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Models of PA: when two elements are necessarily order automorphic

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We are interested in the question of how much the order of a non-standard model of PA can determine the model. In particular, for a model M, we want to characterize the complete types $p(x, y)$ of non-standard elements (a, b) such that the linear orders $\{x : x < a\}$ and $\{x : x < b\}$ are necessarily isomorphic. It is proved that this set includes the complete types $p(x, y)$ such that if the pair (a, b) realizes it (in M) then there is an element c such that for all standard n, $c^n < a$, $c^n < b$, $a < bc$, and $b < ac$. We prove that this is optimal, because if $\diamondsuit_{\aleph_1}$ holds, then there is M of cardinality $\aleph_1$ for which we get equality. We also deal with how much the order in a model of PA may determine the addition.

1 Introduction

Let M be a model of Peano Arithmetic (PA). We write $M^<$ and $M^{<,+}$ for the $\{<\}$-reduct and the $\{<, +\}$-reduct of the model M, respectively. For an $a \in M$, we write $M_{<a} := \{c \in M : M \models c < a\}$ with the inherited linear order. For any pair (a, b) of non-standard elements of M, we consider conditions

$$(M_{<a}, <_M) \cong (M_{<b}, <_M); \qquad (*)_{M,a,b}$$

$$\text{for every } N, \text{ if } M \prec N, \text{ then } (*)_{N,a,b}; \text{ and} \qquad (*)^{\mathrm{pot}}_{M,a,b}$$

$$\text{for every model } N, \text{ if } M \equiv N \text{ and for some } M_0 \prec N, \text{ we have } \{a, b\} \subseteq M_0 \prec M, \text{ then } (*)_{N,a,b}. \qquad (*)^{\mathrm{tp}}_{M,a,b}$$

The main aim of this paper is to solve the following questions for models of PA:

Question 1.1 What is the set of complete types $p(x, y)$ such that if (a, b) realizes $p(x, y)$, then we have $(*)_{M,a,b}$? Given a model M and a and $b \in M$, when do we have $(*)^{\mathrm{pot}}_{M,a,b}$ or just $(*)^{\mathrm{tp}}_{M,a,b}$? What if we restrict the first question to $\aleph_1$-saturated models?

For the problem as stated, on the one hand we give a sufficient condition, and on the other hand, for $(*)^{\mathrm{tp}}_{M,a,b}$, we prove its necessity, assuming $\diamondsuit_{\aleph_1}$.

Question 1.2 How much does the linear order of a non-standard model M of PA determine M? Is there a non-standard model M of PA such that for every model N of PA, if $M^< \cong N^<$, then $M \cong N$?

We discussed those problems with Gregory Cherlin and he proposed:

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Task 1.3 (Cherlin) Show that $\{M^< : M \models \mathsf{PA}\}$ is complicated.

This question is too vague for our taste. Recall [1, Problem 14] asked by Friedman:

Question 1.4 Is there a model M of PA such that for every model N of PA, if $M^< \cong N^<$, then $M \equiv N$?

Question 1.4 is of a different character since it only deals with the theory of N. Of course, a positive answer to Question 1.2 would also give an answer to Question 1.4.

We may go half way: maybe the linear order of M does not determine M, say up to isomorphism, but just the additive structure (from which the order is definable). This means

Question 1.5 How much does the order of a non-standard model M of a completion T of PA determine the isomorphism type of $(|M|, <_M, +_M)$?

A more general version of our question is

Question 1.6 Can we construct a non-standard model M of PA with few order automorphisms (for some meaning of "few")?

Recall that any countable non-standard model M of PA is recursively saturated and hence has many order automorphisms (cf. [1]). Much has been done on other classes of structures: for Abelian groups and modules, cf. [2]; for general first order structures, cf. [3]. In particular, there are non-standard models of PA with no automorphism, this motivating the "order-automorphism" in Question 1.6. Now answer to Question 1.1 sheds some light on Question 1.5.

Let us discuss the content of the present paper: First, in § 2, we introduce and deal with the equivalence relations E^ℓ_M (for $0 \le \ell \le 6$), and in Theorem 2.6 it is proved that $aE^2_M b$ implies $(*)_{M,a,b}$ so $aE^2_M b$ is a sufficient condition for a positive answer to Question 1.1, while for the so called 2-order rigid models M, we prove that the isomorphism type of $M^<$ determines that of $M^{<,+}$ but only up to almost isomorphism, shedding some light on Question 1.5.

Then, in § 3, we get that even $aE^3_M b$ implies $(*)_{M,a,b}$. This shows that Theorem 2.6 is not interesting in its own right, but its proof is a warm-up for § 3. Moreover this is only part of the picture, cf. § 5. In § 3, we also show that if M is 3-order rigid then $M^{<,+}$ is unique up to almost isomorphism.

In § 4, we show that E^3_M is the right notion as if $\diamondsuit_{\aleph_1}$ holds then every countable model of PA has elementary extension M of cardinality $\aleph_1$ such that for a and $b \in M \backslash \mathbb{N}$ we have $aE^3_M b$ if and only if $M_{<a} \cong M_{<b}$. We comment there on the case $\neg aE^4_M b$.

For most results, some weaker version of PA suffices. We comment on this in § 5; so usually when a result supercedes an earlier one, normally it has a harder proof and really uses more axioms of PA.

Convention 1.7 Models are models of PA. We call a model M of PA ordinary if $\mathbb{N} \subseteq M$. Unless otherwise specified, all models are ordinary models of PA.

2 Somewhat rigid order

We define some equivalence relations E^ℓ_M for models M (of PA). We shall deal with their basic properties in Claim 2.4 and Observation 2.8, with the relations between them in Claim 3.1, with cofinalities of equivalence classes in Observations 2.8 and 3.4, with question of order isomorphism and almost $\{<, +\}$-isomorphism (cf. Definition 2.1) in Claims 2.5 and 3.3. We also note that we have $E^\ell_M \subseteq E^5_M$ for ℓ-order rigid models. Finally, we prove versions of "if $M_1^<$ and $M_2^<$ are isomorphic then $M_1^{<,+}$ and $M_2^{<,+}$ are almost isomorphic", cf. Theorems 2.6 & 3.5.

Definition 2.1 A function $f : M \to N$ is called an *almost* $\{<, +\}$*-isomorphism* if it is an isomorphism from $M^<$ onto $N^<$ and for all a and $b \in M$ there is an $n \in \mathbb{N}$ such that the distance between $f(a +_M b)$ and $f(a) +_N f(b)$ is n. If there is such a function, M and N are called *almost* $\{<, +\}$*-isomorphic*.

Definition 2.2 An equivalence relation E on a model M is called *convex* if $a <_M b <_M c$ and aEc implies aEb. If $x, y \in M$, we say that x is *multiplicatively small relative to* y if for all $n \in \mathbb{N}$, we have that $x \times_M n <_M y$.

If $x, y \in M$, we say that x is *exponentially small relative to* y if for all $n \in \mathbb{N}$, we have that $x^n <_M y$. We define the following equivalence relations on $M \backslash \mathbb{N}$:

(a) We say $a E^0_M b$ if and only if there is an $n \in \mathbb{N}$ such that $a <_M b +_M n$ and $b <_M a +_M n$.
(b) We say that $a E^1_M b$ if and only if there is some $c \in M$ that is multiplicatively small relative to both a and b such that $a <_M b +_M c$ and $b <_M a +_M c$.
(c) We say $a E^2_M b$ if and only if there is an $n \in \mathbb{N}$ such that $a <_M b \times_M n$ and $b <_M a \times_M n$.
(d) We say that $a E^3_M b$ if and only if there is some $c \in M$ that is exponentially small relative to both a and b such that $a <_M b \times_M c$ and $b <_M a \times_M c$.
(e) We say $a E^4_M b$ if and only if there is an $n \in \mathbb{N}$ such that $a <_M b^n$ and $b <_M a^n$.
(f) We say $a E^5_M b$ if and only if there is an order-automorphism of M that maps a to b.
(g) We define E^6_M to be the minimal convex equivalent relation on M which is refined by E^5_M.

