INFINITE ABELIAN GROUPS, WHITEHEAD PROBLEM AND SOME CONSTRUCTIONS

BY

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ABSTRACT

We solve here some problems from Fuchs' book. We show that the answer to Whitehead's problem (for groups of power $\aleph_1$) is independent from the usual axioms of set theory. We prove the existence of large rigid systems for groups of power λ, with no restriction on λ. We also prove that there are many non-isomorphic reduced separable p-groups.

1. Introduction

Here a group means abelian group.

Whitehead's problem (see, for example, [8, Prob. 79, p. 184]) is for which group G does $\mathrm{Ext}(G, Z) = 0$ hold. In other words, if $h : H \to G$ is an epimorphism with kernel Z (the integers) then H can be reconstructed from G in one way only: as a direct sum. More precisely, there is a homomorphism $g : G \to H$, $hg = 1_G$. Such groups are called W-groups (Whitehead groups). By Stein [22] and Rotman [19] (or, for example, [8, 99.1]) each W-group is $\aleph_1$-free and separable. In particular W-groups are torsion-free, and free groups are W-groups. Hence a countable group is a W-group iff it is free. For an (infinite) cardinal λ let:

$$(W_\lambda): \text{each } W\text{-group of power } \lambda \text{ is free.}$$

We prove in Section 3 the independence of $(W_{\aleph_1})$ from the usual axioms of set theory (ZFC: Zermelo-Frenkel with the axiom of choice). But we do not use the methods of Cohen [1] directly. Rather we rely on previous independence proofs; that is, various additional axioms have been shown to be consistent with ZFC (assuming the consistency of ZFC). Now if ZFC + X is consistent and from it we

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can prove a statement X', then of course X' is also consistent with ZFC. Thus we prove that if $V = L$ then $(W_{\aleph_1})$. (We could prove, in fact, that (W_λ) with $\lambda < \aleph_{\omega_1}$ holds. The difficulty in $\aleph_{\omega_1}$ seems to me to be in L.) On the other hand, we prove that if MA (Martin axiom) and $2^{\aleph_0} > \aleph_1$ then $(W_{\aleph_1})$ fails. Thus by Gödel [11] and Martin and Solovay [17], $(W_{\aleph_1})$ is independent of ZFC. The question on the independence of $(W_{\aleph_1})$ from ZFC $+ 2^{\aleph_0} = \aleph_1$ remains open. (As $V = L$ implies GCH, the consistency is clear.) If $V = L$, then every W-group is free.

Fuchs [8, Prob. 52, p. 55] asked for the number of non-isomorphic separable p-groups of cardinality λ. The answer is that the number is 2^λ. For $\lambda = \aleph_0$ this is immediate, and for $\lambda > \aleph_0$ it follows, in fact, from a result of the author on the number of non-isomorphic models of a non-superstable theory (part of it appears in [20, Sect. 3]). In Section 1 we give the proof for regular λ; as for singular cardinals, the proof is complicated. Note that by the construction of (1.1) we can show that if $V = L$ then, by Jensen [14], for every regular non-weakly-compact cardinal λ, there is a separable p-group of power λ which is not the direct sum of cyclic groups, but every subgroup of smaller cardinality is. (For weakly compact cardinals there is no such group; weakly compact cardinals are inaccessible and rare.) This partially answers [8, Prob. 56, p. 55]. (Independently, Mekler [16], Eklof [2] and Gregory obtained similar results, and Eklof [3], Gregory [12] obtained better results. See also Eklof [4]).

The proof indicates to me that separable p-groups cannot be characterized by any reasonable set of invariants. (This answers Problem 51 of Fuchs [8].) Because, first of all, (using (1.2) notation), in order to characterize $G(B)$, we need $B/D(\omega_1)$ and then if $V = L$, as in (3.4), we can define $G'(B)$ in the same way, using only different η_δ's, and obtain $G'(B) \not\cong G(B)$.

A rigid system of groups is a family with only trivial homomorphisms between its members (if $h : G \to H$ then $h = 0$ or $G = H$, $h(x) = rx$ for some rational r). Fuchs, with the help of Corner, proves inductively that for every λ smaller than the first inaccessible cardinal, there is a rigid system of 2^λ torsion-free groups of cardinality λ. In Section 2 we remove the restriction on λ and our proof does not use induction. Note that each member of a rigid system is indecomposable. This answers Problem 21 from [9, p. 183]. Fuchs [10] succeeded in replacing the first inaccessible cardinal by much higher cardinals in an inductive proof. Fuchs kindly draws my attention to the fact that Th. 2.1 also solved Problem 37 in [7, p. 208], that is, there are 2^λ non-isomorphic compact and connected groups of cardinality 2^λ;

with some care we can make them algebraically isomorphic. For this use the duality theorem of Pontryagin, Hewitt, and Ross [13].

In a forthcoming paper we shall prove that there are essentially indecomposable p-groups of arbitrarily large cardinalities (answering positively a question of Pierce [18] repeated in [8, Prob. 55, p. 55]). We shall also prove that for arbitrarily large cardinality λ there is a system of 2^{λ} p-groups such that the homomorphisms between different members are small (for definition see [7, (46.3), p. 195]). (This answers positively Problem 53 of [8, p. 55].) Another construction gives for $\mu = \lambda^{\aleph_0} = 2^{\lambda} > 2^{\aleph_0}$ a family of 2^{μ} separable p-groups of power μ, so that any homomorphism between different members has range of cardinality $\leqq \lambda$.

We assume knowledge in naive set theory, and in separable p-groups as in [7, VI];[8, XI].

Notation. Let λ, μ, κ denote infinite cardinals, α, $\beta, \gamma, \delta, i, j$ ordinals, δ a limit ordinal, k, l, m, n, M, N natural numbers or integers, ω the first infinite ordinal. We let η, τ, v be sequences of ordinals. Let $l(\eta)$ be the length of η, $\eta(i)$ its ith element. Let $\mathrm{cf}[\alpha]$ be the cofinality of α.

