime $p$, and various iterations of coiled groups. Clearly the degree of freeness of the resulting groups is related to the *combinatorial structure* induced on a set of generators $X = \langle x_i : i < \mu \rangle$ of $G$, by the sets of supporting elements $\{x_{j_1}^{i,n} \cdots x_{j l_i}^{i,n}\}$ of $a_i^n$, where the coiling relations $k_n y_i^{n+1} = y_i^n + a_i^n$ are assumed to hold. Moreover the success of using coiling relations in Ext forging hinges on the observation that though any homomorphism $h \in \operatorname{Hom}(G, \mathbf{Z}/p\mathbf{Z})$ has an extension to all of the group $G^*$ coiled over $G$, once the extension has been arbitrarily determined on the $y_i^0$, a homomorphism $f \in \operatorname{Hom}(G, \mathbf{Z})$ may or may not extend to all of $G^*$. This extension depends on the degree to which the divisibility requirements determined by the coiling relations can be met. Thus by suitable arrangements of the $a_i^n$ in the coiling relations, we can expect to determine the number of independent elements of $\operatorname{Hom}(G, \mathbf{Z}/p\mathbf{Z})$ over $H[\operatorname{Hom}(G, \mathbf{Z})]$.

The simplest exploitation of this idea can be seen in our construction where we show that

(5) THEOREM (ZFC + $V = L$). *If $\kappa$ is a non-weakly-compact inaccessible cardinal, then there exists an almost free group G of cardinality $\kappa$ such that* $\operatorname{Ext}_p(G, \mathbf{Z}) = 0$ *for all prime numbers p.*

PROOF. Our proof of this result is based on the following principle, which follows from ZF + $V = L$.

(5.1) *Principle* $\diamondsuit_\kappa^+$: $\forall \alpha < \kappa\ \exists P_\alpha \subseteq \mathscr{P}(\alpha) \cup {}^\alpha\alpha$ with the following properties:

1) $|P_\alpha| \le |\alpha|$,

and

2) $\forall h \in {}^\kappa\kappa$ there exists a closed unbounded subset $C$ of $\kappa$, $(\mathrm{club}_\kappa(C))$, such that $\forall \alpha \in C$, $h \restriction \alpha \in P_\alpha$ and $C \cap \alpha \in P_\alpha$.

The following combinatorial principle will also be employed for this case.

(5.2) DEFINITION. A stationary subset $S$ of $\kappa$ is said to be *nonreflecting* iff every initial segment $S \cap \alpha$, $\alpha < \kappa$, is not stationary in $\alpha$.

Jensen has shown that, in $L$,

(5.3) THEOREM. a) *If $\kappa$ is inaccessible and not weakly compact, then there are nonreflecting $S \subseteq \kappa$ such that all $\alpha \in S$ are strong limit cardinals of a given cofinality.*

b) *If* $\mathrm{cf}(\kappa) = \mu < \kappa$, *then there are nonreflecting stationary subsets S of $\kappa$ such that* $\forall \alpha \in S\ \mathrm{cf}(\alpha) = \mu$.

(5.31) Let $S$ be a fixed nonreflecting stationary subset of the inaccessible non-weakly-compact cardinal $\kappa$, consisting of strong limit cardinals of cofinality $\aleph_0$.

(5.32) DEFINITION. Let $\kappa$ be a regular cardinal and $|A| = \kappa$. A *$\kappa$-filtration* of $A$ is a continuous increasing sequence of sets $A_\alpha$, $\alpha < \kappa$, $|A_\alpha| < H$ and $\bigcup_{\alpha<\kappa} = A$. A $\kappa$-filtration of $A$ as a group is a filtration with $A_\alpha$ as subgroups of $A$ and $A_0 = \langle 0 \rangle$.

We first show how to construct an almost free group of cardinality $\kappa$ such that $\mathrm{Ext}_p(G, \mathbf{Z}) = 0$ for a fixed prime $p$.

We define by induction on $\alpha$ a filtration of groups $G_\alpha$, $\alpha \leq \kappa$, with the following properties:

1. The set of elements of $G_\alpha$ is an ordinal $\chi_\alpha < \kappa$.
2. $G_\alpha$ is free.
3. If $\alpha < \beta$ and $\alpha \notin S$, then $G_\beta/G_\alpha$ is free.
4. If $\alpha \in S$ then $G_{\alpha+1}/G_\alpha$ is of rank 1.
5. We precede this property with the following definition:

(5.4) DEFINITION. a) $(h, C) \in P_\alpha \times P_\alpha$ is an *$\alpha$-candidate* iff $\chi_\alpha = \alpha$, $h \in \mathrm{Hom}(G_\alpha, \mathbf{Z}/p\mathbf{Z})$, and $C$ is a closed subset of $\alpha$ such that $\exists f \in \mathrm{Hom}(G_\alpha, \mathbf{Z})$ for which $f/p\mathbf{Z} = h$.

b) An $\alpha$-candidate $(h', C')$ is said to *extend* a $\beta$-candidate $(h, C)$ iff $h' = h \restriction \beta$, $C' = C \cap \beta$ and $\beta < \alpha$.

We will also define in the induction on $\alpha$ choice functions $\Phi_\alpha$: $\{(h, C):(h, C) \text{ is an } \alpha\text{-candidate}\} \to \mathrm{Hom}(G_\alpha, \mathbf{Z})$ such that $\Phi_\alpha(h, C) = f_{h,C}/p\mathbf{Z} = h$.

Finally, property 5 states: if $\chi_\alpha = \alpha$, and the $\alpha$-candidate $(h, C)$ extends the $\beta$-candidate $(h', C')$ and $\beta \in C$, then $f_{h,C}$ extends $f_{h',C'}$.

6. If $\alpha \in S$ and $(h, C) \in P_\alpha \times P_\alpha$ is an $\alpha$-candidate, then $f_{h,C}$ has a unique extension in $\mathrm{Hom}(G_{\alpha+1}, \mathbf{Z})$.

