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Around random algebra

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Abstract. It is shown that there is a subalgebra of the measure algebra forcing dominating reals. Also results are given about iterated forcing connected with random reals.

Introduction

In this work we will give results connected with random reals. All these results were discovered when the authors worked on building a model for K_B (measure) having cofinality ω (see [BJ 1]) where K_B (measure) is the minimal cardinality of a family of measure zero sets covering the real line. That problem remains open.

In Sect. 1 we will show that under a certain assumption (K_B (category) $= b$), there exists a subalgebra of the Random real algebra forcing dominating reals. We don't know if this is true under ZFC. Our construction is by induction on a dominating family. The first stage is to build a name for the dominating real.

In Sect. 2 we will show that an ω-interation adds a random real which appears only in the limit if and only if there is an intermediate stage containing a perfect set of random reals over the ground model. We will give incisive applications of this last result by answering a question of J. Paulikowski and give more results about finite support iteration and random reals.

The reader may ask what are the connections between Sects. 1 and 2. To supply information we can say that the problems of adding dominating reals and adding perfect sets of random reals are related by the following

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Note. In the first version of this paper we proved only a weak form of the main result of Sect. 2 (see 2.2). Also, the first version contained a third section, but the main result of that version is a weak form of 2.5

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Theorem [BJ]. *If there is a random real r over M and a dominating real over M[r] then there is a perfect tree of random reals over M.*

To get a dominating real from a perfect set of random reals is an open problem.

1 Adding a dominating real

In this section we will show that under *CH*, or *MA* (countable), there is a subalgebra $\mathfrak{B}$ of the measure algebra (Random real algebra) such that forcing with $\mathfrak{B}$ yields a real which eventually dominates every function from ω to ω which is in the ground model. We will assume *CH*; minor changes to this construction will give a proof under *MA* (countable).

We will construct the algebra by induction on some fixed well order of ω^ω of order type ω_1. In stage 0 we will give an ω-sequence of maximal antichains, that we will use for the name of the dominating function. The main definition in this section is

1.1 Definition. $\mathfrak{B}$ is adequate for $\bar{B}=\langle\langle B_{n,\varrho}:\varrho\in{}^n\omega\rangle:n\in\omega\rangle$, if

(a) $\mathfrak{B}$ is a subalgebra of the algebra of the Borel sets.

$$(\mathfrak{B}^+=\{B\in\mathfrak{B}:\mu(B)>0\};\ I=\{B\in\mathfrak{B}:\mu(B)=0\}).$$

(b) For every η in $2^{<\omega}$, $[\eta]$ belongs to $\mathfrak{B}$.

(c) $B_{n,\varrho}$ belongs to $\mathfrak{B}$ for all $n\in\omega$ and $\varrho\in{}^n\omega$. $B_{n+1,\varrho\hat{}(k)}\subseteqq B_{n,\varrho}$ for all $n\in\omega$, $\varrho\in{}^n\omega$, and $k\in\omega$.

(d) For each $n\in\omega$, $\langle B_{n,\varrho}:\varrho\in{}^n\omega\rangle$ is the maximal antichain of $\mathfrak{B}$, and for each $\varrho\in{}^n\omega$

$$\sum_k \mu(B_{n+1,\varrho\hat{}(\kappa)})<\mu(B_{n,\varrho})\cdot 10^{-2}.$$

(e) For every X in $\mathfrak{B}^+$ and all $n\in\omega$, we have

$$(\alpha)_{x,n}: \text{there exists } w\in[{}^n\omega]^{<\aleph_0},\ X\subseteqq\bigcup_{\varrho\in w} B_{n,\varrho}(\mathrm{mod}(I))$$

or

$$(\beta)_{x,n}: \text{there are infinitely many } \varrho\in{}^n\omega \text{ such that } B_{n,\varrho}\cap X\in\mathfrak{B}^+.$$

(f) For every X in $\mathfrak{B}^+$

$$\{n<\omega:(\alpha)_{x,n}\} \text{ is finite}.$$

(g) If $X\in\mathfrak{B}^+$ and $\mu(X)>1-\mu\Big(\bigcup_{\varrho\in{}^n\omega} B_{n,\varrho}\Big)$ then for infinitely many $\varrho\in{}^n\omega$ the set

$$(X\cap B_{n,\varrho})-\cup\{B_{n+1,\eta}:\eta\in{}^{n+1}\omega\}$$

has positive measure.