G,H, and sometimes K, I, R are groups, h, g are homomorphisms, p, q are prime natural numbers, r a rational or sometimes a p-adic integer.

When notation becomes complex, $a_i(j)$ is written as $a[j, i]$, a_i as $a[i]$.

1. There are many separable p-groups

Here a group means a reduced separable p-group, that is, a group G such that for every $a \in G$ ($a \neq 0$) for some n, $p^n a = 0$, and for some n no $b \in G$ satisfies $a = p^n b$.

Definition 1.1. For every limit ordinal α of cofinality bigger than ω,

(a) let $D(\alpha)$ be the filter of subsets of $\alpha = \{\beta : \beta < \alpha\}$ generated by the closed unbounded subsets of α;

(b) we write $A \subseteq B$ [mod $D(\alpha)$] if $\alpha - [A - B] \in D(\alpha)$, similarly for $A = B$, $A \neq B$[mod $D(\alpha)$];

(c) $A \subseteq \alpha$ is called a *stationary* (subset of α) if $A \neq 0$ [mod $D(\alpha)$].

Theorem 1.1 (See Solovay [21]). *If λ is a regular cardinal bigger than $\aleph_0$, A, a stationary subset of λ, then A can be partitioned into λ pairwise disjoint stationary subsets of λ.*

Remark. The particular cases we need can be proven more easily.

Theorem 1.2. *If $\lambda > \aleph_0$ is a regular cardinal, then there is a family of 2^{λ}*

non-isomorphic (reduced) separable p-groups, each of cardinality λ. Moreover no one of them is isomorphic to a subgroup of the other.

PROOF. It is well known that $A = \{\alpha : \alpha < \lambda, \alpha \text{ has cofinality } \aleph_0\}$ is a stationary subset of λ. By (1.1) we can have $A = \bigcup_{i<\lambda} A_i$, A_i stationary and pairwise disjoint. We can find a family $\{J_\beta : \beta < 2^\lambda\}$ of subsets of λ, such that no one is a subset of ithe other. Let, for $\beta < 2^\lambda$, $B_\beta = \bigcup_{i \in J_\beta} A_i$. Then B_β is a set of ordinals less than λ of cofinality $\aleph_0$ and, for $\beta \neq \gamma$, $B_\beta \nsubseteq B_\gamma\,[\mathrm{mod}\ D(\lambda)]$.Choose for each $\delta \in A$ an increasing sequence $\eta = \eta_\delta$ of length ω whose limit is δ.

Let G^c_λ be the torsion completion of $\bigoplus_{i<\lambda, n<\omega} \langle x^n_i \rangle$, where x^n_i has order p^{n+1} (and $\langle x^n_i \rangle$ is generated by x^n_i). (See [8, p. 14].)

For each $B \subseteq A$ we now define a subgroup $G(B)$ of G^c_λ: it is generated by x^n_i for $i < \lambda$, $0 \leqq n < \omega$, and by $y^m_\delta = \sum_{n=m}^{\infty} p^{n-m} x^n_{\eta(n)}$ for each $m < \omega$, $\eta = \eta_\delta$, $\delta \in B$. Clearly each $G(B)$ is a reduced separable p-group.

So it suffices to prove that for $\beta \neq \gamma$, $G(B_\beta)$ is not isomorphic to any subgroup of $G(B_\gamma)$. For this is sufficient to show:

(*) If there is an embedding F of $G(B^1)$ into $G(B^2)$ then $B^1 \subseteq B^2[\mathrm{mod}\ D(\lambda)]$ (where B^1, B^2 are subsets of A).

PROOF OF (*). For each α, β let $G_\alpha(B)$ be the subgroup of $G(B)$ generated by x^n_β, y^m_δ for $\beta, \delta < \alpha$. For each $i < \lambda$ let $f(i)$ be the first limit ordinal γ such that

(1) if $j < i$ then $F(x^n_j) \in G_\gamma(B^2)$;

(2) if $\delta < i$ then $F(y^m_\delta) \in G_\gamma(B^2)$;

(3) if $j < i$ and, for some $a \in G(B^1)$, $F(a) \in G_j(B^2)$ then $a \in G_\gamma(B^1)$.

Clearly $f(i) < \lambda$, and $i \leqq j$ implies $f(i) \leqq f(j)$, and for a limit ordinal δ, $f(\delta) = \bigcup_{i<\delta} f(i)$. Hence

$$C = \{\delta : \delta < \lambda,\ \delta \text{ a limit ordinal},\ f(\delta) = \delta\}$$

is a closed unbounded subset of λ, that is, $C \in D(\lambda)$, so it suffices to prove $B^1 \cap C \subseteq B^2 \cap C$. Let $\delta \in B^1 \cap C$. so $y^0_\delta \in G(B^1)$, and let $b = F(y^0_\delta)$. For every m, $y^0_\delta - p^m y^m_\delta$ belongs to $G_\delta(B^1)$ hence $b - p^m F(y^m_\delta)$ belongs to $G_\delta(B^2)$. So b belongs to the closure of $G_\delta(B^2)$ in $G(B^2)$ (in the p-adic topology), and by (3), $b \notin G_\delta(B^2)$. But if $\delta \notin B^2$, then $G_\delta(B^2)$ is closed in $G(B^2)$ as $G(B^2) = G_\delta(B^2) \oplus H_\delta(B^2)$ where $H_\delta(B^2)$ is generated by x_i, $\delta \leqq i < \lambda$; y^m_α, $\delta < \alpha < \lambda$, $\eta_\alpha(m) \geqq \delta$; $\alpha \in B^2$. Hence $B^1 \cap C \subseteq B^2 \cap C$, so (*) holds, completing the proof.