Note that we used Convention 1.7 (viz. that the model is ordinary) in parts (b) and (c). This extra assumption could have been circumvented, e.g., by defining $a +_M n$ by repeated addition of 1_M and $a \times_M n$ by repeated additions of a, but that is less convenient.

Definition 2.3 For $\ell \in \{0, 1, \dots, 6\}$, M is called *ℓ-order rigid* if for all a and $b \in M$, we have that $(M_{<a}, <_M) \cong (M_{<b}, <_M)$ implies that $a E^\ell_M b$.

While we know that there are rigid linear orders and we know that there are rigid models of PA, it is harder to build ℓ-order rigid models of PA, our relevant result will be Theorem 4.6.

Claim 2.4

(1) For $\ell \in \{0, \dots, 6\}$ the two place relation E^ℓ_M is an equivalence relation on $M \backslash \mathbb{N}$ and is convex except possibly for $\ell = 5$.
(1A) If $a E^3_M b$, then any $c \in M$ is exponentially small relative to a if and only if it is exponentially small relative to b, and the set of these elements is convex and closed under products and sums.
(1B) If $a E^1_M b$, then any $c \in M$ is multiplicatively small relative to a if and only if it is multiplicatively small relative to b, and the set of these elements is convex and closed under sums.
(2) If $a, b \in M \backslash \mathbb{N}$ and $a E^2_M b$, then $(M_{<a}, <_M) \cong (M_{<b}, <_M)$; moreover there is an automorphism of $M^<$ mapping a to b, that is, $a E^5_M b$.
(3) If $a, b \in M \backslash \mathbb{N}$, then $(M_{<a}, <_M) \cong (M_{<b}, <_M)$ if and only if $a E^5_M b$.
(4) We have $a E^6_M b$ if and only if there is $c \leq_M \min\{a, b\}$ and an order-automorphism f of M such that $\max\{a, b\} \leq_M f(c)$ if and only if this holds for $c = \min\{a, b\}$.

Note that Claim 3.3 establishes a stronger version of Claim 2.4(2); however, the proof of Claim 3.3 uses Claim 2.4(2). Note furthermore that Claim 2.4(2) can be proved for weaker versions of PA than the proof of Claim 3.3, cf. § 5.

P r o o f . Parts (1A) and (1B) are easy to check. For (1), let $\ell = 3$. If $a_1 E^3_M a_2$ and let c witness it, then c witnesses also $a_2 E^3_M a_2$ and $a_2 E^3_M a_1$, so reflexivity and symmetry hold. Finally, assume $M \models a_1 < a_2 < a_3$; if $a_k E^3_M a_{k+1}$ and let c_k witness this for $k = 1, 2$, then the product $c_1 c_2$ witness $a_1 E^3_M a_3$ by part (1A) and if $a_1 E^3_M a_3$ then the same witness gives $a_1 E^3_M a_2 \wedge a_2 E^3_M a_3$ so transitivity and convexity holds. For $\ell = 1$ the proof is similar (using part (1B) instead of part (1A)), also for $\ell = 0, 2, 4$ the proof is even easier and for $\ell = 5, 6$ it holds by the definition.

For (2), assume, without loss of generality, that $a < (n-1)a < b < na$, where $2 \leq n \in \mathbb{N}$, and also there is a c such that $(n-1)c = na - b$. Then $c < a$ because otherwise $(n-1)a \leq (n-1)c = na - b$ hence $b \leq a$, contradiction. Let X be a set of representatives for M/E^0_M and, without loss of generality, assume that a and $c \in X$. Now define $f : M \to M$ by first defining it on X and then extending it to all of M in the obvious way. (The obvious way is: if $y = x + k$, where $x \in X$ and $k \in \mathbb{Z}$, then $f(y) = f(x) + k$ and f is the identity on $\mathbb{N}$.) If $x \in X$, then let

$$f(x) = \begin{cases} x & \text{if } x \leq c, \\ n(x-a) + b & \text{otherwise.} \end{cases}$$

Clearly, $f(c) = c$ and $f(a) = b$. Now check that f is as required.

For (3), first notice that if f exemplifies $aE^5_M b$ (i.e., is an automorphism of M mapping a to b), then $f \restriction M_{<a}$ is an isomorphism from $(M_{<a}, <_M)$ onto $(M_{<b}, <_M)$. Second, if f is an isomorphism from $(M_{<a}, <_M)$, onto $(M_{<b}, <_M)$ we define a function $g : M \to M$ by: $g(c)$ is $f(c)$ if $c <_M a$ and is $b + (c - a)$ if $a \leq_M c$. Now check the rest.

For (4), let $E'_M = \{(a, b)$: for some order-automorphism f of M we have $f(\min\{a, b\}) \geq \max\{a, b\}\}$; clearly this is symmetric (by definition), reflexive (using $f =$ the identity) and as f is monotonic also convex (i.e., $a \leq a_1 \leq b_1 \leq b \wedge aE'_M b$ implies $a_1 E'_M b_1$). To prove transitivity it is now enough to show $a_1 < a_2 < a_3 \wedge a_1 E'_M a_2 \wedge a_2 E'_M a_3$ implies $a_1 E'_M a_3$ which hold by composing the automorphisms f_1, f_2 witnessing $a_1 E'_M a_2, a_2 E'_M a_3$ respectively. So E'_M is a convex equivalence relation and obviously $aE^5_M b$ implies $aE'_M b$.

Lastly, E'_M is refined by any convex equivalence relation refining E^5_M, so it follows that $E^6_M = E'_M$ so we are done. □

Claim 2.5 If f is an order-isomorphism from M_1 onto M_2 then f maps $E^0_{M_1}$ onto $E^0_{M_2}$; if f is an almost $\{<, +\}$-isomorphism from M_1 onto M_2 then f maps $E^1_{M_1}$ onto $E^1_{M_2}$ and $E^2_{M_1}$ onto $E^2_{M_2}$.

P r o o f . Straightforward. □

Claim 2.5 is also true for embeddings, but this will not be used in this paper. For the next result, recall Definition 2.3(2):

Theorem 2.6 *If M_1 is 2-order rigid and $M_1^<$, $M_2^<$ are isomorphic, then M_1, M_2 are almost $\{<, +\}$-isomorphic.*

Note that Theorem 2.6 does not say "by the same isomorphism". The assumption of Theorem 2.6 is too strong to be true for models of full PA, but makes sense for weaker versions of PA, cf. Theorem 5.4. Part of the proof serves as proof to Claim 3.6, and so is indirectly part of the proof of Theorem 3.5.

Question 2.7 Are M_1 and M_2 isomorphic when M_1 and M_2 are isomorphic as linear orders? Are M_1 and M_2 isomorphic when M_1 and M_2 are ℓ-order rigid? Are M_1 and M_2 isomorphic when just M_1 is ℓ-order rigid? (The main case is $\ell = 3$.)

For the proof of Theorem 2.6, we shall use the following observation:

Observation 2.8 Assume that $a \in M \backslash \mathbb{N}$. Then:

(1) $\langle a + n : n \in \mathbb{N}\rangle$ is increasing and cofinal in a/E^0_M;
(2) $\langle a - n : n \in \mathbb{N}\rangle$ is decreasing and unbounded from below in a/E^0_M;
(3) $\langle n \times a : n \in \mathbb{N}\rangle$ is increasing and cofinal in a/E^2_M;
(4) $\langle min\{b : n \times_M b \geq a\} : n \in \mathbb{N}\rangle$ is decreasing and unbounded from below in a/E^2_M.
(5) Moreover, there is some b such that $2^b \leq a < 2^{b+1}$; hence, in (3) and (4), we can use the sequence $\langle 2^{b+n} : n \in \mathbb{N}\rangle$, $\langle 2^{b-n} : n \in \mathbb{N}\rangle$ instead.
(6) For $\ell = 1, 2$, assume $M_\ell \models a \times b = c_\ell$ and $M_1^< = M_2^<$. Then $c_1 E^5_{M_2} c_2$.
(7) For $\ell = 1, 2$, assume $M_\ell \models a_1 \times a_2 \times \ldots \times a_m = c_\ell$ and $M_1^< = M_2^<$. Then $c_1 E^5_{M_\ell} c_2$.