We define by induction on $\alpha$, $\chi_\alpha$, $G_\alpha$, and $\Phi_\alpha$:

i) For $\alpha = 0$, $G_0 = 0$ and $\Phi_0 = \varnothing$.

ii) For $\alpha = \beta + 1$ when $\beta \notin S$, we add one free generator to $G_\beta$ and define $\chi_\alpha$ and $G_\alpha$ accordingly. To define $\Phi_\alpha$, let $(h, C)$ be an $\alpha$-candidate. Since $C$ is closed, it has a maximal element $\mu$. Let $(h', C')$ be the $\mu$-candidate such that $(h', C') < (h, C)$. We consider two cases, $\mu \notin S$ and $\mu \in S$.

(a) $\mu \notin S$. In this case $G_\beta/G_\mu$ is free by the induction hypothesis; and by our construction $G_{\beta+1}/G_\beta$ is free. Hence there is an extension $f \in \mathrm{Hom}(G_\alpha, \mathbf{Z})$ of $f_{h',C'}$ such that $f/p\mathbf{Z} = h$. We set $\Phi_\alpha(h, C) = f$.

(b) $\mu \in S$. In this case we apply 6 of the induction hypothesis to obtain a unique extension $f^*$ of $f_{h',C'}$ in $\mathrm{Hom}(G_{\mu+1}, \mathbf{Z})$. Since $\mu + 1 \notin S$, $G_{\beta+1}/G_{\mu+1}$ is free by the induction hypothesis and our construction as in (a). Thus there is an extension $f$ of $f^*$ in $\mathrm{Hom}(G_\alpha, \mathbf{Z})$. We set $\Phi(h, C) = f$. $\chi_\alpha$, $G_\alpha$ and $\Phi_\alpha$ have clearly been defined so as to satisfy 1–6 of the induction hypothesis.

iii) $\alpha$ is a limit ordinal. Since $S$ is nonreflecting, we can choose an increasing continuous sequence $\delta_j$, $j < \eta$, such that $\sup_{j<\eta}(\delta_j) = \alpha$ and $\delta_j \notin S$. Thus $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ is free by 3 of the induction hypothesis. Also, $\chi_\alpha = \bigcup_{\beta<\alpha} \chi_\beta$, and it remains to choose $\Phi_\alpha$. Let $(h, C)$ be an $\alpha$-candidate, and $(h, C)_\delta$ denote the restriction of $(h, C)$ to $\delta$ (i.e., $(h, C)_\delta = (h \restriction G_\delta, C \cap \delta)$). We distinguish two cases:

(a) If $C$ has no last element, we can set

$$f_{(h,C)} = \Phi(h, C) = \bigcup_{\delta \in C} f_{(h,C)_\delta} \in \mathrm{Hom}(C_\alpha, \mathbf{Z}).$$

This is well defined by 5 of the induction hypothesis.

(b) If $C$ has a last element $\mu$, we proceed exactly as in the successor case ii). In any event it is clear that the induction hypothesis is still satisfied for $\beta \leq \alpha$.

Finally the crucial case:

iv) $\alpha = \beta + 1$ when $\beta \in S$.

$G_{\beta+1}$ is to be generated from $G_\beta$ and the single coil $y_\alpha^n$, $n < \omega$, where the coiling relations are

$$py_\alpha^{n+1} = y_\alpha^n + a_\alpha^n; \qquad a_\alpha^n \in G_\beta, \quad n < \omega.$$

The $a_\alpha^n$ are to be chosen in a manner enabling us to define $\Phi_\alpha$ in the required way. $\chi_\alpha$ is clearly no problem. We choose $a_\alpha^n$ so that for every $\beta$-candidate $(h, C)$, $\exists n_{\alpha,h,C} < \omega$ such that

$$f_{h,C}(a_\alpha^n) = 0, \qquad \forall n > n_{\alpha,h,C}.$$

This is done as follows: we first choose a $\beta$-enumeration of all $f_{(h,C)}, (h, C) \in P_\beta \times P_\beta$ a $\beta$-candidate, say $f^i$, $i < \beta$. Then, since elements of $S$ are strong limit cardinals of cofinality $\omega$, we can choose increasing $\beta_n \notin S$, $n < \omega$, such that $\bigcup_{n<\omega} \beta_n = \beta$ and $2^{\beta_n} < \beta_{n+1}$. If $K_n$ are $\beta_{n+1}$ pure independent elements over $G_{\beta_n}$, then by a cardinality argument we are assured of two elements of $K_n$, say $b_1^n$ and $b_2^n$, such that $f^i(b_1^n) = f^i(b_2^n)$, $\forall i < \beta_n$. We set $a_\alpha^n = b_1^n - b_2^n \neq 0$. The $a_\alpha^n$, $n < \omega$, are clearly as required. Moreover, since $G_\beta/G_{\beta_n}$ and $G_\beta$ are free with say $G_\beta = \langle x_i : i < \lambda \rangle$, $B = \{x_i : i < \lambda\}$ a free basis of $G_\beta$, the $a_\alpha^n$ can be chosen so that $\mathrm{supp}_B\,(a_\alpha^n) \cup \mathrm{supp}_B(a_\alpha^m) = \varnothing$, $n \neq m$. Thus the $a_\alpha^n$, $n < \omega$, are a pure independent set of elements of $G_\beta$.

