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WEAK DEFINABILITY IN INFINITARY LANGUAGES

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Abstract. We shall prove that if a model of cardinality κ can be expanded to a model of a sentence ψ of $L_{\lambda^+,\omega}$ by adding a suitable predicate in more than κ ways, *then*, it has a submodel of power μ which can be expanded to a model of ψ in $>\mu$ ways provided that λ, κ, μ satisfy suitable conditions.

§1. Introduction. By Beth's theorem [3] and Svenonius [20] and Kueker [22].

THEOREM. *Let L be a language, P a predicate (one place w.l.o.g.), T a theory in $L + P$, n a natural number; then the following conditions are equivalent for $\kappa \geq |L| + \aleph_0$.* ($(\mathrm{II})_\kappa$ is included only if T is complete.)

$(\mathrm{I})_\kappa$ *For every L-model M of cardinality κ, the number of $P \subseteq |M|$ such that $(M, P) \models T$ is $\leq n$.*

$(\mathrm{II})_\kappa$ *For every $(L + P)$-model (M, P) of T of cardinality κ, the number of images of P under automorphisms of M is $\leq n$.*

(III) *There are formulas $\varphi_i(\bar{x}, \bar{y}) \in L$, $i = 1, \cdots, n$,* and *$\psi(\bar{y})$ such that*

$$T \vdash (\forall \bar{y})\left(\psi(\bar{y}) \rightarrow \bigvee_{i=1}^{n} (\forall x)[\varphi_i(x, \bar{y}) \equiv P(x)]\right) \wedge (\exists \bar{y})(\psi \bar{y}).$$

If we ignore (III) the theorem still tells us that the $(\mathrm{I})_\kappa$ are equivalent for $\kappa \geq |L| + \aleph_0$, and $(\mathrm{I})_\kappa \leftrightarrow (\mathrm{II})_\kappa$.

From Chang [4], Makkai [9], Reyes [12] and Shelah [16], the following theorem arises:

THEOREM. *In the previous theorem's notation, the following conditions are equivalent:*

$(\mathrm{I})_\kappa$ *For every L-model M of cardinality κ there are $\leq \kappa$ $P \subseteq |M|$ such that $(M, P) \models T$.*

$(\mathrm{II})_\kappa$ *For every $(L + P)$-model (M, P) of T of cardinality κ, the number of images of P under automorphisms of M is $\leq \kappa$.*

(III) *There are formulas $\varphi_i(x, \bar{y}) \in L$, $i = 1, 2$, such that*

$$T \vdash \bigvee_{i=1}^{2} (\exists \bar{y})(\forall x)[\varphi_i(x, \bar{y}) \equiv P(x)].$$

In this case, if we ignore (III), the theorem is not trivial. We have a weak generalization of the equivalence of $(\mathrm{I})_\kappa$, $(\mathrm{II})_\kappa$, $\kappa \geq |L| + \aleph_0$, to infinitary languages.

A complete list appears in Shelah [17] (correct there K_1 to K in the first sentence of the definition).

We shall give one of these weak generalizations.

For negative results on the generalization of Craig's and Beth's theorems for infinitary languages see Malitz [10] and Friedman [5]; for positive results, see

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Lopez-Escobar [19] and Malitz [10].

The theorem we shall prove is

MAIN THEOREM 1. *Let ψ be a sentence in $(L + P)_{\lambda^+, \omega}$, $|L| \leq \lambda$, M an L-model of cardinality $\aleph_{\alpha+\beta}$ such that*

$$|\{P: P \subseteq |M|, (M, P) \models \psi\}| > \aleph_{\alpha+\beta}.$$

Assume further that $\beta < \omega_1$, $\aleph_\alpha$ has cofinality $\aleph_0$, $\mu_n \geq \lambda$, $\mu = \Sigma_{n<\omega}\mu_n$, and $\kappa < \aleph_\alpha \Rightarrow \kappa^{\mu_n} < \aleph_\alpha$ for $n < \omega$.