P r o o f . Items (1) to (5) are easy observations. Item (6) is essentially proved in the proof of $(*)_5$ in the proof of Theorem 2.6 below. Item (7) has a similar proof. □

P r o o f o f T h e o r e m 2.6. By Claim 2.4(1), we easily get that

$$\text{every } E^2_{M_\ell}\text{-equivalence class is convex.} \qquad (*)_0$$

Without loss of generality, we can assume that

$$<_{M_1} = <_{M_2} \qquad (*)_1$$

hence M_1 and M_2 have the same universe and we write $M := M_1^{<_1} = M_2^{<_2}$. We also write $<_M$ for $<_{M_1} = <_{M_2}$. Also as usual

$$\mathbb{N} \subseteq M_\ell \text{ for } \ell = 1, 2. \qquad (*)_2$$

Now

$$\text{if } a, b \in M\backslash\mathbb{N} \text{ and } a <_M b \text{ and } M_2 \models a + b = c \text{ and } M_1 \models a' + b = c, \text{ then } a E^2_{M_1} a'. \qquad (*)_3$$

[Why? Since $M_2 \models a + b = c$ and M_2 satisfies PA, we have that $[b, c)_{M_2} = [b, c)_M$ is isomorphic to $[0, a)_{M_2}$ as linear orders, and hence $[b, c)_{M_1}$ is isomorphic to $[0, a)_{M_1}$ as linear orders. Of course $[b, c)_{M_1}$ is isomorphic to $[0, c -_{M_1} b) = [0, a')_{M_1}$ as linear orders. So $[0, a)_{M_1}$, $[0, a')_{M_1}$ are isomorphic as linear orders. But M_1 is 2-order rigid hence $a E^2_{M_1} a'$ as required].

$$\text{If } a <_M b \text{ and } b \in M\backslash\mathbb{N}, \text{ then } b, a +_{M_1} b, \text{ and } a +_{M_2} b \text{ are } E^2_{M_1}\text{-equivalent.} \qquad (*)_4$$

[Why? Similar proof: (using the proof of Claim 2.4(3)) trivially $b E^2_{M_\ell} (a +_{M_\ell} b)$ for $\ell = 1, 2$ and $(a +_{M_1} b) E^5_{M_1} (a +_{M_2} b)$; so use the fact that M_1 is 2-order rigid to deduce $(a +_{M_1} b) E^2_{M_1} (a +_{M_2} b)$. This proves the claim.]

$$\text{If } a, b \in M\backslash\mathbb{N} \text{ and } a \times_{M_\ell} b = c_\ell \text{ for } \ell = 1, 2, \text{ then } c_1 E^2_{M_1} c_2. \qquad (*)_5$$

[Why? For $\ell = 1, 2$, as M_ℓ is a model of PA, it follows that $(M_{<c_\ell}, <_M)$ is isomorphic to $(M_{<a}, <_M) \times (M_{<b}, <_M)$, ordered lexicographically. Hence $((M_1)_{<c_1}, <_M) = (M_{<c_1}, <_M)$ and $((M_2)_{<c_2}, <_M) = (M_{<c_2}, <_M)$ are isomorphic (and trivially $c_1, c_2 \notin \mathbb{N}$) hence by the fact that M_1 is 2-order rigid we have $c_1 E^2_{M_1} c_2$ as promised.]

$$\text{For } a, b \in M\backslash\mathbb{N}, \text{ we have } a E^2_{M_2} b \text{ if and only if } a E^2_{M_1} b. \qquad (*)_6$$

[Why? First, assume $a E^2_{M_2} b$; now, without loss of generality, $a <_M b$, and pick $0 \neq k \in \mathbb{N}$ such that $b < k \times_{M_2} a$; in particular, $b < a +_{M_2} \ldots +_{M_2} a$ (where this is a sum of k summands). By $(*)_4$ we can prove by induction on k that $b E^2_{M_1} a$ as required in the "only if" direction. (Alternatively by Claim 2.4(2) we have $a E^5_M b$ hence by the fact that M_1 is 2-order rigid, we get $b E^2_{M_1} a$.)

Second, assume $\neg(a E^2_{M_2} b)$ and, without loss of generality, $a <_M b$; note that we cannot use the same argument as above by just interchanging M_1 and M_2, because we only assumed that M_1 is 2-order rigid. As $M_2 \models$ PA, there is $c \in M_2$ such that $M_2 \models a \times c \leq b < a \times c + a$. It is clear that $c \notin \mathbb{N}$ because we are assuming $\neg(a E^2_{M_2} b)$. By $(*)_5$ we have $(a \times_{M_1} c) E^2_{M_1} (a \times_{M_2} c)$. But by $(*)_4$, we have that $a \times_{M_2} c$ and $a \times_{M_2} c +_{M_2} a$ are $E^2_{M_1}$-equivalent hence by the choice of c and $(*)_0$ also $a \times_{M_2} c$ and b are $E^2_{M_1}$-equivalent; so together with the previous sentence $(a \times_{M_1} c) E^2_{M_1} b$. Clearly, $c \in M\backslash\mathbb{N}$ satisfies $\neg(a E^2_{M_1} (a \times_{M_1} c))$, so by the definitions everything is as required in the "if" direction.]

$$\text{If } a \in M\backslash\mathbb{N}, \text{ then } (a/E^2_{M_1}, <_M) \text{ has cofinality } \aleph_0 \text{ and also its inverse has cofinality } \aleph_0. \qquad (*)_7$$

[Why? As $M_1 \models$ PA the sequence $\langle a \times_{M_1} 2^n : n \in \mathbb{N}\rangle$ is increasing and its members form an unbounded subset of $a/E^2_{M_1}$; similarly $\langle \min\{b \in M : a \leq b \times_{M_1} 2^n\} : n \in \mathbb{N}\rangle$ is decreasing and its members form a subset of $a/E^2_{M_1}$ unbounded from below. Applying the definition of $E^2_{M_1}$ yields the claim.]

For $\ell = 1, 2$, let $X_\ell = \{(2^b)^{M_\ell} : b \in M_\ell\}$. Then

$$\text{if } a \in M\backslash\mathbb{N}, \text{ then } X_\ell \cap (a/E^2_{M_1}) \text{ has order-type } \mathbb{Z}$$
$$\text{and is unbounded in } (a/E^2_{M_1}, <_M) \text{ from above and from below.} \qquad (*)_8$$

[Why? Fix $\ell \in \{1, 2\}$ and $a' \in M\backslash\mathbb{N}$. Since $M_\ell \models$ PA, there is some b_ℓ such that $M_\ell \models 2^{b_\ell} \leq a' < 2^{b_\ell + 1} = 2^{b_\ell} + 2^{b_\ell}$. The claim follows from the definition of $E^2_{M_\ell}$.]

We might hope that if $a \in M\backslash\mathbb{N}$ and $M_\ell \models 2^a = b_\ell$ for $\ell = 1, 2$, then $b_1 E^2_{M_1} b_2$. For $\ell = 1, 2$, define

(a) $f_\ell : M_\ell \to M_\ell : a \mapsto (2^a)^{M_\ell}$, and

(b) let M^*_ℓ be the model with universe X_ℓ, such that f_ℓ is an isomorphism from M_ℓ onto M^*_ℓ. $\circledast_1$

Then for $\ell = 1, 2$,

$$\text{if } a, b \in X_\ell, \text{ then } a +_{M^*_\ell} b = a \times_{M_\ell} b. \qquad \circledast_2$$

[This follows since PA $\vdash 2^x 2^y = 2^{x+y}$.]

