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COMBINATORIAL PROBLEMS ON TREES: PARTITIONS, Δ-SYSTEMS AND LARGE FREE SUBTREES

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We prove partition theorems on trees and generalize to a setting of trees the theorems of Erdös and Rado on Δ-systems and the theorems of Fodor and Hajnal on free sets. Let μ be an infinite cardinal and T_μ be the tree of finite sequences of ordinals $<\mu$, with the partial ordering of being an initial segment. $\boldsymbol{\alpha} \trianglelefteq \boldsymbol{\beta}$ denotes that $\boldsymbol{\alpha}$ is an initial segment of $\boldsymbol{\beta}$. A subtree of T_μ is a nonempty subset of T_μ closed under initial segments. $T \leqslant T_\mu$ means that T is a subtree of T_μ and $\langle T, \trianglelefteq \rangle \cong T_\mu$. The following are extracts from Section 2, 3 and 4.

Theorem 1 (Shelah). A partition theorem. *Suppose* $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$, $F : T_\mu \to \lambda$, *and for every branch* b *of* T_μ $\mathrm{Sup}(\{F(\boldsymbol{\alpha}) \mid \alpha \in b\}) < \lambda$, *then there is* $T \leqslant T_\mu$ *such that* $\mathrm{Sup}(\{F(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in T\}) < \lambda$.

Theorem 2 (Rubin). A theorem on large free subtrees. *Let* $\lambda^+ \leqslant \mu$, $F : T_\mu \to P(T_\mu)$, *for every branch* b *of* $T_\mu : |\bigcup \{F(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in b\}| < \lambda$, *and for every* $\boldsymbol{\alpha} \in T_\mu$ *and* $\boldsymbol{\beta} \in F(\boldsymbol{\alpha})$, $\boldsymbol{\beta} \ntrianglelefteq \boldsymbol{\alpha}$; *then there is* $T \leqslant T_\mu$ *such that for every* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T : \boldsymbol{\beta} \notin F(\boldsymbol{\alpha})$.

Let $P_\lambda(C)$ denote the ideal in $P(C)$ of all subsets of C whose power is less than λ. Let $\mathrm{cov}(\mu, \lambda)$ mean that μ is regular, $\lambda < \mu$, and for every $\kappa < \mu$ there is $D \subseteq P_\lambda(\kappa)$ such that $|D| < \mu$, and D generates the ideal $P_\lambda(\kappa)$ of $P(\kappa)$. Note that if for every $\kappa < \mu$ $\kappa^{<\lambda} < \mathrm{cf}(\mu) = \mu$, then $\mathrm{cov}(\mu, \lambda)$ holds. Let $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ denote the maximal common initial segment of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Theorem 3 (Shelah). A theorem on Δ-systems. *Suppose* $\mathrm{Cov}(\mu, \lambda)$ *holds*, $F : T_\mu \to P(C)$ *and for every branch* b *of* T_μ: $|\bigcup \{F(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in b\}| < \lambda$, *then there is* $T \leqslant T_\mu$ *and a function* $K : T \to P_\lambda(C)$ *such that for every incomparable* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T : F(\boldsymbol{\alpha}) \cap F(\boldsymbol{\beta}) \subseteq K(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})$.

In 4.12, 4.13, we almost get that $K(\boldsymbol{\alpha}) \cap K(\boldsymbol{\beta}) = K(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$.

1. Introduction

In [9], [10], [11] and [8], we came across combinatorial problems on trees, similar to those described in the abstract. These combinatorial facts were needed in proving results in set theory and model theory, While considering these problems we had the feeling that theorems on trees which concern with

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partitions, Δ-systems, and free subtrees might be applicable in various contexts of set theory and model theory in the same way that their classical counterparts have been.

In addition we realized that these problems are interrelated, there is a unified setting in which they can be discussed, and this setting gives rise to numerous other combinatorial problems. Moreover, it seems that some new ideas are required in order to prove these new theorems — ideas that were not needed in proving their classical special cases.

We start with some terminology. A set of sequences in *hereditary* if it is closed under initial segments. A *tree* is a nonempty hereditary set of sequences partially ordered by $\leqslant$. A tree of finite sequences is classed an *ω-tree,* trees which are not ω-trees are called *high trees.* If T is a tree and $\boldsymbol{\alpha} \in T$, let $\mu_{\boldsymbol{\alpha}}^T$ denote the number of successors of $\boldsymbol{\alpha}$ in T.

A non-empty hereditary subset of a tree T is called a *subtree* of T.

A subtree T' of a tree T is called a *T-large subtree* of T ($T' \leqslant T$) if: (a) for every $\boldsymbol{\alpha} \in T'$: $\mu_{\boldsymbol{\alpha}}^{T'} = \mu_{\boldsymbol{\alpha}}^T$; and (b) if $b \subseteq T'$ is a chain (i.e. for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in b$ either $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ or $\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}$), and $\bigcup \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in b\} \in T$, then $\bigcup \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in b\} \in T'$.

Note that for ω-tree (b) is not needed. Since we do not have much to say about high trees, by a 'tree', we usually mean an ω-tree.

Let Λ denote the empty sequence. In order to obtain classical theorems as special cases of the results presented here, we interpret an infinite set A be the tree $T_a \overset{\text{def}}{=} \{\Lambda\} \cup \{\langle a \rangle \mid a \in A\}$. In this interpretation a subset B of A having the same cardinality as A is represented by a T_A-large subtree of T_A.

Analogues of Theorems 1, 2, 3 from the abstract for high trees, seem to require the assumption that the $\mu_{\boldsymbol{\alpha}}$'s should be either large cardinals or large cardinals in some inner model; but we do not know how to prove that this is really the case. Our knowledge about high trees is summarized in the discussion at the end of Section 2.

In Section 2 we deal mostly with partition theorems, these theorems are later used in Sections 3 and 4. The partition theorems reveal why new ideas are needed when dealing with trees. Let us explain this by an example. Consider Theorem 2.2, or better consider Theorem 1 in the abstract, that already presents all the difficulties that occur in 2.2, if we translate it to a classical problem, i.e. we apply it just to trees of the form T_A, then we obtain the following trivial fact. Let μ and λ be infinite cardinals such that $\mathrm{cf}(\mu) \neq \mathrm{cf}(\lambda)$, then if $F: \mu \rightarrow \lambda$, then there is $A \subseteq \mu$ such that $|A| = \mu$ and $\mathrm{Sup}(\{F(\alpha) \mid \alpha \in A\}) < \lambda$.

The same phenomenon happens with the other theorems and lemmas in Section 2, namely, their classical special case is trivial.

The other main results in Section 2 are Theorem 2.3 and Lemmas 2.15 and 2.16. Theorem 2.3 concerns itself with the following question. Let $M(T, \lambda, \chi)$ mean that for every $F: T \rightarrow P_\lambda(\chi)$ there is $T' \leqslant T$ and $\nu \in \lambda$ such that $\nu \notin \bigcup \{F(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in T'\}$. In Theorem 2.3 we give necessary and sufficient conditions on T, λ, χ to assure that $M(T, \lambda, \chi)$ holds.

Definition. Let $G:T\to P(\chi)$, $b\subseteq\chi$ is called a G-valve if for every $c\in P_\chi(\chi-b)$ there is $T'\leqslant T$ such that $c\cap\bigcup\{G(\boldsymbol{\alpha})\mid\boldsymbol{\alpha}\in T'\}=\emptyset$.

Lemmas 2.15 and 2.16 are elaborations of the following claim. If $M(T,\lambda,\chi)$ holds and $G:T\to P_\lambda(\chi)$, then there is a G-valve $b\subseteq\chi$ such that $|b|<\chi$.

Section 3 is devoted to the proof of one theorem, namely, Theorem 3.1. Theorem 3.1 as well as its special case, Theorem 2 in the abstract, generalize the well known theorem on free sets. (See [2] for previous history.)

Theorem (Fodor [2] for μ regular, and Hajnal [3] for μ singular). *Let μ and λ be infinite cardinals and $\mu\geqslant\lambda^+$. Then if $F:\mu\to P_\lambda(\mu)$ is such that, for every $\alpha\in\mu$, $\alpha\notin F(a)$, then there is $A\subseteq\mu$ such that $|A|=\mu$, and, for every $\alpha,\beta\in A$, $\alpha\notin F(\beta)$.*

In Theorem 3.1 we give a condition $Q_1(T,\lambda)$ which is necessary and sufficient to assume that: for every $F:T\to P_\lambda(T)$ such that for every $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}\lessdot\boldsymbol{\alpha}$ $\boldsymbol{\beta}\notin F(\boldsymbol{\alpha})$, there is $T'\leqslant T$ which is F-free.

$Q_1(t,\lambda)$ involves only conditions on the ordering relation between the $\mu_{\boldsymbol{\alpha}}$'s, the $\mathrm{cf}(\mu_{\boldsymbol{\alpha}})$'s and λ.

Section 3 is concerned with Δ-systems. The classical theorem on Δ-systems is due to Erdös and Rado [1]. A function $F:I\to P(C)$ is a Δ-system if there is a set K such that, for every distinct $i,j\in I$, $F(i)\cap F(j)=K$.

Let $\Phi(\mu,\lambda)\equiv\mu$ is regular $\wedge$ $(\forall\chi<\mu)(\chi^{<\lambda}<\mu)$.

Theorem 1.1 (Erdös, Rado [1]). *Suppose $\Phi(\mu,\lambda)$ holds, then if $F:\mu\to P_\lambda(C)$ then there is $I\subseteq\mu$ such that $|I|=\mu$ and $F\restriction I$ is a Δ-system.*

Our policy is to postpone cardinal arithmetical assumptions as much as possible to the end of the proofs.

With this approach in mind we break 1.1 into two subclaims 1.2 and 1.3. The conclusion of 1.2 assures a weak form of a Δ-system, but the assumption in 1.2 is considerably weaker than $\Phi(\mu,\lambda)$, namely, we assume just $\mathrm{cov}(\mu,\lambda)$. In 1.3 we add the assumption that $2^{<\lambda}<\mu$ and obtain a Δ-system.

Theorem 1.2. On weak Δ-systems. *Suppose* $\mathrm{cov}(\mu,\lambda)$ *holds, then if $F:\mu\to P_\lambda(C)$, then there is $I\subseteq\mu$ and a set $K\in P_\lambda(C)$ such that I has power μ and, for every distinct $i,j\in I$, $F(i)\cap F(j)\subseteq K$.*

$F\restriction I$ is called a weak λ-Δ-system.

Proposition 1.3. *If* $\mathrm{cov}(\mu,\lambda)$ *holds and* $2^{<\lambda}<\mu$, *then in* 1.2 *one can strengthen the conclusion by finding a $K\in P_\lambda(C)$ such that, for every distinct $i,j\in I$, $F(i)\cap F(j)=K$.*

Usually it happens that whenever one applies Theorem 1.1, one can use 1.2 instead.

Certainly $\Phi(\mu, \lambda) \Rightarrow \mathrm{cov}(\mu, \lambda) \wedge 2^{<\lambda} < \mu$, so 1.3 is stronger than 1.1. In the discussion at the end of Section 4 we point out cases when $\mathrm{cov}(\mu, \lambda)$ holds; it follows that $\mathrm{cov}(\mu, \lambda) \wedge 2^{<\lambda} < \mu$ holds in many cases in which $\Phi(\mu, \lambda)$ does not.

For general ω-trees it is impossible to obtain a theorem like Theorem 3, see Example 4.1. This motivates an even weaker notion of a Δ-system for trees.

Definition. Let $F: T \to P(C)$; F is called a λ-weak successor Δ-system (weak λ-S-Δ-system), if there is $G: T \to P_\lambda(C)$ such that, for $\boldsymbol{\alpha} \in T$ and distinct successors $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ of $\boldsymbol{\alpha}$, $G(\boldsymbol{\beta}) \cap G(\boldsymbol{\gamma}) \subseteq G(\boldsymbol{\alpha})$ and $F(\boldsymbol{\alpha}) \subseteq G(\boldsymbol{\alpha})$.

Theorem 4.2 is the main theorem in Section 4. It more or less states, that if, for every $\boldsymbol{\alpha} \in T$, $\mathrm{cov}(\mu^T_{\boldsymbol{\alpha}}, \lambda)$ holds, and $F: T \to P_\lambda(C)$, then there is $T' \leqslant T$ such that $F \restriction T'$ is a weak λ-S-Δ-system.

Theorem 4.7 tells when it is possible to get weak λ-Δ-systems. If T is a tree such that for every $\boldsymbol{\alpha} \in T$ $\mathrm{cov}(\mu^T_{\boldsymbol{\alpha}}, \lambda)$ holds and for every $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ $\mu^T_{\boldsymbol{\alpha}} \leqslant \mu^T_{\boldsymbol{\beta}}$, then for every $F: T \to P_\lambda(C)$ there is $T' \leqslant T$ and $K: T' \to P_\lambda(C)$ such that, for every incomparable $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T'$, $F(\boldsymbol{\alpha}) \cap F(\boldsymbol{\beta}) \subseteq K(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})$.

We have not yet mentioned the definition of Δ-systems for trees; Definition 4.1 is where this notion is defined. Theorem 4.9 states that if we add to the previous assumptions the condition: for every $\boldsymbol{\alpha} \in T$, $2^{<\lambda} < \mu^T_{\boldsymbol{\alpha}}$, then the existence of Δ-systems is assured.

Next we shall describe some questions which are left open.

(1) We have very partial knowledge on high trees. This knowledge is summarized in the discussion at the end of Section 2. The next main question is to prove that partition theorems for high trees imply the existence of large cardinals in some inner model. We did not investigate the existence of large free subtrees and the existence of Δ-systems for high trees.

(2) In Section 4 we define a property denoted by $\mathrm{cov}^*(\mu, \lambda)$. $\mathrm{cov}^*(\mu, \lambda)$ is implied by $\mathrm{cov}(\mu, \lambda)$, and in 4.4 we prove that for well-founded trees cov^* is necessary and sufficient for the existence of weak λ-S-Δ-systems (see the precise formulation in 4.4.) The proof of Theorem 4.2 yields in fact a somewhat stronger result than what is stated in 4.2. In order that for every λ-bounded $F: T \to P_\lambda(A)$ it will be possible to obtain a $T' \leqslant T$ on which F is a weak λ-S-Δ-system it suffices to assume: $\Psi(T) \equiv$ there is $T'' \leqslant T$ such that: (1) for every $\boldsymbol{\alpha} \in T''$, $\mathrm{cov}^*(\mu_\alpha, \lambda)$ holds; (2) on every branch b of T'' there is an unbounded set of $\boldsymbol{\alpha}$'s such that: $\boldsymbol{\alpha}$ is an extreme point (i.e. for every $\boldsymbol{\beta} \geqslant \boldsymbol{\alpha}$, $\mu_{\boldsymbol{\beta}} \geqslant \mu_{\boldsymbol{\alpha}}$), and $\mathrm{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ holds.

There are two questions that arise:

(1) Is it consistent that $\mathrm{cov}^*(\mu, \lambda)$ does not imply $\mathrm{cov}(\mu, \lambda)$?

Note that in view of 4.10(e) one needs at least $0^{\#}$ (and even inner models with a measurable cardinal, etc.) in order to prove this.

(2) Is $\Psi(T)$ a necessary and sufficient condition for the conclusion of 4.2?

Note that $\mathrm{cov}^*(\mu, \lambda) \Rightarrow \mathrm{cov}(\mu, \lambda)$ implies (2). However, a positive answer to question (1) will make (2) meaningful.

We can get close approximations $K(\boldsymbol{\alpha}) \cap K(\boldsymbol{\beta}) = K(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$ provided that $\kappa < \lambda < \kappa^{+\omega}$ (really a weaker condition).

Historical Remarks. Theorem 2.2 is a key theorem in this framework. It was proved by Shelah. Partition theorems of the kind of 2.2 were first proved and used by Shelah in [9] and [10]. There, a special case of 2.2 is generalized in the direction described at the discussion in the end of Section 2, Definitions 2.1 and 2.2.

Before that, Namba [7] used an argument of the same nature in order to prove that Namba forcing does not collapse $\aleph_1$. However, Namba did not isolate the combinatorial part from the rest of the proof.

2.2 is however stronger than the previous results, in that the case when $\mathrm{cf}(\mu_\alpha) < \lambda < \mu_\alpha$ is new, and needs a new trick. The other reason for presenting 2.2 was to make this work self-contained.

Theorem 2.13 represents another direction of generalization.

2.19(a) was proved by Shelah in [11], it was applied there in the construction of many non-isomorphic models. 2.19(b) is a new theorem, and its proof will appear elsewhere; it is also due to Shelah.

The other main theorem of Section 2 is Theorem 2.3. This theorem was proved by Rubin as a step in the proof of 3.1.

The notion of a valve was defined by Rubin. He used it in the proof of Theorem 2 in the abstract. However, valves were defined by Rubin, just for two particular ideals $P_\mu(\mu)$ for μ regular, and the ideal I_χ described in the proof of 3.4(b) when μ is singular. This sufficed for Theorem 2 but not for 3.1. Shelah realized the need and the possibility of defining G, I-valves and proved Lemma 2.16. 2.15 is generalization by Shelah of the formulation but not the proof of a lemma of Rubin.

Lemma 2.13 is due to Rubin, 2.14 is due to Shelah. Theorem 2.17 is a trivial modification of the well known theorem of Levy on the Levy's hierarchy (see e.g. [4]). Theorem 2.18 is due to Rubin.

Example 2.20 is due to Shelah. The suggestion that one might get partition theorems for *PF* functions is due to Rubin. Some easy observations of such partition theorems for large cardinals were noted Magidor. 2.21(b), (c) are due to Shelah. Definition 2.3 is due to Shelah. It non-essentially generalizes a notion invented by Laver [5]. Theorem 2.22(b) is due to Laver [5]. Theorem 2.22(c) is due to Shelah. However the method of proof and the metamathematical principle that it is possible to derive from theorems about measurables, analogues for weakly compacts, in the way that it was done in 2.22(c) was known to other people before.

A special case of 3.1 was proved and used by Rubin in [8]; he used it in a proof that certain forcing notions satisfy some strong form of the c.c.c. The proof however used Δ-systems which cannot be used in general. He then proved Theorem 2. Theorem 3.1 was proved jointly by Rubin and Shelah.

A trivial case of Δ-systems was proved and used by Rubin in [8]. The properties cov and cov* were invented by Shelah. The main theorem of Section 4 Theorem 4.2 as well as its accompanying lemmas, was proved by Shelah. The easy corollaries were concluded jointly by Rubin and Shelah. The lemmas describing the relationship between cov and cov* were proved by Shelah, except of 4.10(e) which is due to Magidor.

Also 4.12, 4.13, 4.14 are due to Shelah[1].

This research was done in winter–spring 1980 except 4.12–4.14 which were done in Dec. '81, Apr.–May '82.

2. Partitions

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be sequences, $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ means that $\boldsymbol{\alpha}$ is an initial segment of $\boldsymbol{\beta}$, $\boldsymbol{\alpha} < \boldsymbol{\beta}$ means that $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, Λ denotes the empty sequence. A set of sequences is hereditary, if it is closed under initial segments; if C is a set of sequences then $H(C)$ denotes its closure under initial segments. $\boldsymbol{\alpha}^\wedge\boldsymbol{\beta}$ denotes the concatenation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\boldsymbol{\alpha}^\wedge B = \{\boldsymbol{\alpha}^\wedge\boldsymbol{\beta} \mid \boldsymbol{\beta} \in B\}$.

A structure $\langle T, \leqslant\rangle$ is called a tree if T is a nonempty hereditary set of finite sequences. A branch of T is a maximal set of pairwise comparable elements of T; $B(T)$ denotes the set of branches of T. T is well founded if every branch of T is finite.

If T is a tree and $\boldsymbol{\alpha} \in T$, then $\mathrm{suc}^T(\boldsymbol{\alpha})$ is the set of successors of $\boldsymbol{\alpha}$ in T, and $\mu^T_{\boldsymbol{\alpha}} = |\mathrm{suc}^T(\boldsymbol{\alpha})|$. $\mu^T_{\boldsymbol{\alpha}}$ is abbreviated by $\mu_{\boldsymbol{\alpha}}$ whenever T is understood from the context. Whenever possible we assume without mentioning, that T is a tree of sequences of ordinals, and that for every $\boldsymbol{\alpha} \in T$ $\mathrm{suc}^T(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha}^\wedge\langle\nu\rangle \mid \nu < \mu^T_{\boldsymbol{\alpha}}\}$.

A nonempty hereditary subset of a tree is called a subtree. From now on we deal only with trees T such that for every $\boldsymbol{\alpha} \in T$: $\mu^T_{\boldsymbol{\alpha}} = 0$ or $\mu^T_{\boldsymbol{\alpha}}$ is infinite.

A subtree $T_1 \subseteq T$ is called T-full if for every $\boldsymbol{\alpha} \in T_1$: $\mu^T_{\boldsymbol{\alpha}} = 0$, or $|\mathrm{suc}^T(\boldsymbol{\alpha}) - \mathrm{suc}^{T_1}(\boldsymbol{\alpha})| < \mu^T_{\boldsymbol{\alpha}}$. Note that the intersection of two T-full subtrees is T-full. $T_1 \leqslant T$ means that T_1 is a T-full subtree of T.

A subtree T_1 of T is T-large, if for every $\boldsymbol{\alpha} \in T_1$: $\mu^{T_1}_{\boldsymbol{\alpha}} = \mu^T_{\boldsymbol{\alpha}}$. Note that if T_1 is T-large and T_2 is T-full, then $T_1 \cap T_2$ is T-large. Let $T_1 \leqslant T$ denote that T_1 is a T-large subtree of T.

If $A \subseteq T$ let $T[A] \overset{\mathrm{def}}{=} H(A) \cup \{\boldsymbol{\beta} \in T \mid H(\boldsymbol{\beta}) \cap A \neq \emptyset\}$, $T[\boldsymbol{\alpha}]$ abbreviates $T[\{\boldsymbol{\alpha}\}]$.

λ's and χ's denote infinite cardinals μ's denote cardinals.

$P_\lambda(C) \overset{\mathrm{def}}{=} \{B \subseteq C \mid |B| < \lambda\}$ and $P(C) = P_{|C|^+}(C)$. A function $F: T \to \lambda$ is λ-bounded (or in short bounded), if for every branch b of T $\mathrm{Sup}(F(B)) < \lambda$. A function $F: T \to P_\lambda(C)$ is λ-bounded if for every branch b of T $|\bigcup F(b)| < \lambda$. Note that if T is well founded, or if $\mathrm{cf}(\lambda) > \aleph_0$, then F is automatically bounded.

[1] We may use systematically tagged trees (instead of trees), i.e., pairs $(T, \mathscr{D})$, T a tree as here, $\mathscr{D}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} \in T)$ a filter on $\mathrm{suc}^T(\boldsymbol{\alpha})$. See after 2.18.

In the sequel we shall define several games; with a few exceptions, they all have the same form. We now describe these games. If T is a tree then $B(T)$ possesses a natural topology τ_T, namely $\tau_T = \{B(T[A]) \mid A \subseteq T\}$. Let $B \subseteq B(T)$ be a Borel set. We describe the game $G^T(B)$. A play in $G^T(B)$ is carried out as follows: let $\boldsymbol{\alpha}_0 = \Lambda$; the moves of the game are $n = 1, 2, 3, \ldots$; in the nth move the second player – the hero, has picked $\boldsymbol{\alpha}_n \in T$. Move $n+1$ $(n \geqslant 0)$: the first player — the villain, picks a subset $A_{n+1} \subseteq \mathrm{suc}^T(\boldsymbol{\alpha}_n)$, such that $|A_{n+1}| < \mu^T(\boldsymbol{\alpha}_n)$ and the hero picks $\boldsymbol{\alpha}_{n+1} \in \mathrm{suc}(\boldsymbol{\alpha}_n) - A_{n+1}$. The play terminates after a branch $b = \{\boldsymbol{\alpha}_n \mid n < \rho \leqslant \omega\}$ has been chosen. The hero wins if $b \in B$, and otherwise the villain wins. We say that the hero (or villain) wins the game if he has a winning strategy.

Theorem 2.1. *Let T be a tree and $B \subseteq B(T)$ be a Borel set. Then*:

(a) *$G^T(B)$ is determined.*

(b) *The hero has a winning strategy iff there is $T_1 \leqslant T$ such that $B(T_1) \subseteq B$.*

(c) *The villain wins iff there is $T_1 \leqslant T$ such that $B(T_1) \cap B = \emptyset$.*

Proof. (a) is true by [6] since $G^T(B)$ is a Borel game. (b) and (c) are easy and are left to the reader. □

Remark. We shall use $G^T(B)$ just for closed B's so in fact the depth of [6] is not needed here.

Theorem 2.2. *Let T be a tree and λ be an infinite cardinal. Let $L_1(T, \lambda)$ mean that $T[\{\boldsymbol{\alpha} \mid \mathrm{cf}(\mu_{\boldsymbol{\alpha}}) = \mathrm{cf}(\lambda)\}]$ does not contain a T-full subtree. Let $L_2(t, \lambda)$ mean that for every bounded $F : T \to \lambda$ there is $T_1 \leqslant T$ such that* $\mathrm{Sup}(F(T_1)) < \lambda$. *Then $L_1(T, \lambda)$ is equivalent to $L_2(T, \lambda)$.*

Proof. W.l.o.g. λ is regular. Suppose $\neg L_1(T, \lambda)$ holds. Let $F : T \to \lambda$ be defined as follows: for every $\boldsymbol{\alpha} \in T$ such that $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}) = \lambda$ and for no $\boldsymbol{\beta} < \boldsymbol{\alpha}$, $\mathrm{cf}(\mu_{\boldsymbol{\beta}}) = \lambda$, let $\{\mathbf{A}_i^\alpha \mid i < \lambda\}$ be a partition of $\mathrm{suc}(\boldsymbol{\alpha})$ such that, for every $i < \lambda$, $|\mathbf{A}_i^\alpha| < \mu_{\boldsymbol{\alpha}}$. Let $\boldsymbol{\beta} \in T$ define $F(\boldsymbol{\beta}) = i$ if for some $\boldsymbol{\alpha} \in T$ $\boldsymbol{\beta} \in \mathbf{A}_i^\alpha$, otherwise define $F(\boldsymbol{\beta}) = 0$. It is easy to see that F exemplifies $L_2(T, \lambda)$.

