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UNIFORMIZATION AND THE DIVERSITY OF WHITEHEAD GROUPS

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ABSTRACT

Techniques of uniformization are used to prove that it is not consistent that the Whitehead groups of cardinality $\aleph_1$ are exactly the strongly $\aleph_1$-free groups. Some consequences of the assumption that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is Whitehead are derived. Other results about uniformization are also proved.

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Introduction

The connections between Whitehead groups and uniformization properties were investigated by the third author in [9]. In particular it was essentially shown there that there is a non-free Whitehead (respectively, $\aleph_1$-coseparable) group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which satisfies 2-uniformization (respectively, ω-uniformization). (See also [5, §XII.3]; definitions are reviewed below.) These techniques allowed also the proof of various independence and consistency results about Whitehead groups, for example that it is consistent that there is a non-free Whitehead group of cardinality $\aleph_1$ but no non-free $\aleph_1$-coseparable group (cf. [5, XII.3.18]).

However, some natural questions remained open, among them the following two, which are stated as problems at the end of [5, p. 454].

- Is it consistent that the class of W-groups of cardinality $\aleph_1$ is exactly the class of strongly $\aleph_1$-free groups of cardinality $\aleph_1$?
- If every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is a W-group, are they also all $\aleph_1$-coseparable?

In this paper we use the techniques of uniformization to answer the first question in the negative and give a partial affirmative answer to the second question. (The third author claims a full affirmative solution to the second question, but it is too complicated to give here.)

More precisely, we have the following two theorems of ZFC.

THEOREM 1: *The following are equivalent:*
(a) *There is an $\aleph_1$-separable Whitehead group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$.*
(b) *There is a strongly $\aleph_1$-free Whitehead group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$.*
(c) *There is a Whitehead group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$.*
(d) *There is a Whitehead group of cardinality $\aleph_1$ which is* not *strongly $\aleph_1$-free.*
(e) *There is a ladder system on $\lim(\omega_1)$ which satisfies 2-uniformization.*

The new part of this result is the proof of (d) from (c); this gives a negative answer to the first question. Given the history of independence results regarding Whitehead groups, it is remarkable that the answer to this question is negative. The partial answer to the second question is contained in the following.

THEOREM 2: *Consider the following hypotheses.*
(1) *Every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is $\aleph_1$-coseparable.*

(2) *Every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is a Whitehead group.*
(3) *Every ladder system on a stationary subset of ω_1 satisfies 2-uniformization.*
(4) *Every ladder system on a stationary subset of ω_1 satisfies ω-uniformization.*
(5) *There is a strongly $\aleph_1$-free group of cardinality $\aleph_1$ which is $\aleph_1$-coseparable but not free.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

The new parts of this theorem are the proofs of (3) from (2) and (5) from (2). We consider the implication from (2) to (4) strong evidence for an affirmative answer to the second question; what is lacking for a complete answer is a proof of (1) from (4).

The last two sections of this paper contain some other results about uniformization, which may be of independent interest.

Preliminaries

Let us review some basic notation and terminology. See [5] for further information; throughout the paper we will usually cite [5] for results we need, rather than the original source.

We will always be dealing with abelian groups or $\mathbb{Z}$-modules; we shall simply say "group". A group A is said to be a **Whitehead group** if $\mathrm{Ext}(A, \mathbb{Z}) = 0$; it is said to be **$\aleph_1$-coseparable** if $\mathrm{Ext}(A, \mathbb{Z}^{(\omega)}) = 0$.

A group A of arbitrary cardinality is **$\aleph_1$-free** if and only if every countable subgroup of A is free; A is **strongly $\aleph_1$-free** if and only if every countable subset is contained in a free subgroup B such that A/B is $\aleph_1$-free. A is **$\aleph_1$-separable** if and only if every countable subset is contained in a free subgroup B such that B is a direct summand of A.

Chase [1] showed that CH implies that every Whitehead group is strongly $\aleph_1$-free. In the third author's original paper, [6], on the independence of the Whitehead Problem, a larger class of groups than the strongly $\aleph_1$-free groups plays a key role, namely the groups which the first author ([4]) later named the **Shelah groups**. These are the $\aleph_1$-free groups A such that for every countable subgroup B there is a countable subgroup $B' \supseteq B$ such that for any countable C satisfying $C \cap B' = B$, C/B is free. In [6] it is proved consistent — in fact a consequence of Martin's Axiom plus $\neg$CH — that every Shelah group of cardinality $\aleph_1$ is $\aleph_1$-coseparable. Later, in [8] it was proved consistent — in fact, again a consequence of Martin's Axiom plus $\neg$CH — that the Whitehead groups of cardinality $\aleph_1$ are the same as the $\aleph_1$-coseparable groups and are precisely the Shelah groups. The first author emphasized the strongly $\aleph_1$-free groups in his expository accounts of this work (e.g. in [3, 4]), as a class of groups more familiar to algebraists, and raised the first question cited above. The answer to that question now given here shows, definitively, that the larger class of Shelah groups is the 'right one' to consider for the Whitehead Problem.

Notions of uniformization (in our sense) were first defined in [2]. Let S be a subset of $\lim(\omega_1)$. If $\delta \in S$, a **ladder on** δ is a function $\eta_\delta: \omega \to \delta$ which is strictly increasing and has range cofinal in δ. A **ladder system on** S is an indexed family $\eta = \{\eta_\delta: \delta \in S\}$ such that each η_δ is a ladder on δ. For a cardinal $\lambda \geq 2$, a λ**-coloring** of a ladder system η on S is a family $c = \{c_\delta: \delta \in S\}$ such that $c_\delta: \omega \to \lambda$. A **uniformization** of a coloring c of a ladder system η on S is a pair $\langle f, f^*\rangle$ where $f: \omega_1 \to \lambda$, $f^*: S \to \omega$ and for all $\delta \in S$ and all $n \geq f^*(\delta)$, $f(\eta_\delta(n)) = c_\delta(n)$. If such a pair exists, we say that c can be uniformized. In order for the pair to exist it is enough to have either member of the pair; i.e., either f so that for all $\delta \in S$, $f(\eta_\delta(n)) = c_\delta(n)$, for all but finitely many n, or f^* so that for all δ, $\alpha \in S$, if $n \geq f^*(\delta)$, $m \geq f^*(\alpha)$ and $\eta_\delta(n) = \eta_\alpha(m)$, then $c_\delta(n) = c_\alpha(m)$. We say that (η, λ)**-uniformization holds** or that η **satisfies** λ**-uniformization** if every λ-coloring of η can be uniformized. We will generalize these (by now, standard) notions in the next section.

If A is an $\aleph_1$-free group of cardinality $\aleph_1$, then (we define) $\Gamma(A) = 1$ if and only if A is the union of a continuous chain of countable subgroups

$$A = \bigcup_{\alpha<\omega_1} A_\alpha$$

such that for all $\alpha \in \lim(\omega_1)$, $A_{\alpha+1}/A_\alpha$ is not free. If A is not strongly $\aleph_1$-free, then $\Gamma(A) = 1$, but the converse is false.

LEMMA 3: *If there is a Whitehead group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$, then there is a ladder system on* $\lim(\omega_1)$ *which satisfies 2-uniformization.*

Proof: We assume familiarity with [5, §XII.3] and sketch the modifications to the proof of Theorem XII.3.1 that are needed. In the proofs of Lemma XII.3.16 and Theorem XII.3.1, $\lim(\omega_1)$ is partitioned into countably many sets E_n; to each of these is associated $\Phi^n = \{\varphi_\alpha : \alpha \in E_n\}$, which is a family with 2-uniformization. As defined there, the range of the φ_α is not a set of ordinals, but it is easy to see that, by a coding argument, we can assume that the range of φ_α is contained in α and, furthermore, that if $\alpha \in E_i$ and $\beta \in E_j$, then the ranges of φ_α and φ_β are disjoint. Finally, if necessary, one modifies each φ_α so that it is a ladder on α (say by using a bijection from $\omega_1 \times \omega_1$ to ω_1). This produces a ladder system on $\lim(\omega_1)$ which has 2-uniformization since the uniformizations of the original Φ^n fit together to give a uniformization of the ladder system. ∎

This proof obviously generalizes to prove that if there is a Whitehead group A of cardinality $\aleph_1$ with $\Gamma(A) = \tilde{S}$, then there is a ladder system on S which satisfies 2-uniformization.