If $\ell = 1, 2$ and $a, b \in M_\ell \backslash \mathbb{N}$, then

(a) $a E^0_{M_\ell} b$ if and only if $f_\ell(a) E^2_{M^*_\ell} f_\ell(b)$;

(b) $a E^1_{M_\ell} b$ if and only if $f_\ell(a) E^3_{M^*_\ell} f_\ell(b)$; and $\circledast_3$

(c) $a E^2_{M_\ell} b$ if and only if $f_\ell(a) E^4_{M^*_\ell} f_\ell(b)$.

[Why? Look at the definitions and do basic arithmetic.]
By $(*)_8$ and $(*)_6$, there is an order isomorphism h from X_1 onto X_2 such that

(a) $h \restriction \{(2^n)^{\mathbb{N}} : n \in \mathbb{N}\}$ is the identity, and $\circledast_4$

(b) if $a \in M \backslash \mathbb{N}$, then h maps $X_1 \cap (a/E^2_{M_1})$ onto $X_2 \cap (a/E^2_{M_2})$.

By $\circledast_3$ and $\circledast_4$ and the definition of $E^0_{M^*_\ell}$, we get that

if $a, b \in X_1$, then $a E^0_{M^*_1} b$ implies that $h(a) E^0_{M^*_2} h(b)$. $\circledast_5$

Suppose that $M^*_\ell \models a_\ell + b_\ell = c_\ell$ for $\ell = 1, 2$ and assume that $h(a_1) = a_2$, and $h(b_1) = b_2$; then

(a) $c_1 E^2_{M_\ell} c_2$ and $\circledast_6$

(b) $c_1 E^0_{M^*_\ell} c_2$.

[Why? If $a_1 \in \mathbb{N}$ or $b_1 \in \mathbb{N}$ the conclusion follows easily, so we can assume $a_1, b_1 \notin \mathbb{N}$. For $\ell = 1, 2$, by $\circledast_2$ we have $M_\ell \models a_\ell \times b_\ell = c_\ell$. Also by $\circledast_4$(b) we have that $x \in X_1 \backslash \mathbb{N}$ implies $x E^2_{M_1} h(x)$. We recall that by $(*)_6$, we have $E^2_{M_1} = E^2_{M_2}$ and can conclude that $a_1 E^2_{M_2} a_2$ and $b_1 E^2_{M_2} b_2$. This implies that for some $n \in \mathbb{N}$, we have $M_2 \models a_1 < n \times a_2 \wedge a_2 < n \times a_1 \wedge b_1 < n \times b_2 \wedge b_2 < n \times b_1$. Therefore $M_2 \models a_1 \times b_1 < n^2 \times a_2 \times b_2 \wedge a_2 \times b_2 < n^2 \times a_1 \times b_1$, and hence $(a_1 \times_{M_2} b_1)$ and $(a_2 \times_{M_2} b_2)$ are $E^2_{M_2}$-equivalent and also are $E^2_{M_1}$-equivalent.

So by $(*)_5$ we have that $(a_1 \times_{M_2} b_1)$ and $(a_1 \times_{M_1} b_1)$ are $E^2_{M_1}$-equivalent and also that $(a_2 \times_{M_1} b_2)$ and $(a_2 \times_{M_2} b_2)$ are $E^2_{M_1}$-equivalent. Together with the previous paragraph, by $(*)_6$, they are $E^2_{M_\ell}$-equivalent. In particular, c_1 and c_2 are $E^2_{M_\ell}$-equivalent as required in clause (a) of $\circledast_6$. By $\circledast_3$, $\circledast_1$(b), and $\circledast_6$(b) the claim follows.]

Concludingly, by $\circledast_4$ and $\circledast_6$(b), we are done. □

3 More for E^3_M

In this section, we say more about the equivalence relations E^ℓ_M. In Claim 3.1 we deal with basic properties: When is $E^\ell_M \subseteq E^{\ell+1}_M$? When does ℓ-order rigidity imply $(\ell+1)$-order rigidity? Which of the relations are preserved under $+$ and $\times$? We also prove one half of our answer to Question 1.1: in Claim 3.3, we prove $a_1 E^3_M b$ implies $a_1 E^5_\mu a_2$. Concerning the weak form of uniqueness of the additive structure, in Theorem 3.5, we prove, e.g., if M_1 and M_2 are order isomorphic and M_1 is 3-order rigid then $M_1^{<,+}$ and $M_2^{<,+}$ are almost isomorphic (i.e., the "error" in $+$ is finite), but this is not necessarily the same isomorphism. We end by relating E^3_M and E^4_M in Claim 3.8.

Claim 3.1

(1) For $\ell = 0, 1, 2, 3, 5$, we have that E^ℓ_M refines $E^{\ell+1}_M$.
(2) If $\ell = 0, 1, 2, 3, 4, 5, 6$ and for $k = 1, 2$, we have $a_k E^\ell_M b_k$, then $(a_1 + a_2) E^\ell_M (b_1 + b_2)$.
(3) If $\ell = 2, 3, 4$ and for $k = 1, 2$, we have $a_k E^\ell_M b_k$, then $(a_1 \times_M a_2) E^\ell_M (b_1 \times_M b_2)$.
(4) Part (3) holds also for $\ell = 5, 6$.
(5) If $\ell = 0, 1, 2, 3$, then ℓ-order rigidity implies $(\ell+1)$-order rigidity.

Proof. Recall that E^3_M refines E^5_M by Claim 2.4(3),(4). Part (1) just requires a careful reading of the definitions. For part (2), firstly, assume $\ell = 0$, so by the assumption for $k = 1, 2$ there are $m_k, n_k \in \mathbb{N}$ such that $M \models a_k + m_k = b_k + n_k$. Now let $m := m_1 + m_2 \in \mathbb{N}$ and $n := n_1 + n_2 \in \mathbb{N}$. Then $M \models (a_1 + a_2) + (m_1 + m_2) = (b_1 + b_2) + (n_1 + n_2)$, and hence $(a_1 + a_2)E^0_M(b_1 + b_2)$ as required.

Secondly, assume $\ell = 2$, so by the assumption, for $k = 1, 2$ there is $n_k \in \mathbb{N}$ such that $M \models a_k < n_k \times b_k \wedge b_k < n_k \times a_k$. Let $n = \max\{n_1, n_2\} \in \mathbb{N}$; then $M \models (a_1 + a_2) < n_1 b_1 + n_2 b_2 \leq nb_1 + nb_2 = n(b_1 + b_2)$ and similarly $M \models (b_1 + b_2) < n(a_1 + a_2)$. Therefore $(a_1 + a_2)E^2_M(b_1 + b_2)$.

Now assume $\ell = 1, 3$; without loss of generality, $a_1 <_M b_1$ and as E^ℓ_M is convex (cf. Claim 2.4(1)), without loss of generality, $a_2 \leq_M b_2$. Let c_k be a witness for $a_k E^\ell_M b_k$ (for $k = 1, 2$); then easily $c = \max\{c_1, c_2\}$ witnesses $(a_1 + a_2)E^\ell_M(b_1 + b_2)$.

The case $\ell = 4$ is easy, too. For the case $\ell = 5$, assume that f_k is an order-automorphism of M mapping a_k to b_k for $k = 1, 2$. Define a function f from M to M by

$$f(x) := \begin{cases} f_1(x) & \text{if } x <_M a_1, \\ b_1 + f_2(x - a_1) & \text{if } a_1 \leq_M x, \end{cases}$$

and check that f does what it is supposed to do.