2. Rigid systems

THEOREM 2.1. *For any λ there is a family $\{G_i : i < 2^\lambda\}$ of torsion-free groups*

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each of power λ, which is a rigid system, that is, if $h: G_i \to G_j$ is a non-zero homomorphism, then $i = j$ and $h(x) = nx$ for some integer n.

PROOF. We restrict ourselves to $\lambda > \aleph_0$ regular. Let all the primes mentioned below be distinct. Let a_i^α, $\alpha < \omega + \omega$, $i < \lambda$ be free generators of a group G. Let A be a subset of λ. We now define some equivalence relations over subsets of $\{a_i^\alpha : \alpha < \omega + \omega, i < \lambda\}$ (we define them by a generating set of pairs):

(1) $E_\alpha^0 = \{\langle a_i^0, a_i^\alpha\rangle : i < \lambda\}$ for $\alpha < \omega + \omega$. Notice each equivalence class has at most two elements.

(2) $E_n^1 : (0 < n < \omega)$ rename $\{a_i^n : i < \lambda\}$ as $\{b_{i,j}^n : i,j < \lambda\}$, and let E_n^1 be generated by $\{\langle a_i^0, b_{i,j}^n\rangle : i < \lambda, j < \lambda\}$

(3) $E_n^2 : (0 < n < \omega)$ is generated by $\{\langle a_i^0, b_{j,i}^n\rangle : i,j < \lambda\}$

(4) $E_n^3 : (0 < n < \omega)$ for each $\delta < \lambda$ of cofinality $\aleph_0$ choose an increasing sequence η_δ of ordinals of length ω whose limit is δ. Let E_n^3 be generated by $\{\langle a_\delta^0, b_{\eta(n),\delta}^n\rangle : \delta < \lambda, \mathrm{cf}\delta = \omega\}$. Notice that every equivalence class has at most two elements.

(5) E_0^4: By Theorem 1.1 we can find λ disjoint stationary subsets J_i, $i < \lambda$ of $\{\delta < \lambda : \mathrm{cf}\,\delta = \omega\}$. Let E_0^4 be generated by $\{\langle a_i^0, a_j^1\rangle : j \in J_i\}$.

(6) $E_n^5 : 0 < n < \omega$ rename $\{a_i^{\omega+n} : i < \lambda\}$ as $\{b_\tau : \tau$ a decreasing sequence of ordinals less than λ of length $n\}$ identifying $a_i^{\omega+1}$ and $b_{\langle i\rangle}$. E_n^5 is generated by

$$\{\langle b_\tau, b_{\tau\wedge\langle i\rangle}\rangle : l(\tau) = n,\ i < \lambda,\ \tau^\wedge\langle i\rangle \text{ is decreasing}\}$$

(7) E_0^6 : is generated by $\{\langle a_i^0, a_j^0\rangle : i,j \in A\}$ (remember A was an arbitrary subset of $\lambda\}$.

Let $J_j^* = \{\delta < \lambda : \mathrm{cf}\,\delta = \omega$, for some $i < \lambda$, $a_\delta^1 = b_{j,i}^1\}$. We can assume $J_j^* \cap J_i$ is stationary for any i, j.

Now we can define a group $G(A)$ containing G, which is contained in the divisible hull of G. $G(A)$ is generated by

(a) G;

(b) $(p_\alpha)^{-l} a_i^\alpha$ (for any $0 < l < \omega$, $\alpha < \omega + \omega$, $i < \lambda$); and

(c) $(p_\gamma^n)^{-l}(a_i^\alpha - a_j^\beta)$ (for any $0 < l < \omega$, when $a_i^\alpha E_\gamma^n a_j^\beta$) (of course p_α, p_γ^n are distinct primes).

We say $x \in G(A)$ is divisible by p^∞ if for any $l < \omega$ for some $y \in G(A)$, $p^l y = x$. Now notice that:

(*) *if $x = \sum r_i^\alpha a_i^\alpha \in G(A)$* (clearly only finitely many r_i^n are $\neq 0$) *then*

(1) x is divisible by $(p_\alpha)^\infty$ iff $\beta \neq \alpha \Rightarrow r_i^\beta = 0$,

(2) x is divisible by $(p_\gamma^n)^\infty$ iff for each a_j^β $\sum\{r_i^\alpha : a_i^\alpha E_\gamma^n a_j^\beta\}$ is zero.

Except for $p = p_0^6$, the divisibility by p^∞ in $G(A)$, $G(B)$ does not depend on the choice of A, so we don't mention it.

It suffices to prove that $\{G(A_i): i < 2^\lambda\}$ is the required family, where $\{A_i : i < 2^\lambda\}$ is a family of subsets of λ, no one contained in the other. So it suffices to prove the following.

Suppose $h : G(A) \to G(B)$ is a non-zero homomorphism; then $A \subseteq B$, and for some integer n, $h(x) = nx$ for every $x \in G(A)$.

Observation I. $h(a_i^\alpha)$ is a linear combination of $\{a_j^\alpha : j < \lambda\}$ with rational coefficients. This is because a_i^α, hence $h(a_i^\alpha)$ is divisible by p_α^∞, and by (* 1).

Observation II. $J^0 = \{\delta < \lambda : \mathrm{cf}\,\delta = \omega, h(a_\delta^0) \neq 0\}$ is stationary and includes J_j^* for some j.

As $h \neq 0$ and $G(B)$ is torsion free, $h(a_j^\beta) \neq 0$ for some β, j. As $a_j^0 E_\beta^0 a_j^\beta$, $(p_\beta^0)^\infty$ divides $a_j^0 - a_j^\beta$, hence it divides $h(a_j^0) - h(a_j^\beta)$, but by Observation I and (* 2), $(p_\beta^0)^\infty$ does not divide $h(a_j^\beta)$, hence $h(a_j^0) \neq 0$. Similarly $(p_1^1)^\infty$ divides $h(a_j^0) - h(b_{j,i}^1)$ for any $i < \lambda$, but by Observation I and (* 2) it does not divide $h(a_j^{0})$ (as for $\alpha \neq \gamma < \lambda$ not $a_\alpha^0 E_1^1 a_\gamma^0$) hence $h(b_{j,i}^1) \neq 0$. As before $a_\alpha^1 = b_{j,i}^1 \Rightarrow h(a_\alpha^1) \neq 0$ $\Leftrightarrow h(a_\alpha^0) \neq 0$ hence by the definition of J_j^* we are finished.