Suppose $L_1(T, \lambda)$ holds. By 2.1, w.l.o.g. we can assume that, for every $\boldsymbol{\alpha} \in T$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}) \neq \lambda$. Let $F : T \to \lambda$ be bounded. For every $v < \lambda$ let G_v be the game $G^T(B_v)$ where $B_v = \{b \in B(T) \mid (\forall \boldsymbol{\alpha} \in b)(F(\boldsymbol{\alpha}) < v)\}$. By 2.1 it suffices to show that for some v the hero has a winning strategy in G_v. Suppose by contradiction, this is not so. So for every v let S_v be a winning strategy for the villain in G_v. For every infinite cardinal μ such that $\mathrm{cf}(\mu) \neq \lambda$, let

$$\kappa(\mu, \lambda) = \begin{cases} 1, & \lambda < \mathrm{cf}(\mu), \\ \mathrm{cf}(\mu), & \mathrm{cf}(\mu) < \lambda < \mu, \\ \mu, & \mu < \lambda. \end{cases}$$

We define by induction a subtree T_1 of T, with the purpose that $|T_1|<\lambda$, and for every $\nu<\lambda$ there is $b\in B(T_1)$ which is the resulting branch in a play of G_ν in which the villain plays according to S_ν. Let $T^0=\{\Lambda\}$. Suppose the nth level T^n of T_1 has been defined. For every $\boldsymbol{\alpha}\in T^n$ such that $\mu_{\boldsymbol{\alpha}}\neq 0$, let $\Gamma_{\boldsymbol{\alpha}}=\{\nu\mid$ there is a play in which the villain plays according to S_ν in which the hero picks $\boldsymbol{\alpha}\}$[2]. There is a set $T^{n+1}(\boldsymbol{\alpha})\subseteq \mathrm{suc}^T(\boldsymbol{\alpha})$ such that $|T^{n+1}(\alpha)|=\kappa(\mu_\alpha,\lambda)$ and such that $\Gamma_{\boldsymbol{\alpha}}=\bigcup\{\Gamma_{\boldsymbol{\beta}}\mid\boldsymbol{\beta}\in T^{n+1}(\boldsymbol{\alpha})\}$. Let $T^{n+1}=\bigcup\{T^{n+1}(\boldsymbol{\alpha})\mid\boldsymbol{\alpha}\in T^n$ and $\mu_{\boldsymbol{\alpha}}\neq 0\}$, and let $T_1=\bigcup_{i<\omega}T^i$. Clearly for every $\nu<\lambda$ there is $b\in B(T_1)$ such that b is a result of a play in which the villain plays according to S_ν. Hence $F(T_1)$ is unbounded in λ. This is a contradiction, since if $\lambda>\aleph_0$ then $|T_1|<\lambda$ and λ is regular, and if $\lambda=\aleph_0$, then T_1 is just a branch in T contradicting the fact that F is bounded. □

We have to consider the following partition property: $M_2(T,\lambda,\chi)\equiv$ for every λ-bounded $F:T\to P_\lambda(\chi)$, there is $T_1\leq T$ such that $\bigcup F(T_1)\neq\chi$.

We shall formulate a condition $M_1(T,\lambda,\chi)$, similar in nature to $L_1(T,\lambda)$, that will be proved to be equivalent to $M_2(T,\lambda,\chi)$. M_1 is, however, less transparent than L_1; to clarify what M_1 means we first present the restriction of it to a special kind of trees.

Let $\rho\leq\omega$, $\boldsymbol{\mu}=\{\mu_i\mid i<\rho\}$ be a sequence of infinite cardinals. The tree $T_{\boldsymbol{\mu}}$ is a tree in which each branch has order type $1+\rho$, and such that if $\boldsymbol{\alpha}\in T$ and has length $i<\rho$ then $\mu^T_{\boldsymbol{\alpha}}=\mu_i$.

For cardinals λ, χ let $\mathrm{Kb}(\lambda,\chi)=\{\kappa\mid\lambda\leq\kappa\leq\chi$, κ is a cardinal, and if $\kappa\neq\lambda$ then κ is a successor$\}$. Let $\mathrm{Mb}(\boldsymbol{\mu},\lambda,\chi)\equiv$ for every $n\in\omega$ and $\kappa_1>\cdots>\kappa_n\in\mathrm{Kb}(\lambda,\chi)$ there are $0\leq i_1<\cdots<i_n<\rho$ such that, for every $j=1,\ldots,n$, $\mathrm{cf}(\mu_{i_j})=\mathrm{cf}(\kappa_j)$. Let $\mathrm{Mb}^*(\boldsymbol{\mu},\lambda,\chi)\equiv(\exists\lambda^*\leq\lambda)\mathrm{Mb}(\boldsymbol{\mu},\lambda^*,\chi)$.

Theorem. (a) *If* $\mathrm{cf}(\lambda)>\aleph_0$ *or* $\lambda=\aleph_0$, *then, for every* $\boldsymbol{\mu}$ *and* χ, $M_2(T_{\boldsymbol{\mu}},\lambda,\chi)$ *iff* $\neg\mathrm{Mb}(\boldsymbol{\mu},\lambda,\chi)$, (b) *For every* $\boldsymbol{\mu}$, χ, λ $M_2(T_{\boldsymbol{\mu}},\lambda,\chi)$ *iff* $\neg\mathrm{Mb}^*(\boldsymbol{\mu},\lambda,\chi)$.

We do not prove the above theorem since it is a special case of 2.3.

We now turn to the general case. We define a game $G(T,\lambda,\chi)$. Let

$$K(\lambda,\chi)=\{\kappa\mid\lambda\leq\kappa\leq\chi \text{ and if } \kappa\neq\lambda \text{ then } \mathrm{cf}(\lambda)>\aleph_0\}.$$

For $\boldsymbol{\alpha}\in T$ let $T(\boldsymbol{\alpha})=\{\boldsymbol{\beta}\mid\boldsymbol{\alpha}{}^\frown\boldsymbol{\beta}\in T\}$. The game $G(T,\lambda,\chi)$ is played as follows. Let $\boldsymbol{\alpha}_0=\Lambda$ and $\chi_0=\chi^+$. The moves of the game are $n=1,2,3,\ldots$. In the n-th move the hero has picked $\chi_n\in K(\lambda,\chi)$, and the villain has picked $\boldsymbol{\alpha}_n\in T$. The $(n+1)$st move $(n\geq 0)$: (i) the hero picks $\chi_{n+1}\in K(\lambda,\chi)$ such that $\chi_{n+1}<\chi_n$; (ii) the villain picks $T_{n+1}\leq T(\boldsymbol{\alpha}_n)$; (iii) the hero chooses a branch b_{n+1} of T_{n+1}; (iv) the villain chooses $\boldsymbol{\beta}_{n+1}\in b_{n+1}$. $\boldsymbol{\alpha}_{n+1}$ is defined to be $\boldsymbol{\alpha}_n{}^\frown\boldsymbol{\beta}_{n+1}$. The play terminates when the hero has no χ_{n+1} to choose $(n\geq 0)$. The villain wins in a play, if for every $n\geq 1$ such that χ_n is defined, $\mathrm{cf}(\boldsymbol{\mu}_{\boldsymbol{\alpha}_n})=\mathrm{cf}(\chi_n)$; otherwise, the hero wins.

Clearly $G(T,\lambda,\chi)$ is determined. Let $M_1(T,\lambda,\chi)$ mean that the hero has a winning strategy in $G(T,\lambda,\chi)$.

[2] Note that for $\nu\in\Gamma_{\boldsymbol{\alpha}}$ there is a unique initial segment of the play of G_ν in which the villain plays according to S_ν, and the hero's last move is $\boldsymbol{\alpha}$.

Note that, for $T=T_{\mu}$, $M_1(T,\lambda,\chi)$ is equivalent to $\neg\,\mathrm{Mb}(\mu,\lambda,\chi)$.

We proceed to define the analogue of $\neg\,\mathrm{Mb}^*$. We now define a game $G^*(T,\lambda,\chi)$. A play in $G^*(T,\lambda,\chi)$ is played as follows: First, the hero and the villain carry out a play p_1 of the game $G(T,\lambda^+,\chi)$. The result of this play is $A=\alpha_0\leqslant\alpha\leqslant\cdots\leqslant\alpha_n$ and $\chi=\chi_0>\chi_1>\cdots>\chi_n=\lambda^+$. Now: (i) the hero picks $T^1\leqslant T(\alpha_n)$; (ii) the villain chooses $\lambda_1\leqslant\lambda$ and $\beta_0\in T^1$; (iii) a play p_2 of $G(T^1(\beta_0),\lambda_1,\lambda)$ is now carried out.

The villain wins in the play described above, if he wins in both p_1 and p_2; otherwise the hero wins.

Let $M_1^*(T,\lambda,\chi)$ mean that the hero has a winning strategy in $G^*(T,\lambda,\chi)$. Note that, for $T=T_{\mu}$, $M_1^*(T,\lambda,\chi)$ is equivalent to $\neg\,\mathrm{Mb}^*(T,\lambda,\chi)$.

The theorem that we shall prove is the following.

Theorem 2.3. (a) *If* $\lambda=\aleph_0$ *or* $\mathrm{cf}(\lambda)>\aleph_0$ *then* $M_1(T,\lambda,\chi)$ *is equivalent to* $M_2(T,\lambda,\chi)$. (b) $M_1^*(T,\lambda,\chi)$ *is equivalent to* $M_2(T,\lambda,\chi)$.

Remarks. (a) One might ask why M_1 or M_1^* are better than M_2 itself. This will be explained at the end of Section 2. For the time being we just mention that if $V\subseteq W$ are two universes with the same cardinals and the same cofinality function then it is not difficult to see that $M_1(T,\lambda,\chi)$ ($M_1^*(T,\lambda,\chi)$) holds in V iff it holds in W. This means, that M_1 or M_1^* do not depend on cardinal arithmetic.

Before proving Theorem 2.3, we need some definitions and observations.

Definition. (a) A subset of T is called an antichain if every two elements of A are incomparable.

(b) An antichain $A\subseteq T$ is a frontier, if, for every $b\in B(T)$, $A\cap b\neq\emptyset$; an antichain $A\subseteq T$ is called an almost frontier, if there is $T_1\leqslant T$ such that $A\cap T_1$ is a frontier of T_1.

Observations 2.4. (a) Let $A\subseteq T$, then A does not contain an almost frontier iff there is $T_1\leqslant T$ such that $T_1\cap A=\emptyset$.

(b) $M_1(T,\lambda,\lambda)$ holds iff $\{\alpha\in T\mid \mathrm{cf}(\mu_\alpha)=\mathrm{cf}(\lambda)\}$ does not contain an almost frontier.

(c) $\neg M_1^*(T,\lambda,\lambda)$ holds iff for every $T_1\leqslant T$ there is $\alpha\in T_1$ and $\lambda_1\leqslant\lambda$ such that $\neg M_1(T_1(\alpha),\lambda_1,\lambda)$ holds; i.e. iff for every $T_1\leqslant T$ there is $\alpha\in T_1$ and $\lambda_1<\lambda$ such that either $\mathrm{cf}(\mu_\alpha)=\mathrm{cf}(\lambda)$ or $\neg M_1(T_1(\alpha),\lambda_1,\lambda)$ holds.

(d) If $T_1\leqslant T$, $\lambda_1\geqslant\lambda$ and $\chi_1\leqslant\chi$, then $\neg M_1(T,\lambda,\chi)\Rightarrow\neg M_1(T_1,\lambda_1,\chi_1)$, and if $T_1\leqslant T$ then $M_1(T,\lambda,\chi)\Leftrightarrow M_1(T_1,\lambda,\chi)$. the sme holds for M_1^*.

(e) If $\chi>\lambda$ and $\mathrm{cf}(\chi)=\aleph_0$, then $\neg M_1(T,\lambda,\chi)\Leftrightarrow\bigwedge_{\chi_1<\chi}\neg M_1(T,\lambda,\chi_1)$ and $\neg M_1^*(T,\lambda,\chi)\Leftrightarrow\bigwedge_{\chi_1<\chi}\neg M_1^*(T,\lambda,\chi_1)$.

(f) Let A be an almost frontier of T, and for every $\alpha\in A$, let A_α be an almost frontier of $T(\alpha)$, then $\bigcup\{\alpha^\wedge A_\alpha\mid\alpha\in A\}$ is an almost frontier of T.

(g) $\neg M_1(T,\lambda,\chi)$ holds iff there is an almost frontier $A\subseteq T$ such that, for every $\alpha\in A$, $\neg M_1(T(\alpha),\lambda,\chi)$ holds. The same is true for M_1^*.

(h) If $\neg M_1(T, \lambda, \chi)$, then, for every $\chi_1 < \chi$, $|\{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in \mathrm{suc}(\Lambda)$ and $M_1(T(\alpha), \lambda, \chi_1)\}| < \mu_\Lambda$. If $\mathrm{cf}(\mu_\Lambda) = \mathrm{cf}(\chi) > \aleph_0$ or $\mathrm{cf}(\chi) = \aleph_0$ and, for every $\chi_1 < \chi$, $|\{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in \mathrm{suc}(\Lambda)$ and $M_1(T(\alpha), \lambda, \chi_1)\}| < \mu_\Lambda$, then $\neg M_1(T, \lambda, \chi)$. When $\chi > \lambda$, the same facts hold for M_1^* too.

(i) If $\mathrm{cf}(\mu_\Lambda) \neq \mathrm{cf}(\chi)$ and $\neg M_1(T, \lambda, \chi)$, then $|\{\boldsymbol{\alpha} \in \mathrm{suc}(\Lambda) \mid M_1(T(\alpha), \lambda, \chi\}| < \mu_\Lambda$. The same holds for M_1^*.

(j) If $\neg M_1(T, \lambda, \chi)$ holds then there is $T_1 \leqslant T$ such that for every $\boldsymbol{\alpha} \in T_1$ either $\neg M_1(T(\boldsymbol{\alpha}), \lambda, \chi)$ or there is $\boldsymbol{\beta} < \boldsymbol{\alpha}$ such that $\mathrm{cf}(\mu_{\boldsymbol{\beta}}) = \mathrm{cf}(\chi)$ and $\neg M_1(T(\boldsymbol{\beta}), \lambda, \chi)$. The same holds for M_1^*.

(k) If $\neg M_1(T, \lambda, \chi)$ holds and for every $\boldsymbol{\alpha} \in T$ and $\lambda_1 < \lambda$ $\neg M_1(T(\boldsymbol{\alpha}), \kappa, \lambda_1)$ holds, then $\neg M_1(T, \kappa, \chi)$ holds.

(l) Let $\neg M_1^*(T, \lambda, \chi)$ hold, and $\mathrm{cf}(\chi) = \mathrm{cf}(\lambda)$, then for every $T_1 \leqslant T$ there is $\boldsymbol{\alpha} \in T_1$ such that either $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}) = \mathrm{cf}(\chi)$ and $\neg M_1^*(T_1(\boldsymbol{\alpha}), \lambda, \chi)$ holds, or there is $\lambda_1 < \lambda$ such that $\neg M_1(T_1(\boldsymbol{\alpha}), \lambda_1, \chi)$.

Lemma 2.5. (a) *Let $\lambda = \aleph_0$ or $\mathrm{cf}(\lambda) > \aleph_0$, then, for every T and χ, $M_1(T, \lambda, \chi)$ is equivalent to $M_1^*(T, \lambda, \chi)$.*

(b) *Suppose that in the definition of the games G and G^* we replace the set $K(\lambda, \chi)$ by the set $\mathrm{Kb}(\lambda, \chi) = \{\kappa \mid \lambda \leqslant \kappa \leqslant \chi$ and if $\kappa \neq \lambda$ then κ is a successor$\}$. We denote the resulting games by* Gb *and* Gb* *respectively; then, for every T, λ, χ, $G(T, \lambda, \chi)$ is equivalent to $\mathrm{Gb}(T, \lambda, \chi)$, and $G^*(T, \lambda, \chi)$ is equivalent to $\mathrm{Gb}^*(T, \lambda, \chi)$.*

Proof. (a) It follows from the definitions that, for every T, λ, χ, $M_1^*(T, \lambda, \chi) \Rightarrow M_1(T, \lambda, \chi)$.

If $\lambda = \aleph_0$, then $M_1(T, \lambda, \chi) \Rightarrow M_1^*(T, \lambda, \chi)$ because there are no infinite cardinals $< \aleph_0$.

Suppose $\mathrm{cf}(\lambda) > \aleph_0$ and $M_1(T, \lambda, \chi)$ holds. Let S be a winning strategy for the hero in $G(T, \lambda, \chi)$. We describe a winning strategy for the hero in $G^*(T, \lambda, \chi)$. The hero plays according to S, as long as he does not have to mention λ. So the resulting play yields sequences $\chi_0 = \chi^+ > \chi_1 > \cdots > \chi_n > \lambda$ and $\boldsymbol{\alpha}_0 = \Lambda \leqslant \boldsymbol{\alpha}_1 \leqslant \cdots \leqslant \boldsymbol{\alpha}_n$. If, for some $i > 0$, $\mathrm{cf}(\chi_i) \neq \mu_{\boldsymbol{\alpha}_i}$, then the hero has already won in the G^* play. Otherwise, $M_1(T(\boldsymbol{\alpha}_n), \lambda, \lambda)$ holds, and since $\mathrm{cf}(\lambda) > \aleph_0$ this means that, for every $\lambda_1 < \lambda$, $M_1(T(\boldsymbol{\alpha}_n), \lambda_1, \lambda)$ holds; so the hero can take $T(\alpha_n)$ to be the subtree he has to pick when playing according to G^*. Thus the hero has a winning strategy in $G^*(T, \lambda, \chi)$.

(b) will not be used in the sequel, so we omit its proof. $\square$

Lemma 2.6. *If $\neg M_1(T, \lambda, \chi)$ holds, then there is a λ-bounded function $F: T \to P_\lambda(\chi)$ such that, for every $T_1 \leqslant T$, $\bigcup F(T_1) = \chi$.*

Proof. The proof is by induction on χ. For $\chi < \lambda$ let $F: T \to P_\lambda(\chi)$ be defined as follows: for every $\boldsymbol{\alpha} \in T$, $F(\boldsymbol{\alpha}) = \chi$. So F is as required.

$\chi=\lambda$: Let $\lambda=\bigcup\{v_i \mid i<\mathrm{cf}(\lambda)\}$, where for every $i<j<\mathrm{cf}(\lambda)$: $v_i<v_j<\lambda$. By 2.4(b) let A be an almost frontier of T such that, for every $\boldsymbol{\alpha}\in A$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}})=\mathrm{cf}(\lambda)$. For every $\boldsymbol{\alpha}\in A$ let $\{A_i^{\boldsymbol{\alpha}} \mid i<\mathrm{cf}(\lambda)\}$ be a partition of $\mathrm{suc}(\boldsymbol{\alpha})$ to subsets of power less than $\mu_{\boldsymbol{\alpha}}$. Define F as follows: $F(\boldsymbol{\beta})=\emptyset$ if $\boldsymbol{\beta}\notin\bigcup\{\mathrm{suc}(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha}\in A\}$, $F(\boldsymbol{\beta})=v_i$ if for some $\boldsymbol{\alpha}\in A$, $\boldsymbol{\beta}\in A_i^{\boldsymbol{\alpha}}$. Clearly F is as required.

$\chi>\lambda$ *and* $\mathrm{cf}(\chi)>\aleph_0$: Let $\chi=\bigcup\{v_i \mid i<\mathrm{cf}(\chi)\}$, where $v_0=0$, for every $i<j<\mathrm{cf}(\chi)$, $v_i<v_j<\chi$. By the definition of $G(T,\lambda,\chi)$ and 2.4(j), there is an almost frontier $A\subseteq T$ such that for every $\boldsymbol{\alpha}\in A$: $\mathrm{cf}(\mu_{\boldsymbol{\alpha}})=\mathrm{cf}(\chi)$, and $\neg M_1(T(\boldsymbol{\alpha}),\chi,\lambda)$. By 2.4(h), for every $\boldsymbol{\alpha}\in A$ there is a partition $\{A_i^{\boldsymbol{\alpha}} \mid i<cf(\chi)\}$ of $\mathrm{suc}(\boldsymbol{\alpha})$ such that for every i: $|A_i^{\boldsymbol{\alpha}}|<\mu_{\boldsymbol{\alpha}}$, and, for every $\boldsymbol{\beta}\in A_i^{\boldsymbol{\alpha}}$, $\neg M_1(T(\boldsymbol{\beta}),\lambda,|v_i|)$ holds.

For every $\boldsymbol{\beta}\in A_i^{\boldsymbol{\alpha}}$ let $F_{\boldsymbol{\beta}}:T(\boldsymbol{\beta})\to P_\lambda(v_i)$ be as assumed by $M_1(T(\boldsymbol{\beta}),\lambda,|v_i|)$ and the induction hypothesis. Let $F:T\to P_\lambda(\chi)$ be defined as follows: let $\boldsymbol{\gamma}\in T$ if for some $\boldsymbol{\beta}\in\bigcup\{\mathrm{suc}(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha}\in A\}$ and for some $\boldsymbol{\delta}:\boldsymbol{\gamma}=\boldsymbol{\beta}^\wedge\boldsymbol{\delta}$, then $F(\boldsymbol{\gamma})=F_{\boldsymbol{\beta}}(\boldsymbol{\delta})$; otherwise $F(\gamma)=\emptyset$. Clearly F is as required.

$\chi>\lambda$ *and* $\mathrm{cf}(\chi)=\mathrm{cf}(\lambda)=\aleph_0$: This is proved as in the previous case, however, the proof that an A as above exists is somewhat different.

$\chi>\lambda$, $\mathrm{cf}(\chi)=\aleph_0$ *and* $\mathrm{cf}(\lambda)>\aleph_0$: Let $\chi=\sum_{i\in\omega}\chi_i$ and, for every $i\in\omega$, $\chi_i<\chi$. By the induction hypothesis for every $i\in\omega$ there is $F_i:T\to P_\lambda(\chi_i)$ as required. For every $\boldsymbol{\alpha}\in T$, let $F(\boldsymbol{\alpha})=\bigcup_{i\in\omega}F_i(\boldsymbol{\alpha})$. Since $\mathrm{cf}(\lambda)>\aleph_0$, F is λ-bounded. It is obvious that F is as required.

This concludes the proof of the lemma. $\square$

Lemma 2.7. *If* $\mathrm{cf}(\lambda)>\aleph_0$, *then* $M_1(T,\lambda,\chi)$ *implies* $M_2(T,\lambda,\chi)$.

Proof. *Claim* 1. Let $\chi\geqslant\lambda$, and $\mathrm{cf}(\chi)>\aleph_0$ or T is well founded; suppose, for every $\boldsymbol{\alpha}\in T$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}^T)\neq\mathrm{cf}(\chi)$, then $M_2(T,\lambda,\chi)$.

Proof of Claim 1. *Case* 1: χ *is regular*. In this case the claim follows easily from Theorem 2.2. For let $F:T\to P_\lambda(\chi)$ for every $\boldsymbol{\alpha}\in T$ let $G(\boldsymbol{\alpha})=\mathrm{Sup}(F(\boldsymbol{\alpha}))$, then $G:T\to\chi$, and hence by 2.2 there is $T_1\leqslant T$ such that $\sup(G(T_1))<\chi$, so certainly $\bigcup F(T_1)\neq\chi$.

Case 2: χ *is singular*. Let $F:T\to P_\lambda(\chi)$. There is $\lambda_1<\chi$ and $T_1\leqslant T$ such that, for every $\boldsymbol{\alpha}\in T_1$, $|F(\boldsymbol{\alpha})|<\lambda_1$. This is trivially true if $\lambda<\chi$, because in this case take T_1 to be equal to T and $\lambda_1=\lambda$. If $\chi=\lambda$, let $\chi=\sum_{i<\mathrm{cf}(\chi)}\chi_i$ and $\chi_i<\chi$. Let $G:T\to\mathrm{cf}(\chi)$ be defined as follows: $G(\boldsymbol{\alpha})=\min(\{i \mid |F(\boldsymbol{\alpha})|<\chi_i\})$, then since $\mathrm{cf}(\chi)>\aleph_0$, or T is well founded, G is $\mathrm{cf}(\chi)$-bounded. So by Theorem 2.2 there is $T_1\leqslant T$ such that $\mathrm{Sup}(G(T_1))=i<\mathrm{cf}(\chi)$. $\lambda_1=\chi_i$ and T_1 are as required.

Let $\chi=\sum_{i<\mathrm{cf}(\chi)}\chi_i$ and, for every i, $\lambda_1<\mathrm{cf}(\chi_i)=\chi_1<\chi$. Let $K:T_1\to\mathrm{cf}(\chi)$ be defined as follows: $k(\boldsymbol{\alpha})=i$, iff $\mathrm{cf}(\mu_{\boldsymbol{\alpha}})=\chi_i$; otherwise $K(\boldsymbol{\alpha})=0$. Since $\mathrm{cf}(\chi)>\aleph_0$ or T is well founded, K is bounded. So by 2.2 there is $T_2\leqslant T_1$ such that $\mathrm{Sup}(K(T_2))<i<\mathrm{cf}(\chi)$. By the first case in the claim there is $T_3\leqslant T_2$ such that $\bigcup F(T_3)\not\supseteq\chi_i$, so certainly $\bigcup F(T_3)\neq\chi$, so T_3 is as required. This concludes the proof of Claim 1.