Now let $(h, C) \in P_{\beta+1}$. We wish to extend $f^* = f_{(h,C)_\beta}$ to $G_{\beta+1}$. Let $m < \omega$ be such that $f^*(a_\alpha^n) = 0\ \forall n \geq m$. If we set $f^*(y_\alpha^n) = 0\ \forall n \geq m$, then also $pf^*(y_\alpha^{n+1}) = f^*(y_\alpha^n) + f^*(a_\alpha^n)\ \forall n \geq m$. Thus it remains to define $f^*$ on $y_\alpha^l$, $l < m$, so as to satisfy $pf^*(y_\alpha^{l+1}) = f^*(y_\alpha^l) + f^*(a_\alpha^l)$, $l < m$. Working downwards we can solve the remaining set of $m$ equations with $m$ unknowns over $\mathbf{Z}$. Thus $f^*$ can be appropriately defined also for $y_\alpha^l$, $l < m$, yielding $f^{**} \in \mathrm{Hom}(G_{\beta+1}, \mathbf{Z})$. We set $\Phi_\alpha(h, C) = f^{**}$. Thus article 5 of the induction hypothesis is satisfied. That $G_{\beta+1}$ is free follows from the hypothesis that $G_\beta$ is free, with, say, basis $\{x_i : i < \lambda\}$. Hence if the $x_{i_j}$, $j < \omega$, are all basis elements involved in $a_\alpha^n$, $n < \omega$, and $G' = \langle y_\alpha^n, x_{i_j} : n < \omega, j < \omega \rangle$, $G'' = \langle \{x_i : i < \lambda, i \neq i_j, j < \omega\} \rangle$, then $G'' \subset G'$ is obviously free and $G'$ is free by Pontryagin's criterion. Moreover $G_{\beta+1} = G' \oplus G''$. That $G_{\beta+1}/G_\delta$ is free for $\delta < \beta$, $\delta \notin S$, follows from our choice of $a_\alpha^n \in G_\beta - G_{\beta_m}$ for $n > m$, again by Pontryagin's criterion. Note however that $G_{\beta+1}/G_\beta \approx \langle \{1/p_l : l < \omega\} \rangle \subset Q$ (the rationals) is not free. The remainder of the induction hypotheses 1–6 are now easily seen to hold.

That $G = G_\kappa$ is not free follows from the fact that for $\alpha \in S$ ($S$ stationary) $G_{\alpha+1}/G_\alpha$ is not free. Hence $G$ is almost free (since every subgroup of lower cardinality is contained in some free $G_\alpha$, $\alpha < \kappa$). Since for every homomorphism $h \in \mathrm{Hom}(G, \mathbf{Z}/p\mathbf{Z})$ there is a closed unbounded set $C \subseteq \kappa$ such that $\forall \alpha \in C, h \restriction \alpha \in P_\alpha$ and $C \cap \alpha \in P_\alpha$, we can now easily piece together $f_{(h,C)_\alpha}$, $\alpha \in C$, $\chi_\alpha = \alpha$, to yield $f \in \mathrm{Hom}(G, \mathbf{Z})$ for which $f/p\mathbf{Z} = h$. $\square$

In order to obtain an almost free group $G$ of cardinality $\kappa$ for which $\mathrm{Ext}_p(G, \mathbf{Z}) = 0$ for all primes $p$, we modify the above construction as follows.

First we distinguish between $(\alpha, q)$-candidates $(h, C) \in P_\alpha \times P_\alpha$ for primes $q$. These are like the $\alpha$-candidates defined above with the further requirement that $h \in \mathrm{Hom}(G_\alpha, \mathbf{Z}/q\mathbf{Z})$. $\Phi_\alpha$ will now have to choose $f_{h,C}$ so that if $(h, C)$ is an $(\alpha, q)$-candidate, then $f_{h,C}/q\mathbf{Z} = h$. The construction is for the most part as above except for the case iv) where $\alpha = \beta + 1$, $\beta \in S$. Let $p_0, p_1, \dots p_n, \dots$ be an enumeration of all the primes. Now $G_{\beta+1}$ is generated from $G_\beta$ and the coil $y_\alpha^n$, $n < \omega$, with the coiling relations

$$\begin{aligned} p_0 y_\alpha^1 &= y_\alpha^0 + a_\alpha^0, \\ p_0 p_1 y_\alpha^2 &= y_\alpha^1 + a_\alpha^1, \\ &\vdots \\ p_0 p_1 \cdots p_n y_\alpha^{n+1} &= y_\alpha^n + a_\alpha^n, \qquad a_\alpha^n \in G_\beta,\ n < \omega. \end{aligned}$$

Using a $\beta$-enumeration, $\beta \in S$, of all $(\beta, p_1)$-candidates $(h, C)$, we can arrange for $f_{h,C}(a_\alpha^n) = 0$ for all $n \geq$ some $n_{\alpha,h,C}$. The remainder of the argument is essentially the same as the above, yielding the required result (see also (6.3) below). □

That $\nu_0(G) = |G|^+ = \kappa^+$ follows from Theorem 1 of Hiller, Huber and Shelah [8], and the easily established fact that for any $\kappa$-generated subgroup $A$ of $G$, $G/A$ is not free. □

We now employ another form of Ext forging to prove Theorem (2). The proof of (4.1) and its corollaries involves an elaborate iterated Ext forging and will be published elsewhere. Theorem (2) was originally obtained by the second-named author in early 1978 (see [8], [17] and [14]). An elegant presentation of that result was given by Eklof and Huber [5] using topological notions going back to some early results of Chase [1]. As mentioned above, our method tends to identify and isolate the combinatorial content of a group theoretic problem. This is not only a matter of taste: it also enables one to readily draw upon a large body of infinite combinatorial knowledge, material which may be naturally associated with a given algebraic subject.

(6) *Notation.* Let $G_1$ be the group which is freely generated by $\{x_n | n < \omega\}$.

(6.1) Definition. For every $a \in G_1$ let $\mathrm{spt}(a)$ be the minimal subset $v$ of $\omega$ for which $a \in \langle x_i : i \in v \rangle \cdot v$ is also called the *support* of $a$.

(6.2) *Notation.* Let $q_0$, $q_1, \dots$ be an enumeration of the primes $p$ for which $\chi(p) = 2^{\aleph_1}$ in increasing order, and let $p_0, p_1, \dots$ be an enumeration of the remaining primes in increasing order. If any of these sequences is finite, we assume that the $p_n$ or $q_n$ are equal to 1 from the appropriate places on. By (4.01) we may assume that $\langle p_n \rangle$ is nonempty without loss of generality.

(6.21) Definition. $A$ and $B$ are said to be *almost disjoint* if $A \cap B$ is finite. $\mathfrak{F}$ is said to be an *almost disjoint family* if $\forall A, B \in \mathfrak{F}$ $A \neq B$ implies that $A \cap B$ is finite. Similarly $A$ is *almost contained* in $B$ if $A - B$ is finite.

(6.3) *Notation and outline of construction.* Let $A_i$, $i < \omega_1$, be a family $\mathfrak{F}$ of almost disjoint subsets of $\omega$, and let $a_i^n$ be such that:

(i) $A_i = \bigcup\{\mathrm{spt}(a_i^n) : n < \omega\}$.