If $\alpha < \beta$ are ordinals, denote by (α, β) the open interval of ordinals between α and β, i.e., the set $\{\gamma: \alpha < \gamma < \beta\}$. Similarly we define the half open interval $[\alpha, \beta)$, etc. We will use $\langle \alpha, \beta \rangle$ to denote the ordered pair of ordinals.

1. The First Question

It is consistent that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is Whitehead (cf. [5, XII.1.12]) and it is consistent that there are non-free Whitehead groups of cardinality $\aleph_1$ and every Whitehead group of cardinality $\aleph_1$ is strongly $\aleph_1$-free (cf. [5, XII.1.9]), but here we show that it's not consistent that the Whitehead groups of cardinality $\aleph_1$ are precisely the strongly $\aleph_1$-free groups.

If $\alpha \in [\omega, \omega_1)$ and $\alpha = \delta + n$, where δ is a limit ordinal and $n \in \omega$, a **ladder on** α is defined to be a ladder on δ. Thus, for example, a ladder on $\omega + 1$ is a strictly increasing ω-sequence approaching ω. If $S \subseteq [\omega, \omega_1)$, a **ladder system on** S is an indexed family $\eta = \langle \eta_\alpha: \alpha \in S \rangle$ such that each η_α is a ladder on α.

Whenever we write an ordinal as $\delta + n$ we mean that $\delta \in \lim(\omega_1)$ and $n \in \omega$. We will always assume in what follows that if $\delta + n \in S$, then $\delta \in S$.

Suppose that H is an indexed family $\langle h_\alpha: \alpha \in S \rangle$ where each h_α is a function: $\omega \to \omega$. If $\eta = \langle \eta_\alpha: \alpha \in S \rangle$ is a ladder system on S, an H-**coloring of** η is an indexed family $c = \langle c_\alpha: \alpha \in S \rangle$ such that for all α, $c_\alpha: \omega \to \omega$ and that for all $n \in \omega$, $c_\alpha(n) < h_\alpha(n)$. We say that (η, H)-**uniformization holds** (or η **satisfies** H-**uniformization**) if whenever c is an H-coloring, there is a pair $\langle f, f^* \rangle$ such that $f: \omega_1 \to \omega$, $f^*: S \to \omega$, and for all $\alpha \in S$, $f(\eta_\alpha(n)) = c_\alpha(n)$ whenever $n \geq f^*(\alpha)$. We say that (η, λ)-uniformization holds if each $h_\alpha \in H$ is the constant function λ; this agrees with the previous definition.

A ladder system $\eta = \langle \eta_\alpha: \alpha \in S \rangle$ is said to be **tree-like** if for all α, $\beta \in S$, if $\eta_\alpha(n) = \eta_\beta(m)$, then $n = m$ and $\eta_\alpha(k) = \eta_\beta(k)$ for all $k \leq n$. Let F be a function from S to ω; say that η is **strongly tree-like w.r.t.** F if η is tree-like and in addition, whenever $\eta_\alpha(n) = \eta_\beta(m)$ for some $\alpha, \beta \in S$ and $n, m \in \omega$, then $F(\alpha) = F(\beta)$.

LEMMA 4: *Suppose that there is a ladder system $\zeta = \langle \zeta_\alpha \colon \alpha \in S\rangle$ on $S \supseteq \lim(\omega_1)$ such that (ζ, H)-uniformization holds. Given a function $F\colon S \to \omega$, there is a ladder system $\eta = \langle \eta_\alpha \colon \alpha \in S\rangle$ such that η is strongly tree-like w.r.t. F and (η, H)-uniformization holds.*

Proof: Choose a one-one onto function θ from $\omega \times {}^{<\omega}\omega_1$ to ω_1 with the property that for all limit δ, $\theta[\omega \times {}^{<\omega}\delta] = \delta$ and for all $k \in \omega$, if t is a sequence which extends s then $\theta(k, s) < \theta(k, t)$. For each α, let $\eta_\alpha(n) = \theta(\langle F(\alpha), \langle \zeta_\alpha(m)\colon m \le n\rangle\rangle)$. Since $\theta(k, s) < \theta(k, t)$, η_α is strictly increasing. If we can show that each ς_α is a ladder on α, then we will be done since, by construction, it is strongly tree-like w.r.t. F. Observe that because $\theta\colon \omega \times {}^{<\omega}\delta \to \delta$ is one-one and onto for limit δ, if μ is a limit ordinal $\le \zeta_\alpha(n)$, then $\theta(\langle F(\alpha), \langle \zeta_\alpha(m)\colon m \le n\rangle\rangle) \ge \mu$. Consider now $\alpha = \delta + n$. Note that η_α has range contained in δ. If δ is a limit of limit ordinals then, by the observation, the range of η_α is cofinal in δ since the range of ζ_α is cofinal. If $\delta = \gamma + \omega$ then there is some k so that $\zeta_\alpha(k) \ge \gamma$. Then for all $m \ge k$, $\gamma \le \eta_\alpha(m) < \delta = \gamma + \omega$. So η_α is cofinal in δ. ∎

LEMMA 5: *Suppose that there is a ladder system $\zeta = \langle \zeta_\alpha \colon \alpha \in S\rangle$ where $S \supseteq \lim(\omega_1)\rangle$ such that $(\zeta, 2)$-uniformization holds. Given $H = \langle h_\alpha\colon \alpha \in [\omega, \omega_1)\rangle$ where each $h_\alpha\colon \omega \to \omega$ and given a function $F\colon [\omega, \omega_1) \to \omega$, there is a ladder system $\eta = \langle \eta_\alpha\colon \alpha \in [\omega, \omega_1)\rangle$ such that (η, H)-uniformization holds and η is strongly tree-like w.r.t. F.*

Proof: We shall give the proof as a series of reductions. First of all, by [5, XII.3.2], $(\zeta, 3)$-uniformization holds. Next, we claim that we can assume that ζ is a ladder system on $[\omega, \omega_1)$. Write ω as the union of $\aleph_0$ disjoint infinite sets Y_n $(n \in \omega)$, and for each n let $\theta_n\colon \omega \to Y_n$ enumerate Y_n in increasing order. For each $\delta \in \lim(\omega_1)$ and $n \in \omega$, define $\zeta'_{\delta+n} = \zeta_\delta \circ \theta_n$. Then it is easy to see that $\langle \zeta'_\alpha\colon \alpha \in [\omega, \omega_1)\rangle$ satisfies 3-uniformization.

So we will now assume that $S = [\omega, \omega_1)$. By Lemma [4], we can assume that ζ is tree-like. For each $\alpha \in S$ define $\psi_\alpha\colon \omega \to \omega$ by $\psi_\alpha(n) = \Sigma_{j \le n} h_\alpha(j)$; so $\psi_\alpha(n) - \psi_\alpha(n-1) = h_\alpha(n)$ for all $n \in \omega$ (where $\psi_\alpha(-1) = 0$). Define $\zeta'_\alpha = \zeta_\alpha \circ \psi_\alpha$. Now we claim that ζ' satisfies H-uniformization. Suppose that $c' = \langle c'_\alpha\colon \alpha \in S\rangle$ is an H-coloring of ζ'. Define a 3-coloring c of ζ as follows. Let $c_\alpha(0) = 2$, and for each $n \in \omega$, and $\alpha \in S$, let $c_\alpha(\psi_\alpha(n)) = 2$. Define $c_\alpha(\psi_\alpha(n-1)+k) = 0$ for $1 \leq k \leq c'_\alpha(n)$, and $c_\alpha(\psi_\alpha(n-1)+k) = 1$ for $c'_\alpha(n) < k < h_\alpha(n)$.