***Then** M has an elementary submodel N of cardinality μ such that*

$$|\{P: P \subseteq |N|, (N, P) \models \psi\}| \geq \mu^{\aleph_0}.$$

Another theorem, which we shall not prove, as its proof is simpler is

THEOREM. *Let $\psi \in (L + P)_{\lambda^+, \omega}$, M an L-model of cardinality κ such that $|\{P: P \subseteq |M|, (M, P) \models \psi\}| > \kappa$. Assume further that $\mu \geq \lambda$, $\kappa^\mu = \kappa$. **Then** M has an elementary submodel N of cardinality μ such that $|\{P: P \subseteq |N|, (N, P) \models \psi\}| \geq \mu^{\aleph_0}$.*

In this context it is interesting to remember the following theorem of Kueker [7] (we omit the part on automorphism).

THEOREM. *If $\psi \in (L + P)_{\omega_1, \omega}$ then the following conditions are equivalent:*

(I) *For every countable L-model M,*

$$|\{P: (M, P) \models \psi\}| \leq \aleph_0.$$

(II) *For every L-model M,*

$$|\{P: (M, P) \models \psi\}| \leq \|M\| + \aleph_0.$$

(III) *There are $\varphi_i(x, \bar{y}) \in L_{\omega_1, \omega}$ such that $\psi \vdash \bigvee_{i<\omega}(\exists \bar{y})(\forall x)[\varphi_i(x, \bar{y}) \equiv P(x)]$.*

In proving our theorem for $\aleph_{\alpha+\beta}$ rather than for $\aleph_\alpha$, we use reasoning similar to Baumgartner [1], [2] and Shelah [13], [14, Lemma 3.3] and [15, §3.3]. Another example is

THEOREM. *If T is a complete theory, $|T| = \lambda^+$, λ regular (for simplicity) and every n-type of cardinality $< \lambda$ can be extended to complete n-type of cardinality $< \lambda$, **then** T has a model in which every finite sequence realizes a complete type of cardinality $< \lambda$.*

NOTATION. We will not distinguish strictly between a predicate, a relation and the set (for a one-place relation). $|M|$ is the universe of M, $|A|$ the cardinality of A; λ, μ, κ cardinals, α, β, γ, i, j, k, ξ ordinals, δ a limit ordinal, n, m natural numbers.

A type is a set of formulas $\varphi(x_1, \cdots, x_n)$ (n fixed); a sequence $\bar{a}$ in a model M realizes the type if $M \models \varphi[\bar{a}]$ for every $\varphi(\bar{x})$ in the type.

§1. A counterexample and conjecture. We should naturally ask whether the restrictions of Theorem 1 are necessary. For this observe the following example:

EXAMPLE 1. Let $\psi \in (L + P)_{\aleph_2, \aleph_0}$ be a sentence saying that $<$ is a partial order of a tree, the order-type of every branch is $\leq \omega_1$, and P is a branch of order-type ω_1.

That is

$$\begin{aligned}\psi = {} & (\forall xyz)[x < y \wedge y < z \rightarrow x < z] \wedge (\forall x)[\neg x < x] \\ & \wedge (\forall xyz)[y < x \wedge z < x \rightarrow z < y \vee y < z \vee y = z] \\ & \wedge (\forall x)\Big[\bigvee_{\alpha<\omega_1} \psi_\alpha(x)\Big] \wedge (\forall xy)[P(x) \wedge P(y) \rightarrow x < y \vee y < x \vee y = x] \\ & \wedge (\forall xy)[x < y \wedge P(y) \rightarrow P(x)] \wedge \bigwedge_{\alpha<\omega_1} (\exists x)[P(x) \wedge \psi_\alpha(x)],\end{aligned}$$

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where $\psi_0(x) = \neg(\exists y)(y < x)$; for δ a limit ordinal

$$\psi_\delta(x) = \bigwedge_{\alpha<\delta} (\exists y)[y < x \wedge \psi_\alpha(y)] \wedge (\forall y)\Big[y < x \to \bigvee_{\alpha<\delta} \psi_\alpha(y)\Big],$$
$$\psi_{\alpha+1}(x) = (\exists y)[y < x \wedge \neg(\exists z)(y < z \wedge z < x) \wedge \psi_\alpha(y)].$$