(ii) $\mathrm{spt}(a_i^n) \cap \mathrm{spt}(a_i^m) = \varnothing$, $m \neq n$.

We consider the groups $G_i$, $1 \leq i \leq \omega_1$, which are generated by $\{x_m, y_j^n : m, n < \omega, j < i\}$, freely except for the coiling relations:

$$\begin{aligned} p_0p_1\cdots p_n y_j^{n+1} &= y_j^n - a_j^n, \qquad n<\omega, \quad j<i,\\ &\vdots\\ p_0p_1p_2y_j^3 &= y_j^2 - a_j^2,\\ p_0p_1y_j^2 &= y_j^1 - a_j^1,\\ p_0y_j^1 &= y_j^0 - a_j^0. \end{aligned}$$

Denote $G = G_{\omega_1}$ and $G_0 = \{0\}$. We note the following facts:

(a) $G_i$, $i < \omega_1$, is an $\aleph_1$-*filtration* of $G$; i.e.

(1) $G_0 = \{0\}$;

(2) $G_j < G_i$, $j < i$, and $G_j$ is a pure subgroup of $G_i$;

(3) $\bigcup_{j<i} G_j = G_i$ for lim($i$);

(4) $|G_j| < \aleph_1$.

(b) The $G_i$, $i < \omega_1$, are free.

Because they are countable, and every finite subset is contained in a pure free subgroup of finite rank (as can easily be seen), the $G_i$ are free by Pontryagin's criterion.

(c) $G$ is not free.

Since for all $i < \omega_1$, $G_{i+1}/G_i$ is isomorphic to a nonfree subgroup of the rationals $Q$ generated by $\{1/(p_i)^m \mid i < \omega, m < \omega\}$, hence $G$ is not free.

(d) Every homomorphism $h: G_1 \to \mathbf{Z}/p_m\mathbf{Z}$ has a unique extension $h^{[i]}: G_i \to \mathbf{Z}/p_m\mathbf{Z}\ \forall i \le \omega_1$.

PROOF. For $l \ge m$ we clearly must have $0 = h(y_j^l) - h(a_j^l)$, $j \le i$. For $l < m$ and $j \le i$,

$$p_0p_1\cdots p_l y_j^{l+1} = y_j^l - a_j^l,$$

and the $p_j$, $j \le l < m$, are prime to $p_m$; hence for any value of $h(y_j^m)$ in $\mathbf{Z}/p_m\mathbf{Z}$ we have a unique solution of the $m$ equations

$$p_0p_1\cdots p_l w^{l+1} = w^l - h(a_j^l)$$

in the field $\mathbf{Z}/p_m\mathbf{Z}$. Since the value of $h^{[i]}(b)$, $b \in G_i$, is uniquely determined by the values $h^{[i]}$ on $G_1$ and the $y_j^n$, $h^{[i]}$ is unique. □

(e) Every homomorphism $h: G_i \to \mathbf{Z}/q_m\mathbf{Z}$ has $2^{|i|}$ distinct extensions $h^{[i]} \in \mathrm{Hom}(G_i, \mathbf{Z}/q_m\mathbf{Z})$ if $i$ is infinite, $i \le \aleph_1$.

PROOF. Since $q_m$ is prime to $p_k$, $k < \omega$, we get that for any determination of $h^{[i]}(y_j^0)$ in $\mathbf{Z}/q_m\mathbf{Z}$ we have a unique solution vector satisfying all the equations

$$p_0p_1\cdots p_n q_n w_{n+1} = w_n - h(a_j^n), \qquad n < \omega,$$

over the field $\mathbf{Z}/q_m\mathbf{Z}$. Thus $h^{[i]}(y_j^{m+1}) = w_{m+1}$ will yield a unique extension of $h$ to $G_j$. There are $q_m$ possible choices for $h^{[i]}(y_j^0)$ $1 \le j < i$; hence the claim follows. □

(f) Every $f \in \mathrm{Hom}(G_i, \mathbf{Z})$ has at most one extension to a homomorphism $f^{[i]}$ of $G_i$ to $\mathbf{Z}$.

PROOF. Clearly, to any choice $f^{[i]}(y_j^0) \in \mathbf{Z}$ there is at most one extension to $\{y_j^n : n < \omega\}$ satisfying all the equations. However, because of the $\aleph_0$ divisibility stipulations imposed by the equations over $\mathbf{Z}$, every choice for $f^{[i]}(y_j^0)$ may not suffice already for the $j$th system of equations, for any arbitrary $j$.

(g) If $h_1, h_2 \in \mathrm{Hom}(G_1, \mathbf{Z})$ or $h_1, h_2 \in \mathrm{Hom}(G_1, \mathbf{Z}/p\mathbf{Z})$, and $h_1^{[i]}$ and $h_2^{[i]}$ are defined, then $(h_1 \pm h_2)^{[i]}$ is defined, and $(h_1 \pm h_2)^{[i]} = h_1^{[i]} \pm h_2^{[i]}$.

PROOF. Clearly $h_1 \pm h_2$ and $h_1^{[i]} \pm h_2^{[i]}$ are homomorphisms. $h_1 \pm h_2$ has a unique extension to $G_i$. Since $(h_1^{[i]} \pm h_2^{[i]}) \restriction G_1 = h_1 + h_2$, the result follows from the uniqueness ((d) or (f)).

(h) If $h \in \mathrm{Hom}(G_i, \mathbf{Z}/p\mathbf{Z})$, $i < \omega_1$, then there is an $f \in \mathrm{Hom}(G_i, \mathbf{Z})$ such that $h = f/p\mathbf{Z}$.

PROOF. $G_i$ is free; hence $\mathrm{Ext}(G_i, \mathbf{Z}) = 0$. In particular, by Theorem 4,

$$0 = \mathrm{Ext}_p(G_i, \mathbf{Z}) = \mathrm{Hom}(G_i, \mathbf{Z})/H[\mathrm{Hom}(G_i, \mathbf{Z})].$$

Thus $h \in \mathrm{Hom}(G_i, \mathbf{Z}/p\mathbf{Z}) = H[\mathrm{Hom}(G_i, \mathbf{Z})]$. $\square$

(i) If $f \in \mathrm{Hom}(G_{j^*}, \mathbf{Z})$ is such that $\forall i > j^* \, \exists m_i$ for which $f(a_i^n) = 0 \; \forall n > m_i$, then $\exists f^* \in \mathrm{Hom}(G, \mathbf{Z})$ such that $f^* = f^{[\omega_1]}$.