As an example, suppose $h_\alpha(0) = 5$, $h_\alpha(1) = 4$, $h_\alpha(2) = 5$ and $h_\alpha(3) = 6$. Then $\psi_\alpha(0) = 5$, $\psi_\alpha(1) = 9$, $\psi_\alpha(2) = 14$ and $\psi_\alpha(3) = 20$. If $c'_\alpha(0) = 4$, $c'_\alpha(1) = 1$, $c'_\alpha(2) = 3$ and $c'_\alpha(3) = 0$, then the values of $c_\alpha(n)$ for $0 \leq n \leq 20$ are:

$$2,0,0,0,0,2,0,1,1,2,0,0,0,1,2,1,1,1,1,1,2.$$

(The blocks of 0's between 2's code the values of c'_α.)

Given $\langle f, f^*\rangle$ which uniformizes c, define $\langle f', f'^*\rangle$ as follows. Let $n \geq f'^*(\alpha)$ if and only if $\psi_\alpha(n-1) \geq f^*(\alpha)$. We need to choose f' so that $f'(\nu) = c'_\alpha(m)$ if $\nu = \varsigma'_\alpha(m)$ and $m \geq f'^*(\alpha)$. To see that there is such an f', suppose β and k are such that also $\nu = \varsigma'_\beta(k)$ where $k \geq f'^*(\beta)$. Since $\langle \zeta_i \colon i \in [\omega, \omega_1)\rangle$ is tree-like we have that $\zeta_\alpha\restriction(\psi_\alpha(m)+1) = \zeta_\beta\restriction(\psi_\alpha(m)+1)$ and $\psi_\alpha(m) = \psi_\beta(k)$. For definiteness assume that $\psi_\alpha(m-1) \leq \psi_\beta(k-1)$. Then since $\psi_\alpha(m-1) \geq f^*(\alpha)$ and $\psi_\beta(k-1) \geq f^*(\beta)$, we have that $c_\alpha(r) = c_\beta(r)$ for all r such that $\psi_\beta(k-1) \leq r \leq \psi_\beta(k)$. By the coding we know that $\psi_\alpha(m-1)$ is the greatest natural number, s, less than $\psi_\alpha(m)$ so that $c_\alpha(s) = 2$. Hence $\psi_\alpha(m-1) = \psi_\beta(k-1)$. Also by the coding we have that $c'_\alpha(m)$ is the number of 0's in c_α between $\psi_\alpha(m-1)$ and $\psi_\alpha(m)$, which is the same as the number of 0's in c_β between $\psi_\beta(k-1)$ and $\psi_\beta(k)$.

Finally, we can apply Lemma 4 to get a strongly tree-like η which satisfies H-uniformization. ∎

LEMMA 6: *Suppose that there is a ladder system $\zeta = \langle \zeta_\delta \colon \delta \in \lim(\omega_1)\rangle$ such that $(\zeta, 2)$-uniformization holds, and suppose we are given a prime p_α for each $\alpha \in [\omega, \omega_1)$. Let $\{x_\nu \colon \nu \in \omega_1\}$ and $\{y_\nu \colon \nu \in \omega_1\}$ be sets of symbols.*

Then there are primes $q_{\alpha,n}$ for each $\alpha \in [\omega, \omega_1)$ and $n \in \omega$ and a ladder system $\eta = \langle \eta_\alpha \colon \alpha \in [\omega, \omega_1)\rangle$ such that given any integers r_α and $t_{\alpha,n}$ for all $\alpha \in [\omega, \omega_1)$ and $n \in \omega$, there is a function

$$\psi \colon \{x_\nu, y_\nu \colon \nu \in \omega_1\} \to \mathbb{Z}$$

such that for all $\alpha \in [\omega, \omega_1)$ and all $n \in \omega$,

$$\psi(x_\alpha) \equiv r_\alpha \pmod{p_\alpha} \text{ and}$$

$$\psi(x_\alpha) - \psi(y_{\eta_\alpha(n)}) \equiv t_{\alpha,n} \pmod{q_{\alpha,n}}.$$

Also, η has the property that if $\eta_\alpha(m) = \eta_\beta(n)$, then $m = n$, $p_\alpha = p_\beta$ and $\langle q_{\alpha,k} : k \le n\rangle = \langle q_{\beta,k} : k \le n\rangle$.

Proof: Define the $q_{\alpha,n}$ so that there is no repetition in the sequence $\langle p_\alpha\rangle \frown \langle q_{\alpha,n} : n \in \omega\rangle$ and such that if $p_\alpha = p_\beta$, then $q_{\alpha,n} = q_{\beta,n}$ for all n. Without loss of generality we can suppose that $r_\alpha \in \{0, \dots, p_\alpha - 1\}$ and $t_{\alpha,n} \in \{0, \dots, q_{\alpha,n} - 1\}$. Fix a bijection θ: ${}^{<\omega}\omega \to \omega$ such that if u: $m \to \omega$ and v: $m \to \omega$ are such that $u(i) \le v(i)$ for all $i < m$, then $\theta(u) \le \theta(v)$. For each α and n, let $h_\alpha(n) = \theta(\langle p_\alpha\rangle \frown \langle q_{\alpha,j} : j \le n\rangle)$. Let F be the function on $[\omega, \omega_1)$ such that $F(\alpha) = p_\alpha$. Apply Lemma 5 to this situation to obtain the ladder system η as in that lemma. Then there is a uniformization $\langle f, f^*\rangle$ for the coloring given by $c_\alpha(n) = \theta(\langle r_\alpha\rangle \frown \langle t_{\alpha,j} : j \le n\rangle)$.

We can assume that $f^*(\alpha)$ is minimal for f, i.e., $f^*(\alpha)$ is the least k so that $f(n) = c_\alpha(n)$, for all $n \ge k$. An immediate consequence of the minimality is that if there exists $n \ge f^*(\alpha), f^*(\beta)$ with $\eta_\alpha(n) = \eta_\beta(n)$ then $f^*(\alpha) = f^*(\beta)$. (The point is that $c_\alpha(n) = c_\beta(n)$ implies that $c_\alpha \restriction n = c_\beta \restriction n$.)

We now define ψ in ω stages. At stage k, we will define $\psi(x_\alpha)$ for all α such that $f^*(\alpha) = k$ and we will define $\psi(y_\nu)$ for all ν of the form $\eta_\gamma(k)$ or of the form $\eta_\alpha(n)$ where $f^*(\alpha) = k$ and $n > k$. First of all, for each ν of the form $\eta_\gamma(k)$ for some γ, let $\psi(y_\nu)$ be arbitrary, if it has not already been defined at a previous stage. [Note that if ν is of this form then k, but not α, is uniquely determined by the tree-like property of η.] For each α such that $f^*(\alpha) = k$, define $\psi(x_\alpha)$ to be the minimal natural number such that

$$\psi(x_\alpha) \equiv r_\alpha \pmod{p_\alpha},$$

$$\psi(x_\alpha) \equiv t_{\alpha,j} + \psi(y_{\eta_\alpha(j)}) \pmod{q_{\alpha,j}} \quad \text{for } j \le k.$$

This is possible by the Chinese Remainder Theorem. Now for each ν such that $\nu = \eta_\alpha(n)$ with $n > f^*(\alpha) = k$, choose $\psi(y_\nu)$ minimal in ω such that $\psi(x_\alpha) - \psi(y_\nu) \equiv t_{\alpha,n} \pmod{q_{\alpha,n}}$; this is well-defined (independent of α) by the tree-like properties of η and the primes, the uniformization, and the minimal choices of $\psi(x_\alpha)$ and $\psi(y_\nu)$. Notice as well that by the minimality of f^* and the remark above, any ν is considered at at most one stage. To finish we let $\psi(y_\nu)$ be arbitrary if ν is not of the form $\eta_\alpha(n)$ for any α or n. ∎

THEOREM 7: *If there is a W-group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$, then there is a W-group G of cardinality $\aleph_1$ which is not strongly $\aleph_1$-free.*

Proof: By Lemma 3, there is a ladder system ζ on $\lim(\omega_1)$ which satisfies 2-uniformization. So we are in a position to appeal to Lemma 6. In fact by successive uses of this lemma, we can define, by induction on $m \in \omega$, sequences of primes $\langle p^m_\alpha : \omega \leq \alpha < \omega_1 \rangle$ and $\langle q^m_{\alpha,n} : \omega \leq \alpha < \omega_1, n \in \omega \rangle$, and ladder systems $\eta^m = \langle \eta^m_\alpha : \alpha \in [\omega, \omega_1) \rangle$ which for each $m \in \omega$ satisfy the properties given in Lemma 6 and moreover are such that for all m, α and n, $p^{m+1}_{\eta^m_\alpha(n)} = q^m_{\alpha,n}$.