We now turn to the proof of the lemma. Let $\mathrm{cf}(\lambda) > \aleph_0$. We prove by induction on χ that $M_1(T, \lambda, \chi) \Rightarrow M_2(T, \lambda, \chi)$.

Case 1: $\chi = \lambda$. So $M_1(T, \lambda, \chi)$ implies that there is $T_1 \leqslant T$ such that, for every $\boldsymbol{\alpha} \in T_1$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}) \neq \mathrm{cf}(\chi)$. It thus follows from Claim 1 that $M_2(T_1, \lambda, \chi)$ holds, hence $M_2(T, \lambda, \chi)$ holds.

Case 2: $\chi > \lambda$ *and* $\mathrm{cf}(\chi) = \aleph_0$. So $M_1(T, \lambda, \chi)$ implies there is $\chi_1 < \chi$ such that $M_1(T, \lambda, \chi_1)$. By the induction hypothesis $M_2(T, \lambda, \chi_1)$ and hence $M_2(T, \lambda, \chi)$.

Case 3: $\chi > \lambda$ *and* $\mathrm{cf}(\chi) > \aleph_0$. Suppose $M_1(T, \lambda, \chi)$ holds. We can assume that there is an almost frontier $A \subseteq T$ such that, for every $\boldsymbol{\alpha} \in A$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}) = \mathrm{cf}(\chi)$; otherwise Claim 1 can be applied to some $T_1 \leqslant T$. Replacing T by $T[A]$ we can further assume that A is a frontier in T. We can also assume that for every $\boldsymbol{\alpha} \in A$ and $\boldsymbol{\beta} < \boldsymbol{\alpha}$: $\mathrm{cf}(\mu_{\boldsymbol{\beta}}) \neq \mathrm{cf}(\chi)$. Let $B = \{\boldsymbol{\alpha} \in A \mid \neg M_1(T(\boldsymbol{\alpha}), \lambda, \chi)\}$. By 2.4(g), B is not an almost frontier, so there is $A_1 \subseteq A - B$ such that $T_1 \stackrel{\mathrm{def}}{=} T[A_1] \leqslant T$. By 2.4(h), for every $\boldsymbol{\alpha} \in A_1$ there is $\chi_{\boldsymbol{\alpha}} < \chi$ and $C_{\boldsymbol{\alpha}} \subseteq \mathrm{suc}(\boldsymbol{\alpha})$ such that $|C_{\boldsymbol{\alpha}}| = \mu_{\boldsymbol{\alpha}}$ and for every $\boldsymbol{\gamma} \in C_{\boldsymbol{\alpha}}$: $M_1(T(\boldsymbol{\gamma}), \lambda, \chi_{\boldsymbol{\alpha}})$ holds.

We now show that $M_2(T_1, \lambda, \chi)$ holds. So let $F: T_1 \to P_\lambda(\chi)$. For every $\nu < \chi$ let $B_\nu \subseteq B(T_1)$ be the set of branches b such that $\nu \in F(b)$. By 2.1 it is sufficient to show that for some $\nu < \chi$, the hero has winning strategy in the game $G^{T_1}(B_\nu)$. Suppose by contradiction that for every $\nu < \chi$ the villain has a winning strategy S_ν in the game $G^{T_1}(B_\nu)$. For every $\boldsymbol{\alpha} \in T_1$ let $\Gamma_{\boldsymbol{\alpha}} = \{\nu \mid$ there is a play in which the villain plays according to S_ν and $\boldsymbol{\alpha}$ belongs the branch picked by the hero$\}$. We claim that for some $\boldsymbol{\alpha} \in A_1$, $|\Gamma_{\boldsymbol{\alpha}}| = \chi$. Suppose not; let $T' = H(A_1)$, and let $G: T' \to P_\chi(\chi)$ be defined as follows: if $\boldsymbol{\alpha} \in A_1$, then $G(\boldsymbol{\alpha}) = \Gamma_{\boldsymbol{\alpha}}$; otherwise $G(\boldsymbol{\alpha}) = \emptyset$. Recall that, for every $\boldsymbol{\alpha} \in A_1$ and $\boldsymbol{\beta} < \boldsymbol{\alpha}$, $\mathrm{cf}(\mu_{\boldsymbol{\beta}}) \neq \mathrm{cf}(\chi)$, so by Claim 1, $M_2(T', \chi, \chi)$ holds. So there is $T'' \leqslant T'$ and $\nu < \chi$ such that, for every $\boldsymbol{\alpha} \in T'' \cap A_1$, $\nu \notin \Gamma_{\boldsymbol{\alpha}}$. T'' is well founded; it follows easily by induction on the rank of $\boldsymbol{\beta} \in T''$ that, for every $\boldsymbol{\beta} \in T''$, $\nu \notin T_{\boldsymbol{\beta}}$. Hence $\nu \notin \Gamma_\Lambda$ which contradicts our assumption that S_ν is a winning strategy for the villain.

So far we have proved that there is $\boldsymbol{\alpha} \in A_1$ such that $|\Gamma_{\boldsymbol{\alpha}}| = \chi$. Let $\chi_1 = \chi_{\boldsymbol{\alpha}}$ and $C = C_{\boldsymbol{\alpha}}$ be as defined in the beginning of the proof of Case 3. It is easy to see that there is $\boldsymbol{\gamma} \in C$ such that $|\Gamma_{\boldsymbol{\gamma}}| \geqslant \chi_1$. Since $\chi_1 \geqslant \lambda$ $|\Gamma_{\boldsymbol{\gamma}} - \bigcup F(H(\boldsymbol{\gamma}))| \geqslant \chi_1$. But by the definition of C and $\chi_1 M_1(T_1(\boldsymbol{\gamma}), \lambda, \chi_1)$, so by the induction hypothesis, there is $\nu \in \Gamma_{\boldsymbol{\gamma}}$ and $\bar{T} \leqslant T(\boldsymbol{\gamma})$ such that $\nu \notin \bigcup F(H(\boldsymbol{\gamma})) \cup \bigcup F(\boldsymbol{\gamma}^\wedge \bar{T})$. This contradicts the definition of $\Gamma_{\boldsymbol{\gamma}}$. So the lemma is proved. $\square$

Lemmas 2.6 and 2.7 constitute a proof for Theorem 2.3(a).

We now turn to prove Theorem 2.3(b). By 2.5 and 2.3(a), it remains to show that if $\lambda > \mathrm{cf}(\lambda) = \aleph_0$, then $M_1(T, \lambda, \chi)$ is equivalent to $M_2(T, \lambda, \chi)$.

Lemma 2.8. $\neg M_1^*(T, \lambda, \chi) \Rightarrow \neg M_2(T, \lambda, \chi)$.

Proof. As we have already mentioned we can assume that $\lambda > \mathrm{cf}(\lambda) = \aleph_0$. The lemma is proved by induction on χ. We distinguish between three cases: $\chi < \lambda$,

$\chi \geq \lambda$ and $\mathrm{cf}(\chi)=\aleph_0$; and $\chi \geq \lambda$ and $\mathrm{cf}(\chi)>\aleph_0$. The first and the third cases are proved exactly as the corresponding cases in Lemma 2.6. So we prove the second case. Let $\chi \geq \lambda$, $\mathrm{cf}(\chi)=\aleph_0$ and $\neg M_1^*(T, \lambda, \chi)$ holds.

We define by induction sequence $\{T_i \mid i<i_0\}$ and antichains $\{A_i \mid i<i_0\}$ such that $T_i \leq T$ and $A_i \subseteq T_i$. Suppose T_i, A_i have been defined for every $i<j$. Let $T_j' = \bigcup\{T' \leq T \mid T' \cap \bigcup_{i<j} A_i = \emptyset\}$. Clearly either $T_j'=\emptyset$ or $T_j' \leq T$. In the first case we define $i_0=j$, and in the second we define $T_j = T_j'$. Suppose T_j has been defined. Let $A_j' = \{\boldsymbol{\alpha} \in T_j \mid \text{either } \mathrm{cf}(\mu_{\boldsymbol{\alpha}})=\aleph_0 \text{ and } \neg M_1^*(T_j(\alpha), \lambda, \chi) \text{ or there is } \lambda_1<\lambda \text{ such that } \neg M_1(T_j(\alpha), \lambda_1, \chi)\}$, and $A_j = \{\boldsymbol{\alpha} \in A_j' \mid \text{for every } \boldsymbol{\beta}<\boldsymbol{\alpha}, \boldsymbol{\beta} \notin A_j'\}$. By 2.4(l), $A_j \neq \emptyset$.

For every $i<i_0$ and $\boldsymbol{\alpha} \in A_i$ there is a λ-bounded $F_{\boldsymbol{\alpha}}: T_i(\boldsymbol{\alpha}) \to P_\lambda(\chi)$ such that for every $T \leq T_i(\boldsymbol{\alpha})$ $\bigcup F_{\boldsymbol{\alpha}}(T')=\chi$. To show this we distinguish between two cases. If there is $\lambda_1<\lambda$ such that $\neg M_1(T_i(\boldsymbol{\alpha}), \lambda_1, \chi)$, then our claim follows from Lemma 2.6. Let $\neg M_1^*(T_i(\boldsymbol{\alpha}), \lambda, \chi)$ and $\mathrm{cf}(\mu_{\boldsymbol{\alpha}})=\aleph_0$. Let $\chi=\sum_{j<\omega}\chi_j$, $\chi_j<\chi$ and $\chi_0=0$. By 2.4(h) there is a partition $\{A_j^{\boldsymbol{\alpha}} \mid j \in \omega\}$ of $\mathrm{suc}^{T_i}(\boldsymbol{\alpha})$ such that for every $j \in \omega$: $|A_j^{\boldsymbol{\alpha}}|<\mu_{\boldsymbol{\alpha}}$, and, for every $\boldsymbol{\beta} \in A_j^{\boldsymbol{\alpha}}$, $\neg M_1^*(T_i(\boldsymbol{\beta}), \lambda, \chi_j)$. By the induction hypothesis for every $j \in \omega$ and $\boldsymbol{\beta} \in A_j^{\boldsymbol{\alpha}}$ there is a λ-bounded $F_{\boldsymbol{\beta}}: T_i(\boldsymbol{\beta}) \to P_\lambda(\chi_j)$ such that for every $T' \leq T_i(\boldsymbol{\beta})$, $\bigcup F_{\boldsymbol{\beta}}(T')=\chi_j$. Let $F_{\boldsymbol{\alpha}}: T_i(\alpha) \to P_\lambda(\chi)$ be defined as follows. $F_{\boldsymbol{\alpha}}(\Lambda)=\emptyset$ and if $\boldsymbol{\gamma}$ is such that for some $\boldsymbol{\beta} \in \mathrm{suc}^{T_i}(\boldsymbol{\alpha})$ and $\boldsymbol{\delta}$: $\boldsymbol{\alpha}^\wedge\boldsymbol{\gamma}=\boldsymbol{\beta}^\wedge\boldsymbol{\delta}$, then $F_{\boldsymbol{\alpha}}(\boldsymbol{\gamma})=F_{\boldsymbol{\beta}}(\boldsymbol{\delta})$. $F_{\boldsymbol{\alpha}}$ is as required.

Let $F_{\boldsymbol{\alpha}}'$ be defined as follows: $\mathrm{Dom}(F_{\boldsymbol{\alpha}}')=\boldsymbol{\alpha}^\wedge\mathrm{Dom}(F_\alpha)$ and $F_{\boldsymbol{\alpha}}'(\boldsymbol{\alpha}^\wedge\boldsymbol{\gamma})=F_{\boldsymbol{\alpha}}(\boldsymbol{\gamma})$. By our definitions if $\boldsymbol{\alpha}_1 \neq \boldsymbol{\alpha}_2$ $\mathrm{Dom}(F_{\boldsymbol{\alpha}_1}') \cap \mathrm{Dom}(F_{\boldsymbol{\alpha}_2}')=\emptyset$. Let $F: T \to P_\lambda(\chi)$ be defined as follows; if, for some $\boldsymbol{\alpha} \in \bigcup_{i<i_0} A_i$, $\boldsymbol{\beta} \in \mathrm{Dom}(F_{\boldsymbol{\alpha}}')$, then $F(\boldsymbol{\beta})=F_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$, and otherwise $F(\boldsymbol{\beta})=\emptyset$.

We show that F is λ-bounded, and, for every $T' \leq T$, $\bigcup F(T')=\chi$. Let $b \in B(T)$. If $b \cap \bigcup_{i<i_0} A_i=\emptyset$ then $\bigcup F(b)=\emptyset$. Otherwise let $i=\min(\{j \mid A_j \cap b \neq \emptyset\})$. Let $\{\boldsymbol{\alpha}\}=A_i \cap b$, then for every $\boldsymbol{\alpha}' \in \bigcup_{i<i_0} A_i - \{\boldsymbol{\alpha}\}$, $\mathrm{Dom}(F_{\boldsymbol{\alpha}}') \cap b \subseteq H(\boldsymbol{\alpha})$. Since $F_{\boldsymbol{\alpha}}'$ is λ-bounded $|\bigcup F(b)|<\lambda$. Hence F is λ-bounded.

Let $T' \leq T$. By the definition of the A_i's, $T' \cap \bigcup_{i<i_0} A_i \neq \emptyset$. Let $i=\min(\{j \mid A_j \cap T' \neq \emptyset\}$ and $\boldsymbol{\alpha} \in T' \cap A_i$. We prove that $\boldsymbol{\alpha}^\wedge T'(\boldsymbol{\alpha}) \subseteq T_i$. The proof is by induction on the length of $\boldsymbol{\beta} \in \boldsymbol{\alpha}^\wedge T'(\boldsymbol{\alpha})$. $\boldsymbol{\alpha} \in T_i$. Suppose $\boldsymbol{\beta} \in T_i$ and $\boldsymbol{\gamma} \in \mathrm{suc}(\boldsymbol{\beta}) \cap \boldsymbol{\alpha} \wedge T'(\boldsymbol{\alpha})$. Suppose by contradiction $\boldsymbol{\gamma} \notin T_i$, then there is no $T'' \leq T(\boldsymbol{\gamma})$ such that $\boldsymbol{\gamma}^\wedge T'' \cap \bigcup_{j<i} A_j=\emptyset$; since $T'(\boldsymbol{\gamma}) \leq T(\boldsymbol{\gamma})$, $\boldsymbol{\gamma}^\wedge T'(\boldsymbol{\gamma}) \cap \bigcup_{j<i} A_j \neq \emptyset$, contradicting the minimality of i. So $\boldsymbol{\gamma} \in T_i$. We have thus proved that $\alpha^\wedge T'(\boldsymbol{\alpha}) \subseteq T_i$, hence by the definition of F, $\bigcup F(\boldsymbol{\alpha}^\wedge T'(\boldsymbol{\alpha}))=\chi$. □

We now turn to the other direction of 2.3(b).

Lemma 2.9. *Let* $\lambda>\mathrm{cf}(\lambda)=\aleph_0$, *then* $M_1^*(T, \lambda, \chi) \Rightarrow M_2(T, \lambda, \chi)$.

Proof. The proof is by induction on χ. We distinguish between the following cases: (i) $\chi=\lambda$; (ii) $\chi>\lambda$ and $\mathrm{cf}(\chi)=\aleph_0$; (iii) $\chi>\lambda$ and $\mathrm{cf}(\chi)>\aleph_0$. Cases (ii) and (iii) are proved exactly like the corresponding cases in 2.7.

To prove the first case it seems convenient to introduce an intermediate step.

Consider the following question. Let T be a tree $\bar{\lambda}$ a function from T to cardinals, and we denote $\bar{\lambda}(\boldsymbol{\alpha})$ by $\lambda_{\boldsymbol{\alpha}}$. A function $F:T\to P(\chi)$ is called λ-bounded if, for every $\boldsymbol{\alpha}\in T$, $|F(\boldsymbol{\alpha})|\leqslant\lambda_{\boldsymbol{\alpha}}$. When is it true, that for every $\bar{\lambda}$-bounded $F:T\to P(\chi)$ there is $T_1\leqslant T$ such that $\bigcup F(T_1)\neq\chi$.

In order to complete the proof of 2.9 we need just a special case of the above question. So we assume $\lambda>\mathrm{cf}(\lambda)=\aleph_0$, $\lambda=\sum_{i\in\omega}\lambda_i$ where $\{\lambda_i \mid i\in\omega\}$ is a strictly increasing sequence of uncountable successor cardinals. (In fact, some of these assumptions might be redundant.) Let $L=\{\lambda_i\mid i\in\omega\}$ and $\bar{\lambda}:T\to L$. Let $M_4(T,\bar{\lambda},\chi)$ mean that for every $\bar{\lambda}$-bounded $F:T\to P(\chi)$, there is $T_1\leqslant T$ such that $\bigcup F(T_1)\neq\chi$.

We are interested in $M_4(T,\bar{\lambda},\lambda)$. Let us define the game $G(T,\bar{\lambda},\lambda)$. Let $\boldsymbol{\alpha}_0=\Lambda$ and $\chi_0=\lambda$. The moves of the game are $n=1,2,\ldots$. In the nth move the hero has picked a cardinal χ_n and the villain has picked $\boldsymbol{\alpha}_n\in T$. *The* $(n+1)$st *move* $(n\geqslant 0)$: (i) the hero picks $\chi_{n+1}<\chi_n$ such that $\chi_{n+1}\geqslant\mathrm{cf}(\chi_{n+1})>\aleph_0$; (ii) the villain picks $T_{n+1}\leqslant T(\alpha_n)$; (iii) the hero chooses a branch b_{n+1} of T_{n+1}, (iv) the villain chooses $\boldsymbol{\beta}_{n+1}\in b_{n+1}$. $\boldsymbol{\alpha}_{n+1}$ is defined to be $\boldsymbol{\alpha}_n{}^\wedge\boldsymbol{\beta}_{n+1}$. The play terminates when the hero has no χ_{n+1} to choose (namely when $\chi_n=\aleph_1$). Suppose that in the play p there were n moves and the sequences chosen were $\chi_1,\ldots,\chi_n$ and $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n$; the villain wins in p if there is $0<k\leqslant n$ such that for every $0<i<k$ $\mathrm{cf}(\mu_{\boldsymbol{\alpha}_i})=\mathrm{cf}(\chi_i)$ and, for some $\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}_k$, $\lambda_{\boldsymbol{\beta}}\geqslant\chi_k$; otherwise the play is won by the hero.

Let $M_3(T,\bar{\lambda},\lambda)$ mean that the hero has a winning strategy in the game $G(T,\bar{\lambda},\lambda)$.

The remaining part of the proof of Lemma 2.9 is broken into the following two lemmas. □

Lemma 2.10. *Let* $\lambda>\mathrm{cf}(\lambda)=\aleph_0$; *if* $M_1^*(T,\lambda,\lambda)$ *holds, then, for every λ-bounded* $\bar{\lambda}:T\to L$, $M_3(T,\bar{\lambda},\lambda)$ *holds.*

Lemma 2.11. *Let* $\lambda>\mathrm{cf}(\lambda)=\aleph_0$ *and, for every* $\boldsymbol{\alpha}\in T$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}})\neq\aleph_0$, *if* $\bar{\lambda}:T\to L$ *then* $M_3(T,\bar{\lambda},\lambda)\ \Rightarrow\ M_4(T,\bar{\lambda},\lambda)$.

Proof of 2.10. Assume $\bar{\lambda}:T\to L$ is λ-bounded and $\neg M_3(T,\bar{\lambda},\lambda)$ holds. We prove that $\neg M_1^*(T,\lambda,\lambda)$ holds.

If $\bar{\lambda}:T\to L$ and $\boldsymbol{\alpha}\in T$, let $\bar{\lambda}^{\boldsymbol{\alpha}}:T(\alpha)\to L$ be defined as follows: $\bar{\lambda}^{\boldsymbol{\alpha}}(\boldsymbol{\beta})=\bar{\lambda}(\boldsymbol{\alpha}{}^\wedge\boldsymbol{\beta})$.

As in 2.4(j) there is $T_1\leqslant T$ such that (*): for every $\boldsymbol{\alpha}\in T_1$: either $\neg M_3(T(\boldsymbol{\alpha}),\bar{\lambda}^{\boldsymbol{\alpha}},\lambda)$ holds, or there is $\boldsymbol{\beta}<\boldsymbol{\alpha}$ such that $\mathrm{cf}(\mu_{\boldsymbol{\beta}})=\aleph_0$ and $\neg M_3(T(\boldsymbol{\beta}),\bar{\lambda}^{\boldsymbol{\beta}},\lambda)$ holds. Note that the analogue of 2.4(d) also holds, namely, if $T_1\leqslant T$, then $\neg M_3(T,\bar{\lambda},\lambda)\ \Rightarrow\ \neg M_3(T_1,\bar{\lambda},\lambda)$. By the analogue of the second part of 2.4(d) and by 2.4(d) itself we can w.l.o.g. assume that T satisfies (*). Let $T_1\leqslant T$, then, since $\bar{\lambda}$ is λ-bounded, there is $\boldsymbol{\alpha}\in T_1$ such that $\mathrm{Sup}(\{\lambda_{\boldsymbol{\gamma}}\mid\boldsymbol{\gamma}\geqslant\boldsymbol{\alpha}\})<\lambda$. If there is $\boldsymbol{\beta}<\boldsymbol{\alpha}$ such that $\mathrm{cf}(\mu_{\boldsymbol{\beta}})=\aleph_0$, then $\neg M_1(T_1(\boldsymbol{\beta}),\lambda,\lambda)$, and we finish; otherwise, by (*), we can assume that $\neg M_3(T_1(\boldsymbol{\alpha}),\lambda^{\boldsymbol{\alpha}},\lambda)$ holds. Let $\lambda_1=$

Sup$(\{\mid \gamma \geq \alpha\})$. It is clear from the definition of the games that $\neg M_1(T_1(\alpha), \lambda_1, \lambda)$ holds. In both cases we found $\gamma \in T_1$ and $\lambda' \leq \lambda$ such that $M_1(T_1(\gamma), \lambda', \lambda)$ holds, by 2.4(c) $\neg M_1^*(T, \lambda, \lambda)$ holds. □

Proof of Lemma 2.11. Suppose $M_3(T, \bar{\lambda}, \lambda)$ holds, and for every $\alpha \in T$ $\mathrm{cf}(\mu_\alpha) \neq \aleph_0$. Let S be a winning strategy for the hero in the game $G(T, \bar{\lambda}, \lambda)$. Let p be a play in which the hero plays according to S, and let $\chi_1, \ldots, \chi_n$, $\alpha_1, \ldots, \alpha_n$ be the sequences picked in this play; p is called a good play, if there is $0 \leq k < n$ such that, for every $1 \leq i \leq k$, $\mathrm{cf}(\mu_{\alpha_i}) = \mathrm{cf}(\chi_i)$, and there is no $T' \leq T(\alpha_k)$ such that for every $b \in B(T')$ there is $\beta \in b$ such that $\mathrm{cf}(\mu_{\alpha_k{}^\wedge\beta}) = \mathrm{cf}(\chi_{k+1})$ or $\sum\{\lambda_\gamma \mid \gamma \leq \alpha_k{}^\wedge\beta\} \geq \chi_{k+1}$. We denote α_k by α_p and χ_{k+1} by χ_p. Let A be the set of α's such that, for some good p, $\alpha = \alpha_p$ and, for no good q, $\alpha_q < \alpha$. For every $\alpha \in A$ let $\chi_\alpha = \chi_p$ for some p such that $\alpha = \alpha_p$. So by our definition for every $\alpha \in A$ there is $T^\alpha \leq T(\alpha)$ such that for every $\beta \in T^\alpha$: $\mathrm{cf}(\mu_\beta^{T^\alpha}) \neq \mathrm{cf}(\chi_\alpha)$, and $\lambda_{\alpha^\wedge\beta} < \chi_\alpha$.

If $T_1 \leq T$ then $T_1 \cap A \neq \emptyset$, since the villain can play in such a way that the resulting play p is good, and all the α_i's chosen in this play belong to T_1. Hence $\alpha_p \in T_1$, so $\emptyset \neq H(\alpha_p) \cap A \subseteq T_1 \cap A$.

It thus follows that there is $A_1 \subseteq A$ such that $T[A_1] \leq T$. Let $T' = \bigcup\{H(\alpha) \cup \alpha^\wedge T^\alpha \mid \alpha \in A_1\}$, clearly $T' \leq T$. By Claim 1 in Lemma 2.7 $M_2(H(A_1), \lambda, \lambda)$ holds.

We prove now that $M_4(T', \bar{\lambda} \restriction T', \lambda)$ holds. Let $F: T' \to P(\lambda)$ be $\bar{\lambda} \restriction T'$-bounded and suppose by contradiction, for every $T'' \leq T'$, $\bigcup F(T'') = \lambda$. Define B_ν, S_ν and Γ_α as in 2.7, Case 3. As in 2.7, Case 3 there is $\alpha \in A_1$ such that $|\Gamma_\alpha| = \lambda$. So $|\Gamma(\alpha) - \bigcup F(H(\alpha))| = \lambda > \chi_\alpha$. But $M_1(T'(\alpha), \chi_\alpha, \chi_\alpha)$ holds and since $\mathrm{cf}(\chi_\alpha) > \aleph_0$, Lemma 2.7 can be applied here. So there is $\bar{T} \leq T'(\alpha)$ and $\nu \in \Gamma(\alpha) - \bigcup F(H(\alpha))$ such that $\nu \notin \bigcup F(\alpha^\wedge \bar{T})$. This is impossible. We have thus proved Lemma 2.11. □

Completion of the proof of 2.9. Let $\lambda > \mathrm{cf}(\lambda) = \aleph_0$ and suppose $M_1^*(T, \lambda, \lambda)$ holds. We show that there is $T_1 \leq T$ such that $M_2(T_1, \lambda, \lambda)$ holds. By 2.4(c) there is $T_1 \leq T$ such that, for every $\alpha \in T_1$ and $\lambda_1 \leq \lambda$, $M_1(T_1(\alpha), \lambda_1, \lambda)$ holds, in particular, for every $\alpha \in T_1$, $\mathrm{cf}(\mu_\alpha) \neq \aleph_0$, and trivially, by 2.4(c), $M_1^*(T_1, \lambda, \lambda)$ holds. Hence, by 2.10, for every λ-bounded $\bar{\lambda}: T_1 \to L$, $M_3(T, \bar{\lambda}, \lambda)$ holds. Let $F: T_1 \to P_\lambda(\lambda)$ be λ-bounded, hence the function $\bar{\lambda}(\alpha) \overset{\text{def}}{=} \min(\{\lambda_i \mid |F(\alpha)| \leq \lambda_i\})$ is λ-bounded and F is $\bar{\lambda}$-bounded. So, by 2.11, there is $T_2 \leq T_1$ such that $\bigcup F(T_2) \neq \lambda$. Hence $M_2(T_1, \lambda, \lambda)$ holds therefore $M_2(T, \lambda, \lambda)$ holds.