1.2 Fact. $\{n\in\omega:(\alpha)_{x,n}\}$ is an initial segment of ω. $\square$

The next stage is to introduce the main tool used in the construction. This is a forcing notion which is essentially the Cohen real forcing.

1.3 Definition.

(a) Let PFT_n be the set of all $t \subseteq {}^{n\geq}2$ satisfying:

(i) $\langle\ \rangle \in t$

(ii) if $\eta \in t$ then $\bigwedge_{\ell<|\eta|} (\eta|\ell \in t)$

(iii) if $\eta \in (t \cap {}^{n>}2)$ then $(\eta^\wedge(0) \in t \vee \eta^\wedge(1) \in t)$

(*b*) Set $PFT = \bigcup_n PFT_n$

(c) $AP_n = \{(t,f) : t \in PFT_n$ and $f : t \to Q^+ \cup \{0\}$ and if $\eta \in t \cap {}^{n>}2$ then

$$f(\eta) = \sum_{\eta^\frown(\ell) \in t} f(\eta^\wedge(\ell)) \text{ and } f(\langle\ \rangle) \neq 0\}.$$

Let $AP = \bigcup_n AP_n$. Define a partial order on AP as follows: If $(t,f) \in AP_n$, $(t',f') \in AP_{n'}$, let $(t,f) \leqq (t',f)$ iff $n \leqq n'$, $t = t' \cap {}^{n\geq}2$, $f \subseteq f'$.

(d) For every Lebesgue measurable $A \subseteq {}^{\omega}2$, we define $f_A : {}^{\omega>}2 \to \mathscr{R}$ by

$$f_A(\eta) = \mu(A \cap |\eta|).$$

(e) If $A = \lim(T)$ (= all branches of the tree T) we will write f_T for f_A. If $(t,f) \in AP_n$ we say that A satisfies (t,f) if for all $\eta \in t$, $f(\eta) < f_A(\eta)$. Set

$$AP_n(A) = \{(t,f) \in AP_n : A \text{ satisfies } (t,f)\}, \quad AP(A) = \bigcup_n AP_n(A).$$

1.4 Fact. If $\mu(A) \neq 0$ and $G \subseteq AP(A)$, and G is sufficiently generic, then $T = \cup\{t : \exists(t,f) \in G\}$ is a perfect tree satisfying

(i) $\mu(\lim T \cap |\eta|) = F(\eta)$ for all $\eta \in T$, where $F = \cup\{f : \exists(t,f) \in G\}$. In particular, $\mu(\lim T) = F(\langle\ \rangle)$ (by the Lebesgue density theorem).

(ii) $\lim T \subseteq A$ (modulo a measure zero set).

(iii) $\forall \eta \mu(A \cap |\eta|) > 0 \to \mu(\lim T \cap [\eta]) < \mu(A \cap [\eta]$. $\square$

The first stage of the proof is to show that we can find $\bar{B} = \langle\langle B_{n,p} : \varrho \in {}^n\omega\rangle : n \in \omega\rangle$ satisfying the conditions on 1.1. This will be used to construct the name for the dominating real.

1.5 Lemma. *There is $\bar{B} = \langle\langle B_{n,p} : \varrho \in {}^n\omega\rangle : n < \omega\rangle$ and $\mathfrak{B}$ such that $\mathfrak{B}$ is adequate for $\bar{B}$.*

Proof. We define, by induction on n, $\mathfrak{B}_n$, and $\langle B_{n,p} : \varrho \in {}^n\omega\rangle$ such that

(i) $\mathfrak{B}_n$ satisfies 1.1 (a), (b).

(ii) $\langle B_{n,p} : \varrho \in {}^n\omega\rangle$, for $m < n$, satisfies 1.1 (c), (d), and (e).

(iii) $\mathfrak{B}_n \subseteq \mathfrak{B}_{n+1}$ and $\mathfrak{B}_n$ is generated by $\{[\eta] : \eta \in {}^{\omega>}2\} \cup \{B_{n,\varrho} : \varrho \in {}^n\omega\}$.

(iv) If $X \in \mathfrak{B}^+_{n+1}$ then $\neg(\alpha)_{x,n}$. Let $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$.

The induction. $n = 0$: $\mathfrak{B}_0$ is the algebra generated by $\{[\eta] : \eta \in {}^{\omega>}2\}$.

Fix $\langle \eta_i : i < \omega\rangle = 2^{<\omega}$, be such that $\{i : \eta_i = \eta\} \in [\omega]^\omega$ for each $\eta \in 2^{<\omega}$.