Let F be the free group on $\{x^m_\alpha : \alpha < \omega_1, m \in \omega\} \cup \{z_{\alpha,m,n} : \omega \leq \alpha < \omega_1, m, n \in \omega\}$ and let K be the subgroup of F generated by $\{w_{\alpha,m,n} : \omega \leq \alpha < \omega_1, m, n \in \omega\}$ where

$$w_{\alpha,m,n} = -q^m_{\alpha,n} z_{\alpha,m,n} + x^m_\alpha - x^{m+1}_{\eta^m_\alpha(n)}.$$

Let G be F/K. In a harmless abuse of notation we shall identify elements of F with their images in $F/K = G$. To see that G is not strongly $\aleph_1$-free, consider the set $Y = \{x^m_\alpha : m < \omega, \alpha < \omega\} \subseteq G$ and show by induction on $\alpha < \omega_1$ that if H is an $\aleph_1$-pure subgroup of G containing Y, then $x^m_\alpha \in H$ for all $m \in \omega$. (The key point is that x^m_α will be divisible by infinitely many primes modulo H since $x^{m+1}_{\eta^m_\alpha(n)} \in H$ by induction.)

To see that G is a W-group, consider $f \in \mathrm{Hom}(K, \mathbb{Z})$. We want to define an extension of f to $g \in \mathrm{Hom}(F, \mathbb{Z})$. The definition of g will take place in ω stages. At the start of stage k, for all α and n we have defined $g(z_{\alpha,m,n})$ for $m \leq k-2$ and $g(x^m_\alpha)$ for $m \leq k-1$, and we have defined r^k_α and committed $g(x^k_\alpha)$ to be r^k_α modulo p^k_α ($= q^{k-1}_{\beta,r}$ where $\eta^{k-1}_\beta(r) = \alpha$).

Apply the uniformization property of Lemma 6 with $r_\alpha = r^k_\alpha$ and $t_{\alpha,n} = f(w_{\alpha,k,n})$. We obtain a function $\psi_k : \{x^k_\nu, x^{k+1}_\nu : \nu \in \omega_1\} \to \mathbb{Z}$ such that $\psi_k(x^k_\alpha) \equiv r^k_\alpha \pmod{p^k_\alpha}$ and

$$\psi_k(x^k_\alpha) - \psi_k(x^{k+1}_{\eta^k_\alpha(n)}) \equiv f(w_{\alpha,k,n}) \pmod{q^k_{\alpha,n}}.$$

Define $g(x^k_\alpha) = \psi_k(x^k_\alpha)$ and let r^{k+1}_α be $\psi_k(x^{k+1}_\alpha)$. Then by induction

$$\begin{aligned} g(x^{k-1}_\alpha) - g(x^k_{\eta^{k-1}_\alpha(n)}) &= \psi_{k-1}(x^{k-1}_\alpha) - \psi_k(x^k_{\eta^{k-1}_\alpha(n)}) \\ &\equiv \psi_{k-1}(x^{k-1}_\alpha) - r^k_{\eta^{k-1}_\alpha(n)} \equiv \psi_{k-1}(x^{k-1}_\alpha) - \psi_{k-1}(x^k_{\eta^k_\alpha(n)}) \\ &\equiv f(w_{\alpha,k-1,n}) \pmod{q^{k-1}_{\alpha,n}}. \end{aligned}$$

So define $g(z_{\alpha,k-1,n})$ to be the unique integer such that

$$g(x^{k-1}_\alpha) - g(x^k_{\eta^{k-1}_\alpha(n)}) - f(w_{\alpha,k-1,n}) = q^{k-1}_{\alpha,n} g(z_{\alpha,k-1,n}).$$

This completes the definition at stage k, and thus completes the proof. ∎

As mentioned before, Chase proved that CH implies that every Whitehead group is strongly $\aleph_1$-free. We can thus derive as a consequence of Theorem 7 that CH implies that every Whitehead group A of cardinality $\aleph_1$ satisfies $\Gamma(A) \neq 1$; this is a complicated way to prove a fact already known, which is derived more easily using the weak diamond principle (cf. [5, XII.1.8]).

The following consequence of the theorem was also already known (see [4, 8.2, p. 74]), but the proof here is more elegant, if less direct.

COROLLARY 8: *There exists a Shelah group of cardinality $\aleph_1$ which is not strongly $\aleph_1$-free.*

Proof: Choose sequences of primes $\langle p^m_\alpha : \omega \leq \alpha < \omega_1 \rangle$ and $\langle q^m_{\alpha,n} : \omega \leq \alpha < \omega_1, n \in \omega \rangle$, and ladder systems $\eta^m = \langle \eta^m_\alpha : \alpha \in [\omega, \omega_1) \rangle$ satisfying all the conditions in Theorem 7 except for the uniformization properties. This can clearly be done in ZFC. Construct G as in Theorem 7. Then, as before, G is not strongly $\aleph_1$-free. We need to show that G is a Shelah group. Note that the property of **not** being a Shelah group of cardinality $\aleph_1$ is absolute for extensions which preserve $\aleph_1$. There is a generic extension of the universe which satisfies MA + $\neg$CH. In this model, every ladder system satisfies $\aleph_0$-uniformization (cf. [5, VI.4.6]), so our ladder systems have the property given in Lemma 6. Then the proof of Theorem 7 applies to show that G is a W-group. But in a model of MA + $\neg$CH, every W-group is a Shelah group (cf. [5, XII.3.20]). So G was a Shelah group to begin with. ∎

Combining Theorem 7 with results from [5, Chapter XII] we have a proof of Theorem 1 stated in the Introduction.

In a similar way one can also prove

THEOREM 9: *The following are equivalent:*

(a) *There is an $\aleph_1$-separable $\aleph_1$-coseparable group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$.*
(b) *There is an strongly $\aleph_1$-free $\aleph_1$-coseparable group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$.*
(c) *There is an $\aleph_1$-coseparable group A of cardinality $\aleph_1$ with $\Gamma(A) = 1$.*
(d) *There is an $\aleph_1$-coseparable group of cardinality $\aleph_1$ which is* **not** *strongly $\aleph_1$-free.*

(e) *There is a ladder system on a stationary subset of* $\lim(\omega_1)$ *which satisfies* ω*-uniformization.*

2. The Second Question

It is consistent that there are non-free Whitehead groups of cardinality $\aleph_1$ but every $\aleph_1$-coseparable group of cardinality $\aleph_1$ is free (see [5, XII.3.18]). Here we shall show that if every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is Whitehead, then every ladder system on a stationary subset of $\lim(\omega_1)$ has ω-uniformization, and hence it follows that there are non-free $\aleph_1$-coseparable groups of cardinality $\aleph_1$.

