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The automorphism tower of a centerless group without Choice

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Abstract For a centerless group G, we can define its automorphism tower. We define G^α: $G^0 = G$, $G^{\alpha+1} = \mathrm{Aut}(G^\alpha)$ and for limit ordinals $G^\delta = \bigcup_{\alpha<\delta} G^\alpha$. Let τ_G be the ordinal when the sequence stabilizes. Thomas' celebrated theorem says $\tau_G < (2^{|G|})^+$ and more. If we consider Thomas' proof too set theoretical (using Fodor's lemma), we have here a more direct proof with little set theory. However, set theoretically we get a parallel theorem without the Axiom of Choice. Moreover, we give a descriptive set theoretic approach for calculating an upper bound for τ_G for all countable groups G (better than the one an analysis of Thomas' proof gives). We attach to every element in G^α, the αth member of the automorphism tower of G, a unique quantifier free type over G (which is a set of words from $G * \langle x \rangle$). This situation is generalized by defining "(G, A) is a special pair".

Keywords Automorphism tower · Axiom of Choice · Centerless group

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1 Introduction

Background Given any centerless group G, we can embed G into its automorphism group $\mathrm{Aut}(G)$ as inner automorphisms. Since $\mathrm{Aut}(G)$ is also without center, we can do

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this again, and again. Thus we can define an increasing continuous sequence $\langle G^\alpha | \alpha \in \mathbf{ord}\rangle$—the automorphism tower. The natural question that arises, is whether this process stops, and when. We define $\tau_G = \min\{\alpha | G^{\alpha+1} = G^\alpha\}$.

In 1939 (see [8]) Wielandt proved that for finite G, τ_G is finite. But there exist examples of centerless infinite groups such that this process does not stop in any finite stage. For example—the infinite dihedral group $D_\infty = \langle x, y | x^2 = y^2 = 1\rangle$ satisfies $\mathrm{Aut}(D_\infty) \cong D_\infty$. So the question remained open until 1984, when Simon Thomas' celebrated work (see [6]) proved that $\tau_G \le (2^{|G|})^+$. He later (see [7]) improved this to $\tau_G < (2^{|G|})^+$.

For a cardinal κ we define τ_κ as the smallest ordinal such that $\tau_\kappa > \tau_G$ for all centerless groups G of cardinality $\le \kappa$. As an immediate conclusion from Thomas' theorem we have $\tau_\kappa < (2^\kappa)^+$.

Notation 1.1 For a group G and a subgroup $H \le G$, let $\mathrm{nor}_G(H)$ be the normalizer of H in G (sometimes denoted $N_G(H)$).

For $H \le G$, we define the normalizer tower $\langle \mathrm{nor}_G^\alpha(H) | \alpha \in \mathbf{ord}\rangle$ of H in G by $\mathrm{nor}_G^0(H) = H$, $\mathrm{nor}_G^{\alpha+1}(H) = \mathrm{nor}(\mathrm{nor}_G^\alpha(H))$ and $\mathrm{nor}_G^\delta(H) = \bigcup\{\mathrm{nor}_G^\alpha(H) | \alpha < \delta\}$ for δ limit. Let $\tau_{G,H} = \min\{\alpha |\, \mathrm{nor}_G^{\alpha+1}(H) = \mathrm{nor}_G^\alpha(H)\}$.
This construction turns out to be very useful, thanks to the following:
For a cardinal κ, let $\tau_\kappa^{\mathrm{nlg}}$ be the smallest ordinal such that $\tau_\kappa^{\mathrm{nlg}} > \tau_{\mathrm{Aut}(\mathfrak{A}),H}$, for every structure $\mathfrak{A}$ of cardinality $\le \kappa$ and every group $H \le \mathrm{Aut}(\mathfrak{A})$ of cardinality $\le \kappa$.
In [1], Just, Shelah and Thomas found a connection between these ordinals: $\tau_\kappa \ge \tau_\kappa^{\mathrm{nlg}}$.
In this paper we deal with an upper bound of τ_κ, but there are results regarding lower bounds as well, and the inequality above is used to prove the existence of such lower bounds by finding structures with long normalizer towers. In [6], Thomas proved that $\tau_\kappa \ge \kappa^+$, and in [1] the authors found that one cannot prove in ZFC a better explicit upper bound for τ_κ than $(2^\kappa)^+$ (using set theoretic forcing). In [4], Shelah proved that if κ is strong limit singular of uncountable cofinality then $\tau_\kappa > 2^\kappa$ (using results from PCF theory).

It remains an open question whether or not there exists a countable centerless group G such that $\tau_G \ge \omega_1$.

In a subsequent paper ([5]) we prove that $\tau_\kappa^{\mathrm{nlg}} \le \tau_\kappa$ is also true without Choice.
Results After dealing with the normalizer tower in Sect. 2, Sect. 3 is devoted to our main theorem: (of course, Thomas did not need to distinguish G and ${}^{\omega>}G$)

Theorem 1.2 *(ZF)$\tau_{|G|} < \theta_{\mathcal{P}({}^{\omega>}G)}$ for a centerless group G. That is, there is a function from $\mathcal{P}({}^{\omega>}G)$ onto τ_G.*
Moreover, there is such a function onto $\bigcup\{\tau_{G'} + 1 | G'$ is centerless and $|G'| \le |G|\}$.

This is essentially Theorem 3.16.
As one can gather from the theorem, here we deal with finding τ_G without Choice. We define an algebraic and absolute property of G and a subset A ((G, A) is special—see Definition 3.6), that allows us to find a bound to τ_G (see 3.15) in terms of A. We do that by attaching to each element of G^α its quantifier free type over A.

As a consequence, we get Thomas' Theorem without Choice in 3.14. Since Thomas used Fodor's lemma (and it is known that its negation is consistent with ZF), our result is a strict generalization.

We conclude absoluteness. I.e. that for every cardinal κ, if $\mathbf{V}'$ is a subclass of $\mathbf{V}$ which is a model of ZF such that $\mathcal{P}(\kappa) \in \mathbf{V}'$, then $\tau_\kappa < (\theta_{\mathcal{P}(\kappa)})^{\mathbf{V}'}$, and so $\tau_{\aleph_0} < \theta_{\mathbb{R}}^{L(\mathbb{R})}$ (see Conclusion 3.19).
Moreover, we give a descriptive set theoretic approach to finding $\tau_{\aleph_0}$ in Sect. 4. We show there that $\tau_{\aleph_0}$ is less than or equal to the inductive ordinal of second order number theory (see that section for the definition, and Conclusion 4.4).
In the last section, we improve the main result in some aspects for a wider class of groups that satisfy a weaker algebraic property, though not absolute ((G, A) is weakly special—see Definition 5.1). There, instead of working with quantifier free types over A, we work with partial functions from ${}^{\omega>}A$ to G, and we reduce the bound in the case where A is finite.
A note about reading this paper How should you read this paper if you are not interested in the Axiom of Choice but only in the new and simpler proof of Thomas' Theorem?
You can read only Sect. 3, and in there, you should read:
Definition 3.6, Claim 3.8, Conclusion 3.10, Claim 3.12, Claim 3.13, and then finally Conclusion 3.14.

Notation 1.3 (1) G, H denote groups.
(2) For a group G, its identity element, will be denoted as $e = e_G$.
(3) if $A \subseteq G$ then $\langle A\rangle_G$ is the subgroup generated by A in G. Similarly, if $x \in G$, $\langle A, x\rangle_G$ is the subgroup generated by $A \cup \{x\}$.
(4) For a group G, and a subset $H \subseteq G$, $H \leq G$ means that H is a subgroup of G.
(5) Let G be a group, and A some subset of G, then $C_G(A)$ is the centralizer of A in G (i.e. $\{x \in G | \forall a \in A[xa = ax]\}$).
(6) The center of G is $Z(G) = C_G(G)$.
(7) The language of a structure is its vocabulary.
(8) $\mathbf{V}$ will denote the universe of sets; $\mathbf{V}'$ will denote a transitive class which is a model of ZF.

2 The normalizer tower without Choice

Definition 2.1 (1) For a group G and a subgroup $H \leq G$, we define $\mathrm{nor}_G^\alpha(H)$ for every ordinal number α by:
- $\mathrm{nor}_G^0(H) = H$.
- $\mathrm{nor}_G^{\alpha+1}(H) = \mathrm{nor}_G(\mathrm{nor}_G^\alpha(H))$ (see 1.1).
- $\mathrm{nor}_G^\delta(H) = \bigcup\{\mathrm{nor}_G^\alpha(H) | \alpha < \delta.\}$, for δ limit.