Observation III. If $l(\tau) = n$, let $h(b_\tau) = r_1 b_{\tau_1} + \cdots + r_l b_{\tau_l}$ $(r_1 \neq 0, \cdots, r_l = 0)$, $l(\tau_1) = \cdots = l(\tau_l) = n$ (by Observation I); *then* $\tau_1(n-1), \cdots, \tau_l(n-1) \geqq \tau(n-1)$. (Maybe $l = 0$.)

We prove this assertion by induction on $\tau(n-1)$. If $\tau(n-1)$ is zero, it is immediate. So suppose $\tau(n-1) = \gamma + 1$; let $\nu = \tau^\wedge\langle\gamma\rangle$, $h(b_\nu) = r^1 b_{\nu_1} + \cdots + r^m b_{\nu_m}$ $(r^k \neq 0, l(\nu_k) = n+1)$. Now $(p_n^5)^\infty$ divides $h(b_\tau) - h(b_\nu)$, so by (* 2) and the way E_n^5 was defined for every j, $1 \leqq j \leqq l$, there is an i, $1 \leqq i \leqq m$, such that τ_j is an initial segment of ν_i. As ν_i is decreasing $\tau_j(n-1) > \nu_i(n) \geqq \nu(n) = \gamma$ (using the nduction hypothesis), hence $\tau_j(n-1) \geqq \gamma + 1 = \tau(n-1)$. So we proved Observation III. If $\tau(n-1)$ is a limit ordinal, the proof is similar.

Observation IV. There is a closed unbounded set $C \subseteq \lambda$ such that if $\alpha < \beta$, $\beta \in C$, $h(a_\alpha^0) = r_1 a_{\alpha(1)}^0 + \cdots + r_n a_{\alpha(n)}^0$ $(r_i \neq 0)$ *then* $\alpha \leqq \alpha(i) < \beta$. (Maybe $n = 0$)

As $(p_{\omega+1}^0)^\infty$ divides $a_\alpha^0 - a_\alpha^{\omega+1} = a_\alpha^0 - b_{\langle\alpha\rangle}$, and the definition of $E_{\omega+1}^0$, necessarily $h(a_\alpha^{\omega+1})$ is $r_1 a_{\alpha(1)}^{\omega+1} + \cdots + r_n a_{\alpha(n)}^{\omega+1}$; hence $\alpha \leqq \alpha(l)$ by the previous observation. Let $f_1(\alpha) = \max\{\alpha(l) + 1 : l\}$, $f_2(\alpha) = \sup\{f_1(\beta) : \beta < \alpha\}$, as λ is regular $\alpha < \lambda \Rightarrow f_2(\alpha) < \lambda$, and as in addition $\lambda > \aleph_0$, $C = \{\alpha : f_2(\alpha) = \alpha\}$ is closed and unbounded.

OBSERVATION V. If $\delta \in C$, $\mathrm{cf}\,\delta = \omega$, then $h(a_\delta^0) = r_\delta a_\delta^0$ for some r_δ. Suppose $h(a_\delta^0) = r_1 a_{i(1)}^0 + \cdots + r_n a_{i(n)}^0$ $(r_i \neq 0)$. By Observation IV, we can assume $\delta \leqq i(1) = i_1 < \cdots < i(n) = i_n < \lambda$. Assume $i_n > \delta$. By the definition of the η_i's we can choose $m < \omega$ such that if $\mathrm{cf}[i_{l+1}] = \omega$ $\eta_{i_{l+1}}(m) > i_l$, and if $i_1 > \delta$, $\mathrm{cf}[i_1] = \omega$ then $\eta_{i_1}(m) > \delta$. Now $(p_m^3)^\infty$ divides $a_\delta^0 - b_{\eta[m,\delta],\delta}^m$, hence $h(a_\delta^0) - h(b_{\eta[\delta,m],\delta}^m)$. By the definition of E_m^3, necessarily $i(1), \cdots, i(n)$ have cofinality ω,and

$$h(b_{\eta[m,\delta],\delta}^m) = r_1 b_{\eta[m,i(1)],i(1)}^m + \cdots + r_m b_{\eta[m,i(n)],i(n)}^m.$$

Also $(p_m^1)^\infty$ divides $a_{\eta[m,\delta]}^0 - b_{\eta[m,\delta],\delta}^m$; hence necessarily by (* 2) and Observation I (as the $\eta[m, i(l)]$ $(1 \leqq l \leqq n)$ are distinct), $h(a_{\eta[m,\delta]}^0) = r_1 a_{\eta[m,i(1)]}^0 + \cdots + r_n a_{\eta[m,i(n)]}^0$. This contradicts Observation IV. So *either* $i_n = \delta$ so $n = 1$ and then $h(a_\delta^0) = r_1 a_\delta^0$ *or* $n = 0$ and then choose $r_\delta = 0$.

OBSERVATION VI. For every i, $h(a_i^0) = r a_i^0$ for some r. Using p_1^0 we see that for $\delta \in C$, $\mathrm{cf}\,\delta = \omega$; $h(a_\delta^1) = r_\delta a_\delta^1$. Now for every $i < \lambda$, there is a $\delta \in C \cap J_i$ (as J_i is stationary) so $\mathrm{cf}\,\delta = \omega$. Now $(p_0^4)^\infty$ divides $a_i^0 - a_\delta^1$, so it divides $h(a_i^0) - r_\delta a_\delta^1$, so by Observation I and the definition of E_0^4 and (* 2), $h(a_i) = r_\delta a_i^0$. So for every i, $h(a_i^0) = r_i a_i^0$.