$n+1$: It is clearly enough to define $\langle B_{n+1,\varrho^\frown(i)} : <\omega\rangle$ for each $\varrho \in {}^n\omega$. For this we remark that

$$\langle X \cap B_{n,\varrho} : X \in \mathfrak{B}^+_n\} = \{X \cap B_{n,\varrho} : X \in \mathfrak{B}_0\}.$$

w.l.o.g. we assume that $[\eta_n] \cap B_{n,\varrho} \in \mathfrak{B}^+_n$.

Let

$$A_0 = [\eta_n] \cap B_{n,\varrho},$$

$$C_0 = [B_{n,\varrho}] \setminus A_0.$$

Let
$$S_k=\{(t,f)\in AP(A):f(\langle\ \rangle)<\mu(A_k)\cdot 10^{-(k+3)}\},$$
$$T_k=\{(t,f)\in AP(A):f(\langle\ \rangle)<\mu(C_k)\cdot 10^{-(k+3)}\},$$
$$R_k=S_k\times T_k.$$
Over a model N, forcing with R_k gives two positive sets, let them be $\bar{A}_k$, $\bar{C}_k$. Now we define the following forcing iteration $\langle P_i, Q_i : i\in\omega\rangle$, satisfying
$$\Vdash_{P_i} \text{“}\underset{\sim}{Q}_i=R_i \quad \text{when} \quad A_i=A_0\setminus\bigcup_{j<i}\bar{A}_j, \qquad C_i=C_0\setminus\bigcup_{j<i}\bar{C}_i$$
Clearly $P_1\cong R_0$.

Let $P_\omega=\varinjlim P_i$. Let $N<\langle H(\aleph_1,\in,\leqq\rangle$ be countable and containing $B_{n,\varrho}$. Then clearly $P_\omega\in N$.

Let G_ω be a generic filter for P_ω over N. Then we obtain a sequence $\langle\langle\bar{A}_i,\bar{C}_i\rangle : i<\omega\rangle$, satisfying

(a) For each $j,k<\omega$
$$\mu(\bar{A}_j\cap\bar{C}_k)=0,$$
$$\mu(\bar{A}_j\cap\bar{A}_k)=0 \quad \text{if} \quad j\neq k,$$
$$\mu(\bar{C}_j\cap\bar{C}_k)=0 \quad \text{if} \quad j\neq k.$$
(b) $\sum_j\mu(\bar{A}_j)+\sum_j\mu(\bar{C}_j)<\mu(B_{n,\varrho})\cdot 10^{-2}$.

(c) If $[\eta]\cap B_{n,\varrho}\in\mathfrak{B}^+$ then there exists infinitely many k's such that
$$\mu([\eta]\cap\bar{A}_k)>0 \quad \text{or} \quad \mu([\eta])\cap\bar{C}_k)>0$$
(by genericity of G_ω).

(d) $\mu([\eta]\cap B_{n,\varrho}-\cup\{\bar{A}_j\cup\bar{C}_j : j<\omega\})>0$.

Let
$$B_{n+1,\hat{}\langle 2k\rangle}=\bar{A}_k,$$
$$B_{n+1,\hat{}\langle 2k+1\rangle}=\bar{C}_k.$$
Then $\langle B_{k+1,\varrho\hat{}\langle k\rangle} : k<\omega\rangle$ satisfies the required conditions.

Fact. $\mathfrak{B}=B_n$ satisfies condition 1.1 (g).

[*Proof.* Use (c), (d), and that for each η, $\{i:\eta=\eta_i\}$ is infinite.]

1.6 Lemma. *Suppose that $\mathfrak{B}$ is adequate for $\bar{B}=\langle\langle B_{n,\varrho} : \varrho\in{}^n\omega\rangle : n\in\omega\rangle$, and assume that $\bar{u}=\langle u_n : n<\omega\rangle$ is such that $u_n\in[{}^n\omega]^{<\aleph_0}$, and there is a Cohen real over a model containing $\mathfrak{B}$ and $\bar{u}$. Then there exists $B\subseteq 2^\omega\setminus\bigcup_{n<\omega}\bigcup_{\varrho\in u_n}B_{n,\varrho}$ such that the Boolean algebra generated by $\mathfrak{B}\cup\{B\}$ is adequate for $\bar{B}$.*