This concludes the proof of Theorem 2.3(b). □

Remark. We have defined $M_3(T, \bar{\lambda}, \chi)$ and $M_4(T, \bar{\lambda}, \chi)$. We know how to prove that $\neg M_3(T, \bar{\lambda}, \chi)$ implies $\neg M_4(T, \bar{\lambda}, \chi)$. We believe but we do not know how to prove the converse.

We shall use some easy equivalences of $M_2(T, \lambda, \chi)$.

Definition. *$A \subseteq T$ is T-small, if it does not contain a T-large subtree of T.*

Lemma 2.12. *The following are equivalent.*

(a) $M_2(T, \lambda, \chi)$.

(b) *If $\{A_i \mid i < \chi\}$ is a family of T-small subsets of T then there is $b \in B(T)$ such that $|\{i \mid b \nsubseteq A_i\}| \geq \lambda$.*

(c) *If $F: T \to P_\lambda(A)$ is λ-bounded and $\{A_i \mid i < \chi\}$ is a partition of A then there is $i < \chi$ and $T_1 \leq T$ such that $\bigcup F(T_1) \cap A_i = \emptyset$.*

Proof. Left to the reader. □

As it was already seen, Theorem 2.2 is basic to all other results in this paper. Theorem 2.3 is needed in the formulation and proof of the generalization of the theorem of Fodor and Hajnal on large free subsets.

We turn now to two partition results needed in the proof of the theorem on Δ-systems in Section 4.

Definition. $\boldsymbol{\alpha}$ is an extreme point of T if, for every $\boldsymbol{\beta} \trianglerighteq \boldsymbol{\alpha}$, $\mu^T_{\boldsymbol{\beta}} \geq \mu^T_{\boldsymbol{\alpha}}$. We denote the set of extreme points of T by $\mathrm{Ext}(T)$.

Clearly every tree has a dense set of extreme points. We need the following.

Lemma 2.13. *For every T there is $T_1 \leq T$ such that, for every $b \in B(T_1)$, $\mathrm{Ext}(T_1) \cap b$ is unbounded in b.*

Proof. Let $\boldsymbol{\alpha} \in T$; $\boldsymbol{\alpha}$ is an almost extreme point if for some $T_1 \leq T(\boldsymbol{\alpha})$, Λ is an extreme point of T_1. Let $B \subseteq B(T)$ be the set of branches which contain an almost extreme point. Suppose the villain has a winning strategy in $G^T(B)$, then there is $T_1 \leq T$ which does not contain any almost extreme point of T. But this is impossible since an extreme point of T_1 in an almost extreme point of T. So the hero has a winning strategy in $G^T(B)$, and hence there is $T_1 \leq T$ such that every $b \in B(T_1)$ contain an almost extreme point of T. For every almost extreme point of T, $\boldsymbol{\alpha} \in T_1$ let $T^{\boldsymbol{\alpha}} \leq T(\boldsymbol{\alpha})$ be such that $\Lambda \in \mathrm{Ext}(T^{\boldsymbol{\alpha}})$, and let $S^{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\,{}^\wedge (T(\boldsymbol{\alpha}) - T^{\boldsymbol{\alpha}})$, let $T_2 = T_1 - \bigcup \{S^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha}$ is an almost extreme point of T that belongs to $T_1\}$. Clearly $T_2 \leq T_1$ and hence $T_2 \leq T$ and, for every $b \in B(T_2)$, $b \cap \mathrm{Ext}(T_2) \neq \emptyset$.

We denote $T_2 = T^{(1)}$ and define by induction $T^{(i)}$ and A_i, $i \in \omega$ such that, for every $i < j$, A_i is a frontier in $T^{(i)}$, $A_i \subseteq \mathrm{Ext}T^{(i)})$, $A_i \subseteq T^{(j)}$, $T^{(j)} \leq T^{(i)}$. Suppose $T^{(i)}$, A_i have been defined. Let $S_i = \bigcup \{\mathrm{suc}^{T^{(i)}}(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in A_i\}$. For every $\boldsymbol{\alpha} \in S_i$, let $T^{\boldsymbol{\alpha}} \leq T^{(i)}(\boldsymbol{\alpha})$ be such that every branch of $T^{\boldsymbol{\alpha}}$ contains an extreme point. Let $T^{(i+1)} = H(A_i) \cup \bigcup \{\boldsymbol{\alpha}\,{}^\wedge T^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in S_i\}$. For every $\boldsymbol{\alpha} \in S_i$ let $A^{\boldsymbol{\alpha}}$ be a frontier of extreme points in $T^{\boldsymbol{\alpha}}$, and let $A_{i+1} = \{\boldsymbol{\alpha} \in A_i \mid \mu_{\boldsymbol{\alpha}} = 0\} \cup \bigcup \{\boldsymbol{\alpha}\,{}^\wedge A^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in S_i\}$. $T^{(i+1)}$, A_{i+1} are as required. Let $T' = \bigcap_{i \in \omega} T^{(i)}$, then T' is as required. □

The following lemma resembles in contents and proof to Theorem 2.2. We leave its proof to the reader.

Lemma 2.14. *The following conditions are equivalent.*

(1) *Let* $\{B_i \mid i<\lambda\}$ *be a set of Borel subsets of* $B(T)$ *such that* $\bigcup_{i<\lambda} B_i = B(T)$, *then there is* $i<\lambda$ *and* $T' \leqslant T$ *such that* $B(T') \subseteq B_i$.

(2) *There is* $T' \leqslant T$ *such that, for every* $\alpha \in T'$, $\mathrm{cf}(\mu_\alpha) > \lambda$.

The last two lemmas in this section deal with another notion that to us seems basic in this framework.

Definition. Let $G: T \to P(A)$ and let I be an ideal on A; $B \subseteq A$ is a G, I-valve if, for every $C \in P(A-B) \cap I$, there is $T' \leqslant T$ such that $C \cap \bigcup G(T') = \emptyset$.

The following lemma will be used in Section 3.

Lemma 2.15. *Suppose* $M_1^*(T, \lambda, \chi)$ *holds,* $G: T \to P_\lambda(A)$ *is* λ*-bounded, and* I *is a* χ*-complete ideal on* A; *then there is* $B \in I$ *such that* B *is a* G, I*-valve.*

Proof. Suppose by contradiction there is no B as required. We define, by induction on $i<\chi$, $B_i \in I$. Let $i<\chi$, and suppose B_j has been defined for every $j<i$. $\bigcup_{j<i} B_j \in I$, so there is $B_i \in P(A - \bigcup_{j<i} B_j) \cap I$ such that, for every $T' \leqslant T$, $B_i \cap \bigcup G(T') \neq \emptyset$. Let $D_i \subseteq T$ be defined as follows: $D_i = \{\alpha \mid G(\alpha) \cap B_i = \emptyset\}$ hence D_i is T-small. By 2.3 and 2.12 there is $b \in B(T)$ such that $|\{i \mid B_i \cap \bigcup G(b) \neq \emptyset\}| \geqslant \lambda$. Since the B_i's are pairwise disjoint, this means that $|\bigcup G(b)| \geqslant \lambda$, i.e. G is not λ-bounded. A contradiction, and the lemma is proved. □

The following lemma refers to the notion $\mathrm{cov}(\mu, \lambda)$ which was defined in the abstract.

Lemma 2.16. *Suppose* $\mathrm{cov}(\chi, \lambda)$ *holds and let* $I = P_\chi(A)$; *let* T *be a tree such that, for every* $\alpha \in T$, $\mathrm{cf}(\mu_\alpha) \geqslant \chi$; *let* $G: T \to P_\lambda(A)$ *be* λ*-bounded then there is a* G, I*-valve* B *such that* $|B| < \lambda$.

Proof. Suppose G, T, λ, χ, A contradict the claim of the lemma.

Since I is χ-complete and $M_1^*(T, \lambda, \chi)$ holds, by 2.15, there is a G, I-valve B_0 such that $|B_0| < \chi$. Let $D \subseteq P_\lambda(B_0)$ generate $P_\lambda(B_0)$ and $|D| < \chi$. So by our assumption, for every $d \in D$, d is not a G, I-valve, hence there is $a_d \in P_\chi(A-d)$ such that, for every $T' \leqslant T$, $a_d \cap \bigcup G(T') \neq \emptyset$. Since χ is regular and $|D| < \chi$, $|\bigcup \{a_d \mid d \in D\}| < \chi$. By the choice of B_0 there is $T' \leqslant T$ such that $(\bigcup \{a_d \mid d \in D\} - B_0) \cap \bigcup G(T') = \emptyset$. For every $d \in D$ let $B_d \subseteq B(T')$ be defined as follows: $B_d = \{b \in B(T') \mid B_0 \cap \bigcup G(b) \subseteq d\}$, B_d is closed. Since G is λ-bounded and D

generates $P_\lambda(B_0) \bigcup \{B_d \mid d \in D\} = B(T')$. Hence by Lemma 2.14 there is $d_0 \in D$ and $T'' \leqslant T'$ such that $B(T'') \subseteq B_{d_0}$, hence $a_{d_0} \cap \bigcup G(T'') = \emptyset$, a contradiction. $\square$

Discussion

We have already mentioned that M_1 and M_1^* are properties that do not depend on cardinal arithmetic. We shall now make this assertion more precise. It turns out that in a sense to be defined $M_1(T, \lambda, \chi)$ and $M_1^*(T, \lambda, \chi)$ are Δ_1-properties of T.

Consider the language in which the primitive predicate symbols are $=$ and $\in$, and in which the primitive function symbols are cf and $|\cdot|$ (= the cardinality function.) The terms and formulas that allow in this language are defined inductively as in first order logic, with the addition of the following formation rules: if $\varphi(x)$ is a formula and v is a variable then $(\exists x \in v)\varphi(x)$, $(\forall x \in v)\varphi(x)$ are formulas, and $\{x \in v \mid \varphi(x)\}$ is a term.

We define inductively the Δ_0-formulas and terms. A variable is a Δ_0-term; if t is a Δ_0-term then $\mathrm{cf}(t)$, $|t|$ are Δ_0-terms; if φ_0 is a Δ_0-formula and v is a variable, then $\{x \in v \mid \varphi_0(x)\}$ is a Δ_0-term. If t_1, t_2 are Δ_0-terms, then $t_1 = t_2$, $t_1 \in t_2$ are Δ_0-formulas; if $\varphi_0(x)$ is a Δ_0-formula, then $(\exists x \in v)\varphi_0(x)$, $(\forall x \in v)\varphi_0(x)$ are Δ_0-formulas. The rest of the hierarchy is defined in the usual way.

The following theorem is trivial.

Theorem 2.17. *Let $V \subseteq W$ be universes of* ZFC *with the same cardinals and the same cofinality function. Let $\varphi_0(x_1 \cdots x_n)$ be a Δ_1-formula then for every $a_1, \ldots, a_n \in V$: $V \models \varphi_0[a_1, \ldots, a_n]$ iff $W \models \varphi_0[a_1, \ldots, a_n]$.*

It is not difficult to prove the following theorem (details are left to the reader).

Theorem 2.18. *There are Δ_1-formulas $\varphi(T, \lambda, \chi)$, and $\varphi^*(T, \lambda, \chi)$ which are equivalent to $M_1(T, \lambda, \chi)$, $M_1^*(T, \lambda, \chi)$ respectively.*

In [10] the partition theorems were generalized in two directions. Let T be a tree and I^T be a function whose domain is T such that for every $\boldsymbol{\alpha} \in T$: if $\mu_{\boldsymbol{\alpha}} \neq 0$, then $I^T(\boldsymbol{\alpha})$ is an ideal on $\mathrm{suc}^T(\boldsymbol{\alpha})$, and otherwise $I^T(\boldsymbol{\alpha}) = \emptyset$. We denote $I^T(\boldsymbol{\alpha})$ by $I^T_{\boldsymbol{\alpha}}$. The pair $\langle T, I^T \rangle$ is called an I-tree. $\langle T, I^T \rangle$ is denoted by $\bar{T}$, $\langle T_1, I^{T_1} \rangle$ is denoted by $\bar{T}_1$ etc.

Definition 2.1. $\bar{T}_1 \leqslant \bar{T}$ if T_1 is a subtree of T, for every $\boldsymbol{\alpha} \in T_1$:

$$\mathrm{suc}^{T_1}(\boldsymbol{\alpha}) \notin I^T_{\boldsymbol{\alpha}} \quad \text{and} \quad I^{T_1}_{\boldsymbol{\alpha}} = \{a \cap \mathrm{suc}^{T_1}(\boldsymbol{\alpha}) \mid a \in I^T_{\boldsymbol{\alpha}}\}.$$

Definition 2.2. Let $\bar{T}_1$, $\bar{T}$ be I-trees and κ be a cardinal, $\bar{T}_1 \leqslant^\kappa \bar{T}$ if $\bar{T}_1 \leqslant \bar{T}$ and for every $\boldsymbol{\alpha} \in T_1$: if $\mu^T_{\boldsymbol{\alpha}} < \kappa$, then $\mathrm{suc}^T(\boldsymbol{\alpha}) = \mathrm{suc}^{T_1}(\boldsymbol{\alpha})$.

Our original definition of $T_1 \leqslant T_2$ is of course a special case of the above more general definition, for if $I^{t_i}(\boldsymbol{\alpha})$ is defined to be $P_{\mu_\alpha}(\mathrm{suc}^{T_i}(\alpha))$, then $T_1 \leqslant T_2$ iff $\bar{T}_1 \leqslant \bar{T}_2$ iff $\bar{T}_1 \leqslant^0 \bar{T}_2$.

Theorems 2.2, 2.3 can be reformulated to apply and to be proved in the case of I-trees. The proofs however do not change. There are obvious provable generalizations of the partition theorems for the notion of $\leqslant^\kappa$.

The next question that we want to consider is what can be said in the following situation. Let $G: T \to \lambda$, and we do not assume anymore that, for every $\boldsymbol{\alpha} \in T$, $\mathrm{cf}(\mu_{\boldsymbol{\alpha}}) \neq \mathrm{cf}(\lambda)$. It turns out, that it is still possible to get a partition theorem. Such a partition theorem was used in [11]. For every λ, let $T_\lambda = \langle {}^{\omega>}\lambda, \trianglelefteq \rangle$.

Theorem 2.19. (a) *Let $\lambda > \mathrm{cf}(\lambda) > \aleph_0$, $\{\lambda_i \mid i < \mathrm{cf}(\lambda)\}$ be a continuous strictly increasing sequence converging to λ, and for every $i < \mathrm{cf}(\lambda)$ such that $\mathrm{cf}(i) = \aleph_0$ let $\{\lambda_{in} \mid n \in \omega\}$ be a strictly increasing sequence of successor cardinals converging to λ_i. Let $F: T_\lambda \to \mathrm{cf}(\lambda)$. Then there is a club $C \subseteq \mathrm{cf}(\lambda)$ such that for every $i \in C$ there is $T_i \leqslant \langle \prod_{n\in\omega} \lambda_{in}, \trianglelefteq \rangle$ such that, for every $\boldsymbol{\alpha} \in T_i$, $F(\boldsymbol{\alpha}) < i$.*

(b) *Let $\lambda > 2^{\mathrm{cf}(\lambda)} > \mathrm{cf}(\lambda) > \aleph_0$, and let $F: T_\lambda \to \lambda$, then there is $T \leqslant T_\lambda$ and a club $C \subseteq \lambda$ such that, for every $i \in C$, i is a cardinal, and, for every $i \in C$, $T \restriction i \leqslant T_i$ and, for every $\boldsymbol{\alpha} \in T \restriction i$, $F(\boldsymbol{\alpha}) < i$.*

An interesting direction of investigation is to seek for partition theorems for trees of height $> \omega$. Our knowledge here is sporadic and can be regarded just a beginning. So we see no point in considering at this stage the most general trees.

Let $T_{\mu,\alpha} = \langle {}^{\alpha>}\mu, \trianglelefteq \rangle$, where ${}^{\alpha>}A$ is the set of sequences of length $<\alpha$ with elements in A. Let T be a tree and T_1 a subtree of T; $T_1 \leqslant T$ is, for every $\boldsymbol{\alpha} \in T_1$, $\mu^{T_1}_{\boldsymbol{\alpha}} = \mu^T_{\boldsymbol{\alpha}}$ and for every chain $\{\boldsymbol{\alpha}_i \mid i < \delta\} \subseteq T_1$, if $\bigcup_{i\in\delta} \boldsymbol{\alpha}_i \in T$, then $\bigcup_{i\in\delta} \boldsymbol{\alpha}_i \in T_1$.

The first fact we point out is that the straightforward generalization of Theorem 2.2 is never true.

Example 2.20. Let μ be regular and $\lambda < \mu$, then there is $F: T_{\mu,\omega+1} \to \lambda$ such that for every $T \leqslant T_{\mu,\omega+1} F(T) = \lambda$.

Proof. Let $\{S_i \mid i < \lambda\}$ be a partition of $\{\alpha < \mu \mid \mathrm{cf}(\alpha) = \omega\}$ into λ stationary subsets. Define F as follows: If $\boldsymbol{\alpha} \notin {}^\omega\mu$ or if $\mathrm{cf}(\bigcup_{n\in\omega} \boldsymbol{\alpha}(n)) \neq \omega$ then $F(\alpha) = 0$; if $\bigcup_{n\in\omega} \boldsymbol{\alpha}(n) \in S_i$ then $F(\boldsymbol{\alpha}) = i$. Let $T \leqslant T_{\mu,\omega+1}$, then there is a club C in μ such that for every $\alpha \in C$ such that $\mathrm{cf}(\alpha) = \omega$, there is $\boldsymbol{\alpha} \in T \cap {}^\omega\mu$ such that $\bigcup_{n\in\omega} \boldsymbol{\alpha}(n) = \alpha$. (To define C let $M = \langle \mu \cup T, \trianglelefteq, <, \mathrm{Con} \rangle$ where $\mathrm{Con}(\boldsymbol{\alpha}, \alpha) = \boldsymbol{\alpha}{}^\wedge\langle \alpha \rangle$, and C be the set of limit points of $\{\alpha \mid M \restriction {<} M\}$).

So, for every $i \in \lambda$, $S_i \cap C \neq \emptyset$, and hence $F(T) \ni i$. $\square$

The above example means that in order to obtain a partition theorem we have to make stronger assumptions on the functions $F: T \to \lambda$.

Definition. (a) Let $f: T_{\mu,\alpha} \to A$, we say that f has a finite character (f is FC), if for every $\boldsymbol{\beta} \in T_{\mu,\alpha}$ there is a finite set $\sigma \subseteq \alpha$ such that for every $\boldsymbol{\gamma} \in T_{\mu,\alpha}$: if $\boldsymbol{\gamma} \restriction \sigma = \boldsymbol{\beta} \restriction \sigma$, then $f(\boldsymbol{\gamma}) = f(\boldsymbol{\beta})$.

(b) Let $F: T_{\mu,\alpha} \to \lambda$; F is pseudo finitary (PF), if there is a family $\{f_i \mid i < |\alpha|\}$ of FC functions from $T_{\mu,\alpha}$ to λ such that, for every $\boldsymbol{\beta} \in T_{\mu,\alpha}$, $F(\boldsymbol{\beta}) = \bigcup \{f_i(\boldsymbol{\beta}) \mid i < |\alpha|\}$.

Let $P(\mu, \alpha, \lambda)$ mean that for every PF function $F: T_{\mu,\alpha} \to \lambda$ there is $T \leqslant T_{\mu,\alpha}$ such that $\bigcup F(T) < \lambda$.

Note that $P(\mu, \omega, \lambda)$ is dealt with in Theorem 2.2.

We know of cases when $\alpha > \omega + 1$ and $P(\mu, \alpha, \lambda)$ holds. In all of these cases μ is a large cardinal or μ is a large cardinal in some inner model.

Definition 2.3. (a) Let μ, λ, χ be cardinals and α be an ordinal; let $\mathcal{F}$ be a set of functions from μ to λ. We define a game $G \overset{\text{def}}{=} G_{\mathcal{F},\alpha,\chi}$. The game G has α moves. In the νth move the villain picks a function f_ν in $\mathcal{F}$, and the hero picks a set $a_\nu \subseteq \mu$ such that $|f_\nu(a_\nu)| < \chi$. The hero wins if $|\bigcap_{\nu<\alpha} a_\nu| = \mu$, otherwise the villain wins.

(b) Let μ, λ, χ, κ be cardinals and α be an ordinal. μ is $\langle \lambda, \chi, \kappa, \alpha \rangle$-Galvin, if for every set $\mathcal{F}$ of functions from μ to λ such that $|\mathcal{F}| = \kappa$, the hero has a winning strategy in the game $G_{\mathcal{F},\alpha,\chi}$.

μ is a λ-Galvin, if it is $\langle \lambda, \lambda, 2^\lambda, \omega \rangle$-Galvin, μ is weakly λ-Galvin if it is $\langle \lambda, \lambda, \lambda, \omega \rangle$-Galvin.

The fact, that we know are summarized in the following theorems.

Theorem 2.21. (a) *If μ is a Ramsey cardinal λ is regular and $\alpha < \lambda < \mu$, then $P(\mu, \alpha, \lambda)$ holds.*

(b) *If μ is λ-Galvin and* $\mathrm{cf}(\lambda) > \aleph_0$, *then for every $\alpha < \aleph_1$ $P(\mu, \alpha, \lambda)$ holds.*

(c) *If μ is weakly λ-Galvin and* $\mathrm{cf}(\lambda) > \aleph_0$, *then $P(\mu, \omega + 2, \lambda)$ holds.*

Theorem 2.22. (a) *If μ is measurable then μ is $\langle \lambda, 1, 2^\mu, \alpha \rangle$-Galvin for every $\alpha, \lambda < \mu$.*

(b) *If W is the universe of a Levy collapse of a measurable μ to $\aleph_2$, then in W, $\aleph_2$ is $\langle \aleph_1, 1, 2^{\aleph_2}, \alpha \rangle$-Galvin for every countable α. (See* [5] *and* [12].)

(c) *If μ is weakly compact then μ is $\langle \lambda, 2, \mu, \alpha \rangle$-Galvin for every λ, $\alpha < \mu$.*

(d) *If W is a universe of a Levy collapse of a weakly compact μ to $\aleph_2$, then in W, $\aleph_2$ is $\langle \aleph_1, 2, \aleph_2, \alpha \rangle$-Galvin for every countable α.*

The above theorems raise the following question.

Question. (a) Does $P(\mu, \omega + 2, \aleph_1)$ imply that μ is large in L? Does it imply that μ is weakly compact in L?

(b) Does $(\forall \lambda, \alpha < \mu)(\alpha < \mathrm{cf}(\lambda) = \lambda < \mu \to P(\mu, \alpha, \lambda))$ imply that μ is a large cardinal?

Proof of Theorem 2.21. The proof of (a) is trivial.

We start with the proof of (c) which is easy. Let $F: T_{\mu,\omega+2} \to \lambda$ be PF. W.l.o.g. there are functions f_n, $n \in \omega$, such that: $\mathrm{Dom}(f_n) = \{g \mid g: n \cup \{\omega\} \to \mu\}$ and, for every $\boldsymbol{\alpha} \in T_{\mu,\omega+2}$, $F(\boldsymbol{\alpha}) = \bigcup \{f_n(\boldsymbol{\alpha} \restriction (n \cup \{\omega\})) \mid \mathrm{Dom}(\boldsymbol{\alpha}) \supseteq n \cup \{\omega\}\}$. For every $n \in \omega$ and $\boldsymbol{\alpha} \in {}^{n>}\mu$ let $f_{\boldsymbol{\alpha}}: \mu \to \lambda$ be defined as follows: $f_{\boldsymbol{\alpha}}(\beta) = f_n(\boldsymbol{\alpha} \cup \{\langle \omega, \beta \rangle\})$, and let $\mathscr{F} = \{f_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in {}^{\omega>}\mu\}$. So the hero has a winning strategy S in the game $G_{\mathscr{F},\omega,\lambda}$.

We show that there is a family $\{s(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in {}^{\omega}\mu\}$ such that, for every $\boldsymbol{\alpha} \in {}^{\omega}\mu$, $s(\boldsymbol{\alpha}) \subseteq \mathrm{suc}(\boldsymbol{\alpha})$, $|s(\boldsymbol{\alpha})| = \mu$ and, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in {}^{\omega}\mu$, $\boldsymbol{\alpha}' \in s(\boldsymbol{\alpha})$, $\boldsymbol{\beta}' \in s(\boldsymbol{\beta})$ and $n \in \omega$: if $\boldsymbol{\alpha} \restriction n = \boldsymbol{\beta} \restriction n$ then $f_n(\boldsymbol{\alpha}' \restriction (n \cup \{\omega\})) = f_n(\boldsymbol{\beta} \restriction (n \cup \{\omega\}))$.