Finally, in the case of $\ell = 6$, let $c_k = \min\{a_k, b_k\}$; by Claim 2.4(4) there is $d_k \geq \max\{a_k, b_k\}$ such that $c_k E^5_M d_k$ so $a_k, b_k \in [c_k, d_k]$ for $k = 1, 2$. But by the result for $\ell = 5$, we have that $(c_1 + c_2)E^5_M(d_1 + d_2)$. Therefore $a_1 + b_1, a_2 + b_2 \in [c_1 + c_2, d_1 + d_2]$ and we are done.

For part (3), assume $\ell = 2$. For $k = 1, 2$ let n_k witness $a_k E^2_M b_k$ and choose $n = n_1 n_2$ noting that $n_1, n_2 > 0$ by Definition 2.2(c). Now $M \models a_1 \times a_2 < (n_1 \times b_1) \times (n_2 \times b_2) = n \times (b_1 \times b_2)$ and similarly $M \models (b_1 \times b_2) < n(a_1 \times a_2)$.

The proof for $\ell = 3$ is easy, too. For $\ell = 4$, we use the convexity of E^4_M: without loss of generality, $a_1 \leq a_2$ and $b_1 \leq b_2$ and so there are n and $m \in \mathbb{N}$ such that $a_2 \leq a_1^n$, $b_2 \leq b_1^m$. Therefore, $a_1 \times b_1 \leq a_2 \times b_2 \leq a_1^n \times b_1^m \leq (a_1 \times b_1)^{n+m}$ and hence $(a_1 \times b_1)E^4_M(a_2 \times b_2)$.

For part (4), we follow he proof of Theorem 2.6. Let $\ell = 5$; for $k = 1, 2$, if $c_k = a_k \times_M b_k$, there is an order isomorphism h_k from $M_{<a_k} \times M_{<b_k}$ onto $M_{<c_k}$ and let f_k be an order automorphism of M mapping a_k to b_k. Combining these, we get an order-isomorphism g_1 from $M_{<c_1}$ onto $M_{<c_2}$. Let g be the order automorphism of M such that g extends g_1 and that $c_1 \leq d \in M$ implies that $g(d) = c_2 + f_1(d - c_1)$; so g witnesses $(a_1 \times b_1)E^5_M(a_2 \times b_2)$ as promised. For $\ell = 6$, the claim of part (4) follows as in the proof of part (3).

Finally, part (5) follows from the definition of m-order rigidity and part (1). □

Question 3.2 Is E^5_M convex for every M?

Claim 3.3 If $a_1 E^3_M a_2$, then there is an order-automorphism of M mapping a_1 to a_2, i.e., $a_1 E^5_M a_2$.

Proof. Without loss of generality, we can assume that $a_1 <_M a_2$. If $a_1 E^2_M a_2$ then $a_1 E^5_M a_2$ by Claim 2.4(2), so, without loss of generality, $\neg(a_1 E^2_M a_2)$ hence $n \times a_1 < a_2$ for $n \in \mathbb{N}$. This means that by the definition of E^3_M and the assumption $a_1 E^3_M a_2$, there are $c \in M \setminus \mathbb{N}$ and $n \in \mathbb{N}$ such that

$$c <_M a_1,\ M \models c^n < a_1, \text{ and } (c - 1) \times_M a_1 <_M a_2 \leq_M c \times_M a_1. \qquad (*)_1$$

Clearly $a_2 E^2_M(c \times_M a_1)$ and thus, again by Claim 2.4(2), we can assume without loss of generality that

$$M \models a_2 = c \times_M a_1. \qquad (*)_2$$

We now define an equivalence relation E on $M \setminus \mathbb{N}$:

$$xEy \text{ if and only if there is an } n \in \mathbb{N} \text{ such that } |x - y| < c^n. \qquad (*)_3$$

Clearly E is a convex equivalence relation. For every $n \in \mathbb{N}$, we have $0 + c^n = c^n < a_1$ and $c^n + a_1 < 2 \times a_1 < a_2$; therefore, we can choose a set X of representatives for E such that $0, a_1, a_2 \in X$. Note that

$$\text{if } b_1, b_2 \in M_{\leq a_1}, \text{ then } b_1 E b_2 \text{ implies } (c \times b_1)E(c \times b_2). \qquad (*)_4$$

[Why? We have $|(c \times b_2) - (c \times b_1)| = c \times (|b_2 - b_1|)$, and so for any $n \in \mathbb{N}$, we have $|(c \times b_2) - (c \times b_1)| < c^{n+1}$ if and only if $|b_2 - b_1| < c^n$. This means that $(*)_4$ is true indeed.]

Now we define a function f from M into M by

$$f(x) := \begin{cases} x & \text{if } xE0; \\ c \times x & \text{if } x \in X \text{ and } x \neq 0; \\ f(y) + (x - y) & \text{if } xEy \in X,\ y \neq 0, \text{ and } y \leq_M x; \text{ and} \\ f(y) + (y - x) & \text{if } xEy \in X,\ y \neq 0, \text{ and } x <_M y. \end{cases} \qquad (*)_5$$

Note that f is well defined and is an order-preserving injective function onto M by $(*)_4$. As $f(a_2) = c \times a_1 = a_2$, we have $a_1 E a_2$. Consequently, we are done. □

If we compare this with $(*)_7$ of the proof of Theorem 2.6, then we can observe the following:

Observation 3.4

(1) For any $a \in M \setminus \mathbb{N}$ we have:
  (a) the sequence $\langle \lfloor a^{1+2^{-n}} \rfloor : n \in \mathbb{N} \rangle$, that is $\langle \max\{b : b \text{ in } M, a \text{ divides } b \text{ and } (\lfloor b/a \rfloor)^{2^n} \leq a\} : n \in \mathbb{N} \rangle$ is a decreasing sequence from $\{b : a' < b \text{ for every } a' \in a/E^3_M\}$ unbounded from below in it
  (b) the sequence $\langle \lfloor a^{1-2^{-n}} \rfloor : n \in \mathbb{N} \rangle$, that is $\langle \max\{b : (\lfloor a/b \rfloor)^{2^n} \leq a\} : n \in \mathbb{N} \rangle$ is an increasing sequence included in $\{b : b < a' \text{ for every } a' \in a/E^3_M\}$ and unbounded from above in it.

(2) For $a \in M \setminus \mathbb{N}$ we have:
  (a) the sequence $\langle \lfloor (1 + 2^{-n})a \rfloor : n \in \mathbb{N} \rangle$, is a decreasing sequence in $\{b \in M : b \text{ above } a/E^1_M\}$ cofinal in it
  (b) the sequence $\langle \lfloor (1 + 2^{-n})a \rfloor : n \in \mathbb{N} \rangle$, is an increasing sequence in $\{b \in M : b \text{ below } a/E^1_M\}$ cofinal in it.

P r o o f. Straightforward. □

Theorem 3.5

(1) If M_1 is 3-order rigid and f is an order-isomorphism from M_1 onto M_2 then f maps $E^k_{M_1}$ onto $E^k_{M_2}$ for $k = 3, 4$.
(2) In part (1), moreover $M_1^{<,+}$, $M_2^{<,+}$ are almost isomorphic.
(3) For any M, let $E^7_M = \{(a, b) : (\lfloor \log_2(a) \rfloor) E^4_{M_1} (\lfloor \log_2(b) \rfloor)\}$. Assume there is an order-isomorphism f from M_1 onto M_2 mapping $E^4_{M_1}$ to $E^4_{M_2}$, e.g., as in the conclusion of part (1) and f maps $E^7_{M_1}$ onto $E^7_{M_2}$, then $M_1^{<,+}$ and $M_2^{<,+}$ are almost isomorphic.

