ELSEVIER Annals of Pure and Applied Logic 85 (1997) 47–68

ANNALS OF PURE AND APPLIED LOGIC

Can a small forcing create Kurepa trees

Renling Jin[a,b,*,1], Saharon Shelah[b,c,2]

[a] *Department of Mathematics, College of Charleston, Charleston, SC 29424, USA*
[b] *Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*
[c] *Institute of Mathematics, The Hebrew University, Jerusalem, Israel*

Received 30 April 1995; revised 7 November 1995
Communicated by T. Jech

Abstract

In this paper we probe the possibilities of creating a Kurepa tree in a generic extension of a ground model of *CH* plus no Kurepa trees by an ω_1-preserving forcing notion of size at most ω_1. In Section 1 we show that in the Lévy model obtained by collapsing all cardinals between ω_1 and a strongly inaccessible cardinal by forcing with a countable support Lévy collapsing order, many ω_1-preserving forcing notions of size at most ω_1 including all ω-proper forcing notions and some proper but not ω-proper forcing notions of size at most ω_1 do not create Kurepa trees. In Section 2 we construct a model of *CH* plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree such that forcing with that Aronszajn tree does create a Kurepa tree in the generic extension. At the end of the paper we ask three questions.

0. Introduction

By a model we mean a model of *ZFC*. By a forcing notion we mean a separative partially ordered set $\mathbb{P}$ with a largest element $1_{\mathbb{P}}$ used for a corresponding forcing extension. Given a model V of CH, one can create a generic Kurepa tree by forcing with an ω_1-closed, ω_2-c.c. forcing notion no matter whether or not V contains Kurepa trees [7]. One can also create a generic Kurepa tree by forcing with a c.c.c. forcing notion provided V satisfies $\square_{\omega_1}$ in addition [16]. Both forcing notions mentioned here have size at least ω_2. The size being at least ω_2 seems necessary in each case for guaranteeing that the generic tree has at least ω_2 branches. On the other hand, a Kurepa tree has a base set of size ω_1, so it seems possible to create a Kurepa tree by a forcing notion of size $\leqslant \omega_1$. In this paper we discuss the following question: Given

* Corresponding author. E-mail: rjin@math.rutgers.edu.

[1] Partially supported by NSF postdoctoral fellowship #DMS-9508887.

[2] Supported by Basic Research Foundation of The Israel Academy of Sciences and Humanities. Publication No. 563.

0168-0072/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved
PII S0168-0072(96)00018-8

Sh:563

a model of CH plus no Kurepa trees, can we find an ω_1-preserving forcing notion of size $\leqslant \omega_1$ such that the forcing creates Kurepa trees?

This question is partially motivated by a result on Souslin trees. Given a ground model V, a Souslin tree could be created by a c.c.c. forcing notion of size ω_1 [14]. There is also an ω_1-closed forcing notion of size ω_1 which creates a Souslin tree provided V satisfies CH [7]. The question whether a Souslin tree could be created by a countable forcing notion (equivalent to adding a Cohen real) turns out to be much harder. It was answered positively by the second author [13] ten years ago.

We call a forcing notion ω_1-preserving if ω_1 in the ground model is still a cardinal in the generic extension. In this paper we consider only ω_1-preserving forcing notion for the following reason. Let V be the Lévy model. In V there are no Kurepa trees and CH holds. Notice also that there is an ω_2-Kurepa tree in V. If we simply collapse ω_1 by forcing with the collapsing order $Coll(\omega, \omega_1)$, the set of all finite partial function from ω to ω_1 ordered by reverse inclusion, in V, then the ω_2-Kurepa tree becomes a Kurepa tree in $V^{Coll(\omega,\omega_1)}$. Notice also that $Coll(\omega, \omega_1)$ has size ω_1 in V. So we require the forcing notions under consideration be ω_1-preserving to avoid this triviality.

In Section 1 we show some evidence that in the Lévy model it is extremely hard to find a forcing notion, if it ever exists, of size $\leqslant \omega_1$ which could create a Kurepa tree in the generic extension. Assume that our ground model V is the Lévy model. We mention first an easy result that any forcing notion of size $\leqslant \omega_1$ which adds no new reals could not create Kurepa trees. Then we show two results: (1) For any stationary set $S \subseteq \omega_1$, if $\mathbb{P}$ is an (S, ω)-proper forcing notion of size $\leqslant \omega_1$, then there are no Kurepa trees in the generic extension $V^{\mathbb{P}}$. Note that all axiom A forcing notions are (ω_1, ω)-proper. (2) Some proper forcing notions including the forcing notion for adding a club subset of ω_1 by finite conditions do not create Kurepa trees in the generic extension.

In Section 2 we show that there is a model of CH plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree T such that forcing with T does create a Kurepa tree in the generic extension. We start with a model V containing a strongly inaccessible cardinal κ. In V we define an ω_1-closed, κ-c.c. forcing notion $\mathbb{P}$ such that forcing with $\mathbb{P}$ creates an ω-distributive Aronszajn tree T and a T-name $\dot{K}$ for a Kurepa tree K. Forcing with $\mathbb{P}$ collapses all cardinals between ω_1 and κ so that κ is ω_2 in $V^{\mathbb{P}}$. Take $\bar{V} = V^{\mathbb{P}}$ as our ground model. Forcing with T in $\bar{V}$ creates a Kurepa tree in the generic extension of $\bar{V}$. So the model $\bar{V}$ is what we are looking for except that we have to prove that there are no Kurepa trees in $\bar{V}$, which is the hardest part of the second section.

We assume the consistency of ZFC plus a strongly inaccessible cardinal. We shall write V, $\bar{V}$, etc. for (countable) transitive models of ZFC. For a forcing notion $\mathbb{P}$ in V we shall write $V^{\mathbb{P}}$ for the generic extension of V by forcing with $\mathbb{P}$. Sometimes, we write also $V[G]$ instead of $V^{\mathbb{P}}$ for a generic extension when a particular generic filter G is involved. We shall fix a large enough regular cardinal λ, e.g. $\lambda = (\beth_{1995}(\kappa))^+$, throughout this paper and write $H(\lambda)$ for the collection of sets hereditarily of power less than λ equipped with the membership relation. In a forcing argument with a forcing notion $\mathbb{P}$ we shall write $\dot{a}$ for a $\mathbb{P}$-name of a and $\ddot{a}$ for a $\mathbb{P}$-name of $\dot{a}$ which is again a $\mathbb{Q}$-name of a for some forcing notion $\mathbb{Q}$. If a is already in the ground model we shall write simply a for a canonical name of a. Let $\mathbb{P}$ be a forcing notion and $p \in \mathbb{P}$. We shall write $q \leqslant p$ to mean $q \in \mathbb{P}$ and q is a condition stronger than p. We shall often write $p \Vdash$"..." for some $p \in \mathbb{P}$ instead of $p \Vdash^V_{\mathbb{P}}$"..." when the ground model V and the forcing notion $\mathbb{P}$ in the argument is clear. We shall also write $\Vdash$"..." instead of $1_{\mathbb{P}} \Vdash$"...". In this paper all of our trees are subtrees of the tree $\langle 2^{<\omega_1}, \subseteq\rangle$. So if C is a linearly ordered subset of a tree T, then $\bigcup C$ is the only possible candidate of the least upper bound of C in T. In this paper all trees are growing upward. If a tree is used as a forcing notion we shall put the tree upside down. Let T be a tree and $x \in T$. We write $ht(x) = \alpha$ if $x \in T \cap 2^\alpha$. We write T_α or $(T)_\alpha$, the αth level of T, for the set $T \cap 2^\alpha$ and write $T \restriction \alpha$ or $(T) \restriction \alpha$ for the set $\bigcup_{\beta<\alpha} T_\beta$. We write $ht(T)$ for the height of T, which is the smallest ordinal α such that T_α is empty. By a normal tree we mean a tree T such that (1) for any $\alpha < \beta < ht(T)$, for any $x \in T_\alpha$ there is a $y \in T_\beta$ such that $x < y$; (2) for any α such that $\alpha + 1 < ht(T)$ and for any $x \in T_\alpha$ there are infinitely many successors of x in $T_{\alpha+1}$. Given two trees T and T', we write $T \leqslant_{\text{end}} T'$ for T' being an end-extension of T, i.e. $T' \restriction ht(T) = T$. By a branch of a tree T we mean a totally ordered set of T which intersects every nonempty level of T. By an ω_1-tree we mean a tree of height ω_1 with each of its levels at most countable. A Kurepa tree is an ω_1-tree with more than ω_1 branches. See [8, 11, 12] for more information on forcing, iterated forcing, proper forcing, etc. and to see [15] for more information on trees.

1. Creating Kurepa trees by a small forcing is hard

First, we would like to state a theorem in [12, 2.11] without proof as a lemma which will be used in this section.

Lemma 1. *In a model V let $\mathbb{P}$ be a forcing notion and let N be a countable elementary submodel of $H(\lambda)$. Suppose $G \subseteq \mathbb{P}$ is a V-generic filter. Then*

$$N[G] = \{\dot{a}_G : \dot{a} \text{ is a } \mathbb{P}\text{-name and } \dot{a} \in N\}$$

is a countable elementary submodel of $(H(\lambda))^{V[G]}$.

We choose the Lévy model $\bar{V} = V^{Lv(\kappa,\omega_1)}$ as our ground model throughout this section, where κ is a strongly inaccessible cardinal in V and $Lv(\kappa,\omega_1)$, the Levy collapsing order, is the set

$$\{p \subseteq (\kappa \times \omega_1) \times \kappa : p \text{ is a countable function and } (\forall(\alpha,\beta) \in dom(p))(p(\alpha,\beta) \in \alpha)\}$$

ordered by reverse inclusion. For any $A \subseteq \kappa$ we write $Lv(A,\omega)$ for the set of all $p \in Lv(\kappa,\omega_1)$ such that $dom(p) \subset A \times \omega_1$.

First, we mention an easy result without proof.