Proof. $A=2^\omega\setminus\bigcup_{n<\omega}\bigcup_{\varrho\in u_n}B_{n,\varrho}$. Set $Q=\{\langle(t_i,f_i) : i<k\rangle : k\in\omega$ and $((t_i,f_i)\in AP(A))$ and $(f_i(\langle\ \rangle)<10^{-(i+1)})\}$. We give the following natural order to Q:
$$\langle(t_i^1,f_i^1) : i<k^1\rangle\leqq\langle(t_i^2,f_i^2) : i<k^2\rangle$$
if $k^1\leqq k^2$ and for each $i<k^1$ there is n_i such that
$$t_i^1\subseteq{}^{n_i\geqq}2,$$
$$t_i^1=t_i^2\cap{}^{n_i\geqq}2,$$
$$f_i^1=f_i^2\restriction t_i^1$$

($\langle Q, \leqq \rangle$ is essentially a finite support product of $AP(A)$. Since it is countable, it is essentially a Cohen real forcing.) Now we define some Q-names: [Write members of Q as $(\bar{t}, \bar{f})$.] Let

$$\underset{\sim}{T}_i = \cup\{t : (\exists(\bar{t}, \bar{f}) \in \underset{\sim}{G})(\bar{t}(i) = t)\},$$

$$\underset{\sim}{f}_i = \cup\{f : (\exists(\bar{t}, \bar{f}) \in \underset{\sim}{G})(\bar{f}(i) = f)\}.$$

Then the following fact holds in a generic extension:

(a) $\underset{\sim}{T}_i$ is a perfect tree

(b) $\underset{\sim}{f}_i(\eta) = \mu(\lim \underset{\sim}{T}_i \cap [\eta])$

(c) if $i < \omega$ and $n \in \omega$ and $\varrho \in u_n$ then

$$\mu(\lim \underset{\sim}{T}_i \cap B_{n,\varrho}) = 0$$

(d) $\underset{\sim}{B} = \bigcup_i \lim \underset{\sim}{T}_i$ satisfies the requirements of the lemma.

Let $\mathfrak{B}'$ be the Boolean algebra generated by $\mathfrak{B} \cup \{B\}$.

We will show only (d). The rest is clear. The conditions 1.1 (a)–(c) are clear. We need to show 1.1 (d), 1.1 (e), 1.1 (f), and 1.1 (g). Clearly 1.1 (d) follows from 1.1 (e). Therefore we will show 1.1 (e), 1.1 (f), and 1.1 (g) for $X \in \mathfrak{B}'^+$. If $X = X_1 \cup X_2 \cup \ldots \cup X_n$ (a disjoint union) then it is enough to show 1.1 (e)–(g) for each X_i. Therefore w.l.o.g. there is $Y \in \mathfrak{B}^+$ such that $X \in \{Y \cap B, Y \setminus B\}$. Now $Y \in \mathfrak{B}^+$, thus Y satisfies 1.1 (e)–(g) and clearly

$$(\alpha)_{Y,n} \to (\alpha)_{X,n}.$$

We start by showing that if for infinitely many ϱ,

$$\mu(Y \cap B_{n,\varrho} - \cup\{\cup\{(B_{j,\eta} : \eta \in {}^j\omega\} : n < j < \omega\}) > 0:$$

then for infinitely many ϱ,

$$\mu(X \cap B_{n,\varrho} - \cup\{\cup\{(B_{j,\eta} : \eta \in {}^j\omega\} : n < j < \omega\}) > 0.$$

This is clear when $X = Y \cap B$, using a density argument. We only recall that for such ϱ

$$\mu(Y \cap B_{n,\varrho} \cap A) > 0.$$

Therefore we may assume $X = Y - B$. We will give a density argument. Let $\langle (t_0, f_0), \ldots, (t_k, f_k) \rangle \in Q$. Let ϱ be such that

$$\mu(B_{n,\varrho}) < \min\{\mu(A \cap [\eta]) - f_i(\eta) : \eta \in t_i \wedge i \leqq k\}$$

and

$$\mu(Y \cap B_{n,\varrho} - \cup\{\cup\{B_{j,\eta} : \eta \in {}^j\omega\} : n < j < \omega\} > 0.$$

Fix $k(*)$ such that $\sum_{i \geqq k(*)} \frac{1}{10^i} < \frac{\varepsilon}{10}$ where

$$\varepsilon = \mu(Y \cap B_{n,\varrho} - \cup\{\cup\{B_{j,\eta} : \eta \in {}^j\omega\} : n < j < \omega\}.$$