PROPOSITION 10: *Assume that every strongly* $\aleph_1$*-free group of cardinality* $\aleph_1$ *is Whitehead. Then for any ladder system* $\eta = \langle \eta_\delta : \delta \in S \rangle$ *on a stationary subset* S *of* $\lim(\omega_1)$*, and any* ω*-coloring* $c = \langle c_\delta : \delta \in S \rangle$ *of* η*, there is a pair* $\langle g, g^* \rangle$ *such that* $g^* \colon S \to \omega$ *and* $g \colon \omega_1 \to \omega$ *such that for all* $\delta \in S$ *and all* $n \in \omega$*, if* $n \geq g^*(\delta)$*, then* $g(\eta_\delta(n)) > c_\delta(n)$*.*

Proof: Given what we are trying to prove, we can assume that each c_δ is a strictly increasing function: $\omega \to \omega$. For each δ, n choose a prime $p_{\delta,n} > 4c_\delta(n)$. Define G to be the free group on $\{y_{\delta,n} : \delta \in S,\ n \in \omega\} \cup \{x_\nu : \nu \in \omega_1\}$ modulo the relations

$$p_{\delta,n} y_{\delta,n+1} = y_{\delta,0} + x_{\eta_\delta(n)}. \tag{1}$$

It is routine to check that G is strongly $\aleph_1$-free. Let H be the free group on $\{y'_{\delta,n} : \delta \in S,\ n \in \omega\} \cup \{x'_\nu : \nu \in \omega_1\} \cup \{z\}$ modulo the relations

$$p_{\delta,n} y'_{\delta,n+1} = y'_{\delta,0} + x'_{\eta_\delta(n)} + c_\delta(n) z. \tag{2}$$

Then there is a homomorphism π of H onto G taking $y'_{\delta,n}$ to $y_{\delta,n}$ and x'_ν to x_ν and which has kernel $\mathbb{Z}z$. By hypothesis, since G is Whitehead, there is a splitting $\varphi \colon G \to H$, i.e., such that $\pi \circ \varphi = 1_G$. In particular, for all $\alpha \in \omega_1$, there is $d(\alpha) \in \mathbb{Z}$ such that $\varphi(x_\alpha) - x'_\alpha = d(\alpha) z$.

Define $g(\alpha) = 2|d(\alpha)|$. Applying φ to equation (1) and subtracting (2), we see that $p_{\delta,n}$ divides

$$\varphi(y_{\delta,0}) - y'_{\delta,0} + \varphi(x_{\eta_\delta(n)}) - x'_{\eta_\delta(n)} - c_\delta(n) z \tag{3}$$

in $\mathbb{Z}z$. Let b be such that $bz = \varphi(y_{\delta,0}) - y'_{\delta,0}$. Define $g^*(\delta)$ so that $c_\delta(g^*(\delta)) > 2b$.

Assume that $n \geq g^*(\delta)$. Then $p_{\delta,n} > 4c_\delta(n) > 8b$. Now consider two cases. The first is that (3) is zero, in which case $d(\eta_\delta(n))z = \varphi(x_{\eta_\delta(n)}) - x'_{\eta_\delta(n)} = c_\delta(n)z - bz$. Since $c_\delta(n) - b > c_\delta(n) - c_\delta(n)/2 = c_\delta(n)/2$, $c_\delta(n) < 2d(\eta_\delta(n))$, and thus $c_\delta(n) < g(\eta_\delta(n))$. In the second case, (3) equals mz where m is at least $p_{\delta,n}$ in absolute value, so $|d(\eta_\delta(n))| + |b - c_\delta(n)| \geq p_{\delta,n}$. But $|b - c_\delta(n)| \leq |b| + |c_\delta(n)| < p_{\delta,n}/2$, so $|d(\eta_\delta(n))| \geq p_{\delta,n}/2 > c_\delta(n)$. Hence $c_\delta(n) < g(\eta_\delta(n))$. ∎

COROLLARY 11: *Assume that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is Whitehead. Given a ladder system $\eta = \langle \eta_\delta : \delta \in S \rangle$ on a stationary subset S of $\lim(\omega_1)$, there is a function $g\colon \omega_1 \to \omega$ such that for all $\delta \in S$, $g(\eta_\delta(n)) \geq n$ for all but finitely many $n \in \omega$.*

Proof: Define an ω-coloring $c = \langle c_\delta : \delta \in S \rangle$ by $c_\delta(n) = n - 1$. There is a pair $\langle g, g^* \rangle$ as in Proposition 10 with respect to c. Clearly g is the desired function. ∎

LEMMA 12: *Given any positive integer k and prime $p > 8k$, there are integers a_0 and a_1 and a function $F\colon \mathbb{Z}/p\mathbb{Z} \to 2$ such that for all $m \in \mathbb{Z}$, if $|m| \leq k$, then $F((m + a_\ell) + p\mathbb{Z}) = \ell$ for $\ell = 0, 1$.*

Proof: Let $a_0 = 0$, $a_1 = 3k$. Then $\{m + a_0 : |m| \leq k\} = [-k, k]$ and $\{m + a_1 : |m| \leq k\} = [2k, 4k]$. Since $p > 8k$, $\{i + p\mathbb{Z} : -k \leq i \leq k\}$ is disjoint from $\{j + p\mathbb{Z} : 2k \leq j \leq 4k\}$, so we can define F as desired. ∎

As mentioned in the Introduction, it was shown in [9] that if there is one strongly $\aleph_1$-free group of cardinality $\aleph_1$ which is not free but Whitehead, then there is some ladder system on a stationary subset of ω_1 which satisfies 2-uniformization. Here we show:

THEOREM 13: *Assume that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is Whitehead. Then every ladder system on a stationary subset of $\lim(\omega_1)$ satisfies 2-uniformization.*

Proof: Given a ladder system $\eta = \langle \eta_\delta : \delta \in S \rangle$, let g be as in Corollary 11. By omitting a finite initial segment of each ladder, we can assume, without loss of generality, that $g(\eta_\delta(n)) \geq n$ for all $n \in \omega$.

For each $\alpha \in \omega_1$, choose a prime $p_\alpha > 8g(\alpha)$. Also, for each $\alpha \in \aleph_1$, choose a function

$$F_\alpha\colon \mathbb{Z}/p_\alpha\mathbb{Z} \to 2$$

and integers a_0^α, a_1^α such that for all $m \in \mathbb{Z}$, if $|m| \leq g(\alpha)$, then $F_\alpha(m + a_\ell^\alpha) = \ell$, for $\ell = 0$, 1. (Here, and hereafter, we write $F_\alpha(k)$ instead of $F_\alpha(k + p_\alpha \mathbb{Z})$.) This is possible by Lemma 12.

Now given a 2-coloring $c = \langle c_\delta \colon \delta \in S \rangle$ of η define, as in Proposition 10, G to be the free group on $\{y_{\delta,n} \colon \delta \in S,\, n \in \omega\} \cup \{x_\nu \colon \nu \in \omega_1\}$ modulo the relations

$$p_{\eta_\delta(n)} y_{\delta,n+1} = y_{\delta,0} + x_{\eta_\delta(n)}. \tag{4}$$

and let H be the free group on $\{y'_{\delta,n} \colon \delta \in S,\, n \in \omega\} \cup \{x'_\nu \colon \nu \in \omega_1\} \cup \{z\}$ modulo the relations

$$p_{\eta_\delta(n)} y'_{\delta,n+1} = y'_{\delta,0} + x'_{\eta_\delta(n)} + a_{\delta,n} z \tag{5}$$

where $a_{\delta,n} = a^{\eta_\delta(n)}_{c_\delta(n)}$. Let $\pi \colon H \longrightarrow G$ be the homomorphism taking $y'_{\delta,n}$ to $y_{\delta,n}$ and x'_ν to x_ν; then there is a splitting $\varphi \colon G \to H$ of π. We shall identify the elements of $\mathbb{Z}z$ with integers; thus, for example, $\varphi(x_\alpha) - x'_\alpha$ is an integer.

Define the uniformizing function $f \colon \aleph_1 \to 2$ by

$$f(\alpha) = F_\alpha(\varphi(x_\alpha) - x'_\alpha).$$

We claim that $f(\eta_\delta(n)) = c_\delta(n)$ when $n \geq |\varphi(y_{\delta,0}) - y'_{\delta,0}|$. As in Proposition 10, by applying φ to (4) and subtracting (5), we get that $\varphi(x_{\eta_\delta(n)}) - x'_{\eta_\delta(n)}$ is congruent to $y'_{\delta,0} - \varphi(y_{\delta,0}) + a_{\delta,n}$ (mod $p_{\eta_\delta}(n)$). Hence

$$f(\eta_\delta(n)) = F_{\eta_\delta(n)}(y'_{\delta,0} - \varphi(y_{\delta,0}) + a_{\delta,n})$$

which equals $c_\delta(n)$ when $|y'_{\delta,0} - \varphi(y_{\delta,0})| \leq g(\eta_\delta(n))$ by choice of $F_{\eta_\delta(n)}$. But in fact this is the case when $n \geq |\varphi(y_{\delta,0}) - y'_{\delta,0}|$ because $g(\eta_\delta(n)) \geq n$. ∎

LEMMA 14: *Given a stationary subset S of $\lim(\omega_1)$, for each $\alpha \in \omega_1$ let $\sigma(\alpha)$ denote the least element of S which is greater than α. Then for each $\alpha \in \omega_1$ there is a ladder ζ_α on $\sigma(\alpha)$ such that $\zeta_\alpha(0) > \alpha$ and such that for all $\alpha \neq \beta$,* $\mathrm{rge}(\zeta_\alpha) \cap \mathrm{rge}(\zeta_\beta) = \emptyset$.