Sh:563

Theorem 2. *Let $\mathbb{P}$ be a forcing notion of size $\leqslant \omega_1$ in $\bar{V}$. If forcing with $\mathbb{P}$ does not add new countable sequences of ordinals, then there are no Kurepa trees in $\bar{V}^{\mathbb{P}}$.*

The proof of Theorem 2 is very simple. Since forcing with $\mathbb{P}$ does not add any new countable sequences of ordinals, then $\mathbb{P}$ is interchangeable with $Lv(\kappa\backslash\eta,\omega_1)$ for some $\eta \in \kappa$. But the forcing notion $Lv(\kappa\backslash\eta,\omega_1)$ could be viewed again as a countable support Lévy collapsing order.

Next we show the result concerning (S,ω)-proper forcing notions with a brief sketch of the proof.

Definition 3. A forcing notion $\mathbb{P}$ is said to satisfy property $(\dagger)$ if for any $x \in H(\lambda)$, there exists a sequence $\langle N_i : i \in \omega\rangle$ of elementary submodels of $H(\lambda)$ such that

(1) $N_i \in N_{i+1}$ for every $i \in \omega$,

(2) $\{\mathbb{P},x\} \subseteq N_0$,

(3) for every $p \in \mathbb{P} \cap N_0$ there exists a $q \leqslant p$ such that q is $(\mathbb{P},N_i)$-generic for every $i \in \omega$.

Lemma 4. *Let V be any model. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions in V such that $\mathbb{P}$ has size $\leqslant \omega_1$ and satisfies property $(\dagger)$, and $\mathbb{Q}$ is ω_1-closed (in V). Suppose T is an ω_1-tree in $V^{\mathbb{P}}$. Then T has no branches which are in $V^{\mathbb{P}\times\mathbb{Q}}$ but not in $V^{\mathbb{P}}$.*

Proof (*sketch*). Suppose, towards a contradiction, that there is a branch b of T in $V^{\mathbb{P}\times\mathbb{Q}}\backslash V^{\mathbb{P}}$. Without loss of generality, we can assume that

$$\Vdash_{\mathbb{P}}\Vdash_{\mathbb{Q}} (\ddot{b} \text{ is a branch of } \dot{T} \text{ in } V^{\mathbb{P}\times\mathbb{Q}}\backslash V^{\mathbb{P}}).$$

Claim 4a. *For any $p \in \mathbb{P}$, $q \in \mathbb{Q}$, $n \in \omega$ and $\alpha \in \omega_1$, there are $p' \leqslant p$, $q_j \leqslant q$ for $j < n$ and $\beta \in \omega_1\backslash\alpha$ such that*

$$p' \Vdash \left((\exists\{t_j : j < n\} \subseteq \dot{T}_\beta)\left((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j<n} (q_j \Vdash t_j \in \ddot{b}) \right)\right).$$

The claim follows from a fact about forcing (see [11, p. 201]).

Claim 4b. *Let $\eta \in \omega_1$ and let $q \in \mathbb{Q}$. There exists a $\nu \leqslant \omega_1$, a maximal antichain $\langle p_\alpha : \alpha < \nu\rangle$ of $\mathbb{P}$, two decreasing sequences $\langle q^j_\alpha : \alpha < \nu\rangle$, $j = 0,1$, in $\mathbb{Q}$ and an increasing sequence $\langle \eta_\alpha : \alpha < \nu\rangle$ in ω_1 such that $q^0_0, q^1_0 < q$, $\eta_0 > \eta$ and for any $\alpha < \nu$*

$$p_\alpha \Vdash ((\exists t_0,t_1 \in \dot{T}_{\eta_\alpha})(t_0 \neq t_1 \wedge (q^0_\alpha \Vdash t_0 \in \ddot{b}) \wedge (q^1_\alpha \Vdash t_1 \in \ddot{b}))).$$

The proof of a similar version of this claim could be found in either [5] or [9]. The size of $\mathbb{P}$ less than or equal to ω_1 is used to ensure $\nu \leqslant \omega_1$.

The lemma follows from above two claims. Let $\langle N_n : n \in \omega\rangle$ witness that $\mathbb{P}$ satisfies property $(\dagger)$. Let $n \in \omega$, $\delta_n = \omega_1 \cap N_n$ and let $\delta = \bigcup_{n\in\omega} \delta_n$. For each $s \in 2^n$ we

Sh:563

construct, in N_n, a maximal antichain $\langle p^s_\alpha : \alpha < \nu_s\rangle$ of $\mathbb{P}$, two decreasing sequences $\langle q^{s\hat{}j}_\alpha : \alpha < \nu_s\rangle$ for $j = 0, 1$, and an increasing sequence $\langle \eta^s_\alpha : \alpha < \nu_s\rangle$ in δ_n such that $\nu_s \leqslant \delta_n$, $q^{s\hat{}j}_0$ are lower bounds of $\langle q^s_\alpha : \alpha < \nu_{s\restriction n-1}\rangle$ for $j = 0, 1$, $\eta^s_0 = \delta^{n-1}$, and

$$p^s_\alpha \Vdash ((\exists t_0, t_1 \in \dot{T}_{\eta^s_\alpha})(t_0 \neq t_1 \wedge (q^{s\hat{}0}_\alpha \Vdash t_0 \in \ddot{b}) \wedge (q^{s\hat{}1}_\alpha \Vdash t_1 \in \ddot{b}))).$$

Each step of the construction uses Claim 4b relative to N_n for some $n \in \omega$. We can choose $q^{s\hat{}0}_0$ and $q^{s\hat{}1}_0$ to be lower bounds of $\langle q^s_\alpha : \alpha < \nu_{s\restriction n-1}\rangle$ in N_n because $\langle q^s_\alpha : \alpha < \nu_{s\restriction n-1}\rangle$ is constructed in N_{n-1} and hence is countable in N_n. Here we use the fact $N_{n-1} \in N_n$.

Let $\bar{p} \leqslant 1_{\mathbb{P}}$ be $(\mathbb{P}, N_n)$-generic for every $n \in \omega$. Since $\mathbb{Q}$ is ω_1-closed in V, for every $f \in 2^\omega$ there is a q_f which is a lower bound of $\langle q^{f\restriction n}_0 : n \in \omega\rangle$. Let $G \subseteq \mathbb{P}$ be a V-generic filter such that $\bar{p} \in G$. It is now clear that T_δ is uncountable because different q_f's force different t_f's into T_δ for $f \in V$. □

A forcing notion $\mathbb{P}$ is called ω-proper if for any ω-sequence $\langle N_n : n \in \omega\rangle$ of countable elementary submodels of $H(\lambda)$ such that $N_n \in N_{n+1}$ for every $n \in \omega$ and $\mathbb{P} \in N_0$, for any $p \in \mathbb{P} \cap N_0$ there is a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N_n)$-generic for every $n \in \omega$. Let S be a stationary subset of ω_1. A forcing notion $\mathbb{P}$ is called S-proper if for any countable elementary submodel N of $H(\lambda)$ such that $\mathbb{P} \in N$ and $N \cap \omega_1 \in S$, and for any $p \in \mathbb{P} \cap N$ there is a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N)$-generic. A forcing notion $\mathbb{P}$ is called (S, ω)-proper if for any ω-sequence $\langle N_n : n \in \omega\rangle$ of countable elementary submodels of $H(\lambda)$ such that $N_n \in N_{n+1}$ for every $n \in \omega$, $N_n \cap \omega_1 \in S$ for every $n \in \omega$, $N \cap \omega_1 \in S$, where $N = \bigcup_{n\in\omega} N_n$, and $\mathbb{P} \in N_0$, for any $p \in \mathbb{P} \cap N_0$ there is a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N_n)$-generic for every $n \in \omega$.

Theorem 5. *Let S be a stationary subset of ω_1 and let $\mathbb{P}$ be an (S, ω)-proper forcing notion of size $\leqslant \omega_1$ in $\bar{V}$. Then there are no Kurepa trees in $\bar{V}^{\mathbb{P}}$.*

Proof (*sketch*). Choose an $\eta < \kappa$ such that S and $\mathbb{P}$ are in $V^{Lv(\eta, \omega_1)}$. Then

$$\bar{V}^{\mathbb{P}} = V^{(Lv(\eta, \omega_1) * \dot{\mathbb{P}}) \times Lv(\kappa \setminus \eta, \omega_1)}.$$

Note that $Lv(\kappa \setminus \eta, \omega_1)$ is ω_1-closed in $V^{Lv(\eta, \omega_1)}$. Now the theorem follows from Lemma 4 by the fact that $\mathbb{P}$ satisfies property (†) in $V^{Lv(\eta, \omega_1)}$. To prove this fact the reader may find that Lemma 1 is needed. □

Remark. (1) The idea of the proof of Lemma 4 is originally from [5]. A version of Theorem 5 for axiom A forcing was proved in [9]. The reader who is familiar with the above two papers may reproduce a complete proof of Theorem 5 without too many difficulties. The proof of Theorem 9 has also some similar ideas.

(2) If $\mathbb{P}$ satisfies axiom A, then $\mathbb{P}$ is ω-proper or (ω_1, ω)-proper. Hence forcing with a forcing notion of size $\leqslant \omega_1$ satisfying axiom A in $\bar{V}$ does not create Kurepa trees.

Sh:563

(3) The ω-properness implies the (S,ω)-properness and the (S,ω)-properness implies the property $(\dagger)$.

Now we prove the result on some non-(S,ω)-proper forcing notions.

The existence of a Kurepa tree implies that there are no countably complete, $\aleph_2$-saturated ideals on ω_1. Therefore, one can destroy all those ideals by creating a generic Kurepa tree [16]. But one does not have to create Kurepa trees for this purpose. Baumgartner and Taylor [4] proved that adding a club subset of ω_1 by finite conditions destroys all countably complete, $\aleph_2$-saturated ideals on ω_1. The forcing notion for adding a club subset of ω_1 by finite conditions has size $\leqslant \omega_1$ and is proper but not (S,ω)-proper for any stationary subset S of ω_1. We are going to prove next that this forcing notion and some other similar forcing notions do not create Kurepa trees if our ground model is the Lévy model $\bar{V}$. Notice also that the ideal of nonstationary subsets of ω_1 could be $\aleph_2$-saturated in the Lévy model obtained by collapsing a supercompact cardinal down to ω_2 [6]. As a corollary we can have a ground model $\bar{V}$ which contains countably complete, $\aleph_2$-saturated ideals on ω_1 such that forcing with some small proper forcing notion $\mathbb{P}$ in $\bar{V}$ destroys all countably complete, ω_2-saturated ideals on ω_1 without creating Kurepa trees.