It is easy to see that there is a model M of cardinality κ for which $|\{P: P \subseteq |M|, (M, P) \vDash \psi\}| > \kappa$ *iff* there is a tree of height ω_1 with κ nodes and $> \kappa$ branches of height (= order-type) ω_1. Assuming GCH, this is equivalent to $\aleph_1 = \mathrm{cf}(\kappa)$ = the cofinality of κ. Moreover, if $\aleph_\alpha$ is a supercompact cardinal in V which satisfies GCH, by Silver [18] there is cardinal-preserving extension V' of V such that $\aleph_\alpha$ is still a measurable cardinal and $2^{\aleph_\alpha} > \aleph_{\alpha+\omega_1}$.

By Prikry [11] we can extend V' to V'' such that the cardinals are preserved, the cofinality of $\aleph_\alpha$ is $\aleph_0$, and $\aleph_\alpha$ is a strong limit cardinal ($\kappa < \aleph_\alpha \to 2^\kappa < \aleph_\alpha$). So in V'' there is a model M of cardinality $\aleph_{\alpha+\omega_1}$ such that $|\{P: P \subseteq |M|, (M, P) \vDash \psi\}| > \aleph_{\alpha+\omega_1}$, $\aleph_\alpha^{\aleph_0} = 2^{\aleph_\alpha} > \aleph_{\alpha+\omega_1} > \aleph_\alpha$; but no strong limit cardinal of cofinality ω satisfies this. This implies that the restrictions in our main theorem are natural. It would be nice to find a corresponding syntactical condition and to generalize the theorem to cardinals of cofinality, e.g., $\aleph_1$, but I am pessimistic. The following conjecture, however, which is from the "other extreme" of the question, seems more hopeful:

CONJECTURE. If $\psi \in (L + P)_{\lambda^+,\omega}$, there is an L-model M of cardinality κ, $\kappa^{\mu(\lambda)} = \kappa$ ($\mu(\lambda)$—the Hanf number of sentences of $L_{\lambda^+,\omega}$) such that

$$|\{P: P \subseteq |M|, (M, P) \vDash \psi\}| > \kappa,$$

then for every $\mu \geq \lambda$ there is an $(L + P)$-model (M, P) of cardinality μ, such that P has $> \mu$ images under automorphisms of M.

It is interesting that this situation has a nontrivial corresponding first-order question. Let $L^* = L + \{P_i: i < i_0\}$, and let T be a theory in L^*. Let K be the class of infinite cardinals $\lambda \geq |L^*|$ such that there is an L-model M of cardinality λ, which is the reduct of $> \lambda$ L^*-models of T. What can K be? It is not hard to check that either $K = \{\lambda: \lambda \geq |L^*| + \aleph_0\}$, or $\lambda^{|i_0|} = \lambda \geq |L^*| + \aleph_0$ implies $\lambda \notin K$. In the second case, assuming GCH, there is a set I of infinite cardinals $\leq |i_0|$ such that $\lambda \in K$ iff $\lambda \geq |L^*| + \aleph_0$ and $\mathrm{cf}(\lambda) \in I$. (Instead of GCH, we can look only at strong limit cardinals.) Small changes (and combinations) of our example show that this result cannot be improved (only if we demand T to be complete; for big I, the answer is not clear to me). On a related problem see [21, p. 330, Conjecture 4E].

§2. Combinatorial lemmas.

LEMMA 1. *If* $\mathrm{cf}(\aleph_\alpha) = \aleph_0, \beta < \omega_1, |A| = \aleph_{\alpha+\beta}$ *then there is a family F of subsets of A each of cardinality* $< \aleph_\alpha$, $|F| = \aleph_{\alpha+\beta}$ *such that every subset of A of cardinality* $< \aleph_\alpha$ *is included in a union of countably many members of the family.*

REMARK. If $\beta < \omega$, $\aleph_{\alpha+\beta}$ countable unions are sufficient.