(2) We define $\tau_{G,H}^{\mathrm{nlg}} = \tau_{G,H} = \min\{\alpha | \mathrm{nor}_G^{\alpha+1}(H) = \mathrm{nor}_G^\alpha(H)\}$.
(3) For a set k, we define $\tau_{|k|}^{\mathrm{nlg}}$ as the smallest ordinal α, such that for every structure $\mathfrak{A}$ of power $\|\mathfrak{A}\| \leq |k|$, $\tau_{\mathrm{Aut}(\mathfrak{A}),H} < \alpha$ for every subgroup $H \leq \mathrm{Aut}(\mathfrak{A})$ of power $|H| \leq |k|$. Note that $\tau_{|k|}^{\mathrm{nlg}} = \bigcup\{\tau_{\mathrm{Aut}(\mathfrak{A}),H} + 1 |$ for such $\mathfrak{A}$ and $H\}$.
(4) For a cardinal number (i.e. some $\aleph$—so an ordinal) κ, define $\tau_\kappa^{\mathrm{nlg}}$ similarly.

Remark 2.2 Note that $\tau^{\mathrm{nlg}}_{|k|}$ is well defined (in ZF) since we can restrict ourselves to structures with a specific (depending only on k) language and universe contained in k. See Observation 2.3.

Observation 2.3 (1) (ZF) For any structure $\mathfrak{A}$ whose universe is $|\mathfrak{A}| = A$ there is a structure $\mathfrak{B}$ such that:

- $\mathfrak{A}, \mathfrak{B}$ have the same universe (i.e. $A = |\mathfrak{B}|$).
- $\mathfrak{A}, \mathfrak{B}$ have the same automorphism group (i.e. $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathfrak{B})$).
- the language of $\mathfrak{B}$ is of the form $L_{\mathfrak{B}} = \{R_{\bar{a}} | \bar{a} \in {}^{\omega>}A\}$ where each $R_{\bar{a}}$ is a $\lg(\bar{a})$ place relation.

(2) (ZFC) If $\mathfrak{A}$ is infinite then the language of $\mathfrak{B}$ has cardinality at most $|A|$.

Proof Define $\mathfrak{B}$ as follows: its universe is $|\mathfrak{A}|$. Its language is $L = \{R_{\bar{a}} | \bar{a} \in {}^{n}A, n < \omega\}$ where $R^{\mathfrak{B}}_{\bar{a}} = o(\bar{a})$, which is defined by $o(\bar{a}) = \{f(\bar{a}) | f \in \mathrm{Aut}(\mathfrak{A})\}$—the orbit of $\bar{a}$ under $\mathrm{Aut}(\mathfrak{A})$. □

Definition 2.4 For a set A, we define $\theta_A = \theta(A)$ to be the first ordinal $\alpha > 0$ such that there is no function from A onto α.

Remark 2.5 (1) $ZFC \vdash \theta_A = |A|^+$.

(2) $ZF \vdash \theta_A$ is a cardinal number (i.e. some $\aleph$), and if A is infinite (i.e. there is an injection from ω into A) then $\theta_A > \aleph_0$.

(3) Usually, we shall consider $\theta^{\mathbf{V}'}_A$ where $\mathbf{V}'$ is a transitive subclass of $\mathbf{V}$ which is a model of ZF.

Claim 2.6 (ZF) If G is a group, $H \leq G$ a subgroup then $\tau_{G,H} < \theta_G$.

Proof If $\tau_{G,H} = 0$ it is clear. If not, define $F : G \to \tau_{G,H}$ by $F(g) = \alpha$ iff $g \in \mathrm{nor}^{\alpha+1}_G(H) \setminus \mathrm{nor}^{\alpha}_G(H)$, and if there is no such α, $F(g) = 0$. By definition of $\tau_{G,H}$, F is onto. From the definition of θ, $\tau_{G,H} < \theta_G$. □

We can do even more:

Claim 2.7 (ZF) For every set k, $\tau^{\mathrm{nlg}}_{|k|} < \theta_{\mathcal{P}({}^{\omega>}k)}$.

Proof We may assume k is not finite (otherwise, $\tau^{\mathrm{nlg}}_{|k|} \leq |k|!$). Let

$$\mathcal{B}_k = \{(\mathfrak{A}, f, x) | \mathfrak{A} \text{ is a structure}, L_{\mathfrak{A}} \subseteq {}^{\omega>}k, |\mathfrak{A}| \subseteq k, f : k \to \mathrm{Aut}(\mathfrak{A}) \subseteq {}^{|\mathfrak{A}|}|\mathfrak{A}|,$$
$$x \in G = \mathrm{Aut}(\mathfrak{A}) \text{ and } H \leq G, H = \mathrm{image}(f)\}$$

And let $\tau^{\mathrm{nlg}-}_{|k|} = \bigcup\{\tau_{G,H} | G = \mathrm{Aut}(\mathfrak{A}), H = \mathrm{image}(f), (\mathfrak{A}, f, x) \in \mathcal{B}_k \text{ for some } x\}$. Let $F : \mathcal{B}_k \to \tau^{\mathrm{nlg}-}_{|k|}$ be the following map: $F(\mathfrak{A}, f, x) = \alpha$ iff $x \in \mathrm{nor}^{\alpha+1}_G(H) \setminus \mathrm{nor}^{\alpha}_G(H)$, and if there is no such α, $F(\mathfrak{A}, f, x) = 0$ (where $G = \mathrm{Aut}(\mathfrak{A})$, and $H = \mathrm{image}(f)$). Since F is onto $\tau^{\mathrm{nlg}-}_{|k|}$, and obviously $\tau^{\mathrm{nlg}}_{|k|} \leq \tau^{\mathrm{nlg}-}_{|k|} + 1$, and $\theta_{\mathcal{P}({}^{\omega>}k)}$ is an infinite cardinal (in particular—a limit ordinal), it's enough to show that there is a one to one function from $\mathcal{B}_k$ to $\mathcal{P}({}^{\omega>}k)$. It is enough to code $\mathfrak{A}$, f and x separately, as $|{}^{\omega>}(\mathcal{P}({}^{\omega>}k))| = |\mathcal{P}({}^{\omega>}k)|$ (this can be proved using the equality

$|^{\omega>}({}^{\omega>}k)| = |^{\omega>}k|$ (which is proved using a definable well known injective function $cd : \omega \times \omega \to \omega$ and the fact that $2 \leq |k|$)).
$x \in G$ and hence $x \subseteq k \times k$.
$f \in {}^k({}^a a)$ for some $a \subseteq k$, and there is a definable bijection ${}^k({}^a a) \to {}^{k\times a}a$, so code f as a subset of $k \times k \times k$.
$\mathfrak{A}$ is a sequence of subsets of ${}^{\omega>}k$, i.e. a function in ${}^{{}^{\omega>}k}\mathcal{P}({}^{\omega>}k)$, and we can encode such a function as a member of $\mathcal{P}({}^{\omega>}k)$. (Why? $|{}^{{}^{\omega>}k}\mathcal{P}({}^{\omega>}k)| = |{}^{{}^{\omega>}k}({}^{{}^{\omega>}k}2)| = |({}^{{}^{\omega>}k}2)| = |\mathcal{P}({}^{\omega>}k)|$). □

Claim 2.8 Assume that $\mathbf{V}'$ is a transitive subclass of $\mathbf{V}$ which is a model of ZF, $G \in \mathbf{V}'$ a group, $H \in \mathbf{V}'$ a subgroup then $\tau^{\mathbf{V}}_{G,H} = \tau^{\mathbf{V}'}_{G,H} < \theta^{\mathbf{V}'}_{G}$.

Proof By Claim 2.6, it remains to show that $\tau^{\mathbf{V}}_{G,H} = \tau^{\mathbf{V}'}_{G,H}$. By induction on $\alpha \in \mathbf{V}'$, one can see that $(\mathrm{nor}^{\alpha}_{G}(H))^{\mathbf{V}} = (\mathrm{nor}^{\alpha}_{G}(H))^{\mathbf{V}'}$ (the formula that says that x is in $\mathrm{nor}_{G'}(H')$ is bounded with the parameters G' and H'). □

It is also true that $\tau^{\mathrm{nlg}}_{|k|}$ is preserved in $\mathbf{V}'$, for every $k \in \mathbf{V}'$, such that $\mathcal{P}({}^{\omega>}k) \in \mathbf{V}'$:

Claim 2.9 Assume that $\mathbf{V}'$ is a transitive subclass of $\mathbf{V}$ which is a model of ZF.