Proof: For each $\gamma \in S$, let γ^+ denote the next largest element of S. Then $\sigma(\alpha) = \gamma^+$ if and only if $\alpha \in [\gamma, \gamma^+)$. It is clear that γ^+ contains the disjoint union of ω sets of order type ω, each of which is cofinal in γ^+:

$$\gamma^+ \supseteq \bigcup_{n<\omega} W_n.$$

Let θ_γ be a bijection of $[\gamma, \gamma^+)$ onto ω. Then if $\alpha \in [\gamma, \gamma^+)$, let ζ_α enumerate $W_{\theta_\gamma(\alpha)} \setminus (\alpha + 1)$ in increasing order. ∎

The following result has been proved in [9, 1. 4, p. 262], but we give a self-contained proof here.

THEOREM 15: *Let S be a stationary subset of $\lim(\omega_1)$. If every ladder system on S satisifes 2-uniformization, then every ladder system on S satisfies ω-uniformization.*

Proof: Consider a ladder system $\eta = \langle \eta_\delta \colon \delta \in S \rangle$ and an ω-coloring $c = \langle c_\delta \colon \delta \in S \rangle$. We are going to define another ladder system $\eta' = \langle \eta'_\delta \colon \delta \in S \rangle$ and a 2-coloring c'. Roughly, and slightly inaccurately, we get η' from η by adding a segment of length $c_\delta(n)$ at each $\eta_\delta(n)$ and then we color the new segment by a binary code for $c_\delta(n)$.

Let the ladders ζ_α be as in Lemma 14. Let η'_δ enumerate the ω-sequence

$$\cup_{n \in \omega} \{\zeta_{\eta_\delta(n)}(k) \colon k \le c_\delta(n)\}.$$

Define $c'_\delta(k) = 0$ if $\eta'_\delta(k) = \zeta_{\eta_\delta(n)}(c_\delta(n))$ for some n, and $c'_\delta(k) = 1$ otherwise.

By hypothesis, there is a uniformization $\langle f, f^* \rangle$ of the coloring c' of η'. Define $g \colon \omega_1 \to \omega_1$ as follows: $g(\alpha)$ equals the number of 1's before the first 0 in $f \restriction \mathrm{rge}(\zeta_\alpha)$. Define $g^* \colon S \to \omega$ by: $g^*(\delta) = m$ if m is minimal such that for every $n < f^*(\delta)$, there exists $k < m$ with $\eta'_\delta(n) \in \zeta_{\eta_\delta(k)}$.

We claim that $\langle g, g^* \rangle$ uniformizes the coloring c of η. Suppose $m \ge g^*(\delta)$. Let $\langle n_i \colon i \le c_\delta(m) \rangle$ enumerate in increasing order the set

$$\{j \colon \eta'_\delta(j) \in \mathrm{rge}(\zeta_{\eta_\delta(m)} \restriction (c_\delta(m) + 1))\}.$$

Then $c'_\delta(n_i) = f(\eta'_\delta(n_i))$ for $i \le c_\delta(m)$. So there are exactly $c_\delta(m)$ 1's before the first 0 in $f \restriction \mathrm{rge}(\zeta_{\eta_\delta(m)})$. ∎

We can now give the proof of Theorem 2 stated in the Introduction: (1) implies (2) is trivial; (2) implies (3) is Theorem 13; (3) implies (4) is Theorem 15; and (4) implies (5) is a consequence of [5, XII.3.1]. ∎

The third author claims to have a proof of (4) implies (1) and hence an affirmative answer to the second question (in the Introduction); but he has not yet been able to convince the first two authors.

3. Uniformization on a cub

The theorems of this section have no direct application to Whitehead groups, but they complete a circle of results regarding uniformizations.

THEOREM 16: *Suppose that S is a stationary subset of $\lim(\omega_1)$ which has the property that for every ladder system $\eta = \langle \eta_\delta : \delta \in S \rangle$ on S and every ω-coloring $c = \langle c_\delta : \delta \in S \rangle$, there is a pair (f, f^*) and a cub C on ω_1 such that for every $\delta \in S \cap C$, $f(\eta_\delta(n)) = c_\delta(n)$ for all $n \geq f^*(\delta)$. Then every ladder system on S satisfies ω-uniformization.*

Proof: Let η be as given and let c be any ω-coloring of η. Let C and (f, f^*) be as in the statement of the theorem. For each $\alpha \in C$, let θ_α be a bijection from ω onto α.

Let $S_1 = C^* \cap S$, where C^* is the set of limit points of C. For each $\delta \in S_1$, let η^1_δ enumerate in increasing order the set $\cup_{n \in \omega} Z_n$, where Z_n is defined as follows. Let $\gamma_n = \min(C \setminus (\eta_\delta(n) + 1))$, i.e., γ_n is the least element of C which is greater than $\eta_\delta(n)$; then $\eta_\delta(n) = \theta_{\gamma_n}(k_n)$ for some unique $k_n \in \omega$. Define

$$Z_n = \{\sigma \colon \sigma = \theta_{\gamma_n}(j) \quad \text{for some } \ j \leq k_n \quad \text{and} \ \ \min(C \setminus (\sigma + 1)) = \gamma_n\}$$

Note that $\eta_\delta(n) \in Z_n$, so the range of η^1_δ includes the range of η_δ. We are going to define a coloring $c^1 = \langle c^1_\delta : \delta \in S_1 \rangle$. It will be convenient to regard c^1_δ as a function whose domain is $\mathrm{rge}(\eta^1_\delta)$ rather than ω; that is, if $\sigma = \eta^1_\delta(k)$, we shall write $c^1_\delta(\sigma)$ instead of $c^1_\delta(k)$. For all $\delta \in S_1$ and $n \in \omega$, if $\eta^1_\delta(n) = \eta_\delta(m)$, then $c^1_\delta(\eta^1_\delta(n))$ is defined to be

$$\langle \langle \eta_\delta(j), r_{\delta,j} \rangle \colon j \leq m \rangle$$

where $r_{\delta,j}$ is the size of the intersection of the open interval $(\eta_\delta(j), \min(C \setminus (\eta_\delta(j) + 1)))$ with $\mathrm{rge}(\eta_\delta)$; $c^1_\delta(\eta^1_\delta(n))$ can be regarded as an element of $\omega \setminus \{0\}$ by a coding argument. Otherwise $c^1_\delta(\eta^1_\delta(n))$ is defined to be 0.

By hypothesis there is a pair (f_1, f^*_1) and a cub C_1 such that for $\delta \in S_1 \cap C_1$ and $n \geq f^*_1(\delta)$, $f_1(\eta^1_\delta(n)) = c^1_\delta(\eta^1_\delta(n))$. Without loss of generality, we can assume that $C_1 \subseteq C^*$.

Define $D_1 = C_1 \cap S$, $D_2 = (C \cap S) \setminus C_1$, $D_3 = S \setminus (D_1 \cup D_2)$. We are going to define the desired uniformization, (f_0, f^*_0), of c by defining $f^*_0 = g^*_1 \cup g^*_2 \cup g^*_3$, where $g^*_i \colon D_i \to \omega$.