PROOF. We shall prove it by induction on β. W.l.o.g. $A = \aleph_{\alpha+\beta}$.

For $\beta = 0$, as $\mathrm{cf}(\aleph_\alpha) = \aleph_0$, there are $\kappa_n < \aleph_\alpha$, $\aleph_\alpha = \bigcup_{n<\omega}\kappa_n$. Let $F = \{\kappa_n: n < \omega\}$.

Suppose we have proved, for each β, $\beta < \beta_0 < \omega_1$. Then, for each ξ, $\aleph_\alpha \leq \xi < \aleph_{\alpha+\beta_0}$, clearly $|\xi| = \aleph_{\alpha+\beta}$ for some $0 \leq \beta < \beta_0$; hence there is a family F_ξ of subsets of ξ, each of cardinality $< \aleph_\alpha$, such that each subset of ξ of cardinality $< \aleph_\alpha$ is included in a countable union of sets from F_ξ. Let $F = \bigcup\{F_\xi : \aleph_\alpha \le \xi < \aleph_{\alpha+\beta_0}\}$. Clearly F satisfies our demands.

LEMMA 2. *If F is a family of subsets of A, $|F| > |A|$, $2^\kappa \le |A|$, then there is $B \subseteq A$, $|B| = \kappa$ and distinct subsets P_i of B $(i < \kappa)$ such that, for each $i < \kappa$,*

$$|\{P : P \in F, P \cap B = P_i\}| > |A|.$$

PROOF. First let κ be regular. Suppose there is no such B, P_i. Then there is no such B with $|B| \le \kappa$. So, for any $B \subseteq A$, $|B| \le \kappa$,

$$|\{P : P \subseteq B, |\{Q : Q \in F, Q \cap B = P\}| > |A|\}| < \kappa.$$

Define B_i, $i \le \kappa$, by induction. $B_0 = \varnothing$ and, for a limit ordinal δ, $B_\delta = \bigcup_{i<\delta} B_i$. If B_i is defined, then for each $P \subseteq B_i$ for which $|\{Q : Q \in F, Q \cap B_i = P\}| > |A|$ there is $a_P^i \in A$ such that $|F_{1,i}^P| > |A|$, $|F_{2,i}^P| > |A|$ where

$$F_{1,i}^P = \{Q : a_P^i \in Q \in F, Q \cap B_i = P\}, \quad F_{2,i}^P = \{Q : a_P^i \notin Q \in F, Q \cap B_i = P\}.$$

We now get B_{i+1} from B_i by adding all the a_P^i. Thus B_κ is defined, $|B_\kappa| \le \kappa$. Let $\{P_i : i < i_0\}$ be the set of $P \subseteq B_\kappa$ for which $|\{Q : Q \in F, Q \cap B_\kappa = P\}| > |A|$. As κ is regular there is $k < \kappa$ such that for $i < j < i_0$, $P_i \cap B_k \ne P_j \cap B_k$. If $a^k_{P_0 \cap B_k} \in P_0$, then as $|F_{2,i}^{P_1 \cap B_k}| > |A|$ there is $Q_0 \subseteq B_k$, such that $Q_0 \cap B_k = P_0 \cap B_k$, $a^k_{P_1 \cap B_k} \notin Q_0$ and $|\{Q : Q \in F, Q \cap B_k = Q_0\}| > |A|$. So there should be $i < i_0$ for which $Q_0 = P_i$, but by the definition of k and Q_0 this leads to contradiction. As $a^k_{P_0 \cap B_k} \notin P_0$ gives a similar contradiction, the case for κ regular is proved.

Now we are left with the case κ is singular. Then for any $\lambda < \kappa$ there is suitable B_λ. $B = \bigcup_{\lambda<\kappa} B_\lambda$ is the desired B.