We first define a property of forcing notions which is satisfied by the forcing notion for adding a club subset of ω_1 by finite conditions.

Definition 6. A forcing notion $\mathbb{P}$ is said to satisfy property (#) if for any $x \in H(\lambda)$ there exists a countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P},x\} \subseteq N$ and for any $p_0 \in \mathbb{P} \cap N$ there exists a $\bar{p} \leqslant p_0$, $\bar{p}$ is $(\mathbb{P},N)$-generic, and there exists a countable subset C of $\mathbb{P}$ such that for any $\bar{p}' \leqslant \bar{p}$ there is a $c \in C$ and a $p' \in \mathbb{P} \cap N$, $p' \leqslant p_0$ such that:

(i) for any dense open subset D of $\mathbb{P}$ below p', $D \in N$, there is a $d \in D \cap N$ such that d is compatible with c, and

(ii) for any $r \in \mathbb{P} \cap N$ and $r \leqslant p'$, r is compatible with c implies r is compatible with $\bar{p}'$.

Let us call the pair (p',c) a related pair corresponding to $\bar{p}'$.

Example 7. The following three examples are forcing notions which satisfy property (#).

(i) Let

$$\mathbb{P} = \{p \subseteq \omega_1 \times \omega_1 : p \text{ is a finite function which can be extended to an increasing continuous function from } \omega_1 \text{ to } \omega_1\}$$

and let $\mathbb{P}$ be ordered by reverse inclusion. $\mathbb{P}$ is one of the simplest proper forcing notions which do not satisfy axiom A [3]. Forcing with $\mathbb{P}$ creates a generic club subset of ω_1 and destroys all countably complete, $\aleph_2$-saturated ideals on ω_1 [4]. It is easy to see that $\mathbb{P}$ satisfies property (#) defined above. For any $x \in H(\lambda)$ we can choose a countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P},x\} \subseteq N$ and $N \cap \omega_1 = \delta$ is an indecomposable ordinal. For any $p_0 \in \mathbb{P} \cap N$ let $\bar{p} = p_0 \cup (\delta, \delta)$ and let $C = \{\bar{p}\}$. Then for any $\bar{p}' \leqslant \bar{p}$ there is a $p' = \bar{p}' \restriction \delta$ and a $c = \bar{p} \in C$ such that all requirements for the definition of property (#) are satisfied.

(ii) Let S be a stationary subset of ω_1. If we define

$$\mathbb{P}_S = \{p : p \text{ is a finite function such that there is an increasing continuous function } f \text{ from some countable ordinal to } S \text{ such that } p \subseteq f\}$$

and let $\mathbb{P}_S$ be ordered by reverse inclusion, then $\mathbb{P}_S$ is S-proper [3]. Forcing with $\mathbb{P}_S$ adds a club set inside S. It is also easy to check that $\mathbb{P}_S$ satisfies (#). For any $x \in H(\lambda)$ let N be a countable elementary submodel of $H(\lambda)$ such that $\{x, \mathbb{P}_S\} \subseteq N$, $N \cap \omega_1 = \delta$ is an indecomposable ordinal and $\delta \in S$. Then for any $p_0 \in \mathbb{P}_S \cap N$ the element $\bar{p} = p_0 \cup \{(\delta, \delta)\}$ is $(\mathbb{P}_S, N)$-generic. Now N, $\bar{p}$ and $C = \{\bar{p}\}$ witness that $\mathbb{P}_S$ satisfies property (#).

(iii) Let T and U be two normal Aronszajn trees such that every node of T or U has infinitely many immediate successors. Let $\mathbb{P}$ be the forcing notion such that $p = (A_p, f_p) \in \mathbb{P}$ iff

(a) A_p is a finite subset of ω_1,

(b) f_p is a finite partial isomorphism from $T \restriction A_p$ into $U \restriction A_p$,

(c) $dom(f_p)$ is a subtree of $T \restriction A_p$ in which every branch has cardinality $|A_p|$.

$\mathbb{P}$ is ordered by $p \leqslant q$ iff $A_p \supseteq A_q$ and $f_p \supseteq f_q$. $\mathbb{P}$ is proper [15]. $\mathbb{P}$ is used in [1] for generating a club isomorphism from T to U. For any $x \in H(\lambda)$, for any countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P}, x\} \subseteq N$ and for any $p_0 \in \mathbb{P} \cap N$, let $\delta = N \cap \omega_1$, let $A_{\bar{p}} = A_{p_0} \cup \{\delta\}$ and let $f_{\bar{p}}$ be any extension of f_{p_0} such that $T_\delta \cap dom(f_{\bar{p}}) \neq \emptyset$. Then $\bar{p} = (A_{\bar{p}}, f_{\bar{p}})$ is a $(\mathbb{P}, N)$-generic condition. Let

$$C = \{d : d \text{ is a finite isomorphism from } T_\delta \text{ to } U_\delta\}.$$

Then C is countable. For any $\bar{p}' \leqslant \bar{p}$ let $c = (f_{\bar{p}'} \restriction T_\delta) \in C$, let $\alpha < \delta$, $\alpha > \max(A_{\bar{p}'} \cap \delta)$ and let

$$g_\alpha = \{(t, u) \in T_\alpha \times U_\alpha : (\exists (t', u') \in (f_{\bar{p}'} \restriction T_\delta))(t < t' \wedge u < u')\}$$

be such that g_α and $f_{\bar{p}'} \restriction T_\delta$ have same cardinality, let $A_{p'} = (A_{\bar{p}'} \cap \delta) \cup \{\alpha\}$, let $f_{p'} = (f_{\bar{p}'} \restriction (\bigcup_{v \in A_{\bar{p}'} \cap \delta} T_v)) \cup g_\alpha$, and let $p' = (A_{p'}, f_{p'})$. Then (p', c) is a related pair corresponding to $\bar{p}'$ [1] and $N, \bar{p}, C$ witness that $\mathbb{P}$ satisfies property (#).

For any stationary set $S \subseteq \omega_1$ an S-proper version of this forcing notion, which satisfies also property (#), could be defined in a similar way.

Lemma 8. *Let V be a model. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions in V such that $\mathbb{P}$ has size $\leqslant \omega_1$ and satisfies property (#), and $\mathbb{Q}$ is ω_1-closed (in V). Suppose T is an ω_1-tree in $V^{\mathbb{P}}$. Then T has no branches which are in $V^{\mathbb{P} \times \mathbb{Q}}$ but not in $V^{\mathbb{P}}$.*

Proof. Suppose, towards a contradiction, that there is a branch b of T in $V^{\mathbb{P} \times \mathbb{Q}} \setminus V^{\mathbb{P}}$. Without loss of generality, we assume that

$$\Vdash_{\mathbb{P}} \Vdash_{\mathbb{Q}} (\ddot{b} \text{ is a branch of } \dot{T} \text{ in } V^{\mathbb{P} \times \mathbb{Q}} \setminus V^{\mathbb{P}}).$$

Following the definition of property (#), we can find a countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P}, \mathbb{Q}, \dot{T}, \ddot{b}\} \subseteq N$, a $\bar{p} \leqslant 1_{\mathbb{P}}$ which is $(\mathbb{P}, N)$-generic and a countable set $C \subseteq \mathbb{P}$ such that N, $\bar{p}$ and C witness that $\mathbb{P}$ satisfies property (#). Let $\langle (p_i, c_i) : i \in \omega \rangle$ be a listing of all related pairs in $(\mathbb{P} \cap N) \times C$ with infinite repetition, i.e. every related pair (p, c) in $(\mathbb{P} \cap N) \times C$ occurs infinitely often in the sequence.

We construct now, in V, a set $\{q_s \in \mathbb{Q} \cap N : s \in 2^{<\omega}\}$ and an increasing sequence $\langle \delta_n : n \in \omega \rangle$ such that

(1) $s \subseteq t$ implies $q_t \leqslant q_s$,

(2) $\delta_n \in \delta = N \cap \omega_1$,

(3) for every $n \in \omega$ there is a $p' \in \mathbb{P} \cap N, p' \leqslant p_n$ such that p' is compatible with c_n, and

$$p' \Vdash \left((\exists \{t_s : s \in 2^n\} \subseteq \dot{T}_{\delta_n}) \left((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^n} (q_s \Vdash t_s \in \ddot{b}) \right) \right).$$

The lemma follows from the construction. Let $G \subseteq \mathbb{P}$ be a V-generic filter and $\bar{p} \in G$. We want to show that

$$V[G] \models T_\delta \text{ is uncountable.}$$

For any $f \in 2^\omega \cap V$ let $q_f \in \mathbb{Q}$ be a lower bound of the set $\{q_{f \upharpoonright n} : n \in \omega\}$ such that there is a $t_f \in T_\delta$ such that $q_f \Vdash t_f \in \dot{b}$. Suppose T_δ is countable. Then there are $f, g \in 2^\omega \cap V$ such that $t_f = t_g$. Let $\dot{t}_f$, $\dot{t}_g$ be $\mathbb{P}$-names for t_f, t_g and let $\bar{p}' \leqslant \bar{p}$ be such that

$$\bar{p}' \Vdash (\dot{t}_f = \dot{t}_g \wedge (q_f \Vdash \dot{t}_f \in \ddot{b}) \wedge (q_g \Vdash \dot{t}_g \in \ddot{b})).$$

Let $m = \min\{i \in \omega : f(i) \neq g(i)\}$. By the definition of property (#) we can find a related pair (p, c) corresponding to $\bar{p}'$. Choose an $n \in \omega$ such that $n \geqslant m$ and $(p, c) = (p_n, c_n)$. Since Definition 6(i) is true, there is a $p' \in \mathbb{P} \cap N$ such that $p' \leqslant p$, p' is compatible with c_n and

$$p' \Vdash \left((\exists \{t_s : s \in 2^n\} \subseteq \dot{T}_{\delta_n}) \left((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^n} (q_s \Vdash t_s \in \ddot{b}) \right) \right).$$

Since $q_f \leqslant q_{f \upharpoonright n}$ and $q_g \leqslant q_{g \upharpoonright n}$, then

$$\bar{p}' \Vdash ((\exists t_0, t_1 \in \dot{T}_{\delta_n})(t_0 \neq t_1 \wedge (q_f \Vdash t_0 \in \ddot{b}) \wedge (q_g \Vdash t_1 \in \ddot{b}))).$$

But also

$$\bar{p}' \Vdash ((\exists t \in \dot{T}_\delta)((q_f \Vdash t \in \ddot{b}) \wedge (q_g \Vdash t \in \ddot{b}))).$$

By the fact that any two nodes in T_{δ_n} which are below a node in T_δ must be same, and that p' is compatible with $\bar{p}'$, we have a contradiction.