$\left[\text{This implies } \mu\left(\bigcup_{i \geqq k(*)} T_i\right) < \frac{\varepsilon}{10}.\right]$ For $k < i \leqq k(*)$, let $t_i\{\langle\ \rangle\}$, $= f_i\{\langle\ \rangle\} < \frac{\varepsilon}{10}$. Now for each $i < k(*)$ we can find $(s_i, g_i) \in AP$ such that $s_i \supseteq t_i$, $g_i \supseteq f_i$, $\langle (s_i, g_i) : i < k(*) \rangle \in Q$ and if T is a perfect tree satisfying for all $\eta \in s_i$

$$\mu(\lim T \cap [\eta]) = g_i(\eta)$$

then

$$\mu(\lim T \cap B_{m,\varrho}) < \frac{\varepsilon}{k(*)10}.$$

(The s_i are obtained by taking a clopen approximation of $B_{m,\varrho}$ with small error, then deleting all this from the t_i by enlarging it to s_i.) Then we get that $(\bar{s}, \bar{g}) \geqq (\bar{t}, \bar{f})$ and

$$(\bar{s}, \bar{g}) \Vdash \mu\left(B_{n,\varrho} \cap \bigcup_i \lim T_i\right) < \frac{\varepsilon}{5}$$

and thus

$$(\bar{s}, \bar{g}) \Vdash \text{“}\mu(X \cap B_{n,\varrho} - \cup\{\cup\{B_{j,\eta} : \eta \in {}^j\omega\} : n < j < \omega\} > 0\text{”}.$$

Claim. The condition 1.1 (g) holds.

[*Proof.* By using the above fact, the case not covered is when $X = B - Y$. But in this case take Y and p and n such that

$$p \Vdash \text{“}\mu(B - Y) - \cup\{(B_{\varrho,n+1} : \varrho \in {}^{n+1}\omega\} : n < j < \omega\} > 0\text{”}.$$

Then p is essentially a clopen set $\bar{p}$ such that $\mu((\bar{p} - Y) - \cup\{(B_{\varrho,n+1} : \varrho \in {}^{n+1}\omega\}) > 0$. The rest follows by a density argument.]

Now we will use this in order to prove

$$(\beta)_{Y,\eta} \rightarrow (\beta)_{X,\eta}.$$

To do this we will use the following

Claim. Let E be in $\mathfrak{B}^+$ and assume that $F = E \cap A$ has positive measure. Then for almost all $n \in \omega$, there are infinitely many $\varrho \in {}^n\omega$ such that $F \cap B_{n,\varrho}$ has positive measure.

[*Proof.* Let $C_n = \bigcup_{m<n} \bigcup_{\varrho \in u_m} B_{\varrho,m}$ and $C = \bigcup_n C_n$. Let n be big enough. Let $E_0 = E \setminus C_n$. Then $E_0 \in \mathfrak{B}^+$ and $\mu(E_0 - F) < 10^{-n+1}$ $\left(\text{because } E_0 - F \subsetneqq \bigcup_{\varrho \in {}^n\omega} B_{\varrho,n}\right)$. Also E_0 satisfies the condition for (g) and so there are infinitely many ϱ's such that $E_0 \cap B_{\varrho,n} - \cup\{B_{\varrho,n+1} : \varrho \in {}^{n+1}\omega\}$ has positive measure. We finish the proof by showing that each one of these ϱ works: Let

$$Z = E_0 \cap B_{\varrho,n} - \cup\{B_{\varrho,n+1} : \varrho \in {}^{n+1}\omega\}.$$

Z has positive measure by the choice of ϱ. Z is disjoint from $\bigcup_{k<n} \bigcup_{\varrho \in u_k} B_{\varrho,k}$ (because $Z \subsetneqq E_0$). Z is disjoint from $\bigcup_{\varrho \in u_n} B_{\varrho,n}$ (because $Z \subsetneqq B_{\varrho,n}$ and we can choose $\varrho \notin u_n$). Z is disjoint from $\bigcup_{\varrho \in u_k} B_{\varrho,k}$ for all $k > n$ (because it is disjoint from $\cup\{B_{\varrho,n+1} : \varrho \in {}^{n+1}\omega\}$). Therefore we conclude that Z is disjoint from C and it is a subset of E. Hence $Z \subsetneqq E - C = F$, and so F has positive measure.]

Claim. The condition 1.1 (e) holds.

[*Proof.* Let $(\bar{t}, \bar{f}) \in Q$ and $k = lg(\bar{t})$.