Notice that $\delta \in J_i \cap C$ implies $r_i = r_\delta$, hence for $\delta(1), \delta(2) \in J_i \cap C$, $r_{\delta(1)} = r_{\delta(2)}$. For any i, consider the homomorphism h_1, $h_1(x) = h(x) - r_i x$. If $h_1 = 0$ we finish, otherwise all our observations can apply to it. As $J_j^* \cap J_i$ is stationary for any i,j, we obtain a contradiction to Observation II. So $r_i = r$. By E_α^α we see that $h(a_i^\alpha) = r a_i^\alpha$.

So now we can finish the proof. Clearly $h(x) = rx$ for every $x \in G(A)$. The rational r should be an integer, as for every p there is an $x \in G(A)$ such that $x/p \notin G(A)$. Using E_0^6 clearly $A \subseteq B$.

QUESTION. Is there a class of torsion-free groups, having 2^λ members in each cardinal λ which is a rigid system?

REMARK 2.1. The completion for singular cardinals and $\aleph_0$ is easy. If λ is singular, $\lambda = \sum_{i<\mathrm{cf}\lambda} \lambda_i$, $\lambda_i < \lambda$, we can combine the constructions for $\mathrm{cf}\,\lambda$, λ_i^+.

REMARK 2.2. It seems that we can make the system homogeneous, for example, of type $(0, 0, \cdots)$ as Corner does to Fuchs' proof.

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3. Whitehead problem

Recall that a group G is called a W-group (Whitehead group) if for every homomorphism $h : H \to G$ onto G with kernel Z (the integers), there is a homomorphism $g : G \to H$, such that $hg = 1_G$ (so H is a direct sum of Z and a copy of G). We deal with a torsion-free group G of cardinality $\aleph_1$, so without loss of generality its universe is $\aleph_1 = \omega_1 = \{\alpha : \alpha < \omega_1\}$, and $G_\delta = \{\alpha : \alpha < \delta\}$ is a pure subgroup of G. It is known that if G is a W-group, it is separable torsion free and $\aleph_1$-free (that is, every countable subgroup is free). We classify the possible G to three possibilities, but first we need a definition.

Definition 3.1. (1) If $L \subseteq G$, $PC(L, G)$ is the smallest pure subgroup of G which contains L. Note that if H is a pure subgroup of G, $L \subseteq H$ then $PC(L, G) = PC(L, H)$. We omit G if it is clear.

(2) If H is a subgroup of G, L a finite subset of G, $a \in G$, we say that $\pi(a, L, H, G)$ if $PC(H \cup L) = PC(H) \oplus PC(L)$ but for no $b \in PC(H \cup L \cup \{a\})$ is $PC(H \cup L \cup \{a\}) = PC(H) \oplus PC(L \cup \{b\})$.

Possibility I. For some $\delta < \omega_1$ there are $a^i_{n[i]} L_i = \{a^i_l : l < n_i\}$, $i < \omega_1$, such that:

(A) $\{a^i_l + G_\delta : i < \omega_1,\ l \leqq n_i\}$ is an independent family in G/G_δ,

(B) $\pi(a^i_{n[i]}, L_i, G_\delta, G)$ holds for any i.

Remark. We can replace G_δ by any countable subgroup of G, and w.l.o.g. $\delta = w$.

Possibility II. Not I, but there are a stationary set $A \subseteq \omega_1$ and for any $\delta \in A$, $L_\delta = \{a^\delta_l : l < n_\delta\}$, $a^\delta_{n(\delta)}$ such that:

(A) $\{a^\delta_l + G_\delta : l \leqq n_\delta\}$ is an independent family in G/G_δ,

(B) $\pi(a^\delta_{n(\delta)}, L_\delta, G_\delta, G)$ holds.

Possibility III. Neither I nor II.

Remark. The classification to the three possibilities depend on G only up to isomorphism. Because if $h : G^1 \to G^2$ is an isomorphism, $C = \{\delta : h$ is an isomorphism from G^1_δ onto $G^2_\delta\}$ is closed and unbounded subset of ω_1. Clearly C is closed, and if $\alpha_0 < \omega$, define by induction $\alpha_n < \omega_1$, $\alpha_{n+1} = \sup\{h(i) : i < \alpha_n\} \cup \{j : h(j) < \alpha_n\}]$. Then $\alpha_{n+1} < \omega_1$ as $\mathrm{cf}\,\omega_1 > \aleph_0$, and similarly $\alpha^* = \bigcup \alpha_n < \omega_1$, $\alpha^* \in C$, so C is unbounded.

Lemma 3.1. (1) *Each possibility is satisfied by some $\aleph_1$-free group (that is, every countable subgroup is free). Of course, the possibilities form a partition.*

(2) *If Possibility I fails, then by renaming the elements of G we can assume*:
(*) *If* $\delta < \omega_1$, $\{\alpha_l + G_l : l \leqq n\}$ *is independent, and* $\pi(a_n, L, G_\rho, G)$ *holds where* $L = \{a_1 : l < n\}$, then $\{a_l + G_{d+w} : l \leqq n\}$ *is not independent.*

PROOF. (1) For Possibility I, let G be generated by $x_n, n < \omega$, and $x^m_\eta = \sum^{\infty}_{n=m}(n!/m!)x_{\eta(n)}$ for each $m < \omega$, $\eta \in C$, where C is a set of $\aleph_1$ increasing sequences of natural numbers of length ω, such that $\eta \neq \tau \in C$ implies the ranges of η, τ have finite intersection. (Take G_δ the subgroup generated by the x_n's, $n_i = 0$ and $a^i_0 = x^m_{\eta_i}$, $C = \{\eta_i : i < \omega_1\}$.)