P r o o f. (1) By the assumption and by Claim 3.3, respectively,

$$\begin{aligned} &\text{(a)} \quad E^3_{M_1} \supseteq E^5_{M_1} \\ &\text{(b)} \quad E^3_{M_\ell} \subseteq E^5_{M_\ell} \text{ for } \ell = 1, 2. \end{aligned} \qquad (*)_0$$

Easily by Claim 2.4(1), we have that

$$\text{every } E^3_{M_\ell}\text{-equivalence class is convex.} \qquad (*)_1$$

Without loss of generality,

$$\begin{aligned} &\text{(a)} \quad <_{M_1} = <_{M_2} \text{ hence} \\ &\text{(b)} \quad M_1 \text{ and } M_2 \text{ have the same universe, and} \\ &\text{(c)} \quad E^5_{M_1} = E^5_{M_2} \text{ so } E^3_{M_2} \subseteq E^5_{M_2} \subseteq E^5_{M_1} = E^3_{M_1}. \\ &\text{(d)} \quad \text{Let } M := M_1^< = M_2^<. \text{ Then } E^5_{M_1} = E^5_M = E^5_{M_2} = E^3_{M_1}. \end{aligned} \qquad (*)_2$$

Also as usual,

$$\mathbb{N} \subseteq M_k \text{ for } k = 1, 2. \qquad (*)_3$$

If $a <_M b$ and $b \in M \backslash \mathbb{N}$, then $b, a +_{M_1} b, a +_{M_2} b$ are $E^3_{M_1}$-equivalent. $(*)_4$

[Why? By the definition of E^3_M.]

If $a \in M, b \in M \backslash \mathbb{N}$ and $a \times_{M_k} b = c_k$ for $k = 1, 2$, then $c_1 E^3_{M_1} c_2$ and $c_1 E^5_M c_2$. $(*)_5$

[By Observation 2.8(6), we have $c_1 E^5_M c_2$ and use $(*)_0$(a) to deduce $c_1 E^3_{M_1} c_2$.]

If $M_\ell \models a_1 \times a_2 \times \ldots \times a_n = b_\ell$ for $\ell = 1, 2$, then $b_1 E^3_{M_1} b_2$ and $b_1 E^5_M b_2$. $(*)_6$

[Why? Similar to $(*)_5$, i.e., by Observation 2.8(7).]

$E^4_{M_1} = E^4_{M_2}$. $(*)_7$

[Why? Let $a, b \in M \backslash \mathbb{N}$ be given. For $\ell = 1, 2$ and $n \in \mathbb{N}$, let $a_{\ell,n}$ be such that $M_\ell \models a^n = a_{\ell,n}$.

First, assume $a E^4_{M_2} b$ and, without loss of generality, $a < b$. So for some $n \in \mathbb{N}$ we have $M_2 \models a < b < a^n$ so $M_2 \models a < b < a_{2,n}$. Also $a_{1,n} E^3_{M_1} a_{2,n}$ by $(*)_6$, so by Claim 3.1(1) we have $a_{1,n} E^4_{M_1} a_{2,n}$. Hence for some m, $M_1 \models (a_{1,n})^m \geq a_{2,n}$, in fact, even $m = 2$ is sufficient. So $M_1 \models b < a_{1,mn}$, but $a <_M b$. Together, this yields $a E^4_{M_1} b$.

Second, assume $\neg(a E^4_{M_2} b)$ and, without loss of generality, $a <_M b$. So for every $n \in \mathbb{N}$, $a_{2,n} <_M b$ and by $(*)_6$, we have $a_{2,n} E^3_{M_1} a_{1,n}$ hence $a_{1,n}/E^3_{M_1}$ has a member $< b$, and so in particular $a_{1,n+1}/E^3_{M_1}$ has a member $< b$, but $a_{1,n}/E^3_{M_1}$ is below $a_{1,n+1}/E^3_{M_1}$ (remember the definitions!) so $a_{1,n} <_M b$. As this holds for every $n \in \mathbb{N}$ we conclude $\neg(a E^4_{M_1} b)$.]

Let

$I^k_d = \{c \in M : M_k \models c^n < d \text{ for every } n \in \mathbb{N}\}$ for $d \in M$ and $k = 1, 2$. $(*)_8$

Now

$I^1_d = I^2_d$ for $d \in M$. $(*)_9$

[Why? By $(*)_7$, since $I^\ell_d = \{c : c/E^4_{M_\ell} \text{ is below } d\}$.]

$E^3_{M_2} = E^3_{M_1}$. $(*)_{10}$

[First, if $a_1 E^3_{M_2} a_2$ then by Claim 2.4(3),(4) we have $a_1 E^5_{M_2} a_2$

We recall that $M^<_1 = M = M^<_2$ and therefore the latter is equivalent to $a_1 E^5_{M_1} a_2$. Since we are assuming that M_1 is 3-order rigid, this implies $a_1 E^3_{M_1} a_2$.

Second, assume $\neg(a_1 E^3_{M_2} a_2)$ and, without loss of generality, $a_1 <_M a_2$. As $M_2 \models \mathsf{PA}$, for some $c \in M$ we have $M_2 \models ca_1 \leq a_2 < (c+1)a_2$ so by the previous sentence $c \notin I^2_{a_1}$ hence by $(*)_9$ also $c \notin I^1_{a_1}$.

By $(*)_5$ we have $(c \times_{M_1} a_1) E^3_{M_1} (c \times_{M_2} a_1)$ and as $c \times_{M_2} a_1 \leq_M a_2$ clearly $(c \times_{M_2} a_1)/E^3_{M_1} \leq a_2/E^3_{M_1}$ so together $(c \times_{M_1} a_1)/E^3_{M_1}$ is smaller or equal to $a_2/E^3_{M_1}$ so for some $a_3 \in a_2/E^3_{M_1}$ we have $c \times_{M_1} a_1 < a_3$. As $c \notin I^1_{a_1}$ this implies a_1, a_3 are not $E^3_{M_1}$-equivalent, so by the choice of a_3 also $\neg(a_1 E^3_{M_1} a_2)$, so we are done proving $(*)_{10}$.]

Hence by $(*)_7$ and $(*)_{10}$, part (1) holds.

Parts (2) and (3) follow from part (1) and Claim 3.6 below for the function $x \mapsto 2^{2^x}$, and the equivalence relation E^4_M. □

Claim 3.6 The models M_1, M_2 are almost $\{<, +\}$-isomorphic when for $k \in \{1, 2\}$,

(a) E_k is a convex equivalence relation on $M_k \backslash \mathbb{N}$;
(b) h is an order-isomorphism from M_1 onto M_2 mapping E_1 onto E_2;

(c) f_k is a function definable in M_k, is increasing, maps $\mathbb{N}$ into $\mathbb{N}$, for each E_k-equivalence class Y the set $\{a \in M : f_k(a) \in Y\}$ has the order type of $\mathbb{Z}$ and is unbounded from below and from above in Y; and
(d) if $a_1 E_k a_2$ and $b_1 E_k b_2$ then $(a_1 + b_1) E_k (a_2 + b_2)$.

Proof. As in the proof of Theorem 2.6. □

Claim 3.7 Assume that h is an order-isomorphism from M_1 onto M_2 and M_1 is 4-order rigid. Then for $a, b \in M$, we have that $h(a) E^4_{M_2} h(b)$ implies $a E^4_{M_1} b$.