§3. Proof of the main theorem. W.l.o.g. μ_n is an increasing sequence and μ_n is regular. By adding relations R_φ for every subformula φ of ψ we get

(i) there is a language $(L_1 + P) \supseteq (L + P)$, $|L_1| \le \lambda$, a (first-order) theory T_1 in $(L_1 + P)$, and a set of types Γ in $(L_1 + P)$, $|\Gamma| \le \lambda$, such that

(A) if (M, P) is an $(L + P)$-model of ψ, and we define $R_\varphi = \{\bar{a} : (M, P) \models \varphi[\bar{a}]\}$, then $(M, \cdots, R_\varphi, \cdots, P)$ (φ runs on subformulas of ψ) is an $(L_1 + P)$-model of T_1 omitting every type in Γ;

(B) if (N, P) is an $(L_1 + P)$-model of $T_1 \cup \{R_\psi\}$ (R_ψ is a zero-place relation = propositional constant) which omits every type in Γ then $(N, P) \models \psi$.

Now we can add to $(L_1 + P)$ its Skolem functions and get

(ii) there is a language $(L_2 + P) \supseteq (L_1 + P)$, $|L_2| \le \lambda$ and a (first-order) theory $T_2 \supseteq T_1$ in $(L_2 + P)$ with Skolem functions such that every $(L_1 + P)$-model of T_1 can be expanded to an $(L_2 + P)$-model of T_2.

From now on M is the L-model given in the theorem. For $P \subseteq |M|$ such that $(M, P) \models \psi$ let N_P be the corresponding $(L_2 + P)$-model of T_2 omitting every type in Γ, and if $(M, P) \models \neg\psi$, let $N_P = \varnothing$. We know that $K = \{P : P \subseteq |M|, N_P \ne \varnothing\}$ has cardinality $> \aleph_{\alpha+\beta} = \|M\|$. For $\gamma \le \omega$, let I_γ be the set of sequences of ordinals η of length $\gamma = l(\eta)$, such that $\eta(n) \le \mu_n$.

Now we define, by induction on n, $A_n \subseteq |M|$, $P_\eta \subseteq A_n$, $K_\eta \subseteq K$ for $\eta \in I_n$, and $B(P, \eta, i) \subseteq |M|$ for $P \in K_\eta$, $i < \omega$, such that

(1) $A_n \subseteq |M|$, $|A_n| = \mu_n$,

(2) for $\eta \in I_n$, $P_\eta \subseteq A_n$ such that, for $\eta \ne \tau \in I_n$, $P_\eta \ne P_\tau$,

(3) for $\eta \in I_n$, $K_\eta \subseteq \{P: P \in K, P \cap A_n = P_\eta\}$, $|K_\eta| > \aleph_{\alpha+\beta}$ and if $m \leq n$ then $K_\eta \subseteq K_{\eta|m}$ [$\eta|m$ is $\langle \eta(0), \cdots, \eta(m-1)\rangle$],

(4) for every $\eta \in I_n$, $P \in K_\eta$, $i < \omega$, $B(P, \eta, i)$ belongs to F (from Lemma 1) (hence $|B(P, \eta, i)| < \aleph_\alpha$) and the Skolem-hull of A_n in N_P, Hull(A_n, N_P), is included in $\bigcup_{i<\omega} B(P, \eta, i)$,

(5) if $m < n$, $i < n$, $P_1, P_2 \in K_\eta$, $\eta \in I_n$ *then*

$$B(P_1, \eta|m, i) = B(P_2, \eta|m, i) \quad \text{and}$$
$$\text{Hull}(A_m, N_{P_1}) \cap B(P_1, \eta|m, i) = \text{Hull}(A_m, N_{P_2}) \cap B(P_1, \eta|m, i),$$

(6) if $m + 1 < n$, $i + 1 < n$, $P \in K_\eta$, $\eta \in I_n$ *then* Hull$(A_m, N_P) \cap B(P, \eta|m, i) \subseteq A_n$,

(7) if $P_1, P_2 \in K_\eta$, $\eta \in I_n$, $\bar{a}$ a finite sequence from A_n, $\varphi(\bar{x})$ a formula in $(L_2 + P)$ *then*

(A) $N_{P_1} \models \varphi[\bar{a}] \Leftrightarrow N_{P_2} \models \varphi[\bar{a}]$,