Now let us inductively construct $\{\delta_i : i \in \omega\}$ and $\{q_s : s \in 2^{<\omega}\}$. Suppose we have had $\{q_s : s \in 2^{\leqslant n}\}$ and $\{\delta_i : i \leqslant n\}$. let $D \subseteq \mathbb{P}$ be such that $r \in D$ iff

(1) $r \leqslant p_n$ (recall that (p_n, c_n) is in the enumeration of all related pairs in $(\mathbb{P} \cap N) \times C$,

(2) there exists $\eta > \delta_n$ and there exists $\{q_s \leqslant q_{s\restriction n} : s \in 2^{n+1}\}$ such that

$$r \Vdash \left((\exists \{t_s : s \in 2^{n+1}\} \subseteq \dot{T}_\eta) \left((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^{n+1}} (q_s \Vdash t_s \in \ddot{b}) \right) \right).$$

It is easy to see that D is open and $D \in N$.

Claim 8a. *D is dense below p_n.*

Proof. Suppose $r_0 \leqslant p_n$. It suffices to show that there is an $r \leqslant r_0$ such that $r \in D$. Applying Claim 4a, for any $s \in 2^n$ we can find $r_s \leqslant r_0$, $\eta_s > \delta_n$ and $\{q_j^s \leqslant q_s : j < 2^{n+1}\}$ such that

$$r_s \Vdash \left((\exists \{t_j : j < 2^{n+1}\} \subseteq \dot{T}_{\eta_s}) \left((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j < 2^{n+1}} (q_j^s \Vdash t_j \in \ddot{b}) \right) \right).$$

Let $\{s_i : i < 2^n\}$ be an enumeration of 2^n. By applying Claim 4a 2^n times as above we obtained $r_0 \geqslant r_{s_0} \geqslant r_{s_1} \geqslant \cdots r_{s_{2^n-1}}$ such that above arguments are true for any $s \in 2^n$. Pick $\eta = \max\{\eta_s : s \in 2^n\}$. Then we extend $r_{s_{2^n-1}}$ to r', and extend q_j^s to $\bar{q}_j^s$ for every such s and j such that for each $s \in 2^n$

$$r' \Vdash \left((\exists \{t_j : j < 2^{n+1}\} \subseteq \dot{T}_\eta) \left((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j < 2^{n+1}} (\bar{q}_j^s \Vdash t_j \in \ddot{b}) \right) \right).$$

Now applying an argument in Claim 4b repeatedly we can choose $\{q_{s^\frown 0}, q_{s^\frown 1}\} \subseteq \{\bar{q}_j^s : j < 2^{n+1}\}$ for every $s \in 2^n$ and extend r' to r'' such that

$$r'' \Vdash \left((\exists \{t_s : s \in 2^{n+1}\} \subseteq \dot{T}_\eta) \left((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^{n+1}} (q_s \Vdash t_s \in \ddot{b}) \right) \right).$$

This showed that D is dense below p_n.

Notice that since N is elementary, then η exists in N and all those q_s's for $s \in 2^{n+1}$ exist in N. Choose $r \in D$ such that r, c_n are compatible and let δ_{n+1} be correspondent η. This ends the construction. □

Theorem 9. *If $\mathbb{P}$ in $\bar{V}$ is a forcing notion defined in* (i)–(iii) *of Example* 7, *then forcing with $\mathbb{P}$ does not create any Kurepa trees.*

Proof. Suppose T is a Kurepa tree in $\bar{V}^{\mathbb{P}}$. Let $\eta < \kappa$ be such that $\mathbb{P}, T \in V^{Lv(\eta,\omega_1)}$. Since the definition of $\mathbb{P}$ is absolute between $\bar{V}$ and $V^{Lv(\eta,\omega_1)}$, then $\mathbb{P}$ satisfies property (#) in $V^{Lv(\eta,\omega_1)}$. Since T has less than κ branches in $V^{Lv(\eta,\omega)*\dot{\mathbb{P}}}$, there exist branches of T in $\bar{V}$ which are not in $V^{Lv(\eta,\omega_1)*\dot{\mathbb{P}}}$. This contradicts Lemma 8. □

Remark. The forcing notions in Example 7, (i)–(iii) are not (S,ω)-proper for any stationary S.

2. Creating Kurepa trees by a small forcing is easy

In this section we construct a model of CH plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree T such that forcing with T does create a Kurepa tree in the generic extension.

Let V be a model and κ be a strongly inaccessible cardinal in V. Let $\mathscr{T}$ be the set of all countable normal trees. Given a set A and a cardinal λ, let $[A]^{<\lambda} = \{S \subseteq A : |S| < \lambda\}$ and $[A]^{\leqslant\lambda} = \{S \subseteq A : |S| \leqslant \lambda\}$. We define a forcing notion $\mathbb{P}$ as following:

Definition 10. p is a condition in $\mathbb{P}$ iff

$$p = \langle \alpha_p, t_p, k_p, U_p, B_p, F_p \rangle$$

where

(a) $\alpha_p \in \omega_1$,

(b) $t_p \in \mathscr{T}$ and $ht(t_p) = \alpha_p + 1$,

(c) k_p is a function from t_p to $\mathscr{T}$ such that for any $x \in t_p$, $ht(k_p(x)) = ht(x) + 1$, and for any $x, y \in t_p$, $x < y$ implies $k_p(x) \leqslant_{\text{end}} k_p(y)$,

(d) $U_p \in [\kappa]^{\leqslant\omega}$,

(e) $B_p = \{b_\gamma^p : \gamma \in U_p\}$ where b_γ^p is a function from t_p to $\omega_1^{<\omega_1}$ such that for any $x \in t_p$, $b_\gamma^p(x) \in (k_p(x))_{ht(x)}$ and for any $x, y \in t_p$, $x \leqslant y$ implies $b_\gamma^p(x) \leqslant b_\gamma^p(y)$,

(f) $F_p = \{f_\gamma^p : \gamma \in U_p\}$ where f_γ^p is a function from $\alpha_p + 1$ to γ

(g) for any $x \in t_p \restriction \alpha_p$, for any finite $U_0 \subseteq U_p$ and for any ε such that $ht(x) < \varepsilon \leqslant \alpha_p$, there exist infinitely many $x' \in (t_p)_\varepsilon$ such that $x' > x$ and for any $\gamma_1, \gamma_2 \in U_0$, $b_{\gamma_1}^p(x) = b_{\gamma_2}^p(x)$ implies $b_{\gamma_1}^p(x') = b_{\gamma_2}^p(x')$.

In the condition (g) of the definition we call each x' a conservative extension of x at level ε with respect to U_0 (or with respect to $\{b_\gamma^p : \gamma \in U_0\}$).

Generally, we need the following notation. Suppose $t \in \mathscr{T}$ and B is a set of functions with $domain(b) = t$ for each $b \in B$. We say t is *consistent with respect to B* if for any $x \in t$, for any finite $B_0 \subseteq B$ and for any ε such that $ht(x) < \varepsilon \leqslant ht(t)$, there exist infinitely many $x' \in (t)_\varepsilon$ such that $x' > x$ and for any $b_1, b_2 \in B_0$, $b_1(x) = b_2(x)$ implies $b_1(x') = b_2(x')$. So $p \in \mathbb{P}$ implies that t_p is consistent with respect to B_p.

For any $p, q \in \mathbb{P}$ we define the order of $\mathbb{P}$ by letting $p \leqslant q$ iff

(1) $\alpha_q \leqslant \alpha_p$, $t_q \leqslant_{\text{end}} t_p$, $k_q \subseteq k_p$ and $U_q \subseteq U_p$,

(2) for any $\gamma \in U_q$, $b_\gamma^q \subseteq b_\gamma^p$ and $f_\gamma^q \subseteq f_\gamma^p$,

For any $\theta < \kappa$ let

$$\mathbb{P}_\theta = \{p \in \mathbb{P} : U_p \subseteq \theta\}.$$

Then the identity embedding of $\mathbb{P}_\theta$ into $\mathbb{P}$ is a complete embedding. For each $p \in \mathbb{P}$ we shall write $p \restriction \mathbb{P}_\theta = q$ if $\langle \alpha_p, t_p, k_p \rangle = \langle \alpha_q, t_q, k_q \rangle$, $U_q = U_p \cap \theta$ and for each $\gamma \in U_q$ we have $b_\gamma^p = b_\gamma^q$ and $f_\gamma^p = f_\gamma^q$.

Remark. In the definition of $\mathbb{P}$ the part t_p is used for creating an ω-distributive Aronszajn tree T. The part k_p is used for creating a T-name of an ω_1-tree K. The part B_p

is used for adding κ branches to K so that K becomes a Kurepa tree in the generic extension by forcing with T. The part F_p is used for collapsing all cardinals between ω_1 and κ.