Proof. Without loss of generality, h is the identity and let $M_1^< = M = M_2^<$. Define $E_M = \{(a, b)$: there are $n \in \mathbb{N}$, $c_1 \in M$, and $c_2 \in M$ such that $c_1 \leq_M \min_M\{a, b\}$ and $\max_M\{a, b\} \leq c_2$ and $(M_{<c_1})^n \cong M_{<c_2}\}$. Easily E_M is a convex equivalence relation, $E^4_{M_\ell} \subseteq E_M$ for $\ell = 1, 2$ and $E^4_{M_1} = E_M$. □

Claim 3.8

(1) Assume that $a \in M$ is non-standard and $I = \{b/E^3_M : b \in a/E^4_M\}$, naturally ordered. Then the linear order I can be embedded into $\mathbb{R}_{>0}$ with dense image.
(2) If M is $\aleph_1$-saturated then the embedding is onto $\mathbb{R}_{>0}$.
(3) Moreover, if we define $+_I$ by $(b_1/E^3_M) +_I (b_2/E^3_M) = (b_3/E^3_M)$ when $b_1 \times_M b_2 = b_3$, then the embedding commutes with addition, so the image is an additive sub-semi-group of $\mathbb{R}$. Also $1_{\mathbb{R}}$ belongs to the image.

Proof. As in Definition 4.1, but we elaborate: Fix $a \in M$ and for $b \in a/E^4_M$ let $\mathscr{S}_b = \left\{\frac{m_1}{m_2} : m_1, m_2 \in \mathbb{N}\backslash\{0\}\right.$ and $\left.M \models b^{m_2} \geq a^{m_1}\right\}$. Clearly, $\mathscr{S}_b$ is a subset of $\mathbb{Q}_{>0}$ and as M is a model of PA, clearly $\mathscr{S}_b$ is an initial segment of $\mathbb{Q}_{>0}$. By the definition of $b \in a/E^4_M$, we necessarily have that $\mathscr{S}_b \neq \varnothing$ and $\mathscr{S}_b \neq \mathbb{Q}_{>0}$, so together $r_b = \sup(\mathscr{S}_b)$ belongs to $\mathbb{R}_{>0}$.

Again by PA,

(a) $r_{b_1} \leq r_{b_2}$ when $b_1 \leq_M b_2$ are from a/E^4_M. [Why? Follows directly from the definition of $\mathscr{S}_{b_1}$ and $\mathscr{S}_{b_2}$.]
(b) $r_{b_1} = r_{b_1}$ if and only if $b_1 E^3_M b_2$ for any b_1 and $b_2 \in a/E^4_M$. [Why? By the definition of E^3_M.]
(c) if $\mathbb{Q} \models \frac{m_1}{m_2} < \frac{m_3}{m_4}$ where $m_\ell \in \mathbb{N}\backslash\{0\}$ for $\ell = 1, 2, 3, 4$ then for some $b \in a/E^4_M$ we have $\mathbb{R} \models \frac{m_1}{m_2} \leq r_b < \frac{m_3}{m_4}$. [Why? Let $n \in \mathbb{N}\backslash\{0\}$ be such that $M \models b < a^n$, exists as $b \in a/E^4_M$. Without loss of generality, $m_2 = m_4$ call it m, so necessarily $m_1 < m_3$ and, without loss of generality, $m_1 + n < m_3$. Now by the definition of $b \mapsto r_b$ the requirement on b means that $M \models b^m \geq a^{m_1}$ and $b^m < a^{m_3}$. Let b be the minimal element of M such that $M \models b^m \geq a^{m_1}$. Then $M \models b^{m-1} < a^{m_1}$, and hence $M \models b^m < a^{m_1} b \leq a^{m_1} a^n = a^{m_1+n} \leq a^{m_3}$; so b is as required.]
(d) $\{r_b : b \in a/E^4_M\}$ is a dense subset of $\mathbb{R}_{>0}$. [Why? By (c).]
(e) If M is $\aleph_1$-saturated then $\{r_b : b \in a/E^4_M\} = \mathbb{R}_{>0}$, i.e., part (2). [Why? For any real r and n we can find $b_1, b_2 \in a/E^4_M$ such that $r - \frac{1}{n} < r_{b_2} < r < r_{b_2} < \frac{1}{n}$ by (d) and "$\mathbb{Q}$ is dense in $\mathbb{R}$".]
(f) Part (3) of the claim holds. [Why? By the rules of exponentiation which can be expressed in PA.]

Together we are done. □

So we can arguably say that the distance between E^3_M and E^4_M is small.

4 Constructing somewhat rigid models

In this section, we assume that λ is regular.

Definition 4.1 For any model $M \models$ PA, we define

(a) $\mathbb{Z}_M = \mathbb{Z}[M]$ to be the ring generated by M (so $a \in \mathbb{Z}_M$ iff $a = b \vee a = -b$ for some $b \in M$, of course $\mathbb{Z}_M$ is determined only up to isomorphism over M; similarly below); when, as usual, M is ordinary, without loss of generality, $\mathbb{Z}_M \supseteq \mathbb{Z}$;

(b) $\mathbb{Q}_M = \mathbb{Q}[M]$ be the field of quotients of $\mathbb{Z}_M$; in fact, it is an ordered field, if M is ordinary then, without loss of generality, $\mathbb{Q}_M \supseteq \mathbb{Q}$; and
(c) $\mathbb{R}_M = \mathbb{R}[M]$ be the closure of $\mathbb{Q}_M$ adding all definable cuts, so in particular it is a real closed field. Note that $\mathbb{R}[M]$ is a sub-field of the Scott-Cauchy completion $\bar{\mathbb{R}}[M]$ of $\mathbb{Q}[M]$ and that for so called "rather classless" models M, $\mathbb{R}[M]$ coincides with $\bar{\mathbb{R}}[M]$. (Cf. more on completions in [4].)
(d) We let $S_M = \mathbb{R}_M^{\text{bd}}/\mathbb{R}_M^{\text{infi}}$ where ("bd" stands for bounded, "infi" stands for infinitesimal)
   1. $\mathbb{R}_M^{\text{bd}} = \{a \in \mathbb{R}_M : \mathbb{R}_M \models -n < a < n \text{ for some } n \in \mathbb{N}\}$,
   2. $\mathbb{R}_M^{\text{infi}} = \{a \in \mathbb{R}_M^{\text{bd}} : \mathbb{R}_M \models -1/n < a < 1/n \text{ for every } n \in \mathbb{N}\}$,
   3. $\mathbf{j}_M$ is the function from $\mathbb{R}_M^{\text{bd}}$ into $\mathbb{R}$ such that $M \models n_1/m_1 < a < n_2/m_2$ implies that $\mathbb{R} \models n_1/m_1 < \mathbf{j}_M(a) < n_2/m_2$ for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ such that $m_1, m_2 > 0$. (Cf. Claim 3.8.)

Definition 4.2 Let $\text{AP} = \text{AP}_\lambda$ be the set of $\mathbf{a}$ such that

(a) $\mathbf{a} = (M, \Gamma) = (M_\mathbf{a}, \Gamma_\mathbf{a})$;
(b) M is a model of PA;
(c) $|M|$, the universe of M, is an ordinal $< \lambda^+$;
(d) Γ is a set of $\leq \lambda$ of types over M;
(e) each $p \in \Gamma$ has the form $\{a_{p,\alpha} < x < b_{p,\alpha} : \alpha < \lambda\}$ where for $\alpha < \beta$, we have that $M \models a_{p,\alpha} < a_{p,\beta} < b_{p,\beta} < b_{p,\alpha}$; and
(f) M omits every $p \in \Gamma$.

In the following, we may write $M^\mathbf{a}_{<b}$ instead $(M_\mathbf{a})_{<b}$. Alternatively, replace M^0_b by $(M_\mathbf{a})_{<b}$ in the proof of Main Claim 4.5 below.

Definition 4.3

(1) We define the binary relation $\leq_{\text{AP}}$ on AP by: $\mathbf{a} \leq_{\text{AP}} \mathbf{b}$ iff $M_\mathbf{a} \prec M_\mathbf{b}$ and $\Gamma_\mathbf{a} \subseteq \Gamma_\mathbf{b}$.
(2) Let $\text{AP}_T = \{\mathbf{a} \in \text{AP} : M_\mathbf{a} \text{ is a model of } T\}$.
(3) Let $\text{AP}^{\text{sat}} = \{\mathbf{a} \in \text{AP} : M_\mathbf{a} \text{ is saturated}\}$ and $\text{AP}^{\text{sat}}_T = \text{AP}_T \cap \text{AP}^{\text{sat}}$.