(B) for every function symbol $f \in (L_2 + P)$ and $i < \omega$,

$$f^{N_{P_1}}(\bar{a}) \in B(P_1, \eta, i) \Leftrightarrow f^{N_{P_2}}(\bar{a}) \in B(P_2, \eta, i).$$

For $n = 0$ there is no problem so suppose we have defined up to n and we want to define for $n + 1$. Let

$$A_n^* = A_n \cup \bigcup\{\text{Hull}(A_m, N_P) \cap B(P, \eta|m, i): i < n, m < n, \eta \in I_n, P \in K_\eta\}$$

(this is for satisfying $(6)_{n+1}$).

By condition $(5)_n$ clearly $|A_n^*| = \mu_n$. By Lemma 2 for each $\eta \in I_n$ there is a set $A_\eta^* \subseteq |M|$, $|A_\eta^*| = \mu_{n+1}$ and distinct sets $P^i \subseteq A_n^*$, for $i \leq \mu_{n+1}$ such that

$$|\{P: P \in K_\eta, P \cap A_\eta^* = P^i\}| > \aleph_{\alpha+\beta}$$

and $i < j \leq \mu_n \rightarrow P^i \neq P^j$.

Define

$$A_{n+1} = A_n^* \cup \bigcup_{\eta \in I_n} A_\eta^*.$$

Clearly $|A_{n+1}| = \mu_{n+1}$ and conditions $(1)_{n+1}$–$(6)_{n+1}$ are satisfied. For $\eta \in I_n$, $i \leq \mu_{n+1}$ let

$$K^1_{\eta^\frown\langle i\rangle} = \{P: P \in K_\eta, P \cap A_\eta^* = P^i\}.$$

So $|K_\tau^1| > \aleph_{\alpha+\beta}$ for each $\tau \in I_{n+1}$. Now for each $P \in K_\tau^1$ $(\tau \in I_{n+1})$ by Lemma 1, we can define $B(P, \tau, i) \in F$ for $i < \omega$ such that Hull$(A_{n+1}, N_P) \subseteq \bigcup_{i<\omega} B(P, \tau, i)$. This will assure us that condition $(4)_{n+1}$ will be satisfied. Now for $\eta \in I_{n+1}$ the number of possible sequences $\{B(P, \eta|m, i): m \leq n, i \leq n\}$ for $P \in K_\eta$ is $\leq |F|^{(n+1)^2} = \aleph_{\alpha+\beta} < |K_\eta^1|$. Hence there is $K_\eta^2 \subseteq K_\eta^1$, $|K_\eta^2| > \aleph_{\alpha+\beta}$ such that for $P_1, P_2 \in K_\eta^2$, $i \leq n$, $m \leq n$, $B(P_1, \eta|m, i) = B(P_2, \eta|m, i)$. This will partly assure $(5)_{n+1}$. Similarly as $|B(P, \eta|m, i)|^{\mu_n} < \aleph_\alpha$ [because $B(P, \eta|m, i) \in F$] and $2^{|A_n|} < \aleph_\alpha$ we can find $K_\eta \subseteq K_\eta^2$, $|K_\eta| > \aleph_{\alpha+\beta}$ so that also $(5)_{n+1}$ and $(7)_{n+1}$ will be satisfied. This completes the inductive definition.

Define $A = \bigcup_{n<\omega} A_n$, and let N be the submodel of M with universe A. Now for each $\eta \in I_\omega$ we define an expansion N^η of N to an $(L_2 + P)$-model by the following: *If* $\bar{a}$ is a sequence from A, $\varphi(\bar{x})$ an atomic formula in $(L_2 + P)$, *then* $N^\eta \models \varphi[\bar{a}]$ iff for every big enough $n < \omega$ and for every $P \in K_{\eta|n}$, $N_P \models \varphi[\bar{a}]$.