For Possibility II, choose for each limit $\delta < \omega_1$ an increasing sequence η_δ of ordinals of length ω, whose limit is δ. Let G be generated by x_α, $\alpha < \omega_1$, and $x^m_\delta = \sum^{\infty}_{n=m}(n!/m!)x_{\eta[n,\delta]}$ ($m < \delta$, $\delta < \omega_1$, δ a limit ordinal). (See [8, (75.1)].)

For Possibility III take the free group with $\aleph_1$ generators.

(2) We define by induction an increasing sequence of limit ordinals $\alpha(i)$, $\alpha(0) = 0, \alpha(\delta) = \bigcup_{i<\delta}\alpha(i)$. If $\alpha(i)$ is defined, as Possibility I fails there is a maximal family $a^j_l, l \leqq n(j)$, $j < j_0 < \omega_1$ such that $\{a^j_l + G_{\alpha(i)} : j, l\}$ is independent, and $\pi[(a^j_{n(j)}, \{a^j_l : l < n(j)\}, G_{\alpha(i)}, G]$. Choose a limit $\alpha(i+1) > \alpha(i)$ so that $a_l \in G_{\alpha(i+1)}$ Now rename $\{a : \alpha(i) \leqq a < \alpha(i+1)\}$ (remember the elements of G are ordinals) by $\{\beta : \omega i \leqq \beta < \omega(i+1)\}$.

LEMMA 3.1(3) *For* $\aleph_1$*–, free G possibility III is equivalent to G being the direct sum of countable groups.*

PROOF. Clearly if $G = \oplus_i G_i$, G_i countable, then we can assume each $a \in G_i$ satisfies $\omega_i \leqq a < \omega(i+1)$, so Possibility III holds. Suppose Possibility III holds; then Possibility I fails, hence we can assume (*) from (2). As Possibility II fails, we can find a closed and unbounded set $C \subseteq \{\delta : \delta < \omega_1\}$ so that for $\delta \in C$ we cannot find a and finite A such that $\pi(a, A, G_\rho, G)$ holds. By renaming we can assume $C = \{\omega i : i < \omega_1\}$. Now for each δ, we can find G^n_δ so that G^n_δ is a pure subgroup of G, $G_{\delta+1} = \bigcup_n G^n_\delta$, $G^0_\delta = G_\delta$ and $G^{n+1}_\delta / G^n_\delta$ has rank one. Now we can define by induction on $n \geqq 1$, $a_n \in G_\delta$ so that $G^n_\delta = G_\delta + PC(a_1, \cdots, a_n)$. By the definition of π this clearly can be done. Let $H_\delta = PC(a_1, a_2, \cdots)$, so $G_{\delta+1} = G_\delta \oplus H_\delta$ hence $G = \oplus_\delta H_\delta \oplus G_\omega$.

DEFINITION 3.2. A (G, Z)-group is a group H whose set of elements is $G \times Z = \{(a, b) : a \in G, b \in Z\}$, and the mapping $h : (a, b) \to a$ is a homomorphism from H to G with kernel $Z = \{0\} \times Z$; and $(a, b) + (0, c) = (a, b + c)$. We denote the h corresponding to H_l by h_l, and a (G_i, Z)-group by H_i.

LEMMA 3.2. (1) *Let G_1 be a subgroup of G_2, H_1 a (G_1, Z)-group; then we can extend H_1 to a (G_2, Z)-group H_2.*

(2) *If $g_1 : G_1 \to H_1$ is a homomorphism, $h_1 g_1 = 1_{G[1]}$ and $\pi(a, A, G_1, G_2)$, where A is finite, then we can extend H_1 to a (G_2, Z)-group H_2 so that for no homomorphism $g_2 : G_2 \to H_2$, $h_2 g_2 = 1_{G[2]}$, and g_2 extends g_1.*

PROOF. This in fact is not new.

(1) This is immediate by 51.3(2) from [7, p. 218]. But we need (3.2) only for G_1 a W-group, so we prove it for this case. By iterating, it suffices to prove two cases:

(a) G_2/G_1 is a free group of rank one, say generated by $a_0 + G_1$. Then every $b_1 \in G_2$ has a unique representation as $na_0 + c$ ($c \in G_1$, n integer). So define

$$\begin{aligned}(b_1, k_1) + (b_2, k_2) &= (n_1 a_0 + c_1; k_1) + (n_2 a_0 + c_2, k_2)\\ &\overset{\mathrm{df}}{=} ((n_1 + n_2)a_0 + c_3, k_3)\end{aligned}$$

where (inH_1) $(c_1, k_1) + (c_2, k_2) = (c_3, k_3)$.

(b) G_2/G_1 is a cyclic group of a prime order p, generated by $a_0 + G_1$. So define, $g_1 : G_1 \to H_1$ as a homomorphism, $h_1 g_1 = 1_{G_1}$, and $g_1(c) = (c, m(c))$.

$$\begin{aligned}(n_1 a_0 + c_1, k_1) + (n_2 a_0 + c_2, k_2) \overset{\mathrm{df}}{=} ((n_1 + n_2)a_0 + c_1 + c_2,\ & k_1 + k_2 - m(c_1)\\ & - m(c_2) + m(c_1 + c_2) + f(n_1 + n_2))\end{aligned}$$

where $0 \leqq n_1, n_2 < p$, and $f(n_1, n_2)$ is 0 when $n_1 + n_2 < p$ and $M \in Z$ otherwise.

(2) Let $A = \{a^1, \cdots, a^m\}$, $G_3 = PC(G_1 \cup A)$, $G_5 = PC(G_1 \cup A \cup \{a\})$, and G_4 the group generated by G_3, a. Now by (1) extend H_1 to a (G_4, Z)-group H_4. Now any homomorphism $g : G_4 \to H_4$ extending g_1, $h_4 g = 1_{G[4]}$ is determined by the values of $g(a^i)$, $g(a)$, $i = 1, \cdots, m$, and as $g(a^i) \in \{(a^i, l) : l \in Z\}$ there are only countably many such g's $: g_n, n < \omega$. Clearly G_5/G_4 should be infinite so either

(A) it contains an infinite direct sum of cyclic groups of prime order, or

(B) it contains a copy of $Z(p^\infty)$ for some p.