Define $g_1^*(\delta)$ to be the maximum of $f^*(\delta)$ and the least m such that $\eta_\delta(m) \geq \eta_\delta^1(k)$, where $k \geq f_1^*(\delta)$ and there is $\alpha \in C$ so that $\eta_\delta^1(k-1) \leq \alpha < \eta_\delta^1(k)$. Thus if $n \geq g_1^*(\delta)$, $f(\eta_\delta(n)) = c_\delta(n)$ and $f_1(\eta_\delta(n)) = c_\delta^1(\eta_\delta(n))$. Let

$$A = \{\eta_\delta(n): \delta \in D_1, n \geq g_1^*(\delta)\}.$$

Let $\alpha < \beta$ be two successive members of C (so, in particular, $(\alpha,\beta] \cap C_1 = \emptyset$, since $C_1 \subseteq C^*$). Notice that, by the last clause in the definition of g_1^*, if for some δ and n, $\eta_\delta(n) \in A \cap (\alpha,\beta]$, then $c_\delta^1(\gamma) = f_1(\gamma)$ for all $\gamma \in (\alpha,\beta] \cap \mathrm{rge}(\eta_\delta^1)$. We claim that there exists $\delta \in S_1$ such that $\delta > \beta$ and $A \cap (\alpha,\beta)$ is contained in $\mathrm{rge}(\eta_\delta)$; this implies that $A \cap (\alpha,\beta)$ is finite. It suffices to show that for any δ_1, δ_2 in $D_1 \setminus (\beta+1)$, if $\mathrm{rge}(\eta_{\delta_\ell}) \cap A \cap (\alpha,\beta) \neq \emptyset$ for $\ell = 1,2$, then $\mathrm{rge}(\eta_{\delta_1}) \cap \beta = \mathrm{rge}(\eta_{\delta_2}) \cap \beta$. Now, for each ℓ, n such that $\eta_{\delta_\ell}(n) \in (\alpha,\beta)$, note that $\min(C \setminus (\eta_{\delta_\ell}(n)+1)) = \beta$, so there is a $k_{\ell,n}$ such that $\theta_\beta(k_{\ell,n}) = \eta_{\delta_\ell}(n)$ and a set

$$Z_n^\ell = \{\sigma: \sigma = \theta_\beta(j) \text{ for some } j \leq k_{\ell,n} \text{ and } \min(C \setminus (\sigma+1)) = \beta\} \subseteq \mathrm{rge}(\eta_{\delta_\ell}^1).$$

The sets Z_n^ℓ are linearly ordered by inclusion, so for each ℓ, there is a largest one, which we shall denote Z^ℓ. Without loss of generality, $Z^1 \subseteq Z^2$.

Also by the choice of g_1, we know for each ℓ that $f_1(\sigma) = c_{\delta_\ell}^1(\sigma)$ for $\sigma \in Z^\ell$. So for any $\sigma \in \mathrm{rge}(\eta_{\delta_1}) \cap (\alpha,\beta)$, $0 \neq c_{\delta_1}^1(\sigma) = f_1(\sigma) = c_{\delta_2}^1(\sigma)$. Hence $\mathrm{rge}(\eta_{\delta_1}) \cap \beta \subseteq \eta_{\delta_1} \cap \beta$. Finally if we choose m maximal so that $\eta_{\delta_1}(m) \in (\alpha,\beta)$ then $|(\eta_{\delta_1}(m), \min(C \setminus (\eta_{\delta_1}(m)+1)) \cap \mathrm{rge}(\eta_{\delta_1})| = 0$. Then $c_{\delta_1}^1(\eta_{\delta_1}(m)) = f_1(\eta_{\delta_1}(m)) = c_{\delta_2}^1(\eta_{\delta_1}(m))$, so by definition of c^1, $\eta_{\delta_1}(j) = \eta_{\delta_2}(j)$ for all $j \leq m$ and $\eta_{\delta_2}(m)$ is the largest element of $\mathrm{rge}(\eta_{\delta_2}) \cap \beta$. So we are done.

Define $h_2^*: D_2 \to \omega$ such that if $\alpha < \beta$ are successive members of $C_1 \cup \{0\}$, and if we define

$$B_{\alpha,\beta} = \{\eta_\delta(n): \delta \in D_2 \cap (\alpha,\beta), h_2^*(\delta) \leq n < \omega\},$$

then $B_{\alpha,\beta} \subseteq (\alpha,\beta)$ and for any $\delta \in D_3$, $\sup(B_{\alpha,\beta} \cap \delta) < \delta$. This is not hard to do. Now define $g_2^*(\delta) = \max\{h_2^*(\delta), f_1^*(\delta)\}$ for all $\delta \in D_2$. Let

$$B = \{\eta_\delta(n): \delta \in D_2, g_2^*(\delta) \leq n < \omega\}.$$

Thus for any $\delta \in D_3$, $B \cap \delta$ and $A \cap \delta$ are bounded in δ (the latter because there are successive elements $\alpha < \beta$ in C such that $\alpha < \delta < \beta$ — since $\delta \notin C$). Define $g_3^*: D_3 \to \omega$ such that for all $\delta \in D_3$, $\{\eta_\delta(n): g_3^*(\delta) \leq n < \omega\} \cap (A \cup B) = \emptyset$.

Then let $f_0^* = g_1^* \cup g_2^* \cup g_3^*$. We can then let $f_0 \restriction A \cup B = f_1 \restriction A \cup B$ and easily define f_0 on $\omega_1 \setminus (A \cup B)$ to take care of those δ in D_3. ∎

We shall abbreviate the property given in the hypothesis of Theorem 16 by saying "every ladder system on S satisfies ω-uniformization on a cub". Combining the results of this section with those of the previous section we have the following.

THEOREM 17: *Let S be a stationary subset of* $\lim(\omega_1)$. *Consider the following hypotheses.*

(1) *Every strongly $\aleph_1$-free group A of cardinality $\aleph_1$ with $\Gamma(A) \subseteq S$ is $\aleph_1$-coseparable.*
(2) *Every strongly $\aleph_1$-free group of cardinality $\aleph_1$ with $\Gamma(A) \subseteq S$ is Whitehead.*
(3) *Every ladder system on S satisfies 2-uniformization.*
(4) *Every ladder system on S satisfies ω-uniformization.*
(5) *Every ladder system on S satisfies 2-uniformization on a cub.*
(6) *Every ladder system on S satisfies ω-uniformization on a cub.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6).

Proof: (1) implies (2), (4) implies (3), (6) implies (5) and (4) implies (6) are trivial. Inspection of the proof of Theorem 13 shows that it "localizes" to S, so (2) implies (3). The implication from (3) to (4) is Theorem 15; the proof of (5) implies (6) is exactly the same. That (6) implies (4) is Theorem 16. ∎

4. Topological Considerations

Uniformization results have been associated with the construction of interesting normal spaces. From the existence of a ladder system with 2-uniformization it is easy to construct a normal space which is not metrizable. In fact this is how the consistency with GCH of the failure of the normal Moore space conjecture was established [7]. (See [10] for more information about the normal Moore space conjecture.) A key difference between the Whitehead problem and the construction of normal spaces from ladder systems is that in the topological case the proof of the normality of the space does not require the full power of 2-uniformization, but only requires uniformization of monochromatic colourings. However, by considering a large collection of spaces built from ladder systems, we can get topological equivalents to uniformization principles. We would like to thank Frank Tall for looking at this section, saving us from an elementary error in topology, and providing information about the normal Moore space conjecture.

Recall that if $\alpha = \delta + n$ where $\delta \in \lim(\omega_1)$, then a ladder on α is defined to be a ladder on δ. Suppose that $E \subseteq [\omega, \omega_1)$ and η is a ladder system on E. Then we define a topological space $X(\eta)$ on ω_1 by defining by induction on $\alpha < \omega_1$ a neighborhood base of α. Let α be isolated if $\alpha \notin E$. If $\alpha \in E$, then a neighbourhood base of α is formed by the sets $\{\alpha\} \cup \bigcup_{n \le m} u_m$ where u_m is a neighbourhood of $\eta_\alpha(m)$ and $n < \omega$.