For every $\boldsymbol{\alpha} \in {}^{\omega}\mu$ we define a play $B_{\boldsymbol{\alpha}}$ in the game $G_{\mathscr{F},\omega,\lambda}$, such that if $\boldsymbol{\alpha} \restriction n = \boldsymbol{\beta} \restriction n$ then the first n moves of $B_{\boldsymbol{\alpha}}$ are the same as the first n moves of $B_{\boldsymbol{\beta}}$. In the nth move the villain picks the function $f_{\boldsymbol{\alpha} \restriction n}$ and the hero answers by a choice of $a_{\boldsymbol{\alpha},n} \subseteq \mu$ according to his winning strategy S. Let $s(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha}^\wedge \langle \beta \rangle \mid \beta \in \bigcap_{n \in \omega} a_{\boldsymbol{\alpha},n}\}$. It is easy to see that $s(\boldsymbol{\alpha})$ is as required.

Now we define $G: {}^{\omega>}\mu \to \lambda$. Let $\boldsymbol{\alpha} \in {}^{n}\mu$; let $\boldsymbol{\beta} \rhd \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in {}^{\omega}\mu$ and let $\boldsymbol{\gamma} \in s(\boldsymbol{\beta})$; define $G(\boldsymbol{\alpha}) = f_n(\boldsymbol{\gamma})$, clearly the definition does not depend on the choice of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$.

By Theorem 2.2 there is $T \leqslant \langle {}^{\omega>}\mu, \trianglelefteq \rangle$ such that $\bigcup G(T) < \lambda$. Let $T_1 = \{H(s(\boldsymbol{\alpha})) \mid \boldsymbol{\alpha} \in {}^{\omega}\mu$ and, for every $n \in \omega$, $\boldsymbol{\alpha} \restriction n \in T\}$, hence $\bigcup F(T_1) < \lambda$. □

Proof of (b). Let μ be λ-Galvin, $\mathrm{cf}(\lambda) > \aleph_0$, and α be a countable ordinal. W.l.o.g. λ is regular. Let $F: T_{\mu,\alpha} \to \lambda$ be a PF function. W.l.o.g. there is a sequence $\{\langle \sigma_n, f_n \rangle \mid n \in \omega\}$ such that, for every $n \in \omega$, $\sigma_n \subseteq_{n+1}$, $\bigcup_{n \in \omega} \sigma_n = \alpha$, for every $n \in \omega$, $\mathrm{Dom}(f_n) = \{g \mid g: \sigma_n \to \mu\}$, and, for every $\boldsymbol{\alpha} \in T_{\mu,\alpha}$, $F(\boldsymbol{\alpha}) = \bigcup \{f_n(\boldsymbol{\alpha} \restriction \sigma_n) \mid \sigma_n \subseteq \mathrm{Dom}(\boldsymbol{\alpha})\}$.

Let $\mathscr{F}$ be the set of all functions from μ to λ, and S be a winning strategy for the hero in the game $G_{\mathscr{F},\omega,\lambda}$.

For every $\boldsymbol{\alpha} \in T_{\mu,\alpha}$, we define a play $B_{\boldsymbol{\alpha}}$ in the game $G_{\mathscr{F},\omega,\lambda}$ in which the hero plays according to S, and a sequence $\{v^n_{\boldsymbol{\alpha}} \mid n \in \omega\}$ of ordinals $<\lambda$.

We define the plays by induction on $n \in \omega$ simultaneously for all $\boldsymbol{\alpha} \in T_{\mu,\alpha}$. Our indiction hypothesis is that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same length and $\boldsymbol{\alpha} \restriction \sigma_n = \boldsymbol{\beta} \restriction \sigma_n$, then after the nth step of the induction the same number of moves were defined in both $B_{\boldsymbol{\alpha}}$ and $B_{\boldsymbol{\beta}}$ and these moves are the same in both plays.

Step n: In this step we define an additional move in the play $B_{\boldsymbol{\alpha}}$ for every $\boldsymbol{\alpha}$ such that $\mathrm{Dom}(\boldsymbol{\alpha}) \in \sigma_n$.

Let $\sigma_n = \{\alpha_0, \ldots, \alpha_k\}$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_k$. By a downward induction on $i \leqslant k$, we define for every $\boldsymbol{\beta}$ such that $\mathrm{Dom}(\boldsymbol{\beta}) = \alpha_i$: an additional move in the play $B_{\boldsymbol{\beta}}$, and an ordinal $v_{\boldsymbol{\beta}} < \lambda$.

$i = k$: Let $\mathrm{Dom}(\boldsymbol{\beta}) = \alpha_k$; let $g_{\boldsymbol{\beta}}: \mu \to \lambda$ be defined as follows: $g_{\boldsymbol{\beta}}(v) = f_n(\boldsymbol{\beta} \restriction \sigma_n \cup \{\langle \alpha_k, v \rangle\})$. The additional move in $B_{\boldsymbol{\beta}}$ is the following: the villain picks $g_{\boldsymbol{\beta}}$, and the hero answers according to his winning strategy S. Let $v^n_{\boldsymbol{\beta}} = \bigcup g_{\boldsymbol{\beta}}(a_{\boldsymbol{\beta},n})$, where $a_{\boldsymbol{\beta},n}$ is the set picked to by the hero at the last move.

$i=j-1$: Suppose that ν_γ has been defined for every γ such that $\mathrm{Dom}(\gamma)=\alpha_j$, and we assume by induction that if $\gamma \restriction \sigma_n = \restriction \sigma_n$ then $\nu^n_\gamma = \nu^n_\delta$. Let $\mathrm{Dom}(\boldsymbol{\beta}) = \alpha_{j-1}$. Let $g_{\boldsymbol{\beta}}: \mu \to \lambda$ be defined as follows: $g_\beta(\xi)=\nu$ is for some $\gamma \geq \boldsymbol{\beta}^\wedge\langle\xi\rangle$ such that $\mathrm{Dom}(\gamma)=\alpha_j$, $\nu^n_\gamma=\nu$. The definition of $g_{\boldsymbol{\beta}}(\xi)$ does not depend on the choice of γ. The additional move in $B_{\boldsymbol{\beta}}$ is the following: the villain picks $g_{\boldsymbol{\beta}}$ and the hero answers according to S. Let $\nu^n_{\boldsymbol{\beta}} = \bigcup g_{\boldsymbol{\beta}}(a_{\boldsymbol{\beta},n})$, where $a_{\boldsymbol{\beta},n}$ is the set picked by the hero in the last move. This completes the definition of the $B_{\boldsymbol{\alpha}}$'s.

Let $a_{\boldsymbol{\beta}} \subseteq \mu$ be the intersection of the sets picked by the hero in $B_{\boldsymbol{\beta}}$, hence $|a_{\boldsymbol{\beta}}|=\mu$. Let T be the subtree of $T_{\mu\alpha}$ that has the following properties: T is closed under limits of chains, and if $\boldsymbol{\alpha}\in T$, then $\mathrm{suc}^T(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha}^\wedge\langle\nu\rangle \mid \nu \in a_{\boldsymbol{\alpha}}\}$. It is easy to see that $T \leq T_{\mu\alpha}$.

For every n, let $\boldsymbol{\beta}$ be such that $\mathrm{Dom}(\boldsymbol{\beta}) = \min(\sigma_n)$. It is easy to see that $\bigcup\{f_n(\boldsymbol{\alpha}\restriction\sigma_n) \mid \boldsymbol{\alpha}\in T \text{ and } \sigma_n \subseteq \mathrm{Dom}(\boldsymbol{\alpha})\} < \nu^n_{\boldsymbol{\beta}}$. So since $\mathrm{cf}(\lambda)>\aleph_0$, $\bigcup F(T) < \lambda$. □

3. Large free subtrees

Let F be a function whose range is a set of sets; $A \subseteq \mathrm{Dom}(F)$ is called F-free if, for every $a, b \in A$, $a \notin F(b)$.

Let $Q_2(T,\lambda)$ denote the following property; for every λ-bounded $F: T \to P_\lambda(T)$ such that, for every $\boldsymbol{\alpha}\in T$, $H(\boldsymbol{\alpha}) \cap F(\boldsymbol{\alpha}) = \emptyset$, there is $T_1 \leq T$ such that T_1 is F-free.

Let $Q_1(T,\lambda)$ mean that there is $T_1 \leq T$ such that for every $\boldsymbol{\alpha}\in T_1$ either $\mu_{\boldsymbol{\alpha}}=0$ or $M_1^*(T_1(\boldsymbol{\alpha}),\lambda,\mu_{\boldsymbol{\alpha}})$ holds.

The aim of this section is to prove the following equivalence.

Theorem 3.1. *$Q_2(T,\lambda)$ is equivalent to $Q_1(T,\lambda)$.*

Proof. We first prove that $Q_1(T,\lambda)$ implies $Q_2(T,\lambda)$. We start with an easy case.

Lemma 3.2. *Let $\neg M_1^*(T,\lambda,\mu^T_\lambda)$ hold, then there is $F: T\to P_\lambda(T)$ such that F is λ-bounded, and for no $T_1 \leq Q$, T_1 is F-free.*

Proof. By 2.8, there is a λ-bounded $F': T \to P_\lambda(\mu^T_\lambda)$ such that, for every $T_1 \leq T$, $\bigcup F'(T_1) = \mu^T_\lambda$. Let $F: T \to P_\lambda(\mathrm{suc}^T(\Lambda))$ be defined as follows: $F(\boldsymbol{\alpha}) = \{\langle\nu\rangle \mid \nu \in F'(\boldsymbol{\alpha}) \text{ and } \langle\nu\rangle \nleq \boldsymbol{\alpha}\}$. (Recall that we assume that, for every $\boldsymbol{\alpha}\in T$, $\mathrm{suc}^T(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha}^\wedge\langle\nu\rangle \mid \nu \in \mu^T_{\boldsymbol{\alpha}}\}$.)

We show that F is as required. Suppose by contradiction $T_1 \leq T$ is F-free. Let $\langle\nu\rangle \in \mathrm{suc}^{T_1}(\Lambda)$, and let $T_2 = T_1 - \langle\nu\rangle^\wedge T_1(\langle\nu\rangle)$. Clearly $T_2 \leq T$, but $\bigcup F'(T_2) \not\ni \nu$. This is in contradiction to the choice of F'.

This concludes the proof of the lemma. □

We now turn to the general case. Suppose $\neg Q_1(T,\lambda)$ holds. We have to construct $F:T\to P_\lambda(T)$ that will show that $Q_2(T,\lambda)$ does not hold. This is done in a way similar to the construction in 2.8. We define by induction sequences $\{T_i \mid i<i_0\}$ and $\{A_i \mid i<i_0\}$ such that: $T_i \leqslant T$, $A_i \subseteq T_i$ and A_i is an antichain in T_i. Suppose T_j, A_j have been defined for every $j<i$. Let $T_i' = \bigcup\{T' \leqslant T \mid T' \cap \bigcup_{j<i} A_j = \emptyset\}$. Clearly either $T_i' = \emptyset$ or $T_i' \leqslant T$. In the first case we define $i_0 = i$, and in the second case we define $T_i = T_i'$. Let $A_i' = \{\boldsymbol{\alpha} \in T_i \mid \neg M_1^*(T_i(\boldsymbol{\alpha}), \lambda, \mu_{\boldsymbol{\alpha}})\}$, and $A_i = \{\boldsymbol{\alpha} \in A_i' \mid \text{for every } \boldsymbol{\beta} < \boldsymbol{\alpha}\ \boldsymbol{\beta} \notin A_i'\}$. Since $\neg Q_1(T,\lambda)$ holds $A_i \neq \emptyset$.

By 3.2 for every $i<i_0$ and $\boldsymbol{\alpha} \in A_i$ there is a λ-bounded function $F_{\boldsymbol{\alpha}}: T_i(\boldsymbol{\alpha}) \to P_\lambda(T_i(\boldsymbol{\alpha}))$ such that there is no $T' \leqslant T_i(\boldsymbol{\alpha})$ which is $F_{\boldsymbol{\alpha}}$-free.

We can now define $F:T\to P_\lambda(T)$ which refutes $Q_2(T,\lambda)$. Let $\boldsymbol{\beta} \in T$, if for some $\boldsymbol{\alpha} \in \bigcup_{i<i_0} A_i$ and some $\boldsymbol{\gamma}\boldsymbol{\beta} = \boldsymbol{\alpha}{}^\wedge\boldsymbol{\gamma}$ then $F(\boldsymbol{\beta}) = \boldsymbol{\alpha}{}^\wedge F_{\boldsymbol{\alpha}}(\boldsymbol{\gamma})$; otherwise $F(\boldsymbol{\beta}) = \emptyset$.

The arguments that F is λ-bounded, and that no $T_1 \leqslant T$ is F-free, are the same as the corresponding arguments in 2.8.

$Q_1(T,\lambda) \Rightarrow Q_2(T,\lambda)$: We divide the proof into several subclaims.

Lemma 3.3. (a) *Suppose T is a tree and for every $\boldsymbol{\alpha} \in T$ either $\mu_{\boldsymbol{\alpha}} = 0$ or $\mu_{\boldsymbol{\alpha}}^T > \lambda$; let $F:T\to P_\lambda(T)$, then there is $T_1 \leqslant T$ such that if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T_1$ and $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ then $\boldsymbol{\beta} \notin F(\boldsymbol{\alpha})$.*

(b) *Let $F:T\to P_\lambda(T)$ be λ-bounded for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T$ if $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$, then $\boldsymbol{\alpha} \notin F(\boldsymbol{\beta})$ and $\boldsymbol{\beta} \notin F(\boldsymbol{\alpha})$; then there is λ-bounded $F':T\to P_\lambda(T)$ such that for every $\boldsymbol{\alpha} \in T$ and $\boldsymbol{\beta} \in F'(\boldsymbol{\alpha})$ there is v and $\boldsymbol{\gamma} < \boldsymbol{\alpha}$ such that $\boldsymbol{\beta} = \boldsymbol{\gamma}{}^\wedge\langle v\rangle$, and for every subtree T' of T if T' is F'-free, then T' is F-free.*

For the following definition recall the convention that, for every $\boldsymbol{\alpha} \in T$, $\operatorname{suc}^T(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha}{}^\wedge\langle v\rangle \mid v \in \mu_{\boldsymbol{\alpha}}^T\}$.

Definition. Let $G:T\to P(\mu_\Lambda^T)$ and let $A \subseteq T$, then: (a) A is G-free if, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A - \{\Lambda\}$, $\boldsymbol{\alpha}(0) \notin G(\boldsymbol{\beta})$. (b) A is downwards G-free if, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A - \{\Lambda\}$, if $\boldsymbol{\alpha}(0) > \boldsymbol{\beta}(0)$ then $\boldsymbol{\alpha}(0) \notin G(\boldsymbol{\beta})$. A is upwards G-free if, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A - \{\Lambda\}$, if $\boldsymbol{\alpha}(0) > \boldsymbol{\beta}(0)$ then $\boldsymbol{\alpha}(0) \notin G(\boldsymbol{\beta})$.

Lemma 3.4 (Main Lemma). (a) *Suppose $M_1^*(T,\lambda,\mu_\Lambda^T)$ holds and $G:T\to P_\lambda(\mu_\Lambda^T)$ is λ-bounded, then there is $T_1 \leqslant T$ such that T_1 is downwards G-free.*

(b) *Let T, λ and G be as in* (a), *then there is $T_1 \leqslant T$ such that T_1 is upwards G-free.*

Both parts of Lemma 3.3 are trivial so they are left to the reader. Lemma 3.4 is where the main point of the proof lies. We postpone the proof of 3.4 for a while, and we first prove that $Q_1(T,\lambda)$ implies $Q_2(T,\lambda)$, assuming 3.3 and 3.4.

So suppose $Q_1(T,\lambda)$ holds and let $F:T\to P_\lambda(T)$ be a λ-bounded function such that, for every $\boldsymbol{\alpha} \in T$, $F(\boldsymbol{\alpha}) \cap H(\boldsymbol{\alpha}) = \emptyset$. Hence there is $T_1 \leqslant T$ such that for every $\boldsymbol{\alpha} \in T_1$ wither $\mu_{\boldsymbol{\alpha}} = 0$ or $M_1^*(T_1(\boldsymbol{\alpha}), \lambda, \mu_{\boldsymbol{\alpha}})$ holds; and therefore, for every $\boldsymbol{\alpha} \in T_1$ either $\mu_{\boldsymbol{\alpha}} = 0$ or $\mu_{\boldsymbol{\alpha}} > \lambda$. Let $T_2 \leqslant T_1$ be as assured by 3.3(a). Thus for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T_2$ if $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ then $\boldsymbol{\alpha} \notin F(\boldsymbol{\beta})$ and $\boldsymbol{\beta} \notin F(\boldsymbol{\alpha})$. Let F' be as assured by 3.3(b). For every $\alpha \in T_2$ let $F'_\alpha : T_2(\alpha) \to \mu_\alpha$ be defined as follows: $F'_\alpha(\gamma) = \{\nu \mid \alpha^\wedge \langle \nu \rangle \in F'(\alpha^\wedge \gamma)\}$. Clearly, for every $\alpha \in T_2$, F'_α is λ-bounded.

We now define by induction on $n \in \omega$ sets $A_n \subseteq T_2$ and for every $\alpha \in A_n$ we define some $T_\alpha \leqslant T_2(\alpha)$. A_n will be a subset of the nth level of T_2, and T_α will be F'_α-free. Our purpose is that $\bigcup_{n\in\omega} A_n$ will be an F-free T_2-large subtree of T_2.

Let $A_0 = \{\Lambda\}$ and let $T_\Lambda \leqslant T_2$ be F'_Λ-free. Such T_Λ exists by Lemma 3.4. Suppose A_n has been defined, and for every $\alpha \in A_n$, T_α has been defined. Let $A_{n+1} = \bigcup \{\alpha^\wedge \mathrm{suc}^{T_\alpha}(\Lambda) \mid \alpha \in A_n\}$, for every $\beta \in A_{n+1}$ let $T'_\beta = (\bigcup \{\alpha^\wedge T_\alpha \mid \alpha \in A_n\})(\beta)$. Clearly $T'_\beta < T_2(\beta)$. By Lemma 3.4 there is $T_\beta \leqslant T'_\beta$ which is F'_β-free, this concludes the definition of the A_n's and the T_α's.

It is easy to see that $\bigcup_{n\in\omega} A_n \leqslant T_2$ and is F-free.

We can now turn to the proof of 3.4.

Proof of (3.4(a)). *Case* 1: *μ_Λ is regular.* Let $M_1^*(T, \lambda, \mu_\Lambda)$ hold and $G : T \to P_\lambda(\mu_\Lambda)$ be λ-bounded. So $\mu_\Lambda > \lambda$. By the second part of 2.4(h), we can assume that there is $\chi < \mu_\Lambda$ such that, for every $\alpha \in \mathrm{suc}^T(\Lambda)$, $M_1^*(T(\alpha), \lambda, \chi)$ holds.

Suppose by contradiction no T-large subtree of T is downwards G-free. We define, by induction on $i < \chi$, $A_i \subseteq \mu_\Lambda$. Suppose, for every $j < i$, A_j has been defined. Let $A_i \subseteq \mu_\Lambda$ be the maximal set with the following property: for every $\alpha \in A_i$, α is the minimal ordinal in the set $\{\alpha' \mid \alpha' \in \mu_\Lambda - \bigcup_{j<i} A_j - A_i \cap \alpha'$ and there is $T' \leqslant T(\langle \alpha' \rangle)$ such that $\bigcup G(\langle \alpha \rangle {}^\wedge T') \cap A_i \cap \alpha' = \emptyset\}$. For every $\alpha \in A_i$ let $T^i_\alpha \leqslant T(\langle \alpha \rangle)$ be such that $\bigcup G(\langle \alpha \rangle {}^\wedge T^i_\alpha) \cap A_i \cap \alpha = \emptyset$. Clearly, for every $i < \chi$, $T^i \overset{\mathrm{def}}{=} \bigcup \{\langle \alpha \rangle {}^\wedge T^i_\alpha \mid \alpha \in A_i\}$ is downward G-free. So by our assumption T^i is not T-large and hence $|A_i| < \mu_\Lambda$.

Since μ_Λ is regular and $\chi < \mu_\Lambda$, $\bigcup_{i<\chi} A_i \neq \mu_\Lambda$. Let $\alpha \in \mu_\Lambda - \bigcup_{i<\chi} A_i$, so by the definition of the A_i's, for every $i < \chi$, the set $\{\beta \in T(\langle \alpha \rangle) \mid G(\langle \alpha \rangle {}^\wedge \beta) \cap A_i = \emptyset\}$ is $T(\langle \alpha \rangle)$-small. So, by 2.12 and 2.3(*b*), there is $b \in B(T(\langle \alpha \rangle))$ such that $|\{i \mid (\exists \beta \in b)(G\langle \alpha \rangle {}^\wedge \beta) \cap A_i \neq \emptyset\}| \geqslant \lambda$; since the A_i's are pairwise disjoint, this means that $|\bigcup G(b)| \geqslant \lambda$, i.e. G is not λ-bounded. A contradiction.

Case 2: *μ_Λ is singular.* Let $M_1^*(T, \lambda, \mu_\Lambda)$ hold and suppose by contradiction $G : T \to P_\lambda(\mu_\Lambda)$ is a counter-example to the claim of the lemma. By $M_1^*(T, \lambda, \mu_\Lambda)$, $\mu_\Lambda > \lambda$. By the second part of 2.4(h), we can assume that there is $\chi < \mu_\Lambda$ such that, for every $\alpha \in \mathrm{suc}^T(\Lambda)$, $M_1^*(T(\alpha), \lambda, \chi)$ holds. If we enlarge χ, M_1^* still holds, so we can further assume that $\chi > \lambda + \mathrm{cf}(\mu_\Lambda)$ and is regular. Let $\kappa = \mathrm{cf}(\mu_\Lambda)$, and $\{\mu_i \mid i < \kappa\}$ be a strictly increasing sequence of successor cardinals converging to μ_Λ, and $\chi^+ < \mu_0$. for every $i < \kappa$ let $E_i = \{\alpha \mid \bigcup_{j<i} \mu_j \leqslant \alpha < \mu_i\}$, hence $|E_i| = \mu_i$. For every $i < \kappa$ let $\{E_{i,\nu} \mid \nu < \chi^+\}$ be a partition of E_i to sets of cardinality μ_i. For a set of ordinals A, let $\hat{A} = \{\langle \alpha \rangle \mid \alpha \in A\}$. By (2.4(h), $M_1^*(T[\hat{E}_{i\nu}], \lambda, \mu_i)$ holds for every i and ν. Since each μ_i is regular we can apply the previous case, and can thus can assume that each $T(\hat{E}_{i\nu}]$ is downwards G-free.

If $A \subseteq \mu_\Lambda$, let

$$R(A) = \{\alpha \in \mu_\Lambda \mid \alpha \notin A \text{ and } (\exists T' < T(\langle \alpha \rangle))(A \cap \bigcup G(\langle \alpha \rangle {}^\wedge T') = \emptyset\}.$$

We regard $\kappa\times\chi^+$ as being ordered lexicographically, i.e. $\langle i, v\rangle<\langle i', v'\rangle$ iff $i<i'$ or $i=i'$ and $v<v'$.

We now define by induction on $\eta<\chi$ sets $A_\eta\subset\mu_\Lambda$. Each A_η will be defined as follows: we define a sequence of sets $\{A_{\eta\theta}\mid\theta<\theta_\eta\}$ and A_η will be defined to be $\bigcup\{A_{\eta\theta}\mid\theta<\theta_\eta\}$. Suppose $A_{\eta'}$ has been defined for every $\eta'<\eta$, and $A_{\eta\theta'}$ has been defined for every $\theta'<\theta$. Let $P(i, v)$ denote the following property:

$$\bigcup_{\theta'<\theta}A_{\eta\theta'}\cap E_i=\emptyset\quad\text{and}\quad\left|R\Big(\bigcup_{\theta'<\theta}A_{\eta\theta'}\Big)\cap\Big(E_{iv}-\bigcup_{\eta'<\eta}A_{\eta'}\Big)\right|=\mu_i.$$

If, for no $\langle i, v\rangle$, $P(i, v)$ holds, then we define $\theta_\eta=\theta$ and $A_\eta=\bigcup_{\theta'<\theta}A_{\eta\theta'}$; and thus we move to define $A_{\eta+1,0}$. Otherwise let $\langle i_0, v_0\rangle$ be the first element in $\kappa\times\chi^+$ to satisfy P. Let

$$A_{\eta\theta}=R\Big(\bigcup_{\theta'<\theta}A_{\eta\theta'}\Big)\cap\Big(E_{i_0v_0}-\bigcup_{\eta'<\eta}A_{\eta'}\Big).$$

This concludes the definition of the A_η's. First, note that every A_η is downwards G-free; second, $|A_\eta|=\sum\{\mu_i\mid\exists v(E_{iv}\cap A_\eta\neq\emptyset)\}$, but since by what we assumed on G, $|A_\eta|<\mu_\Lambda$, it must happen that for every $\eta<\chi$ there is $i(\eta)<\kappa$ such that $A\cap\bigcup_{i(\eta)\leqslant i}E_i=\emptyset$. It follows, that, for some $i_0<\kappa$, $|\{\eta\mid i(\eta)=i_0\}|=\chi$. Let $J=\{\eta\mid i(\eta)=i_0\}$.

By our definitions for every i and η there is at most one v such that $E_{iv}\cap A\neq\emptyset$; and since there are χ^+ v's and just χ η's, for every $i<\kappa$, $|E_i-\bigcup_{\eta<\chi}A_\eta|=\mu_i$. Let $E=E_{i_0}-\bigcup_{\eta<\chi}A_\eta$. Clearly if $\eta\in J$ then $|R(A_\eta)\cap E|<\mu_{i_0}$. Since $|E|=\mu_{i_0}>\chi=|J|$, and μ_{i_0} is regular, there is $\alpha\in E-\bigcup\{R(A_\eta)\mid\eta\in J\}$. So, for every $\eta\in J$ the set

$$D_\eta\overset{\text{def}}{=}\{\boldsymbol{\beta}\in T(\langle\alpha\rangle)\mid G(\langle\alpha\rangle{}^\frown\boldsymbol{\beta})\cap A_\eta=\emptyset\}$$

is $T(\langle\alpha\rangle)$-small. Recall that $M_1^*(T(\langle\alpha\rangle), \lambda, \chi)$ holds, hence by 2.12 and 2.3(b), there is $b\in B(T(\langle\alpha\rangle))$ such that $|\{\eta\mid b\nsubseteq D_\eta\}|\geqslant\lambda$. The A's are pairwise disjoint, hence $|\bigcup G(b)|\geqslant\lambda$, i.e. G is not λ-bounded. A contradiction so 3.4(a) is proved. $\square$

Proof of 3.4(b). We first note that the following claim is true.