For any $\varepsilon \in \omega_1$, $\gamma \in \kappa$ and $\eta \in \gamma$, let

$$D^1_\varepsilon = \{p \in \mathbb{P} : \alpha_p \geqslant \varepsilon\},$$

$$D^2_\gamma = \{p \in \mathbb{P} : \gamma \in U_p\},$$

$$D^3_{\eta,\gamma} = \{p \in \mathbb{P} : \gamma \in U_p \text{ and } \eta \in range(f^p_\gamma)\},$$

Lemma 11. *The sets D^1_ε, D^2_γ and $D^3_{\eta,\gamma}$ are open dense in $\mathbb{P}$.*

Proof. It is easy to see that all three sets are open.

Given $p_0 \in \mathbb{P}$. We need to find a $p \leqslant p_0$ such that $p \in D^1_\varepsilon$. Pick an $\alpha_p \geqslant \varepsilon$ such that $\alpha_p \geqslant \alpha_{p_0}$. Let $t_p \in \mathscr{T}$ be such that $ht(t_p) = \alpha_p + 1$ and $t_{p_0} \leqslant_{\text{end}} t_p$. For each $x \in (t_{p_0})_{\alpha_{p_0}}$ choose a $t_x \in \mathscr{T}$ such that $ht(t_x) = \alpha_p + 1$ and $k_{p_0}(x) \leqslant_{\text{end}} t_x$. For any $x' \in t_p$, $x' > x$, define $k_p(x') = t_x \restriction ht(x') + 1$ and define $k_p \restriction t_{p_0} = k_{p_0}$. Let $U_p = U_{p_0}$. For each $x \in (t_{p_0})_{\alpha_{p_0}}$ let

$$\{b^{p_0}_\gamma(x) : \gamma \in U_p\} = \{y_n : n < l\}$$

with y_n's being distinct, for some $l \leqslant \omega$. For each y_n we choose a $z_n \in (t_x)_{\alpha_p}$ such that $y_n \leqslant z_n$. Then for each $x' \in t_p$, $x' > x$ and for each $\gamma \in U_p$ we define $b^p_\gamma(x') = y \in (t_p)_{ht(x')}$ such that $y_n < y \leqslant z_n$ where $b^p_\gamma(x) = y_n$. Also let $b^p_\gamma \restriction t_{p_0} = b^{p_0}_\gamma$. For each $\gamma \in U_p$ let f^p_γ be any extension of $f^{p_0}_\gamma$ to $\alpha_p + 1$ complying with Definition 10(f). It is easy to see that $p \in \mathbb{P} \cap D^1_\varepsilon$ and $p \leqslant p_0$.

Given $p_0 \in \mathbb{P}$. We need to find a $p \leqslant p_0$ such that $p \in D^2_\gamma$. If $\gamma \in U_{p_0}$, let $p = p_0$. Assume that $\gamma \notin U_{p_0}$. If $U_{p_0} = \emptyset$, then let b^p_γ and f^p_γ be any functions complying with Definition 10(e) and (f), respectively. If $U_{p_0} \neq \emptyset$, then pick any $\gamma' \in U_{p_0}$ and let $b^p_\gamma = b^{p_0}_{\gamma'}$, and let f^p_γ be any function complying with Definition 10(f). Let the rest of p be same as p_0. Then $p \in \mathbb{P} \cap D^2_\gamma$ and $p \leqslant p_0$.

Given $p_0 \in \mathbb{P}$. We need to find a $p \leqslant p_0$ such that $p \in D^3_{\eta,\gamma}$. Without loss of generality, we assume that $p_0 \in D^2_\gamma$. Let $p \leqslant p_0$ be chosen as in the proof of the denseness of D^1_ε with $\alpha_p > \alpha_{p_0}$ except that we require $f^p_\gamma(\alpha_{p_0}) = \eta$. Then $p \in \mathbb{P} \cap D^3_{\eta,\gamma}$ and $p \leqslant p_0$. $\square$

Lemma 12. *$\mathbb{P}$ is ω_1-closed.*

Proof. Let $\{p_n : n \in \omega\}$ be a decreasing sequence in $\mathbb{P}$. If $\{\alpha_{p_n} : n \in \omega\}$ has a largest element $\alpha = \alpha_{p_{n_0}}$, then we can just let

$$\alpha_p = \alpha,\ t_p = t_{p_{n_0}},\ k_p = k_{p_{n_0}},\ U_p = \bigcup_{n \in \omega} U_{p_n},$$

$$B_p = \{b^{p_n}_\gamma : \gamma \in U_{p_n}, n \geqslant n_0\} \text{ and } F_p = \{f^{p_n}_\gamma : \gamma \in U_{p_n}, n \geqslant n_0\}.$$

Then p is a lower bound of p_n's.

Sh:563

Assume that $\alpha = \bigcup_{n\in\omega} \alpha_{p_n}$ is a limit ordinal. Let

$$t = \bigcup_{n\in\omega} t_{p_n},\ k = \bigcup_{n\in\omega} k_{p_n} \text{ and } U = \bigcup_{n\in\omega} U_{p_n}.$$

For each $\gamma \in U$ let

$$b_\gamma = \bigcup\{b_\gamma^{p_n} : \gamma \in U_{p_n}\}$$

and let

$$f_\gamma = \bigcup_{n\in\omega} \{f_\gamma^{p_n} : \gamma \in U_{p_n}\}.$$

For each $x \in t$ and each finite set $\Delta \subseteq U$ we can choose a countable set of branches $\{c_{x,\Delta,n} : n \in \omega\}$ of t passing through x such that for each $y \in c_{x,\Delta,n}$ and $y > x$, y is a conservative extension of x with respect to Δ. Now let $\alpha_p = \alpha$ and let

$$t_p = t \cup \{y : x \in t, \Delta \in [U]^{<\omega}, n \in \omega, \text{ and } y = \bigcup c_{x,\Delta,n}\}.$$

Let $U_p = U$. For each $\gamma \in U_p$ let b_γ^p be an extension of b_γ such that for each $y \in (t_p)_{\alpha_p}$

$$b_\gamma^p(y) = \bigcup\{b_\gamma(z) : z \in t, z < y\}.$$

For each $\gamma \in U_p$ let f_γ^p be an extension of f_γ such that for each $y \in (t_p)_{\alpha_p}$

$$f_\gamma^p(y) = \bigcup\{f_\gamma(z) : z \in t, z < y\}.$$

Finally, let k_p be an extension of k such that $k_p(y) \in \mathscr{T}$ for each $y \in (t_p)_{\alpha_p}$ such that $ht(k_p(y)) = \alpha_p + 1$,

$$\bigcup\{k(z) : z \in t, z < y\} \leqslant_{\text{end}} k_p(y)$$

and

$$\{b_\gamma^p(y) : \gamma \in U_p\} \subseteq k_p(y).$$

It is easy to see that p is a lower bound of p_n's in $\mathbb{P}$. $\square$

Lemma 13. $\mathbb{P}$ *satisfies* κ*-c.c.*.

Proof. Let $\{p_\eta : \eta \in \kappa\} \subseteq \mathbb{P}$. By a cardinality argument and Δ-system lemma there is an $S \subseteq \kappa$, $|S| = \kappa$ and there is a triple $\langle \alpha_0, t_0, k_0\rangle$ such that for every $\eta \in S$

$$\langle \alpha_{p_\eta}, t_{p_\eta}, k_{p_\eta}\rangle = \langle \alpha_0, t_0, k_0\rangle,$$

and $\{U_{p_\eta} : \eta \in S\}$ forms a Δ-system with the root U_0. Furthermore, we can assume that for each $\gamma \in U_0$,

$$b_\gamma^{p_\eta} = b_\gamma^{p_{\eta'}} \quad \text{and} \quad f_\gamma^{p_\eta} = f_\gamma^{p_{\eta'}}$$

Sh:563

for any $\eta, \eta' \in S$. Since there are at most $(|\omega_1^{\leqslant \alpha_0}|^{|t_0|})^\omega$ sequences of length ω of the functions from t_0 to $\omega_1^{\leqslant \alpha_0}$, there are $\eta, \eta' \in S$ such that

$$\{b_\gamma^{p_\eta} : \gamma \in U_{p_\eta} \backslash U_0\} \quad \text{and} \quad \{b_\gamma^{p_{\eta'}} : \gamma \in U_{p_{\eta'}} \backslash U_0\}$$

are same set of functions. It is easy to see now that the element

$$p = \langle \alpha_0, t_0, k_0, U_{p_\eta} \cup U_{p_{\eta'}}, B_{p_\eta} \cup B_{p_{\eta'}}, F_{p_\eta} \cup F_{p_{\eta'}} \rangle$$

is a common lower bound of p_η and $p_{\eta'}$. □

Lemma 14. *All cardinals between ω_1 and κ in V are collapsed in $V^{\mathbb{P}}$.*

Proof. Using F_p-part of the conditions together with a density argument in $\mathbb{P}$. □

Remark. By Lemmas 12–14 we have

$$V^{\mathbb{P}} \models (2^\omega = \omega_1^V = \omega_1 \text{ and } 2^{\omega_1} = \kappa = \omega_2).$$

Lemma 15. *Let $G \subseteq \mathbb{P}$ be a V-generic filter and let $T_G = \bigcup\{t_p : p \in G\}$. Then T_G is an ω-distributive Aronszajn tree in $V[G]$.*