PROOF. Set $f'(y_i^n) = 0 \; \forall i > j^*, n > m_i$. We then set $f'(y_i^{m_i}) = f(a_i^{m_i})$, and working downwards we set

$$f'(y_i^l) = p_0 p_1 \cdots p_l f'(y_i^{l+1}) + f(a_i^l).$$

Clearly then $f'$ extends to a homomorphism $f^*$ of $G$ into $\mathbf{Z}$. $\square$

As a corollary we get:

(j) If $h = f/p\mathbf{Z} \in \mathrm{Hom}(G_{j^*}, \mathbf{Z}/p\mathbf{Z})$ is such that $\forall i > j^* \, \exists m_i < \omega$ so that $\forall n > m_i \, f(a_i^n) = 0$, then there is an $f^* \in \mathrm{Hom}(G, \mathbf{Z})$ extending $f$ such that $h^{[\omega_1]} = f^*/p\mathbf{Z}$.

PROOF. Trivial with (i); consider the cases $k \geq m_i$ and $k < m_i$.

Our objective is to determine the $a_i^n$ in such a way that, by using properties (d)–(j), we can control the number of $h \in \mathrm{Hom}(G, \mathbf{Z}/p\mathbf{Z})$ which do not lift to homomorphisms of $G$ into $\mathbf{Z}$, i.e. $h \notin H[\mathrm{Hom}(G, \mathbf{Z})]$, for the various $p$.

(7) DEFINITION. In order to facilitate our construction, we will choose besides the almost disjoint sets $A_i$ with properties (i), (ii) also $A_i^*$ which will almost contain all $A_j$, $j < i$, and be almost disjoint from $A_i$. Thus we shall have $A_i^*$ such that $A_j^* - A_i^*$ is finite for $j \leq i$, $A_j \subseteq A_{j+1}^*$, and $A_i^* \cap A_i$ is finite. Note that this implies that $A_i \cap A_j^*$ is finite for $i \geq j$. We define $I_i$, $i \leq \omega_1$, to be the ideal generated by the finite subsets of $\omega$ together with $\{A_j, A_j^* : j < i\}$. For any such $I_i$ it is a trivial matter to find an $A_i^*$ with the above properties. However, $A_i^*$ will be required to satisfy additional conditions stipulated in (10)–II below.

Clearly then, for any such $I = I_{\omega_1}$ and $\chi$, the $\omega_1$-filtration defined above has the properties (a)–(j).

(8) *Notation.* Let

$${}^pJ_i = \{h : h \in \mathrm{Hom}(G_1, \mathbf{Z}/p\mathbf{Z}), \exists B \in I_i, \forall n \in \omega - B, h(x_n) = 0\}$$

and ${}^pJ_{\omega_1} = {}^pJ$. Also ${}^{p_n}J_i = {}^nJ_i$, when no confusion can arise.

(9) LEMMA. $\{h^{[\omega_1]} \mid h \in {}^pJ\} \subseteq H[\mathrm{Hom}(G, \mathbf{Z})]$.

PROOF. Let $h \in {}^pJ$; we wish to show that $h^{[\omega_1]} \in H[\mathrm{Hom}(G, \mathbf{Z})]$. There exist $A_j^*$ and finite $C$ such that $h(x_n) = 0$ for all $n \notin C \cup A_j^*$. Let $B = C \cup A_j^*$. By assumption, $\forall i < j$, $A_i$ is almost contained in $A_j^*$ and, $\forall i \geq j$, $A_i \cap A_j^*$ is finite. There is a function $\varphi : j \to \omega$ for which:

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1) $D_i = \bigcup_{n \geq \varphi(i)} \mathrm{spt}(a_i^n) \subseteq A_j^*$;
2) $D_{i_1} \cap D_{i_2} = \varnothing$, $i_1 \neq i_2 < j$.
Let

$$K_1 = \langle x_n : n \notin B \rangle, \qquad K_2 = \langle \{x_n : n \in B\} \cup \{y_i^n : i < j, n \geq \varphi(i)\}\rangle.$$

Clearly $K_1 \cap K_2 = \{0\}$. We claim that $K_1 + K_2 = G_j$. It suffices to show that also for $n < \varphi(i)$, $y_i^n \in K_1 + K_2$, $i < j$. This is done by a descending induction on $n \leq \varphi(i)$, $i < j$. By assumption $y_i^n \in K_1 + K_2$ for $n = \varphi(i)$. Assume that, for $m \leq n \leq \varphi(i)$, $y_i^n \in K_1 + K_2$; we then show that $y_i^{m+1} \in K_1 + K_2$. We have $p_0 p_1 \cdots p_{m-1} y_i^m + a_i^{m-1} = y_i^{m-1}$; and since the two elements on the left of the equality are in $K_1 + K_2$, so is $y_i^{m-1}$, $i < j$. Now since $\mathrm{Ext}_p(G_j, \mathbf{Z}) = 0$, there is an $f_2 \in \mathrm{Hom}(K_2, \mathbf{Z})$ such that $f_2/p\mathbf{Z} = h^{[j]} \restriction K_2$; and since $h^{[j]} \restriction K_1 = 0$, $f_1 = 0 \in \mathrm{Hom}(K_1, \mathbf{Z})$ satisfies $f_1/p\mathbf{Z} = h^{[j]} \restriction K_1$. Thus if $f = f_1 \oplus f_2$, $f/p\mathbf{Z} = h^{[j]}$. We can now apply (6.3)–(j) and the fact that $A_i \cap A_j^*$ is finite to obtain $f^* \in \mathrm{Hom}(G, \mathbf{Z})$ extending $f$ for which $f^*/p\mathbf{Z} = h^{[\omega_1]}$. $\square$