(*) For every $\chi<\mu_\Lambda$ there is a χ-complete normal ideal I_χ on μ_Λ, such that for every $A\subseteq\mu_\Lambda$ such that $|A|=\mu_\Lambda$, there is $B\subseteq A$ such that $|B|=\mu_\Lambda$ and $B\in I$.

(*) certainly holds when μ_Λ is regular, for I_χ can always be taken to be the ideal of nonstationary subsets of μ_Λ.

If μ_Λ is singular let $\{\mu_i\mid i<\mathrm{cf}(\mu_\Lambda)\}$ be strictly increasing sequence of regular cardinals converging to μ_Λ such that $\mu_0\geqslant\chi$. Let

$$I_\chi=\{B\subseteq\mu_\Lambda\mid\text{for every } i<\mathrm{cf}(\mu_\Lambda),\ B\cap\mu_i\text{ is nonstationary in }\mu_i\}.$$

It is easy to see that I_χ is as required.

Suppose that $M_1^*(T, \lambda, \mu_\Lambda)$ holds, and let $G: T\to P_\lambda(\mu_\Lambda)$ be λ-bounded. By

2.4(h), we can w.l.o.g. assume that there is $\chi < \mu_\Lambda$ such that, for every $\boldsymbol{\alpha} \in \operatorname{suc}^T(\Lambda)$, $M_1^*(T(\boldsymbol{\alpha}), \lambda, \chi)$ holds.

So, by 2.15, for every $\alpha \in \mu_\Lambda$ there is $B_\alpha \in I$ such that for every $E \in I \cap P(\mu_\Lambda - B_\alpha)$ there is $T' \leqslant T(\langle\alpha\rangle)$ such that $E \cap \bigcup G(\langle\alpha\rangle \,^\wedge T') = \emptyset$. Let $C = \Delta_{\alpha<\mu_\Lambda}(\mu_\Lambda - B_\alpha)$ be the diagonal intersection of the $\mu_\Lambda - B_\alpha$'s. Let $E \in P(C) \cap I$ and $|E| = \mu_\Lambda$. Hence for every $\alpha \in \mu_\Lambda$, and in particular, for every $\alpha \in E$ there is $T_\alpha \leqslant T(\langle\alpha\rangle)$ such that $E \cap \bigcup G(\langle\alpha\rangle \,^\wedge T_\alpha) \subseteq \alpha + 1$. Hence $T' \stackrel{\text{def}}{=} \bigcup \{\langle\alpha\rangle \,^\wedge T_\alpha \mid \alpha \in E\} \leqslant T$, and is upwards G-free.

This concludes the proof of 3.4(b), so Theorem 3.2 is proved. □

Remark. Note that we did not really need the normality of I. Instead we could have required that I is χ-complete, and that the following holds: if $\{\beta_\alpha \mid \alpha < \mu_\Lambda\} \subseteq I$, then there is $B \in I$ such that: (1) $|B| = \mu_\Lambda$; (2) if $\alpha \in B$, then $B \cap B_\alpha \subseteq \alpha + 1$.

Question. Generalize 3.2 to high trees.

4. Δ-systems

Let $\boldsymbol{\beta} \wedge \boldsymbol{\gamma}$ denote the maximal common initial segment of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$.

We first define the straight-forward generalization of the classical notion of a Δ-system.

Definition 4.1. $G: T \to P(A)$; G is a Δ-system if there is $K: T \times \omega \times \omega \to P(A)$ such that, for all incomparable $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T$, $G(\boldsymbol{\alpha}) \cap G(\boldsymbol{\beta}) = K(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}, \operatorname{length}(\boldsymbol{\alpha}), \operatorname{length}(\boldsymbol{\beta}))$. K is called the kernel of G.

The classical theorem of Erdös and Rado on Δ-systems states, that if μ is regular and, for every $\kappa < \mu$, $\kappa^{<\lambda} < \mu$, then for every A and $G: \mu \to P_\lambda(A)$ there is $M \subseteq \mu$, $|M| = \mu$ such that $G \restriction M$ is a Δ-system, i.e. there is $K \in P_\lambda(A)$ such that, for all distinct $\alpha, \beta \in M$, $G(\alpha) \cap G(\beta) = K$.

The above theorem can be divided into two subclaims, the first of which requires weaker assumptions; moreover, in most applications the first subclaim can replace the theorem on Δ-systems. Let us describe it.

Definition. $\operatorname{cov}^*(\mu, \lambda)$ means that μ is regular, $\mu > \lambda$, and if $\{a_i \mid i < \mu\} \subseteq P_\lambda(A)$ and $|\bigcup_{i<\mu} a_i| < \mu$, then there is $M \subseteq \mu$, $|M| = \mu$ such that $|\bigcup_{i \in M} a_i| < \lambda$.

The subclaim that we mentioned is the following. Suppose $\operatorname{cov}^*(\mu, \lambda)$ holds and $G: \mu \to P_\lambda(A)$, then there is $K \in P_\lambda(A)$ and $M \subseteq \mu$, $|M| = \mu$ such that, for all distinct $\alpha, \beta \in M$, $G(\alpha) \cap G(\beta) \subseteq K$.

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This gives rise to a weaker notion of a Δ-system.

Definition. Let $G:T\to P_\lambda(A)$; G is a weak λ-Δ-system if there is $K:T\to P_\lambda(A)$ such that, for all incomparable $\boldsymbol{\alpha}, \boldsymbol{\beta}\in T$, $G(\boldsymbol{\alpha})\cap G(\boldsymbol{\beta})\subseteq K(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$. K is called a weak λ-kernel of G.

An equivalent definition of a weak λ-Δ-system is given in the following proposition.

Proposition 4.1. *Let $G:T\to P_\lambda(A)$. Then G is a weak λ-Δ-system iff there is G' such that* (1) $G':T\to P_\lambda(A)$; (2) *for every $\boldsymbol{\alpha}\in T$, $G(\boldsymbol{\alpha})\subseteq G'(\boldsymbol{\alpha})$; and* (3) *for every $\boldsymbol{\alpha}, \boldsymbol{\beta}\in T$, $G'(\boldsymbol{\alpha})\cap G'(\boldsymbol{\beta})\subseteq G'(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$.* (3) *can also be replaced by: for every $\boldsymbol{\alpha}, \boldsymbol{\beta}\in T$, $G'(\boldsymbol{\alpha})\cap G'(\boldsymbol{\beta})=G'(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$.*

The generalizations that we are seeking can be summarized in the following questions. Under what conditions on T and λ is it true that for every λ-bounded $G:T\to P_\lambda(A)$ there is $T'\leqslant T$ such that $G\restriction T'$ is a Δ-system or a weak λ-Δ-system?

It turns out, as will be shown by the following example, that when dealing with trees there is another weakening of the notion of a Δ-system that should be considered.

Example 4.1. $\mu_1>\mu_2$ and let T be a tree such that $\mu^T_\Lambda=\mu_1$, for every $\alpha\in\mu_1$, $\mu^T_{\langle\alpha\rangle}=\mu_2$, and for every $\boldsymbol{\alpha}\in\mu_1\times\mu_2$, $\mu^T_{\boldsymbol{\alpha}}=0$. Let $G:T\to P_2(\mu_2)$ be such that, for every $\alpha\in\mu_1$, $G\restriction \operatorname{suc}^T(\langle\alpha\rangle)$ is a 1–1 function into $P_2(\mu_2)$. Clearly, if $\lambda\leqslant\mu_2$ then for no $T'\leqslant T$, $G\restriction T'$ is a weak λ-Δ-system.

So, we have two options either to discard with all trees T for which $\operatorname{Ext}(T)\neq T$, or else to weaken our requirement in the definition of a Δ-system. The second option is of course stronger since we get another intermediate subclaim.

Definition. Let $G:T\to P(A)$. (a) G is called a successor Δ-system (S-Δ-system), if there is $K:T\to P(A)$ such that, for every $\boldsymbol{\alpha}\in T$ and distinct $\boldsymbol{\beta},\boldsymbol{\gamma}\in\operatorname{suc}^T(\boldsymbol{\alpha})$, $G(\boldsymbol{\beta})\cap G(\boldsymbol{\gamma})=K(\boldsymbol{\alpha})$.

(b) G is called a weak λ-S-Δ-system if there is $G':T\to P_\lambda(A)$ such that: for every $\boldsymbol{\alpha}\in T$, $G(\boldsymbol{\alpha})\subseteq G'(\boldsymbol{\alpha})$, and for every $\boldsymbol{\alpha}\in T$ and distinct $\boldsymbol{\beta},\boldsymbol{\gamma}\in\operatorname{suc}^T(\boldsymbol{\alpha})$, $G'(\boldsymbol{\beta})\cap G'(\boldsymbol{\gamma})\subseteq G'(\boldsymbol{\alpha})$.

Note that in the definition of a weak λ-S-Δ-system we had two options, according to the two equivalent conditions in Proposition 4.1. For successor Δ-systems these two conditions are no longer equivalent, and we chose as a definition the stronger between the two options.

Note that an S-Δ-system may fail to be a weak S-Δ-system (even for trees of height 3).

Recall that $\mathrm{cov}(\mu, \lambda)$ means that μ is regular $\mu > \lambda$ and for every $\kappa < \mu$ there is $D \subseteq P_\lambda(\kappa)$ such that $|D| < \mu$ and for every $a \in P_\lambda(\kappa)$ there is $d \in D$ such that $a \subseteq d$.

We are now in a position to state the main theorem of this section.

Theorem 4.2. *Let T be a tree such that for every $\boldsymbol{\alpha} \in T$ either $\mu_{\boldsymbol{\alpha}} = 0$ or $\mathrm{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ holds. Then for every A and λ-bounded $G: T \to P_\lambda(A)$ there is $T' \leqslant T$ such that $G \restriction T'$ is a weak λ-S-Δ-system.*

As easy corollaries of 4.2 we shall get conditions on T and λ which respectively assure that for every λ-bounded $G: T \to P_\lambda(A)$ there is $T' \leqslant T$ on which G is an S-Δ-system, or a weak λ-Δ-system or a Δ-system.

Theorem 4.2 has two shortcomings. The first one is that we assume $\mathrm{cov}(\mu_\alpha, \lambda)$ which might be somewhat too strong, hence we cannot get a necessary and sufficient condition on T and λ.

In the classical case $\mathrm{cov}^*(\mu, \lambda)$ is a necessary and sufficient condition, but we shall discuss this in more detail later.

Note that though we have to assume in our proof that G is λ-bounded we do not assure that the G' which exemplifies that $G \restriction T'$ is a weak λ-S-Δ-system is also λ-bounded. Now if $\lambda > \aleph_0$ we can use 2.2 on $|G(s)|$ to replace T by $T_1 \leqslant T$ s.t. $\sup_{s \in T_1} |G(s)| < \lambda$. Also, for $\lambda = \aleph_0$ we can insure the conclusion.

In our formulation we deal only with regular $\mu_{\boldsymbol{\alpha}}$'s. The reason for this is, that we do not know any information on the singular case, except that which follows quite trivially from what we prove for regular $\mu_{\boldsymbol{\alpha}}$'s. This is true even when dealing just with the classical case.

Definition. Let $G: T \to P_\lambda(A)$, T' a subtree of T, $G': T' \to P_\lambda(A)$, for every $\boldsymbol{\alpha} \in T'$, $G(\boldsymbol{\alpha}) \subseteq G'(\boldsymbol{\alpha})$, and for every $\boldsymbol{\alpha} \in T'$ and distinct $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathrm{suc}^{T'}(\boldsymbol{\alpha})$, $G'(\boldsymbol{\beta}) \cap G'(\boldsymbol{\alpha}) \subseteq G'(\boldsymbol{\alpha})$; then G' is called a λ-approximation of G on T'.

Lemma 4.3. $\mathrm{cov}(\mu, \lambda)$ *implies* $\mathrm{cov}^*(\mu, \lambda)$.

The proof is trivial.

Theorem 4.4. *Let T be well founded and, for every $\boldsymbol{\alpha} \in T$, $\mu_{\boldsymbol{\alpha}} = 0$ or $\mu_{\boldsymbol{\alpha}}$ is regular, then the following conditions are equivalent.*

(1) *For every A and $G: T \to P_\lambda(A)$ there is $T' \leqslant T$ such that $G \restriction T'$ is a weak λ-S-Δ-system.*

(2) *There is $T' \leqslant T$ such that, for every $\boldsymbol{\alpha} \in T'$, $\mu_{\boldsymbol{\alpha}} = 0$ or $\mathrm{cov}^*(\mu_{\boldsymbol{\alpha}}, \lambda)$ holds.*

Proof. It is trivial to show that $\neg(2) \Rightarrow \neg(1)$. In order to prove that $(2) \Rightarrow (1)$, we first prove the following special case.

Lemma 4.5. *Suppose $\mathrm{cov}^*(\mu, \lambda)$ holds and $G: \mu \to P_\lambda(A)$, then there is $M \subseteq \mu$, $|M| = \mu$ and $K \in P_\lambda(A)$ such that, for every distinct $\alpha, \beta \in M$, $G(\alpha) \cap G(\beta) \subseteq K$.*

Proof. For some $\theta<\lambda$, $S=\{i<\mu: |G(i)|\leqslant\theta\}$ has power μ (as μ is regular $>\lambda$). W.l.o.g. $S=\mu$. By Fodor's theorem on regressive functions there is $\gamma\in\mu$ and $M'\subseteq\mu$, $|M'|=\mu$ such that, for every $\alpha\in M'$, $G(\alpha)\cap\alpha\subseteq\gamma$. Hence $|\bigcup_{\alpha\in M'}\alpha\cap G(\alpha)|<\mu$, and by $\mathrm{cov}^*(\mu,\lambda)$, there is $M''\subseteq M'$, $|M''|=\mu$, such that $|\bigcup_{\alpha\in M''}\alpha\cap G(\alpha)|<\lambda$. Let $K=\bigcup_{\alpha\in M''}\alpha\cap G(\alpha)$. Let C be a club such that, for every $\beta\in C$ and $\alpha<\beta$, $G(\alpha)\subseteq\beta$, and let $M\subseteq M''$ be such that for every $\alpha,\beta\in M$, $\alpha<\beta$ there is $c\in C$ such that $\alpha<c\leqslant\beta$, and $|M|=\mu$. Then M, K are as required.

We now return to the proof of (2) $\Rightarrow$ (1) in 4.4. Let T be a well founded tree such that T and λ satisfy (2). W.l.o.g. for every $\boldsymbol{\alpha}\in T$, $\mu_{\boldsymbol{\alpha}}=0$ or $\mathrm{cov}^*(\mu_{\boldsymbol{\alpha}},\lambda)$. Let $r(\boldsymbol{\alpha})$ denote the rank of $\boldsymbol{\alpha}$ in T, and

$$E_\nu \overset{\text{def}}{=} \{\boldsymbol{\alpha}\in T \mid r(\boldsymbol{\alpha})\leqslant\nu \text{ and if } \boldsymbol{\alpha}\in\mathrm{suc}(\boldsymbol{\beta}) \text{ then } r(\boldsymbol{\beta})>\nu\}.$$

Clearly, for every ordinal ν, E_ν is a frontier in T.

Let $G:T\to P_\lambda(A)$. We define by induction on $\nu\leqslant r(\Lambda)$ a set B_ν and a function $G_\nu:B_\nu\to P_\lambda(A)$ such that:

(1) $E_\nu\subseteq B_\nu\subseteq\{\boldsymbol{\beta}\mid(\exists\boldsymbol{\alpha}\in E_\nu)(\boldsymbol{\beta}\geqslant\boldsymbol{\alpha})\}$;

(2) if $\nu<\xi\leqslant r(\Lambda)$, then $B_\xi\supseteq\{\boldsymbol{\beta}\mid\boldsymbol{\beta}\in B_\nu \text{ and } (\exists\boldsymbol{\alpha}\in B_\xi\cap E_\nu)(\boldsymbol{\beta}\geqslant\boldsymbol{\alpha})\}$;

(3) for every $\boldsymbol{\alpha}\in B_\nu$, $\{\boldsymbol{\gamma}\mid\boldsymbol{\alpha}^\wedge\gamma\in B_\nu\}\leqslant T(\boldsymbol{\alpha})$;

(4) for every $\boldsymbol{\alpha}\in B_\nu$, $G(\boldsymbol{\alpha})\subseteq G_\nu(\boldsymbol{\alpha})$;

(5) for every $\boldsymbol{\alpha}\in B_\nu$ and distinct $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathrm{suc}^T(\boldsymbol{\alpha})\cap B_\nu$, $G_\nu(\boldsymbol{\beta})\cap G_\nu(\boldsymbol{\gamma})\subseteq G_\nu(\boldsymbol{\alpha})$; and

(6) if $\boldsymbol{\alpha}\in B_\nu\cap B_\xi$ then $G_\nu(\boldsymbol{\alpha})=G_\xi(\boldsymbol{\alpha})$.

Clearly, $B_{r(\Lambda)}\leqslant T$ and $G_{r(\Lambda)}$ is a λ-approximation of G on $B_{r(\Lambda)}$, so $B_{r(\Lambda)}$, $G_{(\Lambda)}$ fulfill what the theorem requires.

Let $B_0=E_0$ and $G_0=G\restriction E_0$. Suppose B_ν, G_ν have been defined.

Let $\boldsymbol{\alpha}\in E_{\nu+1}-E_\nu$, then $\mathrm{suc}(\boldsymbol{\alpha})\subseteq E_\nu$; hence by 4.5, there is $M_{\boldsymbol{\alpha}}\subseteq\mathrm{suc}(\boldsymbol{\alpha})$ and $K_{\boldsymbol{\alpha}}\in P_\lambda(A)$ such that $|M_{\boldsymbol{\alpha}}|=\mu_{\boldsymbol{\alpha}}$, and, for every distinct $\boldsymbol{\beta},\boldsymbol{\gamma}\in M_{\boldsymbol{\alpha}}$, $G_\nu(\boldsymbol{\beta})\cap G_\nu(\boldsymbol{\gamma})\subseteq K_{\boldsymbol{\alpha}}$. Let

$$\begin{aligned}B_{\nu+1}=\{\boldsymbol{\alpha}\in B_\nu\mid H(\boldsymbol{\alpha})\cap(E_{\nu+1}-E_\nu)=\emptyset\}\cup(E_{\nu+1}-E_\nu)\\ \cup\{\boldsymbol{\alpha}\in B_\nu\mid(\exists\boldsymbol{\beta}\in E_{\nu+1}-E_\nu)(M_{\boldsymbol{\beta}}\cap H(\boldsymbol{\alpha})\neq\emptyset)\},\end{aligned}$$

and let $G_{\nu+1}(\boldsymbol{\alpha})=G_\nu(\boldsymbol{\alpha})$ if $\boldsymbol{\alpha}\in B_{\nu+1}\cap B_\nu$, and $G_{\nu+1}(\boldsymbol{\alpha})=K_{\boldsymbol{\alpha}}\cup G(\boldsymbol{\alpha})$ if $\boldsymbol{\alpha}\in E_{\nu+1}-E_\nu$.

Suppose δ is a limit ordinal and, for every $\nu<\delta$, B_ν, G_ν have been defined. Let $B'_\delta=\bigcup_{i<\delta}(\bigcap_{i<\nu<\delta}B_\nu)$ and $G'_\delta=(\bigcup_{\nu<\delta}G_\nu)\mid B'_\delta$. Define B_δ and G_δ from B'_δ and G'_γ is the same way that $B_{\nu+1}$, $G_{\nu+1}$, were defined from B_ν and G_ν in the previous case. $\square$

Let us add a parameter in the definition of a valve. Let T be a tree, $G:T\to P(A)$, I an ideal on A, and $T'\leqslant T$; then $b\in P(A)$ is a G, I, T'-valve, if for every $c\in I\cap P(A-b)$, there is $T''\leqslant T'$ such that $c\cap\bigcup G(T'')=\emptyset$.

If $G: T \to A$ and $\boldsymbol{\alpha} \in T$, let $G_{\boldsymbol{\alpha}}: T(\boldsymbol{\alpha})$ denote the following function: $G_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = G(\boldsymbol{\alpha}\hat{\ }\boldsymbol{\beta})$.

Lemma 4.6. *Let T be a tree, and for every $\boldsymbol{\alpha} \in T$ either $\mu_{\boldsymbol{\alpha}} = 0$ or $\operatorname{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ holds. Let $E \subseteq \operatorname{Ext}(T) - \{\Lambda\}$ be a frontier in T, $G: T \to P_\lambda(A)$ be λ-bounded and $a_0 \in P_\lambda(A)$ be a G, $P_{\mu_\Lambda}(A)$, T-valve. Then there are a subtree T' of T and a λ-approximation G' of G on $H(E \cap T')$ that satisfy the following conditions*: (1) $T' \leqslant T$; (2) $G'(\Lambda) = a_0$; *and* (3) *for every $\boldsymbol{\alpha} \in E \cap T'$ either $\mu^T_{\boldsymbol{\alpha}} = 0$ or $G'(\alpha)$ is a $G_{\boldsymbol{\alpha}}$, $P_{\mu_{\boldsymbol{\alpha}}}(A)$, $T'(\boldsymbol{\alpha})$-valve.*

Proof. We define by induction on $v < \mu_\Lambda$: $\boldsymbol{\alpha}_v \in \operatorname{suc}(\Lambda)$, a subtree T_v of T and a λ-approximation G^v of G on T_v such that: (i) T_v, G^v satisfy conditions (2) and (3) in the conclusion of the lemma; (ii) $T_v \cap \operatorname{suc}(\Lambda) = \{\boldsymbol{\alpha}_\xi \mid \xi < v\}$; and (iii) if $\xi < \zeta < v$, then $T_\zeta(\boldsymbol{\alpha}_\xi) = T_v(\boldsymbol{\alpha}_\xi) \leqslant T(\boldsymbol{\alpha}_\xi)$.

$T_0 = \{\Lambda\}$ and $G^0(\Lambda) = a_0$. If δ is a limit, and T_v, G^v have been defined for every $v < \delta$ let $T_\delta = \bigcup_{v<\delta} T_v$ and $G^\delta = \bigcup_{v<\delta} G^v$.

Suppose T_v, G^v and $\{\boldsymbol{\alpha}_\xi \mid \xi < v\}$ have been defined, we define $\boldsymbol{\alpha}_v$, T_{v+1} and G^{v+1}. Let $c = \bigcup_{\xi<v} G^v(\boldsymbol{\alpha}_\xi)$. Since a_0 is a G, $P_{\mu_\Lambda}(A)$, T-valve and $|c| < \mu_\Lambda$, there is $T' \leqslant T$ such that $c \cap \bigcup G(T') \subseteq a_0$. Let $\boldsymbol{\alpha}_v \in T' \cap (\operatorname{suc}(\Lambda) - \{\boldsymbol{\alpha}_\xi \mid \xi < v\})$. By Lemma 2.16, for every $\boldsymbol{\alpha} \in T'[\alpha_v] \cap E$ there is $a_{\boldsymbol{\alpha}} \in P_\lambda(A)$ such that $a_{\boldsymbol{\alpha}}$ is a $G_{\boldsymbol{\alpha}}$, $P_{\mu_{\boldsymbol{\alpha}}}(A)$, $T'(\boldsymbol{\alpha})$-valve; and since $c \cap \bigcup G(T') \subseteq a_0$ we can further assume that for each $a_{\boldsymbol{\alpha}}$: $a_{\boldsymbol{\alpha}} \cap c \subseteq a_0$. Let $\bar{G}^v: H(T' \cap E)(\boldsymbol{\alpha}_v) \to P_\lambda(A)$ be defined as follows: if $\boldsymbol{\alpha}_v\hat{\ }\boldsymbol{\beta} \in E$ then $\bar{G}^v(\boldsymbol{\beta}) = a_{\boldsymbol{\alpha}_v\hat{\ }\boldsymbol{\beta}}$, and otherwise $\bar{G}^v(\boldsymbol{\beta}) = G(\boldsymbol{\alpha}_v\hat{\ }\boldsymbol{\beta})$. By Theorem 4.4 there is $\tilde{T}^v \leqslant H(T' \cap E)(\boldsymbol{\alpha}_v)$ and $\tilde{G}^v$ such that $\tilde{G}^v$ is a λ-approximation of $\bar{G}^v$ on $\tilde{T}^v$. Since $c \cap \bigcup \bar{G}(H(T' \cap E)(\boldsymbol{\alpha}_v)) \subseteq a_0$ we can further assume that for every $\boldsymbol{\beta} \in \tilde{T}^v$, $c \cap \tilde{G}^v(\boldsymbol{\beta}) \subseteq a_0$.

Let $T_{v+1} = T_v \cup T'[\boldsymbol{\alpha}_v\hat{\ }\tilde{T}^v]$, and let G^{v+1} be defined as follows: if $\boldsymbol{\alpha} \in \operatorname{Dom}(G^v)$ then $G^{v+1}(\boldsymbol{\alpha}) = G^v(\boldsymbol{\alpha})$, and if $\boldsymbol{\alpha} \in \boldsymbol{\alpha}_v\hat{\ }\tilde{T}^v$ then $G^{v+1}(\boldsymbol{\alpha}) = \tilde{G}^v(\boldsymbol{\beta})$ where $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{v+1}\hat{\ }\boldsymbol{\beta}$. It is easy to see that $T_{v+1} G^{v+1}$ satisfies the desired requirements.