Proof. It is obvious that T_G is an ω_1-tree. Suppose there is a $p_0 \in \mathbb{P}$ such that

$$p_0 \Vdash \dot{B} \text{ is a branch of } T_G.$$

We construct $p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$ such that $\langle \alpha_{p_n} : n \in \omega \rangle$ is strictly increasing and

$$p_{n+1} \Vdash z_n \in \dot{B} \cap (t_{p_n})_{\alpha_{p_n}}$$

for some $z_n \in \omega_1^{\alpha_{p_n}}$. Let p be the lower bound of p_n's such that $\alpha_p = \bigcup_{n\in\omega} \alpha_{p_n}$ and let p' be same as p except that $t_{p'} = t_p \backslash \{\bigcup_{n\in\omega} z_n\}$. Then p' is still a lower bound of p_n's. But now we have

$$p' \Vdash \dot{B} \subseteq t_{p'}.$$

Next we prove that T_G is ω-distributive. Let $\mathbb{Q} = \langle T_G, \leqslant' \rangle$ be the forcing notion by reversing tree order ($\leqslant' = \geqslant_{T_G}$). Given any $\tau \in 2^\omega$ in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. It suffices to show that $\tau \in V$. We construct a decreasing sequence

$$\langle p_0, \dot{x}_0 \rangle \geqslant \langle p_1, \dot{x}_1 \rangle \geqslant \langle p_2, \dot{x}_2 \rangle \geqslant \cdots$$

in $\mathbb{P} * \dot{\mathbb{Q}}$ such that

$$\langle p_0, \dot{x}_0 \rangle \Vdash \dot{\tau} \text{ is a function from } \omega \text{ to } 2,$$

$$p_{n+1} \Vdash \dot{x}_n = \bar{x}_n$$

Sh:563

for some $\bar{x}_n \in \omega_1^{\alpha_{p_n}}$ and

$$\langle p_{n+1}, \dot{x}_{n+1} \Vdash \dot{\tau}(n) = l_n$$

for some $l_n \in \{0,1\}$. Let $x = \bigcup_{n\in\omega} \bar{x}_n$ and let p be a lower bound of p_n's such that $\alpha_p = \bigcup_{n\in\omega} \alpha_{p_n}$ and $x \in t_p$. Then $\langle p, x\rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ and

$$\langle q, x\rangle \Vdash \dot{\tau} = \sigma$$

for $\sigma = \langle l_0, l_1, \ldots\rangle \in 2^\omega$ in V. $\square$

Lemma 16. *Let $G \subseteq \mathbb{P}$ be a V-generic filter and let $k_G = \bigcup\{k_p : p \in G\}$. Let T_G and $\mathbb{Q}$ be same as in Lemma 15. Suppose $H \subseteq \mathbb{Q}$ is a $V[G]$-generic filter. Then $K_H = \bigcup\{k_G(x) : x \in H\}$ is a Kurepa tree in $V[G][H]$.*

Proof. It is easy to see that K_H is an ω_1-tree. For any $\gamma \in \kappa$ let

$$b_\gamma = \bigcup\{b_\gamma^p : p \in G \text{ and } \gamma \in U_p\}.$$

Then b_γ is a function with domain T_G. Let

$$W_\gamma = \bigcup\{b_\gamma(x) : x \in H\}.$$

Then it is easy to see that W_γ is a branch of K_H. We need now only to show that W_γ and $W_{\gamma'}$ are different branches for different $\gamma, \gamma' \in \kappa$. Given distinct γ and γ' in κ. Let

$$D^4_{\gamma,\gamma'} = \{p \in \mathbb{P} : (\forall x \in t_p \restriction \alpha_p)(\exists y \in t_p)(y \geqslant x \text{ and } b_\gamma^p(y) \neq b_{\gamma'}^p(y))\}.$$

Claim 16a. *The set $D^4_{\gamma,\gamma'}$ is dense in $\mathbb{P}$.*

Proof. Given $p_0 \in \mathbb{P}$. Without loss of generality, we assume that $p_0 \in D^2_\gamma \cap D^2_{\gamma'}$. First, we extend p_0 to p' such that

$$\alpha_{p'} = \alpha_{p_0} + 1.$$

For each $x \in (t_{p_0})_{\alpha_{p_0}}$ we add one extra successor node y_x of x to $(t_{p'})_{\alpha_{p'}}$ to form t_p. Let $\alpha_p = \alpha_{p'}$, $U_p = U_{p'}$ and let $f^p_{\gamma''} = f^{p'}_{\gamma''}$ for all $\gamma'' \in U_p$. By complying with Definition 10 we arbitrarily extend $k_{p'}$ to k_p on t_p and extend $b^{p'}_{\gamma''}$ to $b^p_{\gamma''}$ on t_p for all $\gamma'' \in U_p$ except that we require $b^p_\gamma(y_x) \neq b^p_{\gamma'}(y_x)$. Then $p \in \mathbb{P} \cap D^4_{\gamma,\gamma'}$ and $p \leqslant p_0$. This ends the proof of the claim. $\square$

We need to prove W_γ and $W_{\gamma'}$ are different branches of K_H in $V[G][H]$. Suppose $x \in H$ and

$$x \Vdash \dot{W}_\gamma = \dot{W}_{\gamma'}$$

Sh:563

in $V[G]$. Let $p_0 \in G$ be such that $x \in t_{p_0}$. By the claim we can find a $p \leqslant p_0$ and $p \in G \cap D^4_{\gamma,\gamma'}$ such that $\alpha_p > ht(x)$. Then we can choose $y \in t_p$ and $y > x$ such that $b^p_\gamma(y) \neq b^p_{\gamma'}(y)$. Therefore

$$y \Vdash \dot{W}_\gamma \neq \dot{W}_{\gamma'},$$

which contradicts that

$$x \Vdash \dot{W}_\gamma = \dot{W}_{\gamma'}. \qquad \square$$

The proof of next lemma is probably the hardest part of this section.

Lemma 17. *There are no Kurepa trees in $V^{\mathbb{P}}$.*

Proof. Suppose

$$\Vdash_{\mathbb{P}} \dot{T} \text{ is a Kurepa tree.}$$

Since $\mathbb{P}$ has κ-c.c., there exists a regular uncountable cardinal $\theta < \kappa$ such that $\dot{T}$ is a $\mathbb{P}_\theta$-name. Because of $2^{\omega_1} < \kappa$ in $V^{\mathbb{P}_\theta}$, there exists a set of $\mathbb{P}$-names $\dot{\mathscr{C}} = \{\dot{c}_\beta : \beta \in \kappa\}$, where $\dot{c}_\beta$'s are $\mathbb{P}$-names of different branches of T in $V^{\mathbb{P}} \setminus V^{\mathbb{P}_\theta}$. We want to show that $V^{\mathbb{P}_\theta} \models T_\delta$ is uncountable for some $\delta \in \omega_1$.

For each $\beta \in \kappa$ with $cof(\beta) = (2^\theta)^+$ we choose an elementary submodel $\mathfrak{A}_\beta$ of $H(\lambda)$ such that

(a) $|\mathfrak{A}_\beta| \leqslant 2^\theta$,

(b) $\{\dot{T}, \dot{\mathscr{C}}, \mathbb{P}, \beta\} \cup \theta \subseteq \mathfrak{A}_\beta$,

(c) $[\mathfrak{A}_\beta]^{\leqslant \theta} \subseteq \mathfrak{A}_\beta$.

We shall not distinguish a model from its base set. By the Pressing Down Lemma we can find a stationary set

$$S' \subseteq \{\beta \in \kappa : cof(\beta) = (2^\theta)^+\}$$

and a $\bar{\beta} \in \kappa$ such that for any $\beta \in S'$ we have

$$\bigcup(\beta \cap \mathfrak{A}_\beta) = \bar{\beta}.$$

Then by the Δ-System Lemma we can find an $S \subseteq S'$ such that $|S| = \kappa$,

$$\{\mathfrak{A}_\beta : \beta \in S\}$$

forms a Δ-system with common root $\mathfrak{B}$ and $\beta \cap \mathfrak{A}_\beta \subseteq \mathfrak{B}$ for each $\beta \in S$. Furthermore, we can assume that for any $\beta, \beta' \in S$ there is an isomorphism $h_{\beta,\beta'}$ from $\mathfrak{A}_\beta$ to $\mathfrak{A}_{\beta'}$ such that $h_{\beta,\beta'} \restriction \mathfrak{B}$ is an identity map. Note that $\omega_1^{<\omega_1} \cup \mathbb{P}_\theta \subseteq \mathfrak{B}$ and $\mathbb{P}_\theta \in \mathfrak{B}$. Note also that for any $\beta, \beta' \in S$ we have $h_{\beta,beta'}(\beta) = \beta'$.

Let $\beta_0 = \min S$. In $\mathfrak{A}_{\beta_0}$ we want to construct inductively the sequences

$\langle p_n \in \mathbb{P}_\theta : n \in \omega \rangle$,

$\langle p_s \in \mathbb{P} \cap \mathfrak{A}_{\beta_0} : s \in 2^{<\omega} \rangle$,

$\langle \eta_n \in \omega_1 : n \in \omega \rangle$

Sh:563

and

$\langle x_s \in \omega_1^{<\omega_1} : s \in 2^{<\omega} \rangle$

such that for any $n \in \omega$ and any $s, s' \in 2^{<\omega}$

(1) $p_{n+1} < p_n$ and $\alpha_{p_n} < \alpha_{p_{n+1}}$,

(2) $p_s \leqslant p_{s'}$ if $s' \subseteq s$,

(3) $p_s \restriction \mathbb{P}_\theta = p_n$,

(4) $\eta_n < \eta_{n+1}$,

(5) $x_{s'} \leqslant x_s$ if $s' \subseteq s$,

(6) $\eta_{n-1} \leqslant ht(x_s) < \eta_n$ if $s \in 2^n$,

(7) x_s and $x_{s'}$ are incompatible if s and s' are incompatible,

(8) $p_s \Vdash x_s \in \dot{c}_{\beta_0}$ for every $s \in 2^{<\omega}$,

(9) each function in $\bigcup_{r \in 2^n} B_{p_r}$ has a copy in B_{p_n} (note that (9) is stronger than that 't_{p_n} is consistent with respect to $\bigcup_{r \in 2^n} B_{p_r}$').