Case 1. $(\bar{t}, \bar{f}) \Vdash$ “$X = Y \cap B$” for some $Y \in \mathfrak{B}^+$. Let n such that $(B)_{Y,n}$ holds. Clearly we may assume that $Y \cap A$ has positive measure. Then there are two cases

(i) $\{\varrho : \mu(Y \cap A \cap B_{\varrho,n}) > 0\}$ is finite.

In this case we have $(\alpha)_{x,n}$. By previous claim, this case holds only for finitely many n. Therefore for almost all n we have

(ii) $$\{\varrho : \mu(Y \cap A \cap B_{\varrho,n}) > 0\} \text{ is infinite}.$$

Let ϱ be such that $\mu(Y \cap A \cap B_{\varrho,n}) > 0\}$ and $\mu(B_{\varrho,n}) < \frac{1}{10^{k+1}}$. Then find $(t_k, f_k) \in AP$ such that if T is any perfect tree satisfying "for all $\eta \in t_k$

$$\mu(\lim T \cap T[\eta]) = f_k(\eta)\text{"}$$

then $\mu(B_{n,\varrho} \Delta \lim(T)) < \frac{1}{10^{k+50}}$. Therefore $(\bar{t}, \bar{f})^\wedge(t_k, f_k)$ forces that

$$\mu(B_{n,\varrho} \cap X) > \tfrac{1}{2}\mu(B_{n,\varrho} \cap Y) > 0.$$

Case 2. $(\bar{t}, \bar{f}) \Vdash$ "$X = Y - B$" for some $Y \in \mathfrak{B}^+$. Like Case 1, we assume $(\beta)_{Y,n}$. Pick ϱ such that

$$\mu(B_{n,\varrho}) < \min\{\mu(A \cap [\eta] - f_i(\eta) : \eta \in t_i \text{ and } i < k\} = \varepsilon.$$

Fix $k(*)$ such that

$$\sum_{i \geq k(*)} \frac{1}{10^i} < \frac{1}{10}\mu(B_{n,\varrho} \cap Y).$$

[This will imply $\mu\left(\bigcup_{i \geq k(*)} T_i\right) < \frac{1}{10}\mu(B_{n,\varrho} \cap Y)$.] Now for each $i < k(*)$ we can find $(s_i, g_i) \in A$ such that $s_i \Delta t_i$, $g_i \Delta f_i$, $\langle (s_i, g_i) : i < k(*) \rangle \in Q$ and if T is any perfect tree satisfying for all $\eta \in s_i$

$$\mu(\lim T \cap [\eta]) = g_i(\eta)$$

then

$$\mu(\lim T \cap B_{n,\varrho}) < \frac{\varepsilon}{k(*)10}.$$

(We can get this condition by using a clopen approximation to $B_{n,\varrho} \cap Y$ and then we extend t_i by deleting this clopen set.) Then clearly $(\bar{s}, \bar{g}) \geq (\bar{t}, \bar{f})$ and

$$(\bar{s}, \bar{g}) \Vdash \mu(Y \cap B_{n,\varrho} \cap B) < \frac{\varepsilon}{5}.$$

Therefore

$$(\bar{s}, \bar{g}) \Vdash \text{"}\mu(Y - B) > \frac{\varepsilon}{4} > 0\text{"}].$$

This finishes the proof of the lemma. □

1.7 Theorem (CH). *There exists $\mathfrak{B}$, a subalgebra of the measure algebra, such that*

$$V^{\mathfrak{B}} \models (\exists f\, \forall g \in \omega^\omega \cap V)(\exists n \forall m \geq n)(g(m) < f(m)).$$

Proof. Let $\langle f_i : i < \omega_1 \rangle$ be such that

$$(\forall i < j < \omega_1)(\exists n \forall m \geq n)(f_i(m) < f_j(m))$$

and $(\forall f \in \omega^\omega \exists i \in \omega_1)(\exists n \forall m \geq n)(f(m) < f_i(m))$. We will get $\mathfrak{B}$ by transfinite induction on ω_1.

Stage 0: Let $\mathfrak{B}_0, \bar{B}$ be given by 1.5. Let $\langle B_n^i : i<\omega\rangle$ be an enumeration of $\langle B_{n,\varrho} : \varrho \in {}^n\omega\rangle$.