Using (4), (5), (6), (7) we can prove inductively that this holds for every $\varphi \in (L_2 + P)$. [Notice that if $\bar{a}$ is from A_n, f a function symbol then, for each

$P \in K_{\eta|n}$, by (4), there is $i = i(P)$ such that $f^{N_P}(\bar{a}) \in B(P, \eta|n, i)$; and by (7)(B), $i(P) = i_0$ for each $P \in K_P$; hence by (6) for every $P \in K_{\eta|m}$ $(m \geq i_0 + 2, m \geq n + 2)$, $f^{N_P}(\bar{a}) \in A_m$, and so by (7)(A), there is $b \in A_m$ such that $N_P \models f(\bar{a}) = b$ for every $P \in K_{\eta|m}$.]

REFERENCES

[1] J. E. BAUMGARTNER, *Results and independence proofs in combinatorial set-theory*, Ph.D. thesis, University of California, Berkeley, 1970.

[2] ———, *On the cardinality of dense subsets of linear orderings*. I. ***Notices of the American Mathematical Society***, vol. 15 (1968), p. 935. Abstract #68T-E33.

[3] E. W. BETH, *On Padoa's method in the theory of definitions*, ***Indagationes Mathematicae***, vol. 15 (1953), pp. 330–339.

[4] C. C. CHANG, *Some new results in definability*, ***Bulletin of the American Mathematical Society***, vol. 70 (1964), pp. 808–813.

[5] H. FRIEDMAN, *Back and forth*, $L(Q)$, $L_{\infty,\omega}(Q)$ *and Beth theorem*, mimeograph, Stanford University, November 1971; ***Israel Journal of Mathematics*** (to appear).

[6] H. J. KEISLER, *Model theory for infinitary logic*, North-Holland, Amsterdam, 1971.

[7] D. KUEKER, *Definability, automorphisms and infinitary languages, the syntax and semantics of infinitary languages* (J. Barwise, Editor), ***Lecture Notes in Mathematics***, Springer-Verlag, Berlin, 1972, pp. 152–165.

[8] K. KUNEN, *Implicit definability and infinitary language*, this JOURNAL, vol. 33 (1968), pp. 446–451.

[9] M. MAKKAI, *A generalization of a theorem of E. W. Beth*, ***Acta Mathematica Academiae Scientiarum Hungaricae***, vol. 15 (1964), p. 227.

[10] J. MALITZ, *Infinitary analogs of theorems from first order model theory*, this JOURNAL, vol. 36 (1971), pp. 216–228.

[11] K. PRIKRY, *Changing measurable into accessible cardinals*, ***Dissertationes Mathematicae Rozprawy Matematyczne***, No. 68, Warsaw, 1970.

[12] G. E. REYES, *Local definability theory*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 95–137.

[13] S. SHELAH, *Generalizations of saturativity*, ***Notices of the American Mathematical Society***, vol. 18 (1971), p. 258. Abstract #71T–E2.

[14] ———, *The number of nonisomorphic models of an unstable first order theory*, ***Israel Journal of Mathematics***, vol. 9 (1971), pp. 473–487.

[15] ———, *Notes in combinatorial set theory*, ***Israel Journal of Mathematics*** (to appear).

[16] ———, *Remark to "Local definability theory" of Reyes*, ***Annals of Mathematical Logic***, vol. 2 (1971), pp. 441–447.

[17] ———, *Weak definability for infinitary languages*, ***Notices of the American Mathematical Society***, vol. 17 (1970), p. 834. Abstract #70T–E57.

[18] J. SILVER (to appear).

[19] E. LOPEZ-ESCOBAR, *An interpolation theory for denumerably long sentences*. ***Fundamenta Mathematicae***, vol. 57 (1965), pp. 253–272.

[20] L. SVENONIUS, *A theorem on permutations in models*, ***Theoria***, vol. 25 (1959), pp. 173–178.

[21] S. SHELAH, *Stability, the* f.c.p. *and superstability, model-theoretic properties of formulas in first order theory*, ***Annals of Mathematical Logic***, vol. 3 (1971), pp. 262–271.

[22] D. W. KUEKER, *Generalized interpolation and definability*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 423–468.

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