It is easy to see that $T_{\mu_\Lambda} \stackrel{\text{def}}{=} \bigcup_{v<\mu_\Lambda} T_v$ and $G^{\mu_\Lambda} \stackrel{\text{def}}{=} \bigcup_{v<\mu_\Lambda} G^v$ satisfy the conditions required in the lemma. $\square$

Proof of Theorem 4.2. Let T be a tree such that for every $\boldsymbol{\alpha} \in T$ either $\mu_{\boldsymbol{\alpha}} = 0$ or $\operatorname{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ holds. By Lemma 2.13 w.l.o.g., for every $b \in B(T)$, $\operatorname{Ext}(T) \cap b$ is unbounded in b. We do not lose generality if we assume that every branch of T is infinite, so, to simplify the notation we assume it. Let $E_n = \operatorname{Ext}(T) \cap \{\boldsymbol{\beta} \in T \mid |\operatorname{Ext}(T) \cap H(\boldsymbol{\beta})| = n + 1$. Clearly each E_n is a frontier in T.

Let $G: T \to P_\lambda(A)$ be λ-bounded. By applying Lemma 4.6 it is easy to define, by induction on $n \in \omega$, $T_n \leqslant T$, $T^n \leqslant H(E_n)$ and G^n such that: G^n is a λ-approximation of G on T^n, $T^n = H(E_n \cap T_n)$, $T^n \subseteq T^{n+1}$, $T_n \supseteq T_{n+1}$, $G^n \subseteq G^{n+1}$, and, for every $\boldsymbol{\alpha} \in T^n \cap E_n$, $G^n(\boldsymbol{\alpha})$ is a $G^n_{\boldsymbol{\alpha}}$, $P_{\mu_{\boldsymbol{\alpha}}}(A)$, $T_n(\boldsymbol{\alpha})$-valve.

Let $T' = \bigcup_{n\in\omega} T^n$ and $G' = \bigcup_{n\in\omega} G^n$, then clearly, $T' \leqslant T$ and G' is a λ-approximation of G on T'. This proves the theorem. $\square$

Now we strengthen the requirements on T in order to assure the existence of weak λ-Δ-systems.

Theorem 4.7. *Let T be a tree such that for every $\boldsymbol{\alpha} \in T$, $\mu_{\boldsymbol{\alpha}} = 0$ or* $\operatorname{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ *holds. Then the following conditions are equivalent.*

(a) *For every A and λ-bounded $G: T \to P_\lambda(A)$ there is $T' \leqslant T$ such that $G \restriction T'$ is a weak λ-Δ-system.*

(b) *There is $T' \leqslant T$ such that for every $\boldsymbol{\alpha} \in T'$ and $\mu < \mu_{\boldsymbol{\alpha}}$,*

$$|\{\boldsymbol{\beta} \mid \boldsymbol{\beta} \in \operatorname{suc}^{T'}(\boldsymbol{\alpha}) \text{ and } 0 < \mu_{\boldsymbol{\beta}} \leqslant \mu\}| < \mu_{\boldsymbol{\alpha}}.$$

Proof. $\neg(b) \Rightarrow \neg(a)$. Suppose T satisfies $\neg(b)$. It is easy to see that there is $T' \leqslant T$, a frontier E of T', and, for every $\boldsymbol{\alpha} \in E$, $0 < \kappa_{\boldsymbol{\alpha}} < \mu_{\boldsymbol{\alpha}}$ such that, for every $\boldsymbol{\beta} \in \operatorname{suc}^{T'}(\boldsymbol{\alpha})$, $0 < \mu_{\boldsymbol{\beta}} \leqslant \kappa_{\boldsymbol{\alpha}}$. For every $\boldsymbol{\alpha} \in E$ and $\boldsymbol{\beta} \in \operatorname{suc}^{T'}(\boldsymbol{\alpha})$, let $G_{\boldsymbol{\beta}}$ be a 1–1 function from $\operatorname{suc}^{T}(\boldsymbol{\beta})$ to $P_2(\kappa_{\boldsymbol{\alpha}})$. Let $G: T \to P_2(\bigcup_{\boldsymbol{\alpha} \in E} \kappa_{\boldsymbol{\alpha}})$ be defined as follows: let $\boldsymbol{\gamma} \in T$; if for some $\boldsymbol{\alpha} \in E$ and $\boldsymbol{\beta} \in \operatorname{suc}^{T}(\boldsymbol{\alpha})$: $\boldsymbol{\gamma} \in \operatorname{suc}^{T}(\boldsymbol{\beta})$, then $G(\boldsymbol{\gamma}) = G_{\boldsymbol{\beta}}(\boldsymbol{\gamma})$; and if such $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ do not exist then $G(\gamma) = \emptyset$.

It is easy to see that G is as required.

We shall now make some preparations in order to prove that (b) $\Rightarrow$ (a). T is called a regular tree if, for every $\boldsymbol{\alpha} \in T$, $\mu_{\boldsymbol{\alpha}} = 0$ or $\mu_{\boldsymbol{\alpha}}$ is regular. T is called a monotonic tree if, for every $\boldsymbol{\alpha} \in T$ and $\mu < \mu_{\boldsymbol{\alpha}}$, $|\{\boldsymbol{\beta} \in \operatorname{suc}(\boldsymbol{\alpha}) \mid 0 < \mu_{\boldsymbol{\beta}} \leqslant \mu\}| < \mu_{\boldsymbol{\alpha}}$. Let T be a tree and T' be a subtree of T; T' is T-balanced if, for every $\mu > 0$, $|\{\boldsymbol{\beta} \in T' \mid \mu_{\boldsymbol{\beta}}^{T} = \mu\}| \leqslant \mu$. T is balanced if it is T-balanced. Note that, for $T' \leqslant T$, T' is balanced iff it is T-balanced.

$\boldsymbol{\alpha} \in T$ is a reflection point of T, if $\mu_{\boldsymbol{\alpha}}$ is a regular limit cardinal, $\mu_{\boldsymbol{\alpha}} = \bigcup \{\mu_{\boldsymbol{\beta}} \mid \boldsymbol{\beta} \in \operatorname{suc}(\boldsymbol{\alpha})\}$, and, for distinct $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \operatorname{suc}(\boldsymbol{\alpha})$, $\mu_{\boldsymbol{\beta}} \neq \mu_{\boldsymbol{\gamma}} < \mu_{\boldsymbol{\alpha}}$. Let

$$R(T, \lambda) \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} \in T \mid \mu_{\boldsymbol{\alpha}} = \lambda \text{ and } \boldsymbol{\alpha} \text{ is a reflection point of } T\},$$

and $R(T) = \bigcup \{R(T, \lambda) \mid \lambda \text{ is a cardinal}\}$. T is called a nice tree, if it is regular, and for every $\boldsymbol{\alpha} \in T$: either $\boldsymbol{\alpha}$ is a reflection point of T or

$$(\forall \boldsymbol{\beta} \in \operatorname{suc}(\boldsymbol{\alpha}))(\neg(0 < \mu_{\boldsymbol{\beta}} < \mu_{\boldsymbol{\alpha}})).$$

Lemma 4.8. *If T is a regular monotonic tree then there is $T' \leqslant T$ such that T' is balanced.*

Proof. Clearly, every regular monotonic tree T contains a nice $T' \leqslant T$. So, it suffices to prove that every nice T contains a balanced $T' \leqslant T$. If T is a nice tree and T', T'' are subtrees of T let $T' \leqslant_R^T T''$ mean that $T' \subseteq T''$, for every $\boldsymbol{\alpha} \in T' - R(T)$, $\operatorname{suc}^{T'}(\boldsymbol{\alpha}) = \operatorname{suc}^{T''}(\boldsymbol{\alpha})$, and, for every $\boldsymbol{\alpha} \in T' \cap R(T)$, $|\operatorname{suc}^{T'}(\boldsymbol{\alpha})| =$

$|\mathrm{suc}^{T''}(\boldsymbol{\alpha})|$. Obviously $T' \leqslant_R^T T$ implies $T' \leqslant T$. In fact we shall prove that if T is nice, then there is a balanced $T' \leqslant_R^T T$.

Let T be nice; for every cardinal λ, let

$$T^{\lambda} = \{\boldsymbol{\alpha} \in T \mid (\forall \kappa \geqslant \lambda)(\forall \boldsymbol{\beta} < \boldsymbol{\alpha})(\boldsymbol{\beta} \notin R(T, \kappa))\}.$$

Let $\Pr(\lambda)$ be the following claim: for every nice T, for every $\kappa < \lambda$, and for every T-balanced $T' \leqslant_R^T T^{\kappa}$, there is T-balanced $T'' \leqslant_R^T T^{\lambda}$ such that $T' \subseteq T''$, and, for every $\boldsymbol{\alpha} \in T'' - T'$ and $\boldsymbol{\beta} \in T'$, $\neg(0 < \mu_{\boldsymbol{\beta}}^T = \mu_{\boldsymbol{\alpha}}^T < \kappa)$.

We prove $\Pr(\lambda)$ by induction on λ. For $\lambda = \aleph_0$ there is nothing to prove, since, for every $\kappa < \lambda$, $T^{\kappa} = T^{\lambda}$.

Let λ be a limit cardinal and suppose for every $\lambda' < \lambda$, $\Pr(\lambda')$ holds. Let $\kappa < \lambda$, T be a nice tree, and $T' \leqslant_R^T T^{\kappa}$ be T-balanced. Let $\{\lambda_i \mid i < \mathrm{cf}(\lambda)\}$ be a strictly increasing continuous sequence such that $\lambda_0 = \kappa$ and $\sum_{i<\mathrm{cf}(\lambda)} \lambda_i = \lambda$. We define by induction on $i \leqslant \mathrm{cf}(\lambda)$ subtrees T_i of T. Our induction hypothesis are: (1) $T_i \leqslant_R^T T^{\lambda_i}$; (2) T_i is T-balanced; (3) if $i < j$, then $T_i \subseteq T_j$; and (4) if $i < j$, $\boldsymbol{\alpha} \in T_j - T_i$ and $\boldsymbol{\beta} \in T_i$, then $\neg(0 < \mu_{\boldsymbol{\beta}}^T = \mu_{\boldsymbol{\alpha}}^T < \lambda_i)$. Let $T_0 = T'$, use the induction hypothesis $\Pr(\lambda_{i+1})$ to obtain T_{i+1} from T_i, and if i is a limit ordinal let $T_i = \bigcup_{j<i} T_j$. It is trivial to see that $T_{\mathrm{cf}(\lambda)}$ is as required.

Let $\lambda = \mu^+$ and suppose $\Pr(\mu)$ holds. If μ is not a regular limit cardinal, then there is nothing to prove since always, $T^{\lambda} = T^{\mu}$. So, suppose μ is a regular limit cardinal, and let T be nice. By $\Pr(\mu)$, it suffices to show that if $T' \leqslant_R^T T^{\mu}$, is T-balanced then there is a T-balanced T'' such that $T' \subseteq T'' \leqslant_R^T T^{\lambda}$, and, for every $\boldsymbol{\alpha} \in T'' - T'$ and $\boldsymbol{\beta} \in T'$, $\neg(0 < \mu_{\boldsymbol{\beta}}^T = \mu_{\boldsymbol{\alpha}}^T < \mu)$.

Let $T' \leqslant_R^T T^{\mu}$ be T-balanced. By niceness we see that $|R(T, \mu) \cap T^{\lambda}| \leqslant \mu$. If $R(T, \mu) \cap T^{\lambda} = \emptyset$, then $T^{\lambda} = T^{\mu}$ and we can define $T'' = T'$. Otherwise, let $\{\boldsymbol{\alpha}_i \mid i < \mu\}$ be an enumeration $R(T, \mu) \cap T^{\lambda}$ such that, for every $\boldsymbol{\alpha} \in R(T, \mu) \cap T^{\lambda}$, $|\{i \mid \boldsymbol{\alpha}_i = \boldsymbol{\alpha}\}| = \mu$.

We now define by induction on $i \leqslant \mu$ subtrees T_i of T^{λ} and cardinals μ_i such that (1) for every $i < \mu$: T_i is T-balanced, $\mu_i < \mu$ and for every $\boldsymbol{\alpha} \in T_i$ either $\mu_{\boldsymbol{\alpha}}^T < \mu_i$ or $\mu \leqslant \mu_{\boldsymbol{\alpha}}^T$; and (2) if $i < j < \mu$, $\boldsymbol{\alpha} \in T_j$ and $\mu_{\boldsymbol{\alpha}}^T < \mu_i$, then $\boldsymbol{\alpha} \in T_i$.

Let $T_0 = T'$ and $\mu_0 = \mathrm{Sup}(\{\mu_{\boldsymbol{\alpha}}^T \mid \boldsymbol{\alpha} \in T' \text{ and } \mu_{\boldsymbol{\alpha}}^T < \mu\})$. It is easy to see that $\mu_0 < \mu$.

If δ is a limit ordinal and T_i, μ_i have been defined for every $i < \delta$, let $T_\delta = \bigcup_{i<\delta} T_i$ and $\mu = \bigcup_{i<\delta} \mu_i$.

Suppose T_i, μ_i have been defined and we wish to define T_{i+1}, μ_{i+1}. If $\boldsymbol{\alpha}_i \notin T_i$ let $T_{i+1} = T_i$, $\mu_{i+1} = \mu_i$. Suppose $\boldsymbol{\alpha}_i \in T_i$. Let $\boldsymbol{\beta} \in \mathrm{suc}^T(\boldsymbol{\alpha}_i)$ be such that $\mu_{\boldsymbol{\beta}}^T > \mu_i$. Obviously, $T(\boldsymbol{\beta})$ is nice. It is easy to see that there is $\bar{T} \leqslant_R^{T(\boldsymbol{\beta})} T(\boldsymbol{\beta})$ such that, for every $\boldsymbol{\gamma} \in T$, $\mu^{T(\boldsymbol{\beta})}(\boldsymbol{\gamma}) > \mu_i$. $\bar{T}^{\aleph_0}$ is certainly $\bar{T}$-balanced, so by $\Pr(\mu)$ there is $\bar{T}$-balanced $\tilde{T} \leqslant_R^T \bar{T}^{\mu}$ which contains $T^{\aleph_0}$. Let $T_{i+1} = T_i \cup \boldsymbol{\beta} \hat{\ } \tilde{T}$, and $\mu_{i+1} = \mathrm{Sup}(\{\mu_{\boldsymbol{\alpha}}^T \mid \boldsymbol{\alpha} \in T_{i+1} \text{ and } \mu_{\boldsymbol{\alpha}}^T < \mu\})$. It is easy to see that T_{i+1}, μ_{i+1} are as required.

It is easy to see that $T_\mu \leqslant_R^T T^{\lambda}$ and by the induction hypotheses that we carried $T' \subseteq T_\mu$, T_μ is balanced and for every $\boldsymbol{\alpha} \in T_\mu - T'$, $\boldsymbol{\beta} \in T'$: $\neg(0 < \mu_{\boldsymbol{\alpha}}^T = \mu_{\boldsymbol{\beta}}^T < \mu)$.

We have thus proved that $\Pr(\lambda)$ holds for every λ. Let T be nice, hence, for some λ, $T=T^\lambda$. $T^{\aleph_0}$ is balanced and $T^{\aleph_0} \leqslant_R^T T^{\aleph_0}$; by $\mathrm{Pr}(\lambda)$ there is a balanced $T' \leqslant_R^T T^\lambda = T$ containing $T^{\aleph_0}$. This proves Lemma 4.8. $\square$

Continuation of the proof of 4.7. b $\Rightarrow$ a. Let T be a monotonic tree such that, for every $\boldsymbol{\alpha} \in T$, $\mathrm{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ holds. Let $G: T \to P_\lambda(A)$ be λ-bounded. By 4.2 and 4.8, w.l.o.g. T is balanced and G is a weak λ-S-Δ-system. Let G' be a λ-approximation of G on T. We shall show that there is $T' \leqslant T$ such that, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T'$, $G'(\boldsymbol{\alpha}) \cap G'(\boldsymbol{\beta}) \subseteq G'(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})$.

Since T is balanced there is an enumeration $\{\boldsymbol{\alpha}_i \mid i < |T|\}$ such that: (1) for every $\boldsymbol{\alpha} \in T$, $|\{i \mid \boldsymbol{\alpha}_i = \boldsymbol{\alpha}\}| = \mu_{\boldsymbol{\alpha}}$; (2) if $\boldsymbol{\beta} < \boldsymbol{\alpha}_i$ then, for some $j<i$, $\boldsymbol{\alpha}_j = \boldsymbol{\beta}$; and (3) $(\forall i < |T|)(i < \mu_{\boldsymbol{\alpha}_i})$.

We define by induction on $i \leqslant |T|$ a subtree T_i of T. The induction hypotheses are $|T_i| \leqslant |i| + 1$, and if $i<j$ then $T_i \subseteq T_j$.

To $T_0 = \{\Lambda\}$. If δ is a limit ordinal then $T_\delta = \bigcup_{i<\delta} T_i$. Suppose T_i has been defined. If $\alpha_i \notin T_i$ let $T_{i+1} = T_i$. Suppose $\boldsymbol{\alpha}_i \in T_i$; let $C = \bigcup G'(T_i)$. $\lambda < \mu_{\boldsymbol{\alpha}_i}$ and $|T_i| \leqslant |i| + 1 < \mu_{\boldsymbol{\alpha}_i}$, hence $|C| < \mu_{\boldsymbol{\alpha}_i}$. Since G' is a λ-approximation, there is $\boldsymbol{\beta} \in \mathrm{suc}^T(\boldsymbol{\alpha}_i) - T_i$ such that $G'(\boldsymbol{\beta}) \cap C \subseteq G'(\boldsymbol{\alpha}_i)$. Let $T_{i+1} = T_i \cup \{\boldsymbol{\beta}\}$. It is easy to see that $T_{|T|} \leqslant T$, and, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T_{|T|}$, $G'(\boldsymbol{\alpha}) \cap G'(\boldsymbol{\beta}) \subseteq G'(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})$. So $G \restriction T_{|T|}$ is a weak λ-Δ-system. $\square$

Finally we examine when it is possible to obtain S-Δ-systems and Δ-systems.

Theorem 4.9. (a) *Let T be a tree such that, for every $\boldsymbol{\alpha} \in T$,* $\mathrm{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ *and* $2^{<\lambda} < \mu_{\boldsymbol{\alpha}}$; *then for every A and $G: T \to P_\lambda(A)$ there is $T' \leqslant T$ such that $G \restriction T$ is an S-Δ-system.*

(b) *Let T be a monotonic tree, for every $\boldsymbol{\alpha} \in T$,* $\mathrm{cov}(\mu_{\boldsymbol{\alpha}}, \lambda)$ *and* $2^{<\lambda} < \mu_{\boldsymbol{\alpha}}$; *then for every A and λ-bounded $G: T \to P_\lambda(A)$ there is $T' \leqslant T$ such that $G \restriction T'$ is a Δ-system.*

Proof. (a) is trivial. We prove (b). Let T, λ, G be as in (b). By 4.7, w.l.o.g. G is a weak λ-Δ-system. Let $K: T \to P_\lambda(A)$ be a weak λ-kernel for G, i.e. for every incomparable $\boldsymbol{\alpha}, \boldsymbol{\beta} \in T$, $G(\boldsymbol{\alpha}) \cap G(\boldsymbol{\beta}) \subseteq K(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})$. We define, by induction on $n \in \omega$, $T_n \leqslant T$. Let $T_0 = T$. Suppose T_n has been defined. For every $\boldsymbol{\alpha} \in T_n$ such that length $(\boldsymbol{\alpha}) = n$, let $F_\alpha: T_n(\boldsymbol{\alpha}) \to P(K(\boldsymbol{\alpha}))$ be defined as follows: $F_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = G(\boldsymbol{\alpha} {}^\frown \boldsymbol{\beta}) \cap K(\boldsymbol{\alpha})$. By a variant of 2.2 which is proved in [9] there is $T_{\boldsymbol{\alpha}} \leqslant T_n(\boldsymbol{\alpha})$ and $K_{\boldsymbol{\alpha}}: \omega \to P(K(\boldsymbol{\alpha}))$ such that, for every $\boldsymbol{\beta} \in T_{\boldsymbol{\alpha}}$, $F_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = K_{\boldsymbol{\alpha}}(\mathrm{length}(\boldsymbol{\beta}))$. Let $T_{n+1} = H(\bigcup \{\boldsymbol{\alpha} {}^\frown T_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in T_n$ and $\mathrm{length}(\boldsymbol{\alpha}) = n\})$. Let $T' = \bigcap_{n \in \omega} T_n$. If $\boldsymbol{\alpha} \in T'$ let $K'(\boldsymbol{\alpha}, n, m) = K_{\boldsymbol{\alpha}}(n) \cap K_{\boldsymbol{\alpha}}(m)$. It is easy to see that $T' \leqslant T$, $G \restriction T'$ is a Δ-system and K' is its kernel. $\square$

Discussion. We shall now investigate the question under what conditions do $\mathrm{cov}(\mu, \lambda)$ and $\mathrm{cov}^*(\mu, \lambda)$ hold.

Lemma 4.10. *Let $\mu^{(-)}$ be the largest limit cardinal which is less than or equal to μ. Let $\lambda < \mu = \mathrm{cf}(\mu)$.*

(a) *If $\mu^{(-)} \leq \lambda$ and $\neg(\lambda = \mu^{(-)} \neq \mathrm{cf}\, \mu^{(-)})$, or if for every $\kappa < \mu$: $\kappa^{<\lambda} < \mathrm{cf}(\mu) = \mu$, then* $\mathrm{cov}(\mu, \lambda)$ *holds.*

(b) *If $\mu > 2^{<\lambda}$ then* $\mathrm{cov}(\mu, \lambda)$ *is equivalent to* $\mathrm{cov}^*(\mu, \lambda)$.

(c) $\mathrm{cov}(\mu, \lambda)$ *implies* $\mathrm{cov}(\mu^+, \lambda)$.

(d) *If $\mu = \mu^{(-)+}$ and $\lambda > \mathrm{cf}(\mu^{(-)})$ then* $\neg\mathrm{cov}(\mu, \lambda)$.

(e) (Magidor). *Suppose $V \subseteq W$ are universes of* ZFC; *$V \vDash$* GCH *and V, W satisfy the covering theorem*; *i.e. for every set of ordinals $a \in W$ there is $b \in V$ such that $a \subseteq b$ and $|b|^W = |a|^W + \aleph_1$. Then in W the following are equivalent*: (1) $\mathrm{cov}(\mu, \lambda)$; (2) $\mathrm{cov}^*(\mu, \lambda)$; *and* (3) $\neg(\mu = \mu^{(-)+} \wedge \mathrm{cf}(\mu^{(-)}) < \lambda)$.

We leave it to the reader to verify these claims.

The following questions remain open. Is it consistent with ZFC that $\mathrm{cov}^*(\mu, \lambda) \not\Rightarrow \mathrm{cov}(\mu, \lambda)$? In order to prove this one has, of course, to assume the existence of large cardinals. In particular is it consistent that $\mu = \mu^{(-)+} \wedge \mathrm{cf}(\mu^{(-)}) < \lambda$ and $\mathrm{cov}^*(\mu, \lambda)$ holds; e.g. is $\mathrm{cov}^*(\aleph_{\omega+1}, \aleph_\omega)$ consistent.

We shall now remark on the existence of Δ-systems when T is not necessarily regular. In fact, we do not know anything that does not follow easily from what we have proved for regular trees. The theorem that we can prove is the following.

Theorem 4.11. *Let T be a tree such that for every $\alpha \in T$ either $\mu_\alpha = 0$; or μ_α is regular and* $\mathrm{cov}(\mu_\alpha, \lambda)$ *holds*; *or μ_α is singular,* $\mathrm{cov}^*(\mathrm{cf}(\mu_\alpha), \lambda)$ *holds, and μ_α is a limit of cardinals μ that satisfy* $\mathrm{cov}(\mu, \lambda)$; *then if $G: T \to P_\lambda(A)$ is λ-bounded, then there is $T' \leq T$ such that $G \restriction T'$ is a weak λ-S-Δ-system.*

If the requirement that $\mathrm{cov}^*(\mathrm{cf}(\mu_\alpha), \lambda)$ holds is omitted then one can still obtain the existence of some weaker notion of a Δ-system.

It is worthwhile to remark that we do not know an exact condition that answers the existence of Δ-systems when μ is singular, even for the classical case of trees of height one.

For simplicity we concentrate below on the tree $T = {}^{\omega>}\lambda$.

Notation. $P_\kappa(\lambda) = \{A : A \subseteq \lambda, |A| < \kappa\}$.

$\mathcal{D}_{<\kappa}(\lambda)$ the filter on $P_\kappa(\lambda)$ generated by ($<\kappa$)-closed (i.e., closed under union of increasing sequence of length $<\kappa$) unbounded (every $s \in P_\lambda(\lambda)$ is included in some member) subfamilies of $P_\kappa(\lambda)$.