Stage α: Let $\mathfrak{B}_\alpha^* = \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. By induction hypothesis $\mathfrak{B}_\alpha^*$ is adequate for $\bar{B}$. Let

$$u_n = \{B_n^i : i \leqq f_\alpha(n)\},$$

and let N be a countable model for ZFC* (a sufficiently rich part of set theory) containing $\mathfrak{B}_\alpha^*, \bar{B}, \langle u_n : n<\omega\rangle$, etc. Then by 1.6 (because N is countable) there is B_α such that

(i) $\mu\left(B_\alpha \cap \bigcup_n \bigcup_{i \leqq f_\alpha(n)} B_n^i\right) = 0$.

(ii) The algebra generated by $\mathfrak{B}_\alpha^* \cup \{B_\alpha\}$ is adequate.

This finishes the construction. Let $\mathfrak{B} = \bigcup_\alpha \mathfrak{B}_\alpha$. Let $\underset{\sim}{f}$ be the following $\mathfrak{B}$-name

$$\{\langle\langle n,i\rangle, B_n^i\rangle : i<\omega, n<\omega\}.$$

Then clearly

$$B_n^i \Vdash_{\mathfrak{B}} \text{“}\underset{\sim}{f}(n) = i\text{”}.$$

By construction we have that

$$(*) \qquad B_\alpha \Vdash_{\mathfrak{B}} \text{“}(\forall n)(f_\alpha(n) < \underset{\sim}{f}(n)\text{”}.$$

We will show that

$$0 \Vdash_{\mathfrak{B}} \forall\alpha\exists n \forall m \geqq n)(f_\alpha(m) < \underset{\sim}{f}(m))\text{”}.$$

Clearly that is enough.

If this claim is false then for almost all $\alpha < \omega_1$ there exists

$$A_\alpha \in \mathfrak{B} \quad \text{such that} \quad A_\alpha \Vdash (\exists^\infty n)(\underset{\sim}{f}(n) \leqq f_\alpha(n))$$

Let $\{A_{\alpha_i} : i<\omega\}$ be a maximal subset of $\mathfrak{B}$ satisfying

$$\mu(\underset{\sim}{A}_{\alpha_i} \cap A_{\alpha_j}) = 0, \qquad i \neq j,$$
$$A_{\alpha_i} \Vdash (\exists_n^\infty)(\underset{\sim}{f}(n) \leqq f_{\alpha_i}(n)).$$

Claim. $\{A_{\alpha_i} : i<\omega\}$ is a maximal antichain.

[*Proof.* If not there is $A \in B$ s.t. over A, $\mathfrak{B}$ forces dominating reals, use then the fact that A is isomorphic to the measure algebra.]

Let $\alpha > \sup\{\alpha_i : i \in \omega\}$.

Claim. $0 \Vdash_{\mathfrak{B}} \text{“}(\exists^\infty n)(\underset{\sim}{f}(n) \leqq f_\alpha(n))$.

[*Proof.* If not there exist $A \in \mathfrak{B}$ such that

$$A \Vdash_{\mathfrak{B}} \text{“}(\forall^\infty n)(\underset{\sim}{f_\alpha}(n) < \underset{\sim}{f}(n))\text{”}.$$

Let $i \in \omega$ such that $\mu(A_{\alpha_i} \cap A) > 0$. Then

$$A_{\alpha_i} \cap A \Vdash (\exists_n^\infty)(\underset{\sim}{f}(n) \leqq f_\alpha(n)), \text{ a contradiction.]}$$

This claim contradicts (*).

1.8 Remark. Clearly we can replace CH by $b = K_b(\text{meager}) = MA(\text{countable})$.

2 Adding random reals in ω-stages

In this section we will give a characterization of the property of adding random reals in limit stages of finite support iteration.

2.1 Theorem. *Let $\bar{Q}=\langle P_\alpha, \underset{\sim}{Q}_\alpha:\alpha<\beta\rangle$ be a finite support iterated forcing satisfying* c.c.c. *Let $P_\beta=\lim\bar{Q}$. Then the following are equivalent:*

(i) *There exists $\underset{\sim}{r}$ a P_β-measure such that r is random over V and $\Vdash_{P_\beta}$"$r\notin V^{P_\alpha}$, $\alpha<\beta$".*

(ii) *There exists $\alpha<\beta$ and a perfect tree T in V_α such that $[T]=\{$set of branches of $T\}$ is a perfect tree of random reals over V.*

Proof. (i)$\to$(ii) Clearly it is enough to show the theorem when $V\models CH$, because P_β is the same, and has the same antichains, after the collapse with countable conditions, of $2^{\aleph_0}$ to $\aleph_1$. Also these models have the same Borel measure zero sets.