(1) If $\mathcal{P}({}^{\omega>}k) \in \mathbf{V}'$ then $(\tau^{\mathrm{nlg}}_{|k|})^{\mathbf{V}'} = (\tau^{\mathrm{nlg}}_{|k|})^{\mathbf{V}} < \theta^{\mathbf{V}'}_{\mathcal{P}({}^{\omega>}k)}$.
(2) If $k = \kappa$ a cardinal number and $\mathcal{P}(\kappa) \in \mathbf{V}'$ then $(\tau^{\mathrm{nlg}}_{\kappa})^{\mathbf{V}'} = (\tau^{\mathrm{nlg}}_{\kappa})^{\mathbf{V}} < \theta^{\mathbf{V}'}_{\mathcal{P}(\kappa)}$

Proof (2) follows from (1), as we have an absolute definable bijection $cd : {}^{\omega>}\kappa \to \kappa$. For a set $k \in \mathbf{V}'$, such that $\mathcal{P}({}^{\omega>}k) \in \mathbf{V}'$ let

$$\begin{aligned}\mathcal{A}_k = \{(G, H) | &\text{There is a structure } \mathfrak{A}, \text{ with } |\mathfrak{A}| \subseteq k, \text{ such that} \\ &G = \mathrm{Aut}(\mathfrak{A}) \text{ and } H \leq G, |H| \leq |k|\}\end{aligned}$$

It is enough to prove that $(\mathcal{A}_k)^{\mathbf{V}} = (\mathcal{A}_k)^{\mathbf{V}'}$, because by definition

$$\begin{aligned}(\tau^{\mathrm{nlg}}_{|k|})^{\mathbf{V}} &= \bigcup\{\tau_{G,H} + 1 | (G, H) \in (\mathcal{A}_k)^{\mathbf{V}}\} \\ &= \bigcup\{\tau_{G,H} + 1 | (G, H) \in (\mathcal{A}_k)^{\mathbf{V}'}\} \\ &= (\tau^{\mathrm{nlg}}_{|k|})^{\mathbf{V}'} < \theta^{\mathbf{V}'}_{\mathcal{P}({}^{\omega>}k)}\end{aligned}$$

So let us prove the above equality: $(\mathcal{A}_k)^{\mathbf{V}'} \subseteq (\mathcal{A}_k)^{\mathbf{V}}$, since if $(G, H) \in (\mathcal{A}_k)^{\mathbf{V}'}$ and $\mathfrak{A} \in \mathbf{V}'$ a structure such that $G = \mathrm{Aut}(\mathfrak{A})$ then $\mathfrak{A} \in \mathbf{V}$ and $(\mathrm{Aut}(\mathfrak{A}))^{\mathbf{V}} = (\mathrm{Aut}(\mathfrak{A}))^{\mathbf{V}'}$, because $(\mathrm{Aut}(\mathfrak{A}))^{\mathbf{V}} \subseteq {}^{|\mathfrak{A}|}|\mathfrak{A}| \subseteq \mathcal{P}(k \times k) \in \mathbf{V}'$. So $(G, H) \in (\mathcal{A}_k)^{\mathbf{V}}$, as witnessed by the same structure.
On the other hand, suppose $(G, H) \in (\mathcal{A}_k)^{\mathbf{V}}$. So let $\mathfrak{A}$ be a structure on $s \subseteq k$ such that $G = \mathrm{Aut}(\mathfrak{A})$. By Observation 2.3, we may assume that $L_{\mathfrak{A}} = \{R_{\bar{a}} | \bar{a} \in {}^{\omega>}s\}$, and each $R_{\bar{a}}$ is a $\lg(\bar{a})$ place relation. (This is not necessary, it just makes it more convenient.) Define $X_{\mathfrak{A}} = \{\bar{a}\hat{\ }\bar{b} | \lg(\bar{a}) = \lg(\bar{b}) \wedge \bar{b} \in R^{\mathfrak{A}}_{\bar{a}}\}$. Observe that:

- $X_{\mathfrak{A}} \in \mathbf{V}'$, as $X_{\mathfrak{A}} \subseteq {}^{\omega>}k$.
- $\mathfrak{A}$ can be absolutely defined using $X_{\mathfrak{A}}$ and s: its universe is s, and for each $\bar{a} \in {}^{\omega>}s$, $R_{\bar{a}} = \{\bar{b} | \lg(\bar{b}) = \lg(\bar{a}) \wedge \bar{a}\hat{}\bar{b} \in X_{\mathfrak{A}}\}$.

So in conclusion, $\mathfrak{A} \in \mathbf{V}'$, and so $G \in \mathbf{V}'$ as before. In addition $H \in \mathbf{V}'$, because H is the image of a function in ${}^{k}({}^{s}s)$. But there is an absolute definable bijection ${}^{k\times s}s \to {}^{k}({}^{s}s)$, and ${}^{k\times s}s \subseteq \mathcal{P}(k \times k \times k) \in \mathbf{V}'$. By definition $(G, H) \in (\mathcal{A}_k)^{\mathbf{V}'}$ and we are done. □

3 The automorphism tower without Choice

Definition 3.1 For a centerless group G, we define the sequence $\langle G^\alpha | \alpha \in \mathbf{ord}\rangle$:

- $G^0 = G$.
- $G^{\alpha+1} = \mathrm{Aut}(G^\alpha)$
- $G^\delta = \bigcup\{G^\alpha | \alpha < \delta\}$ for δ limit.

Remark 3.2 By the standard definition of $\mathrm{Aut}(G)$, we do not have that $G \subseteq \mathrm{Aut}(G)$. However, recall that $\mathrm{Inn}(G)$ is the group of all inner automorphisms of G, i.e. conjugations by elements of G. Since G is centerless, this definition makes sense—$G \cong \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$, and $\mathrm{Aut}(G)$ is again without center. So we can uniformly identify $\mathrm{Inn}(G)$ with G, and so $G^\alpha \leq G^{\alpha+1}$. This sequence is therefore monotone and continuous.

Definition 3.3

(1) Define the ordinal τ_G by $\tau_G = \min\{\alpha | G^{\alpha+1} = G^\alpha\}$. We shall show below that τ_G is well defined.
(2) For a set k, we define $\tau_{|k|}$ to be the smallest ordinal α such that $\alpha > \tau_G$ for all groups G with power $\leq |k|$.
(3) For a cardinal number κ (i.e. some $\aleph$), define τ_κ similarly.

Definition 3.4 For a group G (not necessarily centerless) and a subset A, we define an equivalence relation $E_{G,A}$ on G by $x E_{G,A} y$ iff $\mathrm{tp}_{\mathrm{qf}}(x, A, G) = \mathrm{tp}_{\mathrm{qf}}(y, A, G)$ where $\mathrm{tp}_{\mathrm{qf}}(x, A, G) =$

$$\{\sigma(z, \bar{a}) |\ \bar{a} \in {}^{n}A, n < \omega, \sigma \text{ a term in the language of groups (i.e. a word) with parameters from } A, z \text{ is its only free variable and } G \models \sigma(x, \bar{a}) = e\}$$

Remark 3.5

(1) Note that $x E_{G,A} y$ iff there is an isomorphism between $\langle A, x\rangle_G$ and $\langle A, y\rangle_G$ taking x to y and fixing A pointwise.
(2) The relation $E_{G,A}$ is definable and absolute (since $\mathrm{tp}_{\mathrm{qf}}(x, A, G)$ is absolute).

Definition 3.6 We say (G, A) is a special pair if $A \subseteq G$, G is a group and $E_{G,A} = \{(x, x) | x \in G\}$ (i.e. the equality).

Example 3.7

(1) If $G = \langle A \rangle_G$ then (G, A) is special.
(2) If $A \trianglelefteq G$ and $C_G(A) = \{e\}$ then (G, A) is special (because for all $g \in G$, $\langle gag^{-1} | a \in A \rangle$ determines g).
(3) By (2) and 3.9 (see below), if G is centerless then $(\mathrm{Aut}(G), G)$ is special. So in general, the converse of (1) is not true.
(4) There is a group G with center such that $(\mathrm{Aut}(G), \mathrm{Inn}(G))$ is special, e.g. $\mathbb{Z}/2\mathbb{Z}$, but
(5) If G is not centerless then (3) is not necessarily true, even if $|Z(G)| = 2$:
It is enough to find a group which satisfies the following properties:
    (a) $Z(G) = \{a, e\}$ where $a \neq e$.
    (b) $H_i \leq G$ for $i = 1, 2$ are two different subgroups of index 2.
    (c) $Z(G) = Z(H_i)$ for $i = 1, 2$
Let π be the homomorphism $\pi : G \to \mathrm{Aut}(G)$ taking g to i_g (where $i_g(x) = gxg^{-1}$). Then $\mathrm{Inn}(G) = \mathrm{image}(\pi)$. We wish to find $x_1 \neq x_2 \in \mathrm{Aut}(G)$ with $x_1 E_{\mathrm{Aut}(G),\mathrm{Inn}(G)} x_2$. So define $x_i(g) = \begin{cases} ag & g \notin H_i \\ g & g \in H_i \end{cases}$. Since $x_i^2 = \mathrm{id}$, $x_i \pi(g) x_i^{-1} = \pi(x_i(g)) = \pi(g)$ and the fact that $x_i \notin \mathrm{Inn}(G)$ (because $Z(G) = Z(H_i)$) it follows that $\mathrm{tp}_{\mathrm{qf}}(x_1, \mathrm{Inn}(G), \mathrm{Aut}(G)) = \mathrm{tp}_{\mathrm{qf}}(x_2, \mathrm{Inn}(G), \mathrm{Aut}(G))$. Now we have to construct such a group. Notice that it is enough to find a centerless group satisfying only the last two properties, since we can take its product with $\mathbb{Z}/2\mathbb{Z}$. So take $G = D_\infty = \langle a, b | a^2 = b^2 = e \rangle$, and $H_a = \ker \varphi_a$ where $\varphi_a : G \to \mathbb{Z}/2\mathbb{Z}$ takes a to 1 and b to 0. In the same way we define H_b, and finish.