(10) Definition. Let $p_n$, $n < \omega$, be as in (6.2) and let $\delta$ be an enumeration of the limit ordinals. For any $i = i^* + n < \omega_1$, $0 \leq i^*$ limit, we will define

(i) $a_i^m$, $m < \omega$,
(ii) ${}^n h_i \in \mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z})$, and
(iii) for $\chi(p_n) = k_n \leq \delta^{-1}(i^*)$, ${}^n f_i \in \mathrm{Hom}(G_1, \mathbf{Z})$

such that the following conditions hold:

I. (1) ${}^n h_i(a) = {}^n f_i(a)/p_n\mathbf{Z}$, $k_n \geq \delta^{-1}(i^*)$.
(2) ${}^n f_i^{[j]}$ is defined for $k_n \leq \delta^{-1}(i^*)$, $i < j$.
(3) For every $h \in \mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z})$ there exists an $i$ for which

$$h \in \langle \{{}^n h_j \,|\, j < i\} \cup {}^n J_i \rangle < \mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z}).$$

We denote

$$H_i = \langle \{{}^n h_j : j < i, \exists k(j = \omega k + n)\}\rangle.$$

(4) If $g \in H_{\delta(k_n)}$ and $f \colon G_1 \to \mathbf{Z}$ are such that $(\forall a \in G_1)[g(a) = f(a)/p_n\mathbf{Z}]$, then for some $i$, $f^{[i]}$ is not defined; i.e. elements of $H_{k_n}$ do not lift to homomorphisms of $G \to \mathbf{Z}$.

Moreover we will require that the ${}^n h_i$ and ${}^n f_i$ be independent in the following strong sense, which, for future purposes, we precede with a general definition.

(10.1) Definition. i) If $G_1$ is freely generated by $X = \{x_i : i < \kappa\}$ and $I$ is an ideal in $\mathscr{P}(X)$ and

$$J_I = \{h \in \mathrm{Hom}(G_1, \mathbf{Z}/p\mathbf{Z}) : \exists B \in I, \forall x \in X - B, h(x) = 0\},$$

then $J_I$ is said to be an *I-nil* subgroup of $\mathrm{Hom}(G_1, \mathbf{Z}/p\mathbf{Z})$. Elements and subgroups of $J_I$ are also said to be *I-nil*.

ii) A subset $K \subseteq \bigcup_p \mathrm{Hom}(G_1, \mathbf{Z}/p\mathbf{Z}) \cup \mathrm{Hom}(G_1, \mathbf{Z})$ is said to be *I-independent* iff $\forall h \in K$ and finite $M \subseteq K - \{h\}$, and $B \in I$, $\exists a \in G_1$ with $\mathrm{spt}(a) \cap B = \varnothing$ and $\{a\}$ generates a pure subgroup of $G_1$ such that $h(a) = 1$ and $h'(a) = 0$, $\forall h' \in M$.

We now complete Definition (10).

II. (0) $\{h_j, f_j : j < i\}$ is $I_i$-independent. Thus
(1) $\{{}^n h_j : j < i, n < \omega\}$ is $I_i$-independent, and
(2) $\{{}^n f_j : n < \omega, j < i\}$ is $I_i$-independent.

Thus if $i_0 = i_0^* + n_0, \ldots, i_m = i_m^* + n_m$, $k_0 \leq \delta^{-1}(i_0^*), \ldots, k_m \leq \delta^{-1}(i_m^*)$, $i_0, \ldots, i_m < i$, and $B \in I$, then there is an $a \in G_1$ with $\mathrm{spt}(a) \cap B = \emptyset$, and $(a)$ generates a pure subgroup of $G_1$ such that ${}^{n_0}f_{i_0}(a) = 1, {}^{n_1}f_{i_1}(a) = 0, \ldots, {}^{n_m}f_{i_m}(a) = 0$.

(10.2) Also note that if ${}^{n_1}f_{i_1}/p_{n_1}\mathbf{Z} = h_{i_1}, \ldots, {}^{n_m}f_{i_m}/p_{n_m}\mathbf{Z} = h_{i_m}$, and $h_{i_0}(a) = 1$, $f_{i_1}(a) = f_i(a) = \cdots = f_{i_m}(a) = 0 = f_{i'_1}(a) = \cdots = f_{i'_m}(a)$, then $h_{i_0}(a) = 1, h_{i_1}(a) = \cdots = h_{i_m}(a) = 0 = f_{i'_1}(a) = \cdots = f_{i'_m}(a)$.

III. ${}^mf_l/p_n\mathbf{Z} \notin \langle\{{}^nh_j : j < i\} \cup {}^nJ_{i+1}\rangle, m \neq n, l < i$, and

$$ {}^nh_{j'} \notin \langle {}^nh_j : j < i, j \neq j'\} \cup \{{}^mf_l/p_n\mathbf{Z} : l < i, m \neq n\} \cup {}^nJ_{i+1}\rangle. $$

We first note that:

(11) Lemma. *If $G = \bigcup_{i<\omega_1} G_i$ satisfies* I, *then $\nu(p) = \chi(p)$ for all primes $p$.*

Proof. Using Theorem 4, this is immediate.

We will construct the $a_i^m$, ${}^nh_i$ and ${}^nf_i$ by induction on $i$. Assuming them defined for $j < i$, we will first define the $a_i^m$ so as to take care of I–(4) for some $f: G \to \mathbf{Z}$, and I–(2). Then ${}^nh_i$ will be chosen to satisfy I–(3), from which we also will get II–(1); and finally we will choose ${}^nf_i$ so as to satisfy both I–(1) and II–(2). If $i = i^* + n$ and $i^*$ is the $k$th limit ordinal with $k < k_n$, then ${}^nf_i$ will not be defined.

Let $\eta$ be a map of all limits ordinals $< \omega_1$ onto $\mathrm{Hom}(G_1, \mathbf{Z})$, where each $f: G_1 \to \mathbf{Z}$ is the image of $\aleph_1$ ordinals.

Let $\sigma$ be an $\omega$-enumeration of $i = \{\sigma(n) : n < \omega\}$ if $i \geq \omega$, and a finite enumeration of $i$ if $i < \omega$.

Let $\psi$ be an $\omega$-enumeration of $\mathbf{Z} = \{\psi(n) : n < \omega\}$.