Case A. Suppose generators of those groups are $a_n + G_4$ ($n < \omega$), of order p_n. Let G_n^* be generated by $G_4, a_0, \cdots, a_{n-1}$. We define by induction (G_n^*, Z)-groups $H_n^*, H_0^* = G_4, H_{n+1}^*$ extending H_n^*. If H_n^* is defined, we define H_{n+1}^* as in (1b) using as constant M_n, and then by (3.2 (1)) find a (G_5, Z)-group H_5 extending all the H_n^*. If $g : G_5 \to H_5$ is a homomorphism, $hg = 1_{G[5]}$, then for some n, g extends g_n. Let $p_n a_n = b_n \in G_4$, $g_n(b_n) = (b_n, k_n)$, and $g(a_n) = (c_n, l_n)$, $(b_n, k_n) = g(b_n) = g(p_n a_n) = p_n g(a_n) = p_n(a_n, l_n) = (b_n, p_n l_n + M_n)$ so $k_n = M_n (\mathrm{mod}\ p_n)$. Hence, if we choose $M_n = k_n + 1$, H_5 satisfies our requirements.

Case B. Similar to Case A; here $pa_{n+1} - a_n = b_{n+1} \in G_4$, $pa_0 = b_0 \in G_4$ and, letting $g(b_n) = (b_n, l_n)$, $g(a_n) = (a_n, k_n)$ we obtain $M_0 + pk_0 = l_0$, $M_{n+1} + pk_{n+1} = k_n + l_{n+1}$. So from the M_n's we can compute the k_n's, and for suitable M_n's this is impossible.

THEOREM 3.3. *Assume CH, that is, $2^{\aleph_0} = \aleph_1$. Then if G satisfies Possibility I, it is not a W-group.*

REMARK. This is not really needed for the independence of $(W_{\aleph_1})$.

PROOF. We shall construct a (G, Z)-group H. Let H_ω be the direct sum of G_ω and Z, and let $\{g^i : i < \omega_1\}$ be the list of all homomorphisms from G_ω into H_ω such that $hg^i = 1_G$ (exists by CH). Let $G_\alpha = G(\alpha) = PC[G_\omega \cup \{a_l^i : l \leqq n(i), i < \alpha\}]$, and we define $a(G_\alpha, Z)$-group H_α by induction on α. For $\alpha = 0$, α limit there are no problems. If we have defined for H_α, define for $H_{\alpha+1}$ so that g^α cannot be extended to a homomorphism g from $G_\alpha^* = PC[G_\omega \cup \{a_l : l \leqq n(\alpha)\}]$ into H_α, satisfying $hg = 1_{G\alpha}^*$ (but $h \restriction G_{\alpha+1}$ is still a homomorphism). This can be done by (3.2 (2)) with trivial changes. When H_{ω_1} is defined we extend the definition to obtain a (G, Z)-group.

If there is $g : G \to H$, $hg = 1_G$, then $g \restriction G_\omega$ is some g^α, so by $g \restriction G_\alpha^*$ we obtain a contradiction.

REMARK. We could prove that there are many non-isomorphic such H's.

THEOREM 3.4. *Assume $V = L$. If G satisfies Possibility I or II then it is not a W-group.*

PROOF. By Gödel [11], as $V = L$, CH holds. So by (3.3) we can deal with Possibility II only, (but in fact the same proof works). We use the notation of Definition 3.1, and let A be a stationary set as in Possibility II.

By Jensen [14] as $V = L$ there are functions $g_\delta : G_\delta \to H_\delta = G_\delta \times Z$ for every $\delta \in A$, such that for any function $g : G \to H$, $\{\delta < \omega_1 : g \restriction G_\delta = g_\delta\}$ is stationary (this is called $_{\diamondsuit\omega_1}$).

Now we define the (G_δ, Z)-group H_δ by induction on δ so that $h \restriction H_\delta$ is a homomorphism. For $\delta = \omega$ we define it arbitrarily, for a limit δ (among the limit ordinals) there is nothing to define. Suppose we have defined H_δ; then we shall define $H_{\delta+\omega}$ so that, if $\delta \in A$, g_δ cannot be extended to homomorphism $g : G_{\delta+\omega} \to H_{\delta+\omega}$, $hg = 1_{G_{\delta+\omega}}$ using (3.2). By the definition of the g_δs, clearly H shows G is not a W-group.

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THEOREM 3.5. *Assume* MA + $2^{\aleph_0} > \aleph_1$ (MA *for Martin Axiom*). *If G has cardinality* $\aleph_1$, *is* $\aleph_1$*-free, and does not satisfy Possibility I then it is a W-group.*

REMARK. We can extend the proof to groups of cardinality less than $2^{\aleph_0}$.

PROOF. If G satisfies Possibility III, then by (3.1 (4)) it is the direct sum of countable groups. Each summand is free by the $\aleph_1$-freeness, so G is free, hence a W-group. So assume G satisfies Possibility II. So let $h : H \to G$ be a homomorphism with kernel Z, the set of elements of H being $G \times Z$, $h((a, b)) = a$. MA says that for any $\lambda < 2^{\aleph_0}$:

(MA_λ): Suppose P is a partial order of power $\leqq \lambda$, and in P there is not any subset of $\aleph_1$ pairwise-contradictory element (a, b are contradictory if they have no common upper bound). Suppose $\{D_i : i < \lambda\}$ are dense subsets of P (that is, $(\forall i < \lambda)(\forall a \in P)(\exists b \in D_i)[a \leqq b]$). *Then* there is a set $\mathbf{G} \subseteq P$ such that $\mathbf{G} \cap D_i \neq \emptyset$ for any $i < \lambda$, and any two members of $\mathbf{G}$ have a common upper bound in $\mathbf{G}$ (such a $\mathbf{G}$ is called generic).