The following is the crucial claim:

Claim 3.8 Assume $G_1 \trianglelefteq G_2$, $C_{G_2}(G_1) = \{e\}$ and that (G_1, A) is a special pair. Then (G_2, A) is a special pair.

Proof First we show that $C_{G_2}(A) = \{e\}$. Suppose that $x \in C_{G_2}(A)$, so $xax^{-1} = a$ for all $a \in A$. Since conjugation by x (i.e. the map $h \mapsto xhx^{-1}$ in G_1) is an automorphism of G_1, (as G_1 is a normal subgroup of G_2), it follows from (G_1, A) being a special pair (by Remark 3.5, Clause (1)) that it must be id. Hence, $x \in C_{G_2}(G_1)$, but we assumed $C_{G_2}(G_1) = \{e\}$ and hence $x = e$.

Next assume that $x E_{G_2,A} y$ where $x, y \in G_2$ and we shall prove $x = y$. There is an isomorphism $\pi : \langle x, A \rangle_{G_2} \to \langle y, A \rangle_{G_2}$ taking x to y and fixing A pointwise. We wish to show that $x = y$, so it is enough to show that $x^{-1}\pi(x) \in C_{G_2}(A)$. This is equivalent to showing $x^{-1}\pi(x)a\pi(x^{-1})x = a$, i.e. $x^{-1}\pi(xax^{-1})x = a$, i.e. $\pi(xax^{-1}) = xax^{-1}$ (remember that $\pi(a) = a$) for every $a \in A$. But xax^{-1} is an element of G_1 (as $G_1 \trianglelefteq G_2$), and $\pi : \langle xax^{-1}, A \rangle_{G_1} \to \langle \pi(xax^{-1}), A \rangle_{G_1}$ must be id because (G_1, A) is a special pair, and we are done. □

Note 3.9 If G is centerless then $G \trianglelefteq \mathrm{Aut}(G)$, and $C_{\mathrm{Aut}(G)}(G) = \{e\}$.

Conclusion 3.10 Assume G is centerless and (G, A) is a special pair. Then:

(1) (G^α, A) is a special pair for every $\alpha \in \mathbf{ord}$.
(2) $C_{G^\alpha}(A) = \{e\}$ for every α.

Proof (2) follows from (1). (why? by the first stage in the proof of Claim 3.8 with $G_1 = G_2$). Prove (1) by induction on α. For limit ordinal, its clear from the definitions, and for successors, the previous claim finishes the job using the above note. □

Conclusion 3.11 Let γ be an ordinal, G a centerless group. Then:

(1) $C_{G^\gamma}(G) = \{e\}$.
(2) $\mathrm{nor}_{G^\gamma}(G^\beta) = G^{\beta+1}$, for $\beta < \gamma$.
(3) $\mathrm{nor}^\beta_{G^\gamma}(G) = G^\beta$ for $\beta \le \gamma$.

Proof (1) Follows from Conclusion 3.10 and from the fact that (G, G) is a special pair.
(2) The direction $\mathrm{nor}_{G^\gamma}(G^\beta) \ge G^{\beta+1}$ is clear from the definition of the action of $G^{\beta+1}$ on G^β. The direction $\mathrm{nor}_{G^\gamma}(G^\beta) \le G^{\beta+1}$ follows from the previous clause: suppose $y \in \mathrm{nor}_{G^\gamma}(G^\beta)$, so conjugation by y is in $\mathrm{Aut}(G^\beta)$. By definition there is $z \in G^{\beta+1}$ such that $yxy^{-1} = zxz^{-1}$ for all $x \in G^\beta$, in particular—for all $x \in G$, So $y = z$ (by (1))
(3) By induction on β.

□

Claim 3.12 If G is centerless and (G, A) is a special pair then:

(1) (ZFC) $|G^\alpha| \le 2^{|A|+\aleph_0}$ for all ordinals α.
(2) (ZF) There is a one to one absolutely definable (with parameters G^α, A, and at most 2 distinguished elements from G) function from G^α into $\mathcal{P}({}^{\omega>}A)$ for each ordinal α.

Proof (1) follows from (2). The natural way to define the function f is $f(g) = \mathrm{tp}_{\mathrm{qf}}(g, A, G^\alpha)$, which is a set of equations. Luckily it is easy to encrypt equations as elements of ${}^{\omega>}A$: We can assume that there are at least two elements in A—a, b (if not, $G = \{e\}$ because $C_G(A) = \{e\}$). Let $\sigma(z, \bar{c})$ be a word over A (i.e. $\bar{c}$ is a sequence in A), so it is of the form $\ldots z^{m_i} c_i^{n_i} z^{m_{i+1}} c_{i+1}^{n_{i+1}} \ldots$ where $n_i, m_i \in \mathbb{Z}$, $c_i \in \bar{c}$, and $i = 0, \ldots, k-1$. First we code the exponents sequence with a natural number, m, using the bijection $cd : {}^{\omega>}\omega \to \omega$, and then we code the sequence of indices where z appears, call it l. Then we encrypt σ by $a^l\hat{\ }b\hat{\ }a^m\hat{\ }b$ and after that—the list of elements of A in σ by order of appearance.
Note that our function is definable as promised. □

Claim 3.13 If G is centerless then:

(1) (ZFC) If $|G^\alpha| \le \lambda$ for all ordinals α, then $\tau_G < \lambda^+$.
(2) (ZF) If $|G^\alpha| \le |A|$ for all ordinals α and a set A, then $\tau_G < \theta_A$. It is enough to assume that there is a function from A onto G^α for each ordinal α.

Proof (1) follows from (2), but with Choice, it is much simpler—$G_{\lambda^+} = \bigcup\{G_\alpha | \alpha < \lambda^+\}$. Since $|G_{\lambda^+}| \le \lambda$ and $\langle G_\alpha \rangle$ is increasing, it follows that there must be some

$\alpha < \lambda^+$ such that $G_\alpha = G_{\alpha+1}$.
(2): Let $\sigma = \theta_A$. Assume that for all $\alpha < \sigma$, $G^\alpha \neq G^{\alpha+1}$. Then the function $f : G^\sigma \to \sigma$ defined by $f(g) = \alpha$ iff $g \in G^{\alpha+1} \backslash G^\alpha$ is onto σ. So $\theta_{G^\sigma} > \sigma$, but by assumption $|G^\sigma| \leq |A|$, so $\theta_A \geq \theta_{G^\sigma} > \sigma = \theta_A$—a contradiction. □

So as promised, here is Thomas' theorem proved in a different way, without Choice:

Conclusion 3.14 (ZFC) Thomas' theorem: if G is a centerless group then $\tau_G < (2^{|G|})^+$. Moreover, $\tau_\kappa < (2^\kappa)^+$.

Proof Taking $A = G$, so that (G, A) is a special pair, applying 3.12 and 3.13 we get the result regarding τ_G. Noting that $(2^\kappa)^+$ is regular and that there are, up to isomorphism, at most 2^κ groups of order κ we are done. □

Now we deal with the case without Choice.

Main theorem 3.15 (ZF) *If (G, A) is a special pair and G is a centerless group, then $\tau_G < \theta_{\mathcal{P}(^{\omega>}A)}$.*

Proof By Claim 3.13, Clause (2), we only need to show that $|G^\alpha| \leq |\mathcal{P}(^{\omega>}A)|$, but this is exactly Claim 3.12, Clause (2). □

Now we shall improve this by:

Main theorem 3.16 (ZF) *For every set k, $\tau_{|k|} < \theta_{\mathcal{P}(^{\omega>}k)}$.*