First, $A_i^*$ is chosen so as to satisfy the requirements of Definition (7) and so that the ideal generated by $I_i \cup \{A_i^*)$ satisfies (10)–II. Since $I_i$, $G_1$ and the number of functions to be considered are countable, this can easily be done.

Next we define, by induction on $m < \omega$, the $a_i^m \in G$ and auxiliary sets $C_m \subseteq \omega$ satisfying the following five stipulations:

$$ (*)_0 \qquad \mathrm{spt}(a_i^m) \text{ is disjoint to } B_m = \bigcup_{l<m} \mathrm{spt}(a_i^l) \cup A_i^* \cup \bigcup_{l<m} C_l. $$

This will already insure that $A_i$ and $I_{i+1}$ are as required in Definition (7). Note also that $B_m \in I_{i+1}$.

$(*)_1$ For every instance of II there are infinitely many $a \in G$, with disjoint supports, and $\mathrm{spt}(a) \cap B = \emptyset$ and a pure subgroup of $G$. We can choose that $C_m$ so as to insure that $A_i$ is almost disjoint to the union of supports for any of the family of $a$'s.

This insures that II holds.

$(*)_2$ If $\sigma(l) = j^* + l'$, $\lim(j^*)$, $l' < \omega$, and $\delta^{-1}(j^*) \geq \chi(p_{l'})$, then ${}^{l'}f_{\sigma(l)}(a_i^k) = 0$ for all $k > l$.

By (i) above this clearly insures that I–(2) will hold, i.e. ${}^{l'}f_{\sigma(l)}^{[j]}$ exists for all $j$, $\sigma(l) < j \leq \omega_1$.

Assume $i = i^* + n$, and $\eta(i^*) = f: G_1 \to \mathbf{Z}$, and that $f/p_n\mathbf{Z} = {}^nh_j$ for some $j < i$, $\delta^{-1}(j^*) < \chi(p_n)$. Thus by (10)–III, $f \neq {}^mf_l, m \neq n$. We want to kill off any possibility for $f$ to have an extension of $f^{[i+1]}: G_{i+1} + \mathbf{Z}$. If $f^{[i]}$ does not exist, we are done. Otherwise denote by $G_i^k$, the group generated by $G_i$ and $\{y_i^l: l < k\}$, and by $f^{[i,k]}$ the corresponding unique extension of $f$ in $\mathrm{Hom}(G_i^k, \mathbf{Z})$, if it exists. Each such possible extension of $f$ is uniquely determined by its value on $y_i^0$. If $f^{[i,m]}$ is not defined for $f^{[i,m]}(y_i^0) = \psi(m)$, it suffices to choose any $a \in G_1$, $\mathrm{spt}(a) \subseteq \omega - B_n$, satisfying $(*)_0$–$(*)_2$. If $f^{[i,m]}$ is defined for $f^{[i,m]}(y_i^0) = \psi(m)$, we arrange for it not to extend to $G_i^{m+1}$ by choosing $a_i^m$ so that

$$(*)_3 \qquad f^{[i,m]}(y_i^m - a_i^m) \text{ is not divisible by } r_0 \cdots r_m.$$

(Here $r_i$ denotes the fixed enumeration of the primes $p$ for which $\chi(p) < 2^{\aleph_1}$, as in (6)–2.)

It suffices to show that $a_i^m$ is not divisible by $p_n$, which may be assumed amongst the $r_0, \ldots, r_m$ (otherwise we adjust and worry about the case when $f'$ extends $f$ and $f'(y_i^0) = \psi(m)$ for larger $m$). However by Lemma (13) below, which by $(*)_1$ can be assumed for $i' < i$ and the ideal $I_i^*$ generated by $I_i \cup \{A_i^*\}$, we can find $a \in G$, $\mathrm{spt}(a) \subseteq \omega - B_m$, such that

$$(*)_4 \qquad {}^{l'}f_{\sigma(l)}(a) = 0 \quad \text{for } l < m, \quad \text{and} \quad \delta^{-1}(j^*) \geq \chi(p_{l'}),$$

where $\sigma(l) = j^* + l'$, and ${}^n h_j(a) = 1$; thus also $f(a) \neq 0 \bmod (p_n)$. Therefore, $f^{[i,m]}(y_i^m - a) - f^{[i,m]}(y_i^m - 2 \cdot a) = f^{[i,m]}(a) \not\equiv 0 \bmod (p_n)$. Thus at least one of $a$, $2a$ *can be chosen for* $a_i^m$ so that $(*)_3$ holds. Thus we can define $a_i^m$, $m < \omega$, so that $(*)_0 - (*)_4$ all hold.

(12) Next we define ${}^n h_i \in \mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z})$ so as to eventually satisfy requirements I–(3), II and III. Now $\mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z})$ is an abelian group, in which every element is of order $p$, hence a vector space over $\mathbf{Z}/p\mathbf{Z}$. Since $G_1$ is free, the cardinality of $\mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z})$ is $2^{\aleph_0} = \aleph_1$; thus there are $\aleph_1$ homomorphisms $h: G_1 \to \mathbf{Z}/p_n\mathbf{Z}$ which are

(i) not in $\langle \{{}^n h_j: j < i\} \cup {}^n J_{i+1} \cup \{{}^m f_l/p_n\mathbf{Z}: m \neq n, l < i\}\rangle$, and

(ii) such that ${}^m f_l/p_n\mathbf{Z} \notin \langle \{{}^n h_j: j < i\} \cup \{h\} \cup {}^n J_{i+1}\rangle$, $m \neq n$, $l < i$.

Choose ${}^n h_i$ to be the first such one according to some fixed $\omega_1$-enumeration of $\mathrm{Hom}(G_1, \mathbf{Z}/p_n\mathbf{Z})$. We have

(12.1) LEMMA. *$H_{i+1}$ satisfies* II–(1).