Suppose that $S \subseteq \lim(\omega_1)$. Let $K_0(S)$ be the set of topological spaces of the form $X(\eta)$ where η is a ladder system on some $E \subseteq \{\delta + n\delta \in S$ and $n \in \omega\}$ which satisfies the additional hypothesis that if $\delta + n, \delta + m \in E$ and $m \neq n$ then $\mathrm{rge}(\eta_{\delta+n}) \cap \mathrm{rge}(\eta_{\delta+m}) = \emptyset$. Let $K_1(S)$ be the subset of $K_0(S)$ consisting of all $X(\eta)$ such that if $\eta = \{\eta_\alpha \colon \alpha \in E\}$, then for all $\alpha \in E$, the range of η_α consists of isolated points (i.e., elements of $\omega_1 \setminus E$).

These classes of spaces can be used to give equivalents to uniformization principles.

THEOREM 18: *Let $S \subseteq \lim(\omega_1)$. The following are equivalent.*

(a) *every ladder system on S satisfies 2-uniformization;*

(b) *every member of $K_0(S)$ is normal;*

(c) *every member of $K_1(S)$ is normal;*

(d) *every ladder system on S satisfies $\aleph_0$-uniformization.*

The equivalence of (a) and (d) has already been established. The rest of the section is devoted to proving the non-trivial implications.

From now on we will assume that every ladder system on a set $E \subseteq [\omega, \omega_1)$ is such that if $\delta + n, \delta + m \in E$ and $m \neq n$ then $\mathrm{rge}(\eta_{\delta+n}) \cap \mathrm{rge}(\eta_{\delta+m}) = \emptyset$. With this assumption, there is a simple connection between uniformization on subsets of $\lim(\omega_1)$ and subsets of $[\omega, \omega_1)$.

PROPOSITION 19: *Suppose $S \subseteq \lim(\omega_1)$ and every ladder system on S satisfies 2-uniformization ($\aleph_0$-uniformization). If $E \subseteq \{\delta + n \colon \delta \in S$ and $n \in \omega\}$, then every ladder system on E satisfies 2-uniformization ($\aleph_0$-uniformization).*

Proof: Given $\{\eta_\alpha \colon \alpha \in E\}$, for each $\delta \in S$ choose η^*_δ so that for all n, if $\delta + n \in E$ then the range of $\eta_{\delta+n}$ is contained, except for a finite set, in the range of η^*_δ. Let $\eta^* = \{\eta^*_\delta \colon \delta \in S\}$. Given a coloring $c = \{c_\alpha \colon \alpha \in E\}$ it is easy to produce a colouring c^* of η^* such that any function which uniformizes c^* also uniformizes c. ∎

If S is a stationary subset of $\lim(\omega_1)$ and η is a ladder system on S such that the ladders consist of successor ordinals, then the space $X(\eta)$ is not metrizable.

The connection with the normal Moore space problem came from the following easy fact.

THEOREM 20: *Suppose $E\subseteq[\omega,\omega_1)$ and η is a ladder system on E which satisfies 2-uniformization where for all $\alpha \in E$ the range of η_α consists of isolated points. Then the space $X(\eta)$ is normal.*

Proof: Suppose A_0 and A_1 are disjoint closed sets. Choose a coloring so that c_α is constantly 0 if $\alpha \in A_0$ and c_α is constantly 1 if $\alpha \in A_1$. Suppose that f uniformizes the coloring. Then we can let $U_0 = A_0 \cup \{\beta\colon f(\beta) = 0 \text{ and } \beta \notin (E \cup A_1)\}$ and $U_1 = A_1 \cup \{\beta\colon f(\beta) = 1 \text{ and } \beta \notin (E \cup A_0)\}$. ∎

Unlike the case of abelian groups, where the group constructed from the ladder system is a Whitehead group if and only if the ladder system has 2-uniformization, we cannot deduce the converse here because in the topological case we only need to deal with monochromatic colorings.

THEOREM 21: *Suppose $S\subseteq \lim(\omega_1)$. If every element of $K_1(S)$ is normal then every ladder system on S satisfies 2-uniformization.*

Proof: Suppose we are given $\eta = \{\eta_\delta\colon \delta \in S\}$ a ladder system on S and $c = \{c_\delta\colon \delta \in S\}$ a coloring of η. Let $\{\varsigma(\alpha)\colon \alpha < \omega_1\}$ enumerate the ordinals equivalent to 2 (mod 3) in increasing order. For $\delta \in S$ and $i = 0, 1$ if there exist infinitely many n so that $c_\delta(n) = i$, let $\eta^*_{\delta+i}$ enumerate $\{\varsigma(\eta_\delta(n))\colon c_\delta(n) = i\}$ in increasing order. Otherwise $\eta^*_{\delta+i}$ is undefined. Let $\eta^* = \{\eta^*_{\delta+i}\colon \eta^*_{\delta+i} \text{ is defined}\}$. It is easy to see that $X(\eta^*) \in K_1(S)$ and so by hypothesis is normal. For $i = 0, 1$, let $A_i = \{\delta + i\colon \delta \in S \text{ and } \eta^*_{\delta+i} \text{ is defined}\}$. Let U_i be as guaranteed by normality and choose f so that $f(\alpha) = i$ if $\varsigma(\alpha) \in U_i$. It is easy to check that f uniformizes c. ∎

The previous two results show that (a) is equivalent to (c). It remains to prove that (a) implies (b).

THEOREM 22: *Suppose $S\subseteq[\omega,\omega_1)$ and every ladder system on S satisfies 2-uniformization. Then every element of $K_0(S)$ is normal.*

Proof: Fix $\eta = \{\eta_\delta\colon \delta \in S\}$. It suffices to show that if C, D are disjoint closed sets, then there exists C' so that: (0) $C\subseteq C'$; (1) $C' \cap D = \emptyset$; (2) C' is closed;

and (3) for all $\alpha \in C$ there is k so that $\eta_\alpha(r) \in C'$ for all $r \geq k$. Before proving that C' exists let us see why the claim suffices.

Given disjoint closed sets A_0 and A_1, let $A_{n0} = A_n$ for $n \in \{0,1\}$. Considering each $n \in \{0,1\}$ alternately, we can inductively define A_{nm}, such that if n is the number considered at stage m, then A_{nm+1} is to $A_{nm}, A_{(1-n)m}$ as C' is to C, D. We also let $A_{(1-n)m+1} = A_{(1-n)m}$. Then let $A^n = \bigcup_{m<\omega} A_{nm}$. To finish the proof we must show that A^n is open. We do this by induction on $\alpha \in A^n$. Suppose $\alpha \in A^n$ and choose a stage m where n is considered and $\alpha \in A_{nm}$. If α is isolated then we are done; otherwise η_α is defined. So for some k and all $r \geq k$, $\eta_\alpha(r) \in A_{nm+1} \subseteq A^n$. By induction, A^n contains an open neighborhood u_r of each $\eta_\alpha(r)$. Hence A^n contains $\{\alpha\} \cup \bigcup_{k \geq r} u_r$, which is an open neighborhood of α.

It remains to show that C' exists. For $\alpha \in S$, define c_α to be constantly 0 if $\alpha \in C$ and let c_α be constantly 1 if $\alpha \notin C$. Choose f which uniformizes the coloring. Let $C' = C \cup \{\alpha\colon f(\alpha) = 0 \text{ and } \alpha \notin D\}$. Requirements (0) and (1) follow from the definition. For clause (2) we must show that the complement of C' is open. By induction we show that if $\beta \notin C'$ then the complement of C' contains an open neighborhood of β. If β is isolated, there is nothing to prove. Otherwise η_β exists and $\beta \notin C$. Since f uniformizes the coloring there is n_0 so that for all $m \geq n_0$, $f(\eta_\beta(m)) = 1$. Furthermore since C is closed there is n_1 so that for all $m \geq n_1$, $\eta_\beta(m) \notin C$. So if we let $n = \max\{n_0, n_1\}$, for all $m \geq n$, $\eta_\beta(m) \notin C'$. By the induction hypothesis there is an open neighborhood of each $\eta_\beta(m)$ contained in the complement of C'. So the complement of C' is open. The verification of (3) is similar to the verification of (2) except we use that D is closed as well as that f uniformizes the coloring. ∎

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