Proof We may assume $2 \leq |k|$. Recall that
$\tau_{|k|} = \bigcup\{\tau_G + 1 | G$ is centerless and $|G| \leq |k|\}$, but we can replace this by
$\tau_{|k|} = \bigcup\{\tau_G + 1 | G \in \mathcal{G}\}$ where
$\mathcal{G} = \{G | G$ is centerless and $G \subseteq k\}$. By the previous Theorem (3.15), we know that $\tau_{|k|} \leq \theta_{\mathcal{P}(^{\omega>}k)}$ (because for all $G \in \mathcal{G}$, as (G, G) is a special pair (see 3.6), $\tau_G < \theta_{\mathcal{P}(^{\omega>}G)} \leq \theta_{\mathcal{P}(^{\omega>}k)}$), but we want a strict inequality.
Let $\tau^-_{|k|} = \bigcup\{\tau_G | G \in \mathcal{G}\}$, clearly $\tau_{|k|} \leq \tau^-_{|k|} + 1$, and since $\theta_{\mathcal{P}(^{\omega>}k)} > \aleph_0$ (see Remark 2.5), it is enough to prove $\tau^-_{|k|} < \theta_{\mathcal{P}(^{\omega>}k)}$.
For each $G \in \mathcal{G}$ we define a function $R_G : \mathcal{P}(^{\omega>}k) \to \tau_G$ which is onto: first we define a function from $\mathcal{P}(^{\omega>}k)$ onto G^{τ_G} (using Claim 3.12), then from G^{τ_G} onto τ_G (using Claim 2.6, and Claim 3.11).
Let $\mathcal{B} = \{(x, G) | G \in \mathcal{G}, x \in \mathcal{P}(^{\omega>}k)\}$. Define a function $R_1 : \mathcal{B} \to \tau^-_{|k|}$ by $R((x, G)) = R_G(x)$, (note—since R_G is definable, there is no use of Choice). By definition, R_1 is onto. Now it is enough to find an injective function $R_2 : \mathcal{B} \to \mathcal{P}(^{\omega>}k)$.
A group $G = \langle |G|, \cdot, \square^{-1} \rangle$ is a triple of nonempty subsets of $^{\omega>}k$ ($|G|$ is the universe set of G). As we already mentioned (see the proof of 2.7), $|^{\omega>}\mathcal{P}(^{\omega>}k)| = |\mathcal{P}(^{\omega>}k)|$ (as $2 \leq |k|$), and we are done. □

We postpone the proof of the following absoluteness lemma to the appendix.

Lemma 3.17 *Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive class which is a model of ZF. Let (G, A) be a special pair, and suppose $G, \mathcal{P}(^{\omega>}A) \in \mathbf{V}'$. Then, for every ordinal $\delta \in \mathbf{V}'$, the automorphism tower $\langle G^\beta | \beta < \delta \rangle$ in $\mathbf{V}'$ is the same in $\mathbf{V}$ (i.e. $\mathbf{V} \models$" $\langle G^\beta | \beta < \delta \rangle$ is the automorphism tower up to δ").*

Using this lemma, we can finally deduce the following theorem.

Theorem 3.18 (1) *Let* $\mathbf{V}' \subseteq \mathbf{V}$ *a transitive subclass, which is a model of* ZF. *If* $\mathcal{P}({}^{\omega>}k) \in \mathbf{V}'$, *then* $(\tau_{|k|})^{\mathbf{V}} = (\tau_{|k|})^{\mathbf{V}'} < \theta^{\mathbf{V}'}_{\mathcal{P}({}^{\omega>}k)}$.

(2) *If* κ *is a cardinal number in* $\mathbf{V}'$ *such that* $\mathcal{P}(\kappa) \in \mathbf{V}'$, *then* $(\tau_\kappa)^{\mathbf{V}} = (\tau_\kappa)^{\mathbf{V}'} < \theta^{\mathbf{V}'}_{\mathcal{P}(\kappa)}$.

(3) *In particular,* $\tau_{\aleph_0} < \theta^{L(\mathbb{R})}_{\mathbb{R}}$.

Proof Obviously, we need only to see (1). Let
$\mathcal{G} = \{G | G$ is a centerless group and $G \subseteq k\}$. By the assumption on k, it is easy to see that $\mathcal{G}^{\mathbf{V}} = \mathcal{G}^{\mathbf{V}'}$. Hence $\tau^{\mathbf{V}}_{|k|} = \bigcup\{\tau_G + 1 | G \in \mathcal{G}^{\mathbf{V}}\} = \bigcup\{\tau_G + 1 | G \in \mathcal{G}^{\mathbf{V}'}\} = \tau^{\mathbf{V}'}_{|k|}$ (the second equality is Lemma 3.17). By Theorem 3.16, we have $\tau^{\mathbf{V}'}_{|k|} < \theta^{\mathbf{V}'}_{\mathcal{P}({}^{\omega>}k)}$. □

If we apply Lemma 1.8 from [1], which says that $\tau^{\mathrm{nlg}}_\kappa \leq \tau_\kappa$ and get:

Main Conclusion 3.19 Let $\mathbf{V}' \subseteq \mathbf{V}$ be as before (but now assume $\mathbf{V} \models ZFC$). If $\mathcal{P}({}^{\omega>}k) \in \mathbf{V}'$, then $\tau^{\mathrm{nlg}}_{|k|} \leq \tau_{|k|} < \theta^{\mathbf{V}'}_{\mathcal{P}({}^{\omega>}k)}$.

Note 3.20 We actually don't need to assume that $\mathbf{V}$ is a model of ZFC and we address this subject in [5]. For a cardinal number κ, we show that $\tau^{\mathrm{nlg}}_\kappa \leq \tau_\kappa$ is true even without Choice, but for a general k, we get $\tau^{nlg'}_{|k^{\langle<\omega\rangle}|} \leq \tau_{|k^{\langle<\omega\rangle}|}$ (see the definitions there).

4 The descriptive set theoretic approach

In this short section we give a descriptive set theoretic approach into finding a bound on $\tau_{\aleph_0}$. We start with the definition.

Definition 4.1 Let $\mathfrak{A}$ be structure with universe $A = |\mathfrak{A}|$.

(1) For a formula $\varphi(x, X)$—a first order formula in the language of $\mathfrak{A}$, where x is a single variable and X is a monadic variable (i.e. serves as a unary predicate—varies on subsets of the structure, so not quantified inside the formula)—we define a sequence $\langle X^\varphi_\alpha \subseteq A | \alpha \in \mathbf{Ord}\rangle$ by:
- $X^\varphi_0 = \emptyset$.
- $X^\varphi_{\alpha+1} = X^\varphi_\alpha \cup \{x \in A | \varphi(x, X^\varphi_\alpha)$ is satisfied in $\mathfrak{A}\}$.
- $X^\varphi_\delta = \bigcup\{X^\varphi_\beta | \beta < \delta\}$ for δ limit.

(2) For such a formula φ, let $\delta_\varphi = \min\{\alpha | X^\varphi_\alpha = X^\varphi_{\alpha+1}\}$.

(3) Let $\delta = \delta(\mathfrak{A})$—the inductive ordinal of the structure—be the first ordinal such that for any such formula φ (allowing members of A as parameters), $\delta_\varphi < \delta$.

For more on this subject see [2], and for more on descriptive set theory, see [3].

Theorem 4.2 *For a centerless group* G *with set of elements* ω *the height of its automorphism tower is smaller than the inductive ordinal of the structure* $\mathfrak{A}$ *with universe* $\omega \cup \mathcal{P}(\omega)$ *the operations of* $\mathbb{N}$, *membership, and* G (*i.e. its product and inverse*).

Remark 4.3 In this version of the theorem we do not need to use parameters in Definition 4.1. However the theorem holds even without assuming that the structure contains G, but then we need parameters (G (as a group) can be encoded as a subset of ω). In that case this is second order number theory. Hence the conclusion is:

Conclusion 4.4 If $\mathfrak{A}$ is the standard model of second order number theory (as above), then $\tau_{\aleph_0} \leq \delta(\mathfrak{A})$.

Remark 4.5 Starting with any group G, we should look at the natural structure with universe $\bigcup\{{}^nG|n < \omega\} \cup \mathcal{P}(\bigcup\{{}^nG|n < \omega\})$, equipped with the group operation and natural "set theoretic" operations.

Proof (of the theorem; sketch) By the definition it is enough to find a formula Δ such that X_α^Δ encodes G^α (including its multiplication and inverse). By (G, G) being special, we know that we can identify members of G^α with sets of finite sequences of ω (see the proof of Claim 3.12). It is well known that the operations of $\mathbb{N}$ allow us to encode finite sequences, and in this structure we can encode finite sequences of subsets of ω.
Hence, much like the proof of Lemma 3.17 (in the appendix—it is advised to read it in order to understand this proof), we can find three formulas as in Definition 4.1—$\Delta'(x, X)$, $\Delta''(x, y, X)$ and $\Delta'''(x, y, z, X)$ such that if X_α^Δ encodes G^α then:

- x satisfies $\Delta'(x, X_\alpha^\Delta)$ iff x encodes a quantifier free type of an element in $G^{\alpha+1}$.
- x, y satisfies $\Delta''(x, y, X_\alpha^\Delta)$ iff $x, y \in G^{\alpha+1}$ and $x \circ y = \mathrm{id}$.
- x, y, z satisfies $\Delta'''(x, y, z, X_\alpha^\Delta)$ iff $x \circ y = z$.

Define $\Delta(x, X_\alpha^\Delta)$ to say that x encodes a triple (a, b, c) where $a \in G^{\alpha+1}$, b encodes a pair (d, d^{-1}) where $d \in G^{\alpha+1}$ and c encodes a triple $(e, f, e \circ f)$ where e and f are from $G^{\alpha+1}$. Now we have successfully encoded $G^{\alpha+1}$ as required. □

5 A relative of the main theorem

Here we improve the main theorem by considering pairs (G, A) that satisfy a weaker condition than being special. Namely, we find a bound on τ_G for centerless groups G with a subset A such that (G, A) is weakly special. This bound, when interpreted in ZFC, is the same bound as one gets using Thomas' proof from [7].