(13) LEMMA. *For every $i_0, \ldots, i_m \leq i$ and $B \in I_{i+1}$ such that $\delta^{-1}(i_j^*) \geq k_{n_j}$, there is an $a \in G_1$, generating a pure subgroup of $G_1$ and* $\mathrm{spt}(a) \cap B = \varnothing$, *such that*:

$$(*) \qquad {}^n h_{i_0}(a) = 1, \qquad {}^{n_1}f_{i_1}(a) = 0, \ldots, {}^{n_m}f_{i_m}(a) = 0.$$

PROOF. If $i_0 < i$, then also $i_1, \ldots, i_m < i$ and the result follows from the induction hypothesis and the construction of $I_{i+1}$. Thus assume $i_0 = i$. First assume $n = n_0 = \cdots = n_m$. By II, the ${}^n f_{i_j}$ map onto $\mathbf{Z}$, and if for all $a$ for which ${}^n f_{i_j}(a) = 0$, with $\mathrm{spt}(a) \subseteq \omega - B$, also ${}^n h_1(a) = 0$, we would get that ${}^n h_i \in \langle \{{}^n h_{i_0}, \ldots, {}^n h_{i_m}\} \cup {}^n J_{i+1}\rangle$, contrary to our choice of $h$. Thus for some $a$ with $\mathrm{spt}(a) \subseteq \omega - B$ we have ${}^n f_{i_j}(a) = 0$, $0 \leq j \leq m$, and ${}^n h_i(a) \neq 0$. A proper multiple of $a$ will now do. Since there are infinitely many such $a$, we can choose $a$'s which generate pure subgroups of $G_1$. If $n$, $n_0, \ldots, n_m$ are not all equal, say $n = n_0 = n_1 = \cdots = n_k$ and then $n_j$ with $k < j \leq m$ are different than $n$, then by the above there are infinitely many $a$'s for which ${}^{n_0}f_{i_0}(a) = \cdots = {}^{n_k}f_{i_k}(a) = 0$ and ${}^n h_i(a) = 1$. Now since the $p_{n_j}$, $k < j \leq m$, are prime to

$p_n$, if for all the $a$'s such that ${}^{n_j}f_{i_j}(a) = 0, 0 < j \le m$, we also have $h(a) = 0$, then

$$h \in \langle \{{}^{n_j}f_{i_j}/p_{n_j}\mathbf{Z} : 0 < j \le m\} \cup J_{i+1} \rangle,$$

contrary to (12)–(i) in our choice of $h$. (Note that if $j \le k$, then $f_{i_j}/p_{n_j}\mathbf{Z} = h_{i'}$ for some $i' < i$.) Moreover, we have infinitely many such $a$'s which also generate pure subgroups of $G_1$. $\square$

By Remark (12.1) we easily see that

(13.1) COROLLARY. *For every $i_0, \ldots, i_m \le i$, $j_1, \ldots, j'_m \le i$ and $B \in I_{i+1}$ such that $\delta^{-1}(j_l^*) \ge k_{n_l}$, there is an $a \in G_1$ generating a pure subgroup of $G_1$ and* $\operatorname{spt}(a) \cap B = \varnothing$ *such that*

$${}^{n}h_{i_0}(a) = 1, \qquad {}^{n_1}h_{i_1}(a) = \cdots = {}^{n_m}h_{i_m}(a) = 0 = {}^{n'_1}f_{j_1}(a) = \cdots = {}^{n'_{m'}}f_{j_{m'}}(a).$$

It remains only to choose ${}^{n}f_i$ so that ${}^{n}f_i(a)/p_n\mathbf{Z} = {}^{n}h_i(a)\ \forall a \in G_1$ and II holds.

From Lemmas (12.1) and (13) above we can find $B \in I_{i+1}$ and $a_m \in G_1$, $m < \omega$, such that:

(1) $A_i^* \subseteq B$;

(2) $\operatorname{spt}(a_m)$ are pairwise disjoint and disjoint to $B$;

(3) $a_m$ generates a pure subgroup of $G_1$; and

(4) for every respective instance of II and $l$, we can find infinitely many $a$'s from the $a_m$'s such that ${}^{n}h_i(a_m) = l/p_n\mathbf{Z}$; without loss of generality we can assume that $B \cup \bigcup_m \operatorname{spt}(a_m) = \omega$.

Thus $G_1$ is the direct sum of $G^m = \langle \{x_l : l \in \operatorname{spt}(a_m)\} \rangle$, $m < \omega$, and

$$G' = \langle \{x_l : l \in B\} \cup \{y_j^k : j < i, k < \omega\} \rangle.$$

Therefore (since $\operatorname{Ext}(G_i, \mathbf{Z})$ is isomorphic to the direct sum of $\operatorname{Ext}(G', \mathbf{Z})$ and $\operatorname{Ext}(\sum_n G^n, \mathbf{Z})$), it suffices to choose functions $f' : G' \to \mathbf{Z}$ and $f^m : G^m \to \mathbf{Z}$ so that $f'(b)/p_n\mathbf{Z} = {}^{n}h_i^{[i]}(b)$ and $f^m(b)/p_n\mathbf{Z} = {}^{n}h_i^{[i]}(b)$, for appropriate $b$. The $G_i$, hence also $G'$, are free; thus such an $f'$ exists. As for $f^m$, we can choose $f^m(a_m)$ arbitrarily from the coset of ${}^{n}h_i(a_m)$. Now $a_m$ generates a pure subgroup of $G_1$ by assumption (3) above; hence if $f_i^*$ is the unique homomorphism from $G_{i+1}$ to $\mathbf{Z}$ extending $f' \cup \bigcup_m f^m$ and $f_i = f_i^* \restriction G_1$, then $f_i = {}^{n}f_i$ is as required. Moreover, since we only have to contend with at most countably many ${}^{l}h_j$, $j < i$, $l \ne n$, we can choose each $f^m(a_m)$ so as to also ensure that ${}^{n}f_i/p_l\mathbf{Z} \ne {}^{l}h_j$, $j < i$, $l \ne n$. $\square$

To see that $\nu_0(G) = 2^{\aleph_1}$, note that for any $\aleph_1$-generated subgroup $A$ of $G$, $G/A$ is not free. Thus by the proof of Theorem 1 in Hiller, Huber and Shelah [8], under the assumption $2^{\aleph_0} < 2^{\aleph_1}$ we get $\nu_0(G) = |G|^+ = 2^{\aleph_1}$. $\square$

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