We need to construct two more sequences and add three more requirements for all the sequences along the construction. Let us fix an onto function $j : \omega \mapsto \omega \times \omega$ such that $j(n) = \langle a, b \rangle$ implies $a \leqslant n$. Let π_1, π_2 be projections with $\pi_1(\langle a, b \rangle) = a$ and $\pi_2(\langle a, b \rangle) = b$ for any pair $\langle a, b \rangle$. Let

$$\xi_n : \omega \mapsto t_{p_n} \times \left(\left[\bigcup_{s \in 2^n} U_{p_s} \right]^{<\omega} \right)$$

and

$$\zeta_n : \omega \mapsto \bigcup_{s \in 2^n} U_{p_s}$$

be two infinite-to-one onto functions for each $n \in \omega$. Let e be a function with $dom(e) = \omega$ and for each $n \in \omega$

$$e(n) = \xi_{\pi_1(j(n))}(\pi_2(j(n))).$$

The functions ξ_n's, ζ_n's and e will be used for bookkeeping purpose. For $s \in 2^m$ and $m < n$ let

$$C_{s,n} = \{ s' \in 2^n : s \subseteq s' \}.$$

For any $m, n \in \omega$, $m \leqslant n$ define

$$\begin{aligned} Z_m^n = & \{ b_\gamma^{p_{s'}} : s \in 2^{\pi_1(j(m))},\ \gamma \in \pi_2(e(m)) \cap U_{p_s} \text{ and } s' \in C_{s,n} \} \\ & \cup \{ b_\gamma^{p_{s'}} : s \in 2^{\pi_1(j(m))},\ \gamma \in U_{p_s} \text{ and } \gamma = \zeta_{\pi_1(j(m))}(i) \text{ for some } i \leqslant n \}. \end{aligned}$$

Note that Z_m^n is finite. For each $m, n \in \omega$, $m \leqslant n$, let

$$Y_m^n = \{ y_{m,i} : m \leqslant i \leqslant n \}.$$

Then Z_m^n's and Y_m^n's and other four sequences should satisfy three more requirements.

(10) $y_{m,m} = \pi_1(e(m))$ and $y_{m,i} \in (t_{p_i})_{\alpha_{p_i}}$ for $m < i \leqslant n$,

(11) $y_{m,i+1}$ is a conservative extension of $y_{m,i}$ with respect to Z_m^{i+1} for $m \leqslant i < n$,

(12) for any $m \leqslant i \leqslant n$ either $y_{m,i} \leqslant y_{n,n}$ or $y_{m,i}$ and $y_{n,n+1}$ are incomparable.

Suppose for some $l \in \omega$ we have found

$$\langle p_n \in \mathbb{P}_\theta : n < l \rangle,$$
$$\langle p_s \in \mathbb{P} \cap \mathfrak{A}_{\beta_0} : s \in 2^{<l} \rangle,$$
$$\langle \eta_n \in \omega_1 : n < l \rangle,$$
$$\langle x_s \in \omega_1^{<\omega_1} : s \in 2^{<l} \rangle,$$
$$\{ Z_m^n : n < l, m \leqslant n \}$$

and

$$\{ Y_m^n : n < l, m \leqslant n \}.$$

Claim 17a. *For any $p \in \mathbb{P} \cap \mathfrak{A}_{\beta_0}$ and for any $\alpha \in \omega_1$ there exists an $\eta \in \omega_1 \setminus \alpha$, there exist $q, q_0, q_1 \in \mathbb{P} \cap \mathfrak{A}_{\beta_0}$ and there exist $x_0, x_1 \in \omega_1^\eta$, $x_0 \neq x_1$, such that $q \in \mathbb{P}_\theta$,*

$$q_0 \restriction \mathbb{P}_\theta = q_1 \restriction \mathbb{P}_\theta = q,$$

$q_0, q_1 \leqslant p$, each function in $B_{q_0} \cup B_{q_1}$ has a copy in B_q, and

$$q_i \Vdash x_i \in \dot{c}_{\beta_0}$$

for $i = 0, 1$.

Proof. Pick a $\beta_1 \in S \setminus \{\beta_0\}$ and let $p' = h_{\beta_0,\beta_1}(p)$. Notice that p and p' are compatible because the part of p not in $\mathfrak{B}$ is completely moved away while the part in $\mathfrak{B}$ is fixed. We construct a common lower bound of p and p'. Let $r = p \restriction \mathbb{P}_\theta$ and let

$$r' = \langle \alpha_{r'}, t_{r'}, k_{r'}, U_{r'}, B_{r'}, F_{r'} \rangle,$$

where

$$\alpha_{r'} = \alpha_r, \qquad t_{r'} = t_r, \qquad k_{r'} = k_r, \qquad U_{r'} = U_p \cup U_{p'},$$

$$B_{r'} = B_p \cup B_{p'} \quad \text{and} \quad F_{r'} = F_p \cup F_{p'}.$$

Then $r' \leqslant p, p'$. Since

$$r' \Vdash \dot{c}_{\beta_0} \neq \dot{c}_{\beta_1},$$

then there exists an $r'' \leqslant r'$, there exist $\eta \in \omega_1 \setminus \alpha$ and there exist $x_0, x_1 \in \omega_1^\eta$, $x_0 \neq x_1$ such that

$$r'' \Vdash x_i \in \dot{c}_{\beta_i}$$

for $i = 0, 1$. By adding in countably many new ordinals in θ to $U_{r''}$ and using those ordinals to index the copies of all functions in $B_{r''}$ we can assume that for any $\gamma \in U_{r''}$

Sh:563

there is a $\gamma' \in U_{r''} \cap \theta$ such that $b^{r''}_{\gamma}$ and $b^{r''}_{\gamma'}$ are same functions. Let $q = r'' \upharpoonright \mathbb{P}_\theta$. Then the following is true.

$$H(\lambda) \models (\exists q_0 \in \mathbb{P})((q_0 \leqslant p) \wedge (q_0 \upharpoonright \mathbb{P}_\theta = q) \wedge (\forall \gamma \in U_{q_0}) \\ (\exists \gamma' \in U_q)(b^{q_0}_\gamma = b^q_{\gamma'}) \wedge (q_0 \Vdash x_0 \in \dot{c}_{\beta_0})).$$

Since $\mathfrak{A}_{\beta_0}$ is an elementary submodel of $H(\lambda)$, there exists a $q_0 \in \mathfrak{A}_{\beta_0}$ such that the above sentence is true in $\mathfrak{A}_{\beta_0}$. By the same reason we can find $q_1' \in \mathbb{P}$ in $\mathfrak{A}_{\beta_1}$ such that

$$\mathfrak{A}_{\beta_1} \models (q_1' \leqslant p') \wedge (q_1' \upharpoonright \mathbb{P}_\theta = q) \wedge (\forall \gamma \in U_{q_1'})(\exists \gamma' \in U_q)(b^{q_1'}_\gamma = b^q_{\gamma'}) \wedge (q_1' \Vdash x_1 \in \dot{c}_{\beta_1}).$$

By the fact that

$$h_{\beta_1,\beta_0}(\beta_1) = \beta_0, \qquad h_{\beta_1,\beta_0}(p') = p, \qquad h_{\beta_1,\beta_0}(q) = q,$$

$$h_{\beta_1,\beta_0}(\mathbb{P}_\theta) = \mathbb{P}_\theta, \qquad h_{\beta_1,\beta_0}(x_1) = x_1, \qquad h_{\beta_1,\beta_0}(\dot{c}_{\beta_1}) = \dot{c}_{\beta_0},$$

and letting $h_{\beta_1,\beta_0}(q_1') = q_1$ we have

$$\mathfrak{A}_{\beta_0} \models (q_1 \leqslant p) \wedge (q_1 \upharpoonright \mathbb{P}_\theta = q) \wedge (\forall \gamma \in U_{q_1})(\exists \gamma' \in U_q)(b^{q_1}_\gamma = b^q_{\gamma'}) \wedge (q_1 \Vdash x_1 \in \dot{c}_{\beta_0}).$$

Clearly, every function in $B_{q_0} \cup B_{q_1}$ has a copy in B_q. It is easy to check that $\eta, q, q_0, q_1, x_0, x_1$ are desired elements.

Claim 17b. *Given $p \in \mathbb{P}$ and $p_0 = p \upharpoonright \mathbb{P}_\theta$. Suppose every function in B_p has a copy in B_{p_0}. Let $q_0 \in \mathbb{P}_\theta$ be such that $q_0 \leqslant p_0$. Then there is a $q \in \mathbb{P}$ such that $q \leqslant p$, $q \upharpoonright \mathbb{P}_\theta \leqslant q_0$ and $U_q \backslash \theta = U_p \backslash \theta$.*

Proof. Let

$$\alpha_q = \alpha_{q_0}, \qquad t_q = t_{q_0}, \qquad k_q = k_{q_0}, \qquad U_q = U_{q_0} \cup U_p.$$

For every $\gamma \in U_{q_0}$ let $b^q_\gamma = b^{q_0}_\gamma$ and let $f^q_\gamma = f^{q_0}_\gamma$. Suppose $\gamma \in U_p \backslash U_{q_0}$. Let f^q_γ be any extension of f^p_γ on $\alpha_q + 1$ complying with Definition 10(f). For b^q_γ we first pick a $\gamma' \in U_{p_0} \subseteq U_{q_0}$ such that $b^p_\gamma = b^{p_0}_{\gamma'}$. Then let $b^q_\gamma = b^{q_0}_{\gamma'}$. Clearly, q is what we want.

We now want to apply Claims 17a and 17b to obtain p_l, $\{p_s : s \in 2^l\}$, η_l and $\{x_s : s \in 2^l\}$ in the inductive construction. Let $2^l = \{s_1, s_2, \ldots, s_{2^l}\}$ and let $p_{l-1} = q^0$. For $n = 1, 2, \ldots, 2^l$ we apply Claims 17a and 17b to construct q^n, $q_{s_n{}^\frown i}$ for $i = 0, 1$, η_{s_n} and $x_{s_n{}^\frown i}$ for $i = 0, 1$, inductively so that for any $n < 2^l$ and any $i = 1, 2$

(1) $q^n \geqslant q^{n+1}$,

(2) $q_{s_n{}^\frown i} \leqslant p_{s_n}$,

(3) $\eta_{s_n} \in \omega_1 \backslash \eta_{l-1}$,

(4) $x_{s_n{}^\frown i} \in \omega_1^{\eta_{s_n}}$ and $x_{s_n{}^\frown 0} \neq x_{s_n{}^\frown 1}$,

(5) $q_{s_n{}^\frown i} \Vdash x_{s_n{}^\frown i} \in \dot{c}_{\beta_0}$.