Definition 5.1

(1) For a centerless group G, and subgroups H_1, H_2, we say that a homomorphism (really a monomorphism) $\varphi : H_1 \to H_2$ is good if there is an automorphism $\psi : G^{\tau_G} \to G^{\tau_G}$ (so actually an inner automorphism of G^{τ_G}) such that $\varphi = \psi \restriction H_1$.
(2) If $A \subseteq G$, let $E_{G,A}^k$ be the equivalence relation on G defined by: $x E_{G,A}^k y$ iff there is a good homomorphism taking x to y and fixing A pointwise.
(3) We say that the pair (G, A) is weakly special if $E_{G,A}^k$ is $\{(x, x)|x \in G\}$.

Remark 5.2 If $x E_{G,A}^k y$ then also $x E_{G,A} y$ but not necessarily the other direction, and so if (G, A) is special, it is also weakly special (so the name is justified).

Claim 5.3 If G is centerless, $G^1 = \mathrm{Aut}(G)$, and (G, A) is weakly special, then so is (G^1, A).

Proof The proof is identical to the proof of 3.8, since conjugation is a good homomorphism, $G \trianglelefteq G^1$ and $G^{\tau_G} = (G^1)^{\tau_{G^1}}$. □

And much like Conclusion 3.10 we have:

Conclusion 5.4 If G is a centerless group and (G, A) is (weakly) special then so is (G^α, A) for every ordinal α, and $C_{G^\alpha}(A) = \{e\}$.

The converse is true as well:

Claim 5.5 If for every ordinal α, $C_{G^\alpha}(A) = \{e\}$ then (G, A) is weakly special (so one can take this as the definition).

Proof Suppose φ is a good homomorphism taking x to y and fixing A pointwise. Let $\psi \in \mathrm{Aut}(G^{\tau_G})$ be such that $\psi \restriction G = \varphi$. Then, by definition, ψ has to be conjugation by some element of G^{τ_G}, and by assumption $C_{G^{\tau_G}}(A) = \{e\}$; hence $\psi = \mathrm{id}$. □

Definition 5.6 Denote by $\mathrm{PF}(A, B)$ the set of all partial functions from A to B (i.e. such that the domain is a subset of A).

Definition 5.7 Say that a set A is pseudo finite if there is no function from A onto ω (i.e. $\theta_A \leq \omega$). Obviously, if A is finite, it's also pseudo finite.

Definition 5.8 Let $w_0(x) = x$, $w_{n+1}(x, \langle y_i | i < n+1\rangle) = w_n(xy_nx^{-1}, \langle y_i | i < n\rangle)$.

Definition 5.9 Call a tuple $(G, A, B, \mathbf{h})$ 2-special if

(1) (G, A) is weakly special.
(2) $A \subseteq B \subseteq G$.
(3) $\mathbf{h}$ is a function, with domain G and if A is pseudo finite then $\mathbf{h} : G \to \bigcup\{\mathrm{PF}({}^{n>}A, B) | n < \omega\}$, and if not, then $\mathbf{h} : G \to \mathrm{PF}({}^{\omega>}A, B)$.
(4) If $g \in G$ and $\bar{a} \in \mathrm{dom}(\mathbf{h}(g))$, then $w_{\lg(\bar{a})}(g, \bar{a}) = \mathbf{h}(g)(\bar{a})$.
(5) If $g \in G$, $g' \in G^{\tau_G}$ and $w_{\lg(\bar{a})}(g', \bar{a}) = \mathbf{h}(g)(\bar{a})$ for all $\bar{a} \in \mathrm{dom}(\mathbf{h}(g))$, then $g = g'$.

Remark 5.10 (1) If $(G, A, B, \mathbf{h})$ is 2-special then by (4) and (5) $\mathbf{h}$ is injective.
(2) So we characterize every $g \in G$ by the set of special equations (with parameters in B) it satisfies. The advantage over 3.12 is that we use fewer equations.

Claim 5.11 Suppose that $(G, A, B, \mathbf{h})$ is 2-special, then there is some $\mathbf{h}_1$ extending $\mathbf{h}$ such that $(\mathrm{Aut}(G), A, B, \mathbf{h}_1)$ is 2-special. Moreover, the function $(G, A, B, \mathbf{h}) \mapsto \mathbf{h}_1$ is definable.

Proof Recall that $G^1 = \mathrm{Aut}(G)$. As (G, A) is weakly special, $C_{G^1}(A) = \{e\}$. Define $\mathbf{h}_1$ as follows:

$\mathbf{h}_1 \restriction G = \mathbf{h}$. For $g \in G^1 \backslash G$, and $a \in A$, let $f_{g,a} = \mathbf{h}(gag^{-1})$ (as $G \trianglelefteq G^1$, this is well defined). Let $\mathrm{dom}(\mathbf{h}_1(g)) = \{\bar{b}\hat{\ }a | \bar{b} \in \mathrm{dom}(f_{g,a}), a \in A\}$, and

$$\mathbf{h}_1(g)(\bar{b}\hat{\ }a) = f_{g,a}(\bar{b}).$$

Now we check that the definition holds: (1) holds by 5.3. (2) is obvious.
(3): Obviously $\mathrm{dom}(\mathbf{h}_1(g)) \subseteq {}^{\omega>}A$, so if A is not pseudo finite we are done. In the case where A is pseudo finite, there is some $n = n(g)$ such that $\mathrm{dom}(f_{g,a}) \subseteq {}^{n>}A$ for all $a \in A$ (otherwise, the function $a \mapsto \min\{n < \omega \mid \mathrm{dom}(f_{g,a}) \subseteq {}^{n>}A\}$ is onto an unbounded subset of ω, and for every such subset there is a function from it onto ω). Hence $\mathrm{dom}(\mathbf{h}_1(g)) \subseteq {}^{n+1>}A$, so $\mathbf{h}_1 : G^1 \to \bigcup\{\mathrm{PF}({}^{n>}A, B) \mid n < \omega\}$.
(4): For $g \in G$, since $\mathbf{h}_1(g) = \mathbf{h}(g)$, there is nothing to prove. Suppose $g \in G^1 \backslash G$ and $\bar{b}\hat{\ }a \in \mathrm{dom}(\mathbf{h}_1(g))$ (recall that by definition, the length of an element from $\mathrm{dom}(\mathbf{h}_1(g))$ is not zero). Let $n = \lg(\bar{b})$. We have

- $w_{n+1}(g, \bar{b}\hat{\ }a) = w_n(gag^{-1}, \bar{b})$ by the definition of w_{n+1}.
- $w_n(gag^{-1}, \bar{b}) = \mathbf{h}(gag^{-1})(\bar{b}) = f_{g,a}(\bar{b})$ by (4) and the choice of $f_{g,a}$.
- $f_{g,a}(\bar{b}) = \mathbf{h}_1(g)(\bar{b}\hat{\ }a)$ by the definition of $\mathbf{h}_1$.

So, $w_{n+1}(g, \bar{b}\hat{\ }a) = \mathbf{h}_1(g)(\bar{b}\hat{\ }a)$.

(5): Suppose $g \in G^1, g' \in G^{\tau_G}$, and $w_{\lg(\bar{a})}(g', \bar{a}) = \mathbf{h}_1(g)(\bar{a})$ for all $\bar{a} \in \mathrm{dom}(\mathbf{h}(g))$. If $g \in G$, then since $\mathbf{h}_1(g) = \mathbf{h}(g)$, $g' = g$ by assumption. Suppose $g \in G^1 \backslash G$. In this case, we have that for all $a \in A$ and $\bar{b} \in \mathrm{dom}(f_{g,a})$,

$$
\begin{aligned}
w_{\lg(\bar{b})}(g'a(g')^{-1}, \bar{b}) &= w_{\lg(\bar{b}\hat{\ }a)}(g', \bar{b}\hat{\ }a) = \mathbf{h}_1(g)(\bar{b}\hat{\ }a) \\
&= f_{g,a}(\bar{b}) = \mathbf{h}(gag^{-1})(\bar{b}).
\end{aligned}
$$

(Why? the first equality is by definition of w_n, the second by assumption, the third by definition of $\mathbf{h}_1$, and the fourth by the choice of $f_{g,a}$).
So $w_{\lg(\bar{b})}(g'a(g')^{-1}, \bar{b}) = \mathbf{h}(gag^{-1})(\bar{b})$ for all $\bar{b} \in \mathrm{dom}(\mathbf{h}(gag^{-1}))$. By (5), $g'a(g')^{-1} = gag^{-1}$ for all $a \in A$. By (1), and by 5.4, $g = g'$. □

Claim 5.12 Let (G, A) be weakly special. The function $\mathbf{h}$ defined by $\mathbf{h}(g) = \{(\langle\rangle, g)\}$ witnesses that $(G, A, G, \mathbf{h})$ is 2-special.