Assuming CH, let $\langle A_\alpha:\alpha<\omega_1\rangle$ be a sequence of measure zero sets satisfying

(i) $A_\alpha\subseteqq A_\beta$, $\alpha\leqq\beta<\omega_1$.

(ii) For every measure zero set A in V there exists $\alpha<\omega_1$ such that $A\subseteqq A_\alpha$. For each α, let $\langle T^n_\alpha:n<\omega\rangle$ be a sequence of positive perfect trees satisfying

$$2-\bigcup_n[T^n_\alpha]=A_\alpha.$$

By assumption

$$0\Vdash_{P_\beta}\text{“}\underset{\sim}{r}\in\bigcup_n[T^n_\alpha]\text{”},\quad\text{for each}\quad\alpha<\omega_1.$$

Again for each $\alpha<\omega_1$ there exist n_α, r_α such that

$$r_\alpha\Vdash_{P_\beta}\text{“}\underset{\sim}{r}\in[T^n_\alpha]\text{”}$$

w.l.o.g. $\beta=\omega$.

There exists $i<\omega$ such that

$$\{\alpha:\sup(r_\alpha)\subseteqq i\}\in[\omega_1]^{\omega_1}$$

By c.c.c. there exists $G_i\subseteqq P_i$ generic over V s.t.

$$B=\{\alpha:r_\alpha\in G_i\}\in[\omega_i]^{\omega_1}$$

Therefore if G_ω is generic over V and $G_\omega|i=G_i$ then

$$V[G_\omega]\models\underset{\sim}{r}[G_\omega]\in\bigcap_{\alpha\in B}[T^{n_\alpha}_\alpha]$$

But $r[G_\omega]\notin V[G_i]$ and

$$V[G_i]\models\bigcap_{\alpha\in B}[T^{n_\alpha}_\alpha]\neq\emptyset$$

Therefore in $V[G_i]$

$$\bigcap_{\alpha\in B}[T_{\alpha^n}]\text{ contains a perfect tree }T.$$

Because $|B|=\aleph_1$ we have that $[T]$ is a perfect tree of random reals.

(ii)$\longrightarrow$(i) is easy, remember that every new real defines a new branch in an old perfect tree.

2.2 Corollary. *The random real algebra cannot be the union of ω-many algebras, each one not adding random reals.*

2.3 Corollary. *There exists two models $M \subsetneqq N$ satisfying*
(i) $(\exists r \in N)$ *(r random over M)*
(ii) $N \models \mu\{r : r$ *random over* $M\} = 0$.

Proof. Let $M = L$ and let r be random over L. Let $N = M[r][c]$, when c is Cohen over $L[r]$. It is enough to show that no new real in N is random over m. For this, by applying 2.1 (remember that every ω-iteration adds a Cohen real), it is enough to show that in $M[r]$ there is not a perfect tree of random reals over M. But this is a well known result of Chichon (see [BJ 2]).

2.4 Remark. 2.3 answers a question Paulikowski. He also showed that if c is Cohen over V and r is random over $V[c]$, then in $V[c][r]$ there are Cohen reals over $V[r]$.

2.5 Corollary. *Let $P_\omega = \lim\langle P_i, Q_i : i < \omega\rangle$ be a finite support iteration of ccc partially ordered sets. Then the following are equivalent:*
(i) *There exists $\underset{\sim}{r}$ a P_ω-name such that $\underset{\sim}{r}$ is random over V^{P_i}, $i < \omega$.*
(ii) *For each $i < \omega$, in V^{P_ω}, the following holds*

$$\mu(\cup\{A : \mu(A) = 0 \wedge A \in V^{P_i}\}) = 0 .$$

Proof. (i)$\rightarrow$(ii). Let $i < \omega$. By applying 2.2 there exists $j > i$ and a perfect tree T in V^{P_j} such that all branches of T are random reals over V^{P_j}. If $\mu([T]) = 0$ then $\underset{\sim}{r}$ is not random over V^{P_j}, therefore

$$\mu([T]) > 0 .$$

Then $\bigcup_{q \in Q} ([T] + q)$ is a measure one set of random reals over V^{P_j}.
(ii)$\rightarrow$(i). Trivial.

2.6 Remark. In [JS] there are necessary and sufficient conditions to ensure that a forcing P adds a measure one set of random reals.

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