Let P be the set of homomorphisms g from finitely generated pure subgroups I of G into H, such that $hg = 1_I$. So P has power $\aleph_1$. The partial order is "extending", that is, $g_1 \leqq g_2$ iff g_2 extends g_1. Let for $i < \omega_1$, $D_i = \{g \in P : i$ is in the domain of $g\}$. If there is a generic $\mathbf{G} \subseteq P$ define $g^*(x) = y$ if for some $g \in \mathbf{G}$ $g(x) = y$. As $\mathbf{G} \cap D_i \neq \emptyset$, $g^*(i)$ is defined at least once, for each $i < \omega_1$. As any two members of $\mathbf{G}$ have a common upper bound, g^* is uniquely defined, and is a homomorphism; also $hg^* = 1_G$, so we are finished. Thus, it suffices to prove:

(1) Each D_i is dense. This is because I is pure and freely generated by some $a_1, \cdots, a_n \in G$; so there is an a_{n+1} such that $a_1, \cdots, a_{n+1}$ freely generate K for any subgroup K of G of rank $n + 1$ which contains I, for example, $PC(a_1, \cdots, a_n, b)$.

(2) Suppose $\{g_i : i < \omega_1\} \subseteq P$; the g_i's are pairwise contradictory and we shall obtain a contradiction.

Let $\mathrm{Dom}\, g_i$ be freely generated by $a_1^i, \cdots, a_{n_i}^i$. As we can replace $\{g_i : i < \omega_1\}$ by any subfamily of the same cardinality thus we can assure $n = n_i$. Let $\{a_1, \cdots, a_m\}$ be a maximal set of elements of G, which generates freely a pure subgroup and $\{a_1, \cdots, a_m\} \subseteq \mathrm{Dom}\, g_i$ for $\aleph_1$ i's. So without loss of generality $a_1, \cdots, a_m \in \mathrm{Dom}\, g$ for every i, hence without loss of generality $a_1^i = a_1, \cdots, a_m^i = a_m$. As the number of g_i's with some fixed domain is at most countable, we can assume $\mathrm{Dom}\, g_i \neq \mathrm{Dom}\, g_j$ for $i \neq j$; hence $m < n$, and also $\{a_1, \cdots, a_m\} \cup \{a_l^i : m < l < n, i < \omega_1\}$ is independent. Now we define a strictly increasing sequence of ordinals $\alpha(i)$, $i < \omega_1$,

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such that $PC[(\mathrm{Dom}\, g_{\alpha(i)} + G_{i\omega}]$ is finitely generated (this is possible as Possibility I fails). Hence for every i there are $c_1^i, \cdots, c_{k(i)}^i \in G_{i\omega}$ such that $PC[(\mathrm{Dom}\, g_{\alpha(i)}) + G_{i\omega}]$ is generated by $G_{i\omega} \cup \{b_1^i, \cdots, b_{k(i)}^i\}$, where $\{b_1^i, \cdots,\} \subseteq PC(a_1^i, \cdots, a_n^i, c_1^i, \cdots, c_{k(i)}^i)$.

Notice that $c_1^i < i\omega$. As the set of $i < \omega_1$ such that $i\omega = i$ is closed and unbounded we can define a decreasing sequence of stationary sets $J_0 \supseteq J_1 \supseteq \cdots \supseteq J_m$ such that for $i \in J_0$, $k(i) = m$ and for all $i \in J_l$, $c_l = c_l^i$for some c_l (by [6]). Now for each $i \in J_m$ we can extend $g_{\alpha(i)}$, so that its domain will be $PC(a_1^i, \cdots, a_n^i, c_1, \cdots, c_m)$ and call the new homomorphism g^i. By changing notation we can assume $\mathrm{Dom}\, g^i$ is again freely generated by $a_1, \cdots, a_m, a_{m+1}^i, \cdots, a_n^i$and $\{a_1, \cdots, a_m\} \cup \{a^i : m < l < n, i \in J_m\}$ is independent. By replacing J_m by a subset of the same cardinality J^* we can assume $g^i(a_l)$ $(l = 1, \cdots, m)$ are fixed. Choose $i < j \in J^*$ so that $a_{m+1}^i, \cdots, a_n^i \in G_j$. By construction, $\mathrm{Dom}\, g^j + G_{j\omega}$ is a pure subgroup of G, so $\mathrm{Dom}\, g^j + \mathrm{Dom}\, g^i$ is a pure subgroup of G, and its domain is freely generated by $\{a_1, \cdots, a_m, a_{m+1}^i, \cdots, a_n^i, a_{m+1}^j, \cdots, a_n^j\}$. As $g^j(a_l) = g^i(a_l)$ a common extension (which belongs to P) exists. Contradiction.

Conclusion 3.6. The statement "Every W-group of cardinality $\aleph_1$ is free" is independent of and consistent with ZFC (Zermelo-Frenkel with the axiom of choice, the usual set of axioms of set theory). (We assume of course the consistency of ZFC.)

Proof. By Stein [22] and Rotman [19] any W-group is $\aleph_1$-free and separable. By Gödel [11], ZFC + $V = L$ is consistent. By (3.4), if $V = L$ any W-group satisfies Possibility III, so by (3.1 (3)), $G = \oplus_i G_i$, G_i countable. As G is $\aleph_1$-free, G_i is free so G is free. Thus ZFC + $V = L$ implies our statement, hence it is consistent.

By Martin and Solovay [17], ZFC + MA + $2^{\aleph_0} > \aleph_1$ is consistent. By (3.5) and (3.1 (1)), ZFC + MA + $2^{\aleph_0} > \aleph_1$ implies the existence of non-free W-groups of cardinality $\aleph_1$.

Open Question. Is it consistent that

(1) every $\aleph_1$-free group of cardinality $\aleph_1$ is a W-group?

(2) there are such groups, satisfying the same possibility, one a W-group, the other not?

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