Proof Checking the definition, all clauses are trivial, for example, (4): for any $g \in G$, $w_0(g) = \mathbf{h}(g)(\langle\rangle) = g$. □

Conclusion 5.13 Assume (G, A) is weakly special. Then for all $\alpha \leq \tau_G$, there is a function $\mathbf{h}_\alpha$ (in fact defined uniformly) that shows that $(G^\alpha, A, G, \mathbf{h}_\alpha)$ is 2-special.

Proof By induction on α we define $\mathbf{h}_\alpha$ such that for $\beta < \alpha$, $\mathbf{h}_\beta \subseteq \mathbf{h}_\alpha$. For $\alpha = 0$, this is exactly the previous claim. For $\alpha = \beta + 1$, this is the definable version of 5.11. For α limit, let $\mathbf{h}_\alpha = \bigcup\{\mathbf{h}_\beta \mid \beta < \alpha\}$. It is easy to see that Definition 5.9 holds. □

We conclude with:

Theorem 5.14 *If (G, A) is a weakly special pair, then*:

(1) (ZF) *If A is pseudo finite, $\tau_G < \theta_{\bigcup\{\mathrm{PF}({}^{n>}A, G) \mid n<\omega\}}$, and if not, $\tau_G < \theta_{\mathrm{PF}({}^{\omega>}A, G)}$.*
(2) (ZF) *If A is finite, $\tau_G < \theta_{\bigcup\{{}^n(G \cup \{g\}) \mid n<\omega\}}$ where $g \notin G$. If, moreover, there is a function from ω onto G, then $\tau_G < \theta_\omega = \aleph_1$.*
(3) (ZFC) $\tau_G < (|G|^{|A|} + \aleph_0)^+$.

Proof (3) follows from (1) by classical cardinal arithmetics (recall that $|\operatorname{PF}(A, B)| = |B|^{|A|}$, provided that $2 \le |B|$).
To prove (1), we use Claim 3.13, Clause (2): from 5.13, we know that for all α, if A is pseudo finite, then $|G^\alpha| \le |\bigcup\{\operatorname{PF}({}^{n>}A, G) | n < \omega\}|$, and if not, $|G^\alpha| \le |\operatorname{PF}({}^{\omega>}A, G)|$ as witnessed by $\mathbf{h}_\alpha$.
(2): $|\operatorname{PF}(A, B)| = |{}^A(B \cup \{b\})|$, where $b \notin B$ so if A is finite, since $|{}^{\omega>}A| = |\omega|$, $\tau_G < \theta_{\bigcup\{{}^n(G\cup\{g\})|n<\omega\}}$ where $g \notin G$. If there is a function from ω onto G, there is such a function from ω onto $G \cup \{g\}$. Since $|\omega| = |{}^{\omega>}\omega|$, there is a function from ω onto $\bigcup\{{}^n(G \cup \{g\}) | n < \omega\}$, and we are done. □

The ZFC version of 5.14 is not really new, although it is not mentioned explicitly in [7]: one can prove it using a slight modification of the proof there (i.e. using Fodor).

From (2) above, we easily get:

Conclusion 5.15 If G is (pseudo) finitely generated then $\tau_G < \theta_\omega = \aleph_1$.

The case where G is finitely generated is interesting, also due to the fact that the tower is absolute:

Lemma 5.16 *Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive class which is a model of ZF. Let G be a centerless group, finitely generated by A. Then, for every ordinal $\delta \in \mathbf{V}'$, the automorphism tower $\langle G^\beta | \beta < \delta\rangle$ in $\mathbf{V}'$ is the same in $\mathbf{V}$ (i.e. $\mathbf{V} \models$ "$\langle G^\beta | \beta < \delta\rangle$ is the automorphism tower up to δ").*

We prove this lemma in the appendix. In conclusion, we have:

Conclusion 5.17 If G is finitely generated then $\tau_G < \aleph_1^L$.

Comparing Theorem 5.14 with Theorem 3.15:
First of all, the condition—(G, A) is weakly special—is weaker than (G, A) is special (note that specialty is absolute, while weak specialty is not) so 5.14 is stronger in that aspect. In 3.15, G does not appear in the bound, only A, so the bounds are not directly comparable. However, the bound in the last theorem is better in the case where A is finite. If $G = A$, then the theorems are the same, because $|\operatorname{PF}({}^{\omega>}G, G)| = |\mathcal{P}({}^{\omega>}G)|$.

6 Appendix: Absoluteness lemmas

Here we shall prove the absoluteness lemmas (Lemma 3.17, 5.16).

Lemma 6.1 *Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive class which is a model of ZF. Let (G, A) be a special pair, and suppose $G, \mathcal{P}({}^{\omega>}A) \in \mathbf{V}'$. Then, for every ordinal $\delta \in \mathbf{V}'$, the automorphism tower $\langle G^\beta | \beta < \delta\rangle$ in $\mathbf{V}'$ is the same in $\mathbf{V}$ (i.e. $\mathbf{V} \models$ "$\langle G^\beta | \beta < \delta\rangle$ is the automorphism tower up to δ").*

Proof Let $\mathfrak{T} = \langle G^\beta | \beta \in \mathbf{ord}^{\mathbf{V}'}\rangle$. We shall prove by induction on $\alpha < \delta$ that $\mathfrak{T} \restriction \alpha + 1$ is the automorphism tower in $\mathbf{V}$ up to $\alpha + 1$.
For $\alpha = 0$ this is clear since $G \in \mathbf{V}'$.
For α limit this follows from the definitions.

Suppose $\alpha = \beta + 1$. By the induction hypothesis $\mathfrak{T} \restriction \alpha$ is the automorphism tower up to α in $\mathbf{V}$, so $(G^\beta)^{\mathbf{V}} = (G^\beta)^{\mathbf{V}'} \in \mathbf{V}'$. For every $\rho \in \mathrm{Aut}(G^\beta) = G^\alpha$ in $\mathbf{V}$, we need to show that $\rho \in (G^\alpha)^{\mathbf{V}'}$.

A short explanation of what follows: by our assumption, the set of quantifier free types over A is in $\mathbf{V}'$. To show that ρ is in $\mathbf{V}'$, we would like to identify, in an absolute way, its quantifier free type over A. In order to do that, we identify small pieces of the action of ρ on G^β: for each $h \in G^\beta$, we find what is $\rho \restriction \langle A \cup \{h\} \rangle$, by describing the quantifier free type of $\langle \rho, h \rangle$ in G^α over A (i.e. a type in 2 variables). These types amount to normal subgroups of $A * \langle x, y \rangle$. After describing the restrictions of ρ to $\langle A \cup \{h\} \rangle$ for all h, we demand that they agree on their common domains, and this allows us to define ρ as their union.

Without loss of generality A is a subgroup of G—if not, replace it with $\langle A \rangle_G$ (we can define a function from ${}^{\omega >}A$ onto ${}^{\omega >}\langle A \rangle_G$ as in Claim 3.12). Let $\mathbf{A} = A * \langle x \rangle$ i.e. the free product of A and the infinite cyclic group. As in 3.12 there is an absolute definable function from ${}^{\omega >}A$ onto $\mathbf{A}$, so $\mathcal{P}(\mathbf{A}) \in \mathbf{V}'$. Let $\mathbf{B} = A * \langle x, y \rangle$, and by the same reasoning $\mathcal{P}(\mathbf{B}) \in \mathbf{V}'$.

For every $g \in G^\alpha$, there is a homomorphism φ_g from $\mathbf{A}$ onto $\langle A \cup \{g\} \rangle_{G^\alpha}$ defined by $x \mapsto g$, and fixing A pointwise. By 3.10 ((G^α, A) is special), $g \mapsto \ker(\varphi_g)$ is injective, and absolutely definable ($\ker(\varphi_g)$ is basically just $\mathrm{tp}_{\mathrm{qf}}(g, A, G^\alpha)$). Note that by the induction hypothesis, $\varphi_g^{\mathbf{V}} = \varphi_g^{\mathbf{V}'}$ for $g \in G^\beta$. Similarly, for $g, h \in G^\alpha$, there is a homomorphism $\varphi_{g,h}$ from $\mathbf{B}$ onto $\langle A \cup \{g, h\} \rangle_{G^\alpha}$ fixing A pointwise and taking x to g and y to h, and $(g, h) \mapsto \ker(\varphi_{g,h})$ is injective.

The following definition allows us to interpret the type of g in the type of some pair (h_1, h_2) (see example below):

Definition 6.2 Let $B \subseteq \mathbf{B}$

(1) For every $\sigma \in \mathbf{B}$, Let $\psi_\sigma : \mathbf{A} \to \mathbf{B}$ be the homomorphism defined by $x \mapsto \sigma$, $\psi_\sigma \restriction A = \mathrm{id}$.
(2) For $g \in G^\beta$ we say that g is affiliated with B (denoted $g \propto B$) if there is a word $\sigma_g = \sigma(x, y, \bar{a}) \in \mathbf{B}$ ($\bar{a}$ are parameters from A) such that $\ker(\varphi_g) = \psi_{\sigma_g}^{-1}(B)$.