Let $p_l = q^{2^l}$. Let

$$\eta_l = \max\{\eta_{s_n} : n = 1,2,\ldots,2^l\} + 1.$$

Now it is easy to apply Claim 17b again to extend $q_{s_n{}^\frown i}$ to $p_{s_n{}^\frown i}$ such that

$$p_{s_n{}^\frown 0} \restriction \mathbb{P}_\theta = p_{s_n{}^\frown 1} \restriction \mathbb{P}_\theta = p_l.$$

We need also define Z^l_m's and Y^l_m's for all $m \leqslant l$. Note that all Z^l_m's are already defined. Let $y_{l,l} = \pi_1(e(l))$. For $m < l$ we choose $y_{m,l} \in (t_{p_l})_{\alpha_{p_l}}$ such that $y_{m,l}$ is a conservative extension of $y_{m,l-1}$ with respect to Z^l_m and for any $m \leqslant i < l$ either $y_{m,i} \leqslant y_{l-1,l-1}$ or $y_{l-1,l}$ is incompatible with $y_{m,i}$. This can be done because $y_{l-1,l-1}$ has infinitely many conservative extension with respect to Z^l_{l-1} at next level. This ends the construction.

Now we conclude the lemma. For each $m \in \omega$ let

$$y_m = \bigcup_{i\in\omega} y_{m,i}.$$

We want to define p_ω and p_τ for each $\tau \in 2^\omega$. Given $\tau \in 2^\omega$. Let

$$t_{p_\omega} = t_{p_\tau} = \left(\bigcup_{n\in\omega} t_{p_n}\right) \cup \{y_m : m \in \omega\}.$$

Clearly, $t_{p_\omega} \in \mathscr{T}$. Let

$$\alpha_{p_\omega} = \alpha_{p_\tau} = \bigcup_{n\in\omega} \alpha_{p_n}.$$

Then $ht(t_{p_\omega}) = \alpha_{p_\omega} + 1$. Let $k' = \bigcup_{n\in\omega} k_{p_n}$ and let $U = \bigcup_{s\in 2^{<\omega}} U_{p_s}$. Let

$$U_{p_\tau} = \bigcup_{n\in\omega} U_{p_{\tau\restriction n}}.$$

For each $\gamma \in U_{p_\tau}$ let

$$b^\tau_\gamma = \bigcup\{b^{p_{\tau\restriction n}}_\gamma : \gamma \in U_{p_{\tau\restriction n}}\}$$

and let

$$b^{p_\tau}_\gamma = b^\tau_\gamma \cup \left\{\left\langle y_m, \bigcup_{y<y_m} b^\tau_\gamma(y)\right\rangle : m \in \omega\right\}.$$

Let

$$f^\tau_\gamma = \left(\bigcup\{f^{p_{\tau\restriction n}}_\gamma : \gamma \in U_{p_{\tau\restriction n}}\}\right) \cup \{\langle \alpha_{p_\omega}, 0\rangle\}.$$

The only things we have not defined are k_{p_ω} and k_{p_τ}'s. Actually we need only k_{p_ω} and let $k_{p_\tau} = k_{p_\omega}$ for every $\tau \in 2^\omega$. First let

$$k_{p_\omega} \restriction \bigcup_{n\in\omega} t_{p_n} = k'.$$

Sh:563

For every $m \in \omega$ we want to define $k_{p_\omega}(y_m) \in \mathscr{T}$ so that

$$ht(k_{p_\omega}(y_m)) = \alpha_{p_\omega} + 1,$$

$$\bigcup_{y < y_m} k'(y) \leqslant_{\text{end}} k_{p_\omega}(y_m)$$

and

$$\{b_\gamma^{p_\tau}(y_m) : m \in \omega, \tau \in 2^\omega\} \subseteq (k_{p_\omega})_{\alpha_{p_\omega}}.$$

For doing this we need only to check that the set $\{b_\gamma^{p_\tau}(y_m) : m \in \omega, \tau \in 2^\omega\}$ is at most countable. This is guaranteed by Definition 10(g) and by the construction of Z_m^n's and Y_m^n's. Since for each $\gamma \in U$ and $m \in \omega$, there exists an $n \in \omega$ such that for any $s, s' \in 2^l$ with $l > n$ and $s \restriction n = s' \restriction n$ we have $b_\gamma^{p_s}(y_{m,l}) = b_\gamma^{p_{s'}}(y_{m,l})$. So for any $\tau, \tau' \in 2^\omega$ we have $\tau \restriction n = \tau' \restriction n$ implies $b_\gamma^{p_\tau}(y_m) = b_\gamma^{p_{\tau'}}(y_m)$. Hence for each $\gamma \in U$ the set

$$\{b_\gamma^{p_\tau}(y_m) : m \in \omega, \tau \in 2^\omega\}$$

is at most countable.

Up to this stage, we have defined p_ω and p_τ such that p_ω is a lower bound of p_n's, p_τ is a lower bound of $p_{\tau\restriction n}$'s and $p_\tau \restriction \mathbb{P}_\theta = p_\omega$ for each $\tau \in 2^\omega$. Let

$$\delta = \bigcup_{n \in \omega} \eta_n.$$

Given $\tau \in \omega$. Let

$$x_\tau = \bigcup_{n \in \omega} x_{\tau \restriction n}.$$

Then $x_\tau \in \omega_1^\delta$ and

$$p_\tau \Vdash x_\tau \in \dot{c}_{\beta_0} \cap \dot{T}_\delta.$$

Note that $\{x_\tau : \tau \in 2^\omega\}$ is an uncountable set. Let $G_\theta \subseteq \mathbb{P}_\theta$ be a V-generic filter such that $p_\omega \in G_\theta$. Then $T \in V[G_\theta]$ because $\dot{T}$ is a $\mathbb{P}_\theta$-name. Note also that the identity map from $\mathbb{P}_\theta$ into $\mathbb{P}$ is a complete embedding. Then

$$V[G_\theta] \models x_\tau \in T_\delta$$

because $V[G_\theta][H] = V[G]$ for some $V[G_\theta]$-generic filter H and some V-generic filter $G \subseteq \mathbb{P}$ such that $p_\tau \in G$. (See [11, p. 244 (D4)] for the details.) So T_δ is uncountable in $V[G_\theta]$. This contradicts the assumption that T is a Kurepa tree in $V^{\mathbb{P}}$. $\square$

3. Questions

We would like to ask some questions.

Question 1. Suppose our ground model is the Lévy model defined in the first section. Can we find a proper forcing notion such that the forcing extension will contain Kurepa

Sh:563

trees? If the answer is 'no', then we would like to know if there are any forcing notions of size $\leqslant \omega_1$ which preserve ω_1 such that the generic extension contains Kurepa trees?

Question 2. Suppose the answer of one of the questions above is 'Yes'. Is it true that given any model of CH, there always exists an ω_1-preserving forcing notion of size $\leqslant \omega_1$ such that forcing with that notion creates Kurepa trees in the generic extension?

Question 3. Does there exist a model of CH, plus no Kurepa trees, in which there is a c.c.c.-forcing notion of size $\leqslant \omega_1$ such that forcing with that notion creates Kurepa trees in the generic extension? If the answer is 'Yes', then we would like to ask the same question with c.c.c. replaced by one of some nicer chain conditions such as $\aleph_1$-caliber, Property K, etc.

Acknowledgements

The first part of this paper was originated in 1993, when the first author was a Morrey Assistant Professor in the University of California-Berkeley, and the first version of the paper was written when the first author visited University of Illinois during the academic year 1994–1995. He thanks H. Woodin for some inspiring discussion, and thanks the Mathematics Departments of the University of Illinois for giving him an opportunity to work there in a friendly scholastic environment. The first author thanks also the Mathematics Department of Rutgers University for offering free housing during his one week visit in October 1994, when the second author made a major contribution to the second part of the paper.

References

[1] U. Abraham and S. Shelah, Isomorphism types of Aronszajn trees, Israel J. Math. 50 (1985) 75–113.
[2] J. Baumgartner, Iterated forcing, in: A.R.D. Mathias, ed., Surveys in Set Theory (Cambridge University Press, Cambridge, 1983) 1–59.
[3] J. Baumgartner, Applications of the proper forcing axiom, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set Theoretic Topology (North-Holland, Amsterdam, 1984) 913–959.
[4] J. Baumgartner and A. Taylor, Saturation properties of ideals in generic extension. II, Trans. Amer. Math. Soc. 271 (2) (1982) 587–609.
[5] K.J. Devlin, $\aleph_1$-trees, Ann. Math. Logic 13 (1978) 267–330.
[6] M. Foreman, M. Magidor and S. Shelah, Martin's maximum, saturated ideals, and non-regular ultrafilters. Part I, Ann. of Math. 127 (1988) 1–47.
[7] T. Jech, Trees, J. Symbolic Logic 36 (1971) 1–14.
[8] T. Jech, Set Theory (Academic Press, New York, 1978).
[9] R. Jin, The differences between Kurepa trees and Jech–Kunen trees, Arch. Math. Logic 32 (1993) 369–379.
[10] R. Jin and S. Shelah, Essential Kurepa trees versus essential Jech–Kunen trees, Ann. Pure Appl. Logic 69 (1994) 107–131.
[11] K. Kunen, Set Theory, An Introduction to Independence Proofs (North-Holland, Amsterdam, 1980).
[12] S. Shelah, Proper Forcing, Lecture Notes in Math. 940 (Springer, Berlin, 1982).
[13] S. Shelah, Can you take Solovay's inaccessible away? Israel J. Math. 48 (1) (1984) 1–47.

Sh:563

[14] R. Solovay and S. Tennenbaum, Iterated Cohen extensions and Souslin's problem, Ann. of Math. 94 (1971) 201–245.

[15] S. Todorčević, Trees and linearly ordered sets, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set Theoretic Topology (North-Holland, Amsterdam, 1984) 235–293.

[16] B. Veličković, Forcing axioms and stationary sets, Adv. Math. 94 (1992) 256–284.