Theorem 4.12. *Let $\lambda > \kappa > \aleph_0$ be regular cardinals. Then, for any $\langle A_\alpha : \alpha \in T_\lambda \rangle$, $(T_\lambda = ({}^{\omega>}\lambda, <))$, $A_\alpha \subseteq \lambda$, $|A_\alpha| < \kappa$, and $C \in \mathcal{D}_{<\kappa}(\lambda)$ there are $T^* \leq T_\lambda$, and A'_α for $\alpha \in T^*$, such that $A_\alpha \subseteq A'_\alpha \in C$, $A'_{\alpha \restriction k} \subseteq A'_\alpha$, and $A'_\alpha \cap A'_\beta = A'_{\alpha \wedge \beta}$, for $\alpha, \beta \in T^*$*

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provided that:

(*) *there is* $S \subseteq P_\kappa(\lambda)$, $S \neq \emptyset \bmod \mathscr{D}_{<\kappa}(\lambda)$ *such that, for every* $\alpha < \lambda$, $S \restriction \alpha = \{A \cap \alpha : A \in S\}$ *has power* $<\lambda$,

or at least

(**) *there are* $S_\alpha \subseteq P_\kappa(\alpha)$ *for* $\alpha < \lambda$, $|S_\alpha| < \lambda$ *such that for any closed unbounded set* C^* *of* λ *there are* $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$ $(n < \omega)$ *in* C^*, *for which*

$$\left\{A \in P_\kappa\left(\bigcup_n \alpha_n\right) : (\forall n)\left[A \cap \alpha_n \in \bigcup_{j<\alpha_n} S_j\right]\right\} \neq \emptyset \bmod \mathscr{D}_{<\kappa}\left(\bigcup_n \alpha_n\right).$$

Remark. The proof shows that we can choose the A'_α in $\bigcup_\alpha S_\alpha$ (when (**) holds).

Proof of Theorem 4.12. Assume (**). We define a game with ω-steps. In the nth step:

Player I chooses $\boldsymbol{\beta}_n \in T_\lambda$, and $B_n \in \bigcup_{i<\lambda} S_i$ such that $\boldsymbol{\beta}_0 = \Lambda$, for $n > 0$, $\boldsymbol{\beta}_n = \boldsymbol{\beta}_{n-1}{}^\wedge\langle \gamma_n\rangle$, $A_{\boldsymbol{\beta}_n} \subseteq B_n$, $B_n \cap \gamma_n = B_{n-1}$, and $\gamma_n > \alpha_{n-1}$.

Player II chooses $\alpha_n < \lambda$, $\alpha_n > \gamma_n$, such that $B_n \subseteq \alpha_n$.

If player I has no legal move he loses instantly. (Player II has always a legal move). Player I wins the play if he never loses it instantly.

Fact A. The game is determined.

Proof. Well known as the game is open.

Fact B. If player I has a winning strategy, then $\langle A'_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in T^*\rangle$ as required exists.

Proof. Let ${}^{\omega>}\lambda = \{\boldsymbol{\beta}_\xi : \xi < \lambda\}$, such that $\boldsymbol{\beta}_\xi \restriction l \in \{\boldsymbol{\beta}_\zeta : \zeta < \xi\}$ when $l <$ length $\boldsymbol{\beta}_\xi$. We define by induction on ξ, $\alpha_\xi < \lambda$, $\boldsymbol{\gamma}_\xi \in {}^{\omega>}\lambda$, A'_ξ such that:

(1) $A_{\boldsymbol{\gamma}_\xi} \subseteq A'_\xi \in \bigcup_{i<\lambda} S_i$;

(2) if $k = l(\boldsymbol{\beta}_\xi)$, then $l(\boldsymbol{\gamma}_\xi) = k$. If $l < l(\boldsymbol{\beta}_\xi)$, $\zeta < \xi$, then $[\boldsymbol{\beta}_\xi \restriction l = \boldsymbol{\beta}_\zeta \Leftrightarrow \boldsymbol{\gamma}_\xi \restriction l = \boldsymbol{\gamma}_\zeta]$. If $\xi > \zeta$, then $\boldsymbol{\gamma}_\xi \neq \boldsymbol{\gamma}_\zeta$, (This means that the mapping $\boldsymbol{\beta}_\zeta \to \boldsymbol{\gamma}_\zeta$ $(\zeta < \xi)$ is an isomorphism from $T_\lambda \restriction \{\boldsymbol{\beta}_\zeta : \zeta < \xi\}$, onto $T_\lambda \restriction \{\boldsymbol{\gamma}_\xi : \zeta < \xi\}$.)

(3) If $\boldsymbol{\beta}_\xi \wedge \boldsymbol{\beta}_\zeta = \boldsymbol{\beta}_\varepsilon$, then $A'_\xi \cap A'_\zeta = A'_\varepsilon$.

(4) α_ξ is increasing, $\bigcup_{\zeta<\xi} A'_\zeta \subseteq \alpha_\xi$, $\bigcup_{\zeta<\xi} \operatorname{Rang} \boldsymbol{\gamma}_\zeta + 1 \subseteq \alpha_\xi$.

(5) If $\boldsymbol{\beta}_{\zeta_l} = \boldsymbol{\beta}_\xi \restriction l$ for $l \leq l(\boldsymbol{\beta}_\xi)$, then $A'_{\zeta_0}, \boldsymbol{\gamma}_{\zeta_0}, \alpha_1, A'_{\zeta_1}, \boldsymbol{\gamma}_{\zeta 1}, \alpha_2, \ldots, A'_{\zeta_l}, \boldsymbol{\gamma}_{\zeta_l}, \alpha_{\zeta_{l+1}}, \ldots,$ is an initial segment of a play of the game in which player I uses his winning strategy.

In the definition in stage ξ first choose α_ξ (see (4)) then $\boldsymbol{\gamma}_\xi$, A'_ξ (see (5)).

Clearly, if we let $A'_{\boldsymbol{\gamma}_\xi} = A'_\xi$, then $T^* = \{\boldsymbol{\gamma}_\xi : \xi < \lambda\}$, $\langle A'_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in T^*\rangle$ are as required.

Fact C. Player II does not have a winning strategy.

Let F be a winning strategy of player II. Let $\langle M_i : i < \lambda\rangle$ be an increasing continuous sequence of elementary submodels of $(H(2^{2^\lambda}), \in)$, such that

$$\|M_i\| < \lambda, \qquad M_i \cap \lambda = \delta_i, \qquad F \in M_0,$$
$$\langle S_\alpha : \alpha < \lambda\rangle \in M_0, \qquad \langle A_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in {}^{\omega>}\lambda\rangle \in M_0, \qquad \langle M_j : j \leq i\rangle \in M_{i+1}.$$

So $\{\delta_i : i < \lambda\}$ is a closed unbounded subset of λ, as is also $\{\delta_i : i < \lambda, \delta_i = i\}$.

Hence by (**) there are $i_0 < i_1 < \cdots < i_l \cdots$ $(l < \omega)$ such that $\kappa < i_0$, $\delta_{i_l} = i_\delta$ and:

$$S^* = \left\{A \subseteq \bigcup_{l<\omega} \delta_{i_l}: (\forall l < \omega) A \cap \delta_{i_l} \in \bigcup_{j<\delta_{i_l}} S_j\right\} \neq \emptyset \bmod \mathcal{D}_{<\kappa}(\bigcup \delta_{i_l}).$$

Now

$$C^* = \left\{A \subseteq \bigcup_l \delta_{i_l}: |A| < \kappa \text{ and } \left(\exists M < \bigcup_l M_{i_l}\right)[M \cap \lambda = A, \text{ and } F, \langle A_{\boldsymbol\alpha}: \boldsymbol\alpha \in T\rangle,$$

$$\langle S_i: i < \lambda\rangle \text{ belongs to } M, \text{ and } (\forall l)(M_{i_l} \in M, i_l \in M, \delta_{i_l} \in M)]\right\} \in \mathcal{D}_{<\kappa}\left(\bigcup_{l<\omega} \delta_{i_l}\right).$$

So $C^* \cap S^* \neq \emptyset$. Choose $A^* \in C^* \cap S^*$, and M^* witnessing $A^* \in C^*$.

Now we simulate a play of the game in which II uses his strategy but I wins. and in the nth player I will choose:

$$B_n = A^* \cap \delta_{i_{2n}} = A^* \cap M_{i_{2n}}, \qquad \boldsymbol\beta_n = \langle \delta_{i_0}, \ldots, \delta_{i_{2n-1}}\rangle$$

We should prove that this is a legal move. Note that $\boldsymbol\beta_n \in M^*$ as all δ_{i_l} belongs to M^* [by the definition of C^*] note also that $\langle A_{\boldsymbol\alpha}: \boldsymbol\alpha \in {}^{\omega>}\lambda\rangle \in M^*$. Now actually both belong also to $M_{i_{2n}}$, so $A_{\boldsymbol\beta_n} \subseteq M^* \cap M_{i_{2n}}$, hence (as $\kappa \leq i_0$, $|A_{\boldsymbol\beta_n}| < \kappa$, $A_{\boldsymbol\beta_n} \subseteq \lambda$) $A_{\boldsymbol\beta_n} \subseteq B_n$.

Now $B_0 \in M_{i_1}$ as $A^* \in S^*$ so $B_0 = A^* \cap \delta_{i_0} \in \bigcup_{j<\delta_{i_0}} S_j \subseteq M_{i_0}$ (as $A^* \in S^*$ and as $\langle S_j: j < \lambda\rangle \in M_{i_0}$ and each S_j has cardinality $<\lambda$ hence $j \in M_{i_0} \Rightarrow S_j \subseteq M_{i_0}$). Now where will α_n be?

$\alpha_0 < \delta_{i_1}$ as $F \in M_{i_0} \subseteq M_{i_1}$, and $\boldsymbol\beta_0$, $B_0 \in M_{i_1}$.

Similarly, for other n's. $\square$

Theorem 4.13. *Let $\lambda > \kappa > \aleph_0$ be regular cardinals. Then for any $\langle A_{\boldsymbol\alpha}: \boldsymbol\alpha \in T_\lambda\rangle$ (where $T_\lambda = ({}^{\omega>}\lambda)$, $A_{\boldsymbol\alpha} \subseteq \lambda$, $|A_{\boldsymbol\alpha}| < \kappa$), and $C \in \mathcal{D}_{<\kappa}(\lambda)$ there are $T^* \leq T_\lambda$, and $A'_{\boldsymbol\alpha}$ for $\eta \in T^*$ such that:*

$A_{\boldsymbol\alpha} \subseteq A'_{\boldsymbol\alpha} \in C$, $|A'_{\boldsymbol\alpha}| < \kappa$, and if $\eta \restriction k \neq \boldsymbol\beta \restriction k$ then $A'_{\boldsymbol\alpha} \cap A'_{\boldsymbol\beta} \subseteq A'_{\alpha \restriction k} \cap A'_{\boldsymbol\beta \restriction k}$ (it is natural to look at the minimal k), provided that:

(***) There are $S_\alpha \subseteq \mathcal{P}_{<\kappa}(\alpha)$ for $\alpha < \lambda$, $|S_\alpha| < \lambda$ such that for any closed unbounded subset C^* of λ there are ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$ $(n < \omega)$ in C^* each of cofinality $\geq \kappa$ for which

$$\left\{A \in \mathcal{P}_{<\kappa}\left(\bigcup_{n<\omega} \alpha_n\right): \textit{for every } n < \omega, A \cap \alpha_n \in S_{\alpha_{n+1}}\right\} \neq \emptyset \bmod \mathcal{D}_{<\kappa}\left(\bigcup_{n<\omega} \alpha_n\right)$$

Remark. The proof shows that we can choose the $A'_{\boldsymbol\alpha}$ in the $\bigcup_{\alpha<\lambda} S_\alpha$.

By 4.14, if $\lambda = \kappa^{+n}$, (***) holds.

Proof. We define a game with ω steps. In the nth step:

Player I chooses $\boldsymbol\beta_n \in T_\lambda$, $B_n \in \bigcup_{i<\lambda} S_i$ and an ordinal $\beta_n < \lambda$ such that $\boldsymbol\beta_0 = \Lambda$, and, for $n > 0$, $\boldsymbol\beta_n = \boldsymbol\alpha_{n-1} {}^\wedge \langle \gamma_n\rangle$, $\beta_n > \alpha_{n-1}$, $\beta_n > \gamma_n$, $\beta_n > \operatorname{Sup} B_n$, $\gamma_n > \alpha_{n-1}$, and $B_n \cap \alpha_l \subseteq B_l$ for every $l < n$ and $\bigwedge_{l<n} B_l \subseteq B_n$, and $A_{\boldsymbol\beta_n} \subseteq A_n$.

Player II: Choose $\alpha_n > \lambda$, $\alpha_n > \beta_n$.

If player I has no legal move he loses instantly (Player II has always a legal move.) Player I wins the play if he never loses it instantly.

Fact A. The game is determined.

Proof. As an open game, well known.

Fact B. If player I has a winning strategy then $\langle A'_{\boldsymbol{\alpha}}: \boldsymbol{\alpha} \in T^* \rangle$ as required exists.

Proof. We call $\langle t, \underset{\sim}{A}, \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle$ an approximation if:

(1) t is a subset of T^* closed under initial segments,

(2) $|t| < \lambda$, $\Lambda \in t$,

(3) $\underset{\sim}{A}$ is a function from t into $P_\kappa(\lambda)$,

(4) $(\forall \boldsymbol{\alpha} \in t)[A_{\boldsymbol{\alpha}} \subseteq \underset{\sim}{A}(\boldsymbol{\alpha})]$,

(5) $(\forall \boldsymbol{\alpha} \in t)(\forall k)[\underset{\sim}{A}(\boldsymbol{\alpha} \upharpoonright k) \subseteq \underset{\sim}{A}(\boldsymbol{\alpha})]$,

(6) $\underset{\sim}{\beta}$ is a function from t to λ, $\underset{\sim}{\alpha}$ is a function from $t - \{\Lambda\}$ to λ,

(7) $(\forall \boldsymbol{\alpha} \in t)(\forall k)[l(\boldsymbol{\alpha}) = k+1 \rightarrow \underset{\sim}{\beta}(\boldsymbol{\alpha} \upharpoonright k) < \underset{\sim}{\alpha}(\boldsymbol{\alpha}) < \underset{\sim}{\beta}(\boldsymbol{\alpha})]$,

(8) $(\forall \boldsymbol{\alpha} \in t)(\forall k < l(\boldsymbol{\alpha}))[\underset{\sim}{A}(\boldsymbol{\alpha}) \cap \underset{\sim}{\alpha}(\boldsymbol{\alpha} \upharpoonright (k+1)) \subseteq \underset{\sim}{\beta}(\boldsymbol{\alpha} \upharpoonright k)]$,

(9) the intervals $[\underset{\sim}{\alpha}(\boldsymbol{\alpha}), \underset{\sim}{\beta}(\boldsymbol{\alpha}))$ are pairwise disjoint,

(10) for each $\boldsymbol{\alpha} \in t$, $l(\boldsymbol{\alpha}) = k$ the following is an initial segment of a play of the game in which player I uses his winning strategy:

First move player I: $\underset{\sim}{A}(\boldsymbol{\alpha} \upharpoonright 0)$, $\boldsymbol{\alpha} \upharpoonright 0$, $\underset{\sim}{\beta}(\boldsymbol{\alpha} \upharpoonright 0)$, *player* II: $\underset{\sim}{\alpha}(\boldsymbol{\alpha} \upharpoonright 1)$.

Second move player I: $\underset{\sim}{A}(\boldsymbol{\alpha} \upharpoonright 1)$, $\boldsymbol{\alpha} \upharpoonright 1$, $\underset{\sim}{\beta}(\boldsymbol{\alpha} \upharpoonright 1)$, *player* II: $\underset{\sim}{\alpha}(\boldsymbol{\alpha} \upharpoonright 2)$.

...

kth move player I: $\underset{\sim}{A}(\boldsymbol{\alpha})$, $\boldsymbol{\alpha}$, $\underset{\sim}{\beta}(\boldsymbol{\alpha})$ (so player II has not yet made the kth move).

There is a natural order on the family of approximations. The following subfacts are clearly enough to prove Fact B.

Subfact a. There is an approximation $\langle t^0, \underset{\sim}{A}^0, \underset{\sim}{\alpha}^0, \underset{\sim}{\beta}^0 \rangle$ with $t_0 = \{\Lambda\}$ (see (10)).

Subfact b. If $\tau_i = \langle t_i, \underset{\sim}{A}^i, \underset{\sim}{\alpha}^i, \underset{\sim}{\beta}^i \rangle$ are approximations for $i < \delta$, $\delta < \lambda$, and $\tau_i < \tau_j$ for $i < j$, then $\langle \bigcup t_i, \bigcup \underset{\sim}{A}^i, \bigcup_i \underset{\sim}{\alpha}^i, \bigcup_i \underset{\sim}{\beta}^i \rangle$ is an approximation $> \tau_j$ for every j.

Subfact c. If $\tau = \langle t, \underset{\sim}{A}, \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle$ is an approximation, and $\boldsymbol{\alpha} \in t_i$, then for some γ there is an approximation

$$\tau' = \langle t', \underset{\sim}{A}', \underset{\sim}{\alpha}', \underset{\sim}{\beta}' \rangle > \tau, \qquad t' = t \cup \{\boldsymbol{\alpha}\char`^\langle \gamma \rangle\}, \quad \boldsymbol{\alpha}\char`^\langle \gamma \rangle \notin t$$

Proof. First choose an ordinal $\alpha < \lambda$ such that it is bigger than $\operatorname{Sup} \bigcup_{\boldsymbol{v} \in t} \underset{\sim}{A}(\boldsymbol{v})$, and than $\operatorname{Sup} \operatorname{Rang} \underset{\sim}{\alpha}$, $\operatorname{Sup} \operatorname{Rang} \underset{\sim}{\beta}$ and $\operatorname{Sup}_{\boldsymbol{v} \in t} [\bigcup \operatorname{Rang} \boldsymbol{v}]$. Consider the initial segment of the play written in condition (10) for our $\boldsymbol{\alpha}$. Then let player II play (in his $l(\boldsymbol{\alpha})$th move) α. Then player I's strategy tell him to choose (in his $l(\boldsymbol{\alpha}) + 1$st move) B, $\boldsymbol{\alpha}\char`^\langle \gamma \rangle$, α. We let

$$t' = t \cup \{\boldsymbol{\alpha}\char`^\langle \gamma \rangle\}, \quad \underset{\sim}{A}'(\boldsymbol{\alpha}\char`^\langle \gamma \rangle) = B, \quad \underset{\sim}{\beta}(\boldsymbol{\alpha}\char`^\langle \gamma \rangle = \beta, \quad \underset{\sim}{\alpha}(\boldsymbol{\alpha}\char`^\langle \gamma \rangle) = \alpha.$$

Fact C. Player II does not have a winning strategy.

Let F be a winning strategy of player II. Let $\langle M_i: i < \lambda \rangle$ be an increasing continuous sequence of elementary submodes of $(H((2^{2\lambda})^+), \in)$, $\|M\| < \lambda$, $\langle M_j: j \leqslant i \rangle \in M_{i+1}$, and the finite set $E = \{F, \langle S_\alpha: \alpha < \lambda \rangle, \langle A_{\boldsymbol{\alpha}}: \boldsymbol{\alpha} \in {}^{\omega >}\lambda \rangle\}$ belongs to M_0 and $\delta_i = M_i \cap \lambda$ (is a limit ordinal).

So $\{\delta_i: i<\lambda\}$ is a closed unbounded subset of λ. Hence by (***) there are $i_0<i_1<\cdots<i_n<i_{n+1}<\cdots (n<\omega)$ such that, let $i(*) \stackrel{\text{def}}{=} \bigcup i_n$, the cofinality of δ_{i_l} is $\geqslant\kappa$, and

$$S^* = \{A \subseteq \delta_{i(*)}: \text{for every } l<\omega,\ A\cap\delta_{i_n}\in S_{\delta_{i_n}}\} \notin \emptyset \operatorname{mod} \mathscr{D}_{<\kappa}(\delta_{i(*)}).$$

Expand $(H((2^{2^\lambda})^+), \in)$ by Skolem functions getting a model $\mathfrak{A}$. Now let

$$\underset{\sim}{C}^* = \{A\subseteq\delta_{i(*)}: M_A, \text{ the Skolem hull of } A \text{ in } \mathfrak{A} \text{ satisfies: } M_A\cap\lambda = A,\ E\in M,$$
$$\text{and } \{i_l, M_{i_l}, \delta_{i_l}: l<\omega\}\subseteq M\}.$$

Clearly $\underset{\sim}{C}^* \in \mathscr{D}_{<\kappa}(\delta_{i(*)})$. So $\underset{\sim}{C}^*\cap S^* \neq \emptyset$ and choose $A^*\in \underset{\sim}{C}^*\cap S^*$, $M^* = M_{A^*}$ be the Skolem hull of A^* in $\mathfrak{A}$.

Now we simulate play of the game in which player II uses his winning strategy but player I wins, a contradiction.

In fact we give directly the move of player II. In the nth move player I plays:

$$\boldsymbol{\beta}_n = \langle\delta_{i_0}, \delta_{i_1}, \ldots, \delta_{i_{n-1}}\rangle \qquad (\text{so } \boldsymbol{\beta}_0 = \Lambda),$$
$$B_n = A_{v_0} \quad \text{if } n=0,$$
$$B_n = (A^*\cap\delta_{i_{n-1}})\cup A_{v_n} \quad \text{if } n>0,$$
$$\beta_n = \operatorname{Sup} A^*\cap\delta_{i_n},$$

Note that for every n, the choices of player I in his first, second . . . and nth move belongs to M_{i_n}. More elaborately

(a) $\beta_n\in M_{i_n}$ (as the cofinality of δ_{i_n} is $\geqslant\kappa$),
(b) $\boldsymbol{\beta}_n\in M_{i_n}$ (as $\delta_{i_0}, \ldots, \delta_{i_{n-1}} < \delta_{i_n} = M_{i_n}\cap\lambda$),
(c) $A_{\boldsymbol{\beta}_n}\in M_{i_n}$ (as $\boldsymbol{\beta}_n\in M_{i_n}$, $\langle A_{\boldsymbol{\alpha}}: \boldsymbol{\alpha}\in T_\lambda\rangle\in M_0\subseteq M_{i_n}$),
(d) $S_{\delta_{i_{n-1}}}\in M_{i_n}$ (as $\langle S_j: j<j\rangle\in E\subseteq M_{i_n}$, and $\delta_{i_{n-1}}\in M_{i_n}$),
(e) $S_{\delta_{i_{n-1}}}\subseteq M_{i_n}$ (as $|S_{\delta_{i_{n-1}}}|<\lambda$ and $S_{\delta_{i_{n-1}}}\in M_{i_n}$ and $M_{i_n}\cap\lambda$ is an ordinal),
(f) hence $A^*\cap\delta_{i_{n-1}}\in M_{i_n}$ (as $A^*\in S^*$, $A^*\cap\delta_{i_{n-1}}\in S_{\delta_{i_{n-1}}}$, but $S_{\delta_{i_{n-1}}}\subseteq M_{i_n}$),
(g) $B_n\in M_{i_n}$ (by (c) and (f).)

So really the choice of α_n in the nth move belong to M_{i_n}. As player II uses his winning strategy F, his nth move α_n belongs to M_{i_n} too (remember $M_{i_l}<M_{i_n}$ for $l<n$). So $\beta_n<\alpha_n<\delta_{i_n}$.

It is easy to check that player II plays legally, hence wins the play though player I has used the strategy F; a contradiction. □

Observations 4.14. (1) $(*)\Rightarrow(**)\Rightarrow(***)$ (See 4.12, 4.13 for definition). (Take $S_\alpha = S\restriction\alpha$, and for every $\alpha_0<\alpha_1<\cdots$, $\{A\in P_\kappa(\bigcup\alpha_n): (\forall n)A\cap\alpha_n\in S_{\alpha_{n+1}}\}\supseteq S\restriction(\bigcup\alpha_n)$ which is $\neq\emptyset \operatorname{mod}\mathscr{D}_{<\kappa}(\lambda)$.)

(2) An example for $(*)$ is: W, V are universes of set theory (with the same ordinals, $W\subseteq V$, we suppose (W,V) satisfies the strong covering lemma for subsets of λ of power $<\kappa$ (see [9, Chapter XIII]) and $W\vDash$GCH, $V\vDash$"$\lambda\in\kappa>\aleph_0$ are regular", then $S=\{A\subseteq\lambda: |A|<\kappa,\ A\in W\}$ is as required provided that $W\vDash$"λ not successor of singular of cofinitely $<\kappa$".

(3) For $(***)$ an example is: $\lambda = \aleph_2$, $\kappa = \aleph_1$, $S_\alpha = \{A_{\alpha,i}: i < \omega_1\}$, $A_{\alpha,i}$ countable increasing continuous, $\bigcup_{i<\omega_1} A_{\alpha,i} = \alpha$. Any $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$ works.

(4) We can prove by induction on n that if κ is regular $> \aleph_0$, $\lambda = \kappa^{+n}$ then $(***)$ holds, hence we get the conclusion of Theorem 4.13.

(5) If $\lambda = \chi^+$, $(**)$ is equivalent to "there is $S \subseteq P_\kappa(\chi)$, $S \neq \emptyset \bmod \mathcal{D}_{<\kappa}(\lambda)$, $|S| \leqslant \chi$".

(6) If $(***)$ holds for λ, κ, it holds for λ^+ κ [easy by (5)].

(7) Note that if $S \subseteq P_\kappa(A)$, $B \subseteq A$, $S \neq \emptyset \bmod \mathcal{D}_{<\kappa}(A)$ then $s \restriction B \neq \emptyset \bmod \mathcal{D}_{<\kappa}(B)$.

(8) If $\langle A_{\boldsymbol\alpha}: \alpha \in {}^{\omega>}\lambda\rangle$ satisfies $A_{\boldsymbol\alpha^\wedge\langle i\rangle} \cap A_{\boldsymbol\alpha^\wedge\langle j\rangle} = A_{\boldsymbol\alpha}$, $|A_{\boldsymbol\alpha}| < \lambda$ then for some $T^* \leqslant {}^{\omega>}\lambda$ for every $\boldsymbol\alpha, \boldsymbol\beta \in T^*$, $A_{\boldsymbol\alpha} \cap A_{\boldsymbol\beta} = A_{\boldsymbol\alpha \wedge \boldsymbol\beta}$.

(9) If $V \vDash \mathrm{ZFC}$ + "κ is strongly inaccessible Mahlo" then, for some forcings notion P of power κ not collapsing $\aleph_1$ and κ, $\Vdash_P$"$\kappa = \aleph_2$ and $(**)$ fail" (as in [10]).

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