Example 6.3 Let $\rho \in G^\alpha$, $h \in G^\beta$. If $B = \ker(\varphi_{\rho,h})$ then for every $g \in G^\beta$, $g \propto B$ iff there exists σ_g such that $\varphi_{\rho,h}(\sigma_g) = g$ (i.e. $g \in \langle A \cup \{h, \rho\} \rangle_{G^\alpha} \cap G^\beta$). It could easily be verified that this is indeed true, using the equality $\varphi_{\varphi_{\rho,h}(\sigma)} = \varphi_{\rho,h} \circ \psi_\sigma$ for every $\sigma \in \mathbf{B}$, and 3.10.

We shall find an absolute first order formula $\Delta(H, \mathcal{P}({}^{\omega >}A), G^\beta)$ that will say "H is a normal subgroup of $\mathbf{A} = A * \langle x \rangle$ and there exists an automorphism $\rho \in \mathrm{Aut}(G^\beta) = G^\alpha$ such that $H = \ker(\varphi_\rho)$".

If we succeed then if $\rho \in (G^\alpha)^{\mathbf{V}}$ then $\Delta(\ker(\varphi_\rho), \mathcal{P}({}^{\omega >}A), G^\beta)$ will hold. Since $\ker(\varphi_\rho) \in \mathbf{V}'$, and Δ was absolute, there is some $\rho' \in (G^\alpha)^{\mathbf{V}'}$ such that $\ker(\varphi_\rho) = \ker(\varphi_{\rho'})$ so $\rho = \rho'$ and we are done.

Let us describe Δ. It will say that H is a normal subgroup of $\mathbf{A}$ and that for each $h \in G^\beta$ there exists a subgroup $B = B_h \leq \mathbf{B}$ with the following properties:

(1) B is a normal subgroup of $\mathbf{B}$.
(2) $H \subseteq B$, and $B \cap \mathbf{A} = H$.
(3) For every $a \in A$, $a \propto B$ and $\sigma_a = a$ (equivalently $B \cap A = \{e\}$)
(4) $h \propto B$ and $\sigma_h = y$.
(5) If $g \propto B$ and both σ_1 and σ_2 witness that, then $\sigma_1 \sigma_2^{-1} \in B$.
(6) If $g_1, g_2 \propto B$ then so is $g_1 g_2$ and $\sigma_{g_1 g_2} = \sigma_{g_1} \sigma_{g_2}$.
(7) If $g \propto B$ then there exists $g' \in G^\beta$ such that $g' \propto B$ and $x \sigma_g x^{-1} = \sigma_{g'}$.

$B = B_h$ induces a monomorphism ρ_B whose domain is $H_B = \{g \in G^\beta | g \propto B_h\}$. It is a subgroup of G^β containing A and h (why? because of the conditions on B_h). For every $g \in H_B$ define $\rho_B(g)$ to be the element $g' \in G^\beta$ as promised from property (7) (so the range of ρ_B is also H_B). Note that for every $g_1, g_2 \in H_B$, if $\sigma_{g_1} \sigma_{g_2}^{-1} \in B$ then $g_1 = g_2$.
Why? Since B is normal, ψ_σ induces $\psi'_\sigma : \mathbf{A} \to \mathbf{B}/B$, and so the condition $g \propto B$ becomes $\ker(\varphi_g) = \ker(\psi'_{\sigma_g})$. Now, if $\sigma_0 \in B$, then $\psi'_{\sigma_0 \sigma} = \psi'_\sigma$ so $\psi'_{\sigma_{g_1}} = \psi'_{\sigma_{g_2}}$ and hence $\ker(\varphi_{g_1}) = \ker(\varphi_{g_2})$.

Now it an easy exercise to see that ρ_B is a well defined monomorphism,
After defining ρ_B we demand that for every $h_1, h_2 \in G^\beta$ and all suitable B_1 and B_2, ρ_{B_1} and ρ_{B_2} agree on their common domain. Thus we can define $\rho_H = \bigcup \{\rho_{B_h} | h \in G^\beta\}$, and demand that ρ_H will be an automorphism (i.e. onto). Now all that is left is to say that $H = \ker(\varphi_{\rho_H})$, and Δ is written.
(There is no problem with writing this in first order. Moreover, the formula is bounded in its parameters.)
Why is Δ correct? because if $\Delta(H, \ldots)$ is true, then $H = \ker(\varphi_{\rho_H})$ by definition. On the other hand, if $H = \ker(\varphi_\rho)$ for some ρ, then:

- For each h, $\ker(\varphi_{\rho,h})$ will be a suitable B_h (by the example above).
- If B satisfies the conditions above, then $\rho_B \restriction A = \rho \restriction A$ because by condition (2) $\ker(\varphi_{\rho_B(a)}) = \psi^{-1}_{xax^{-1}}(H) = \ker(\varphi_{\rho(a)})$. Hence, $\rho^{-1} \circ \rho_B \restriction A = \mathrm{id}$ and by 3.10, $\rho_B \restriction H_B = \rho \restriction H_B$. So all the ρ_Bs agree on their common domains.

In conclusion, the demands on H are satisfied, and we are done. □

Lemma 6.4 *Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive class which is a model of ZF. Let $G \in \mathbf{V}'$ be a centerless group, finitely generated by A. Then, for every ordinal $\delta \in \mathbf{V}'$, the automorphism tower $\langle G^\beta | \beta < \delta \rangle$ in $\mathbf{V}'$ is the same in $\mathbf{V}$ (i.e. $\mathbf{V} \models$ "$\langle G^\beta | \beta < \delta \rangle$ is the automorphism tower up to δ").*

Proof As above, let $\mathfrak{T} = \langle G^\beta | \beta \in \mathbf{ord}^{\mathbf{V}'} \rangle$, and we shall prove by induction on $\alpha < \delta$ that $\mathfrak{T} \restriction \alpha + 1$ is the automorphism tower in $\mathbf{V}$ up to $\alpha + 1$.
For $\alpha = 0$ this is clear since $G \in \mathbf{V}'$.
For α limit this follows from the definitions.
Suppose $\alpha = \beta + 1$. By the induction hypothesis $\mathfrak{T} \restriction \alpha$ is the automorphism tower in $\mathbf{V}$ up to α, so $(G^\beta)^{\mathbf{V}} = (G^\beta)^{\mathbf{V}'} \in \mathbf{V}'$. Let $\rho \in \mathrm{Aut}(G^\beta) = G^\alpha$ in $\mathbf{V}$. We need to show that $\rho \in (G^\alpha)^{\mathbf{V}'}$.
We prove by induction on $j \leq \beta$, that the sequence $I_j = \langle \rho \restriction G^i | i \leq j \rangle$ is in $\mathbf{V}'$. For $j = \beta$, we will have that $\rho = \rho \restriction G^\beta \in \mathbf{V}'$.
$j = 0$: Since A is finite, $({}^A(G^\beta))^{\mathbf{V}} = ({}^A(G^\beta))^{\mathbf{V}'}$, so $\rho \restriction A \in \mathbf{V}'$. $\langle A \rangle = G$, so

$\rho \restriction G \in \mathbf{V}'$. (It is just the set of pairs (g, h) where $g \in G, h \in G^\beta$, and there is a finite sequence $a_0, \ldots, a_{n-1} \in A$ and a function $\varepsilon : n \to \{\pm 1\}$, such that $g = \prod a_i^{\varepsilon(i)}$ and $h = \prod \rho(a_i)^{\varepsilon(i)}$).

j limit: Let $\rho \restriction G^j = \bigcup\{\rho \restriction G^i | i < j\}$. Using it we can define $I_j = \langle \rho \restriction G^i | i \leq j \rangle$.

$j = i + 1$: $\rho \restriction G^j$ is the set of all pairs (g, h) such that

- $g \in G^j, h \in G^\beta$ and for all $g' \in A$, $\rho(gg'g^{-1}) = h\rho(g')h^{-1}$.

Note that this condition is absolute, and hence we are done if it works. Why is that true? Obviously if $(g, h) \in \rho \restriction G^j$, then this condition holds. Conversely, it is enough to show that for each g, there is exactly one h such that (g, h) satisfies this condition. Suppose that for h_1, h_2 we have that for all $g' \in A$, $h_1\rho(g')h_1^{-1} = h_2\rho(g')h_2^{-1}$. So, working in $\mathbf{V}$, we let $h_1' = \rho^{-1}(h_1), h_2' = \rho^{-1}(h_2)$, and we get that conjugation by $(h_1')^{-1}h_2'$ is id on G. By the fact that (G^β, A) is special, this implies that $h_1' = h_2'$, so $h_1 = h_2$.

Note that this proof is simpler than the proof of the previous lemma. This is due to the fact that here, given ρ, we have that $\rho \restriction G$ is "automatically" in $\mathbf{V}'$, while this is not the case in general. $\square$

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