Claim 4.4

(1) The binary relation $\leq_{\text{AP}}$ is a partial order of AP.
(2) If $\langle \mathbf{a}_\alpha : \alpha < \delta \rangle$ is $\leq_{\text{AP}}$-increasing, δ a limit ordinal $< \lambda^+$ then $\mathbf{a}_\delta = \bigcup\{\mathbf{a}_\alpha : \alpha < \delta\}$ defined by $M_{\mathbf{a}_\delta} = \bigcup\{M_{\mathbf{a}_\alpha} : \alpha < \delta\}$, $\Gamma_{\mathbf{a}_\delta} = \bigcup\{\Gamma_{\mathbf{a}_\alpha} : \alpha < \delta\}$, is a $\leq_{\text{AP}}$-lub of $\langle \mathbf{a}_\alpha : \alpha < \delta \rangle$.
(3) Assume $\lambda = \lambda^{<\lambda} > \aleph_0$. If $\mathbf{a} \in \text{AP}$ then there is $\mathbf{b}$ such that $\mathbf{a} \leq_{\text{AP}} \mathbf{b}$ and $M_\mathbf{b}$ is saturated (of cardinality λ*)*.

Proof. Easy. □

Main Claim 4.5 If (A) then (B) where:

(A) (a) $\lambda = \aleph_0$, $\mathbf{a} \in \text{AP}$
   (b) $M_\mathbf{a} \models a_* > b_* > n$ for $n \in \mathbb{N}$ and a_*, b_* are not $E^3_{M_\mathbf{a}}$-equivalent
   (c) F is an order isomorphism from $M^\mathbf{a}_{<a_*}$ onto $M^\mathbf{a}_{<b_*}$
(B) there are $\mathbf{b}, c_*$ satisfying
   (a) $\mathbf{a} \leq_{\text{AP}} \mathbf{b}$
   (b) $c_* <_{M_\mathbf{b}} a_*$ so $c_* \in M_\mathbf{b}$ but $c_* \notin M_\mathbf{a}$
   (c) some $p \in \Gamma_\mathbf{b}$ is equivalent to $\{F(a_1) < x < F(a_2) : a_1, a_2 \in M_\mathbf{a} \text{ and } M_\mathbf{b} \models a_1 < c_* < a_2 \leq a_*\}$ recalling F is the isomorphism from *(A)(c)*.

Proof. The proof of this claim is long and complicated and we believe that it cannot be easily digested. Therefore, we have an informal discussion of the proof after the end of the proof. We recommend that the reader try to look at it from time to time, in particular if he or she has lost track of the main idea of the proof. We hope that this will help.

In this proof, we use a_*, b_*, c_* as fixed and use symbols such as a or a_i as variables.

Stage A. We start with various definitions.

(a) Let $\Phi = \Phi_{\mathbf{a}}$ be the set of formulas $\varphi(x) = \varphi(x, \bar{a})$ with $\varphi(x, \bar{y}) \in \mathcal{L}(\tau_{\mathsf{PA}})$ and $\bar{a} \in {}^{\mathrm{lh}(\bar{y})}(M_{\mathbf{a}})$.
(b) We write $\varphi'(x) \vdash \varphi''(x)$ if both are from Φ and $M_{\mathbf{a}} \models (\forall x)(\varphi'(x) \to \varphi''(x)$.
(c) If $\varphi := \varphi(x) := \varphi(x, \bar{a}) \in \Phi$, we write $\varphi(M_{\mathbf{a}}) = \varphi(M_{\mathbf{a}}, \bar{a}) = \{b \in M_{\mathbf{a}} : M_{\mathbf{a}} \models \varphi[b, \bar{a}]\}$
(d) If $k \in \mathbb{N}$, we define $\Sigma^k_{\mathbf{a}} = \Sigma^k_{M_{\mathbf{a}}}$ as the set of $\sigma(x_0, \ldots, x_{k-1}) := \sigma(x_0, \ldots, x_{k-1}, \bar{a})$ such that $\sigma(\bar{x}, \bar{y})$ is a definable function in $M_{\mathbf{a}}$ and $\bar{a} \in {}^{\mathrm{lh}(\bar{y})}(M_{\mathbf{a}})$. If $k = 1$, we omit it in the notation and write $\sigma(x, \bar{y})$ and $\sigma(x)$ in this case. $\boxplus_1$
(e) We let $\xi(\varphi) = \xi(\varphi(x)) = \mathbf{j}_M(\log_2(|\varphi(M_{\mathbf{a}})|)/\log(a_*))$ for $\varphi := \varphi(x) \in \Phi_{\mathbf{a}}$ such that $\varphi(M) \subseteq [0, a_*)_M$; cf. Definition 4.1(d).
(f) If $\varphi_1, \varphi_2 \in \Phi$, $\varphi_2(M) \neq \varnothing$ and $\sigma \in \Sigma_{\mathbf{a}}$, then we let $\sigma' := \sigma[\varphi_1, \varphi_2]$ be the following function from $\varphi_1(M)$ to $\varphi_2(M) \cup \{0\}$, definable in M: $M \models \sigma'(a) = b$ if and only if $a \in \varphi_1(M)$ and $b = \max\{b :$ either $b = 0$ and $\sigma(a) < \min(\varphi_2(M))$ or $b \in \varphi_2(M)$ and $b \leq \sigma(a)\}$.

In (e), by "$|\varphi(M_{\mathbf{a}})|$" and "$\log_2$", we mean the obvious things. Of course, both are expressible by a formula in $\mathcal{L}(\tau_{\mathsf{PA}})$. If (f), one could say that σ' is such σ restricted to $\varphi_1(M)$ while ensuring that the image lies in $\varphi_2(M) \cup \{0\}$.

We now define $\mathbb{P}$; it will serve as a set of approximations to $\mathrm{tp}(c_*, M_{\mathbf{a}}, M_{\mathbf{b}})$:

We let $\mathbb{P}$ consist of pairs $\bar{\varphi} = (\varphi_1, \varphi_2)$ such that

(a) $\varphi_\ell = \varphi_\ell(x)$ are from Φ;
(b) $\varphi_1(x) \vdash x < a_*$ and $\varphi_2(x) \vdash x < b_*$; $\boxplus_2$
(c) if $a_1 < a_2$ are from $\varphi_1(M_{\mathbf{a}})$, then $[F(a_1), F(a_2)]_M \cap \varphi_2(M) \neq \varnothing$;
(d) $\xi(\varphi_1(M)) > \xi(\varphi_2(M))$.

We let $\bar{\varphi}' \leq \bar{\varphi}''$ if and only if $\varphi''_\ell(x) \vdash \varphi'_\ell(x)$ for $\ell \in \{1, 2\}$.

Note that $(\varphi_1(x, \bar{a}), \varphi_2(x, \bar{a}_2)) \in \mathbb{P}$ is not definable in $M_{\mathbf{a}}$ mainly because of $\boxplus_2$(c). Also observe that

$$\text{if } \bar{\varphi} \in \mathbb{P} \text{ and } \varphi(x) \in \Phi \text{ then for some } \mathbf{t} \in \{0, 1\} \text{ we have } (\varphi_1(x) \wedge \varphi(x)^{\mathbf{t}}, \varphi_2(x)) \in \mathbb{P}$$
$$(\text{and is } \leq_{\mathbb{P}} \text{-above } \bar{\varphi};\ \text{recall that } \varphi^{\mathbf{t}} \text{ is } \varphi \text{ is } \mathbf{t} = 1 \text{ and is } \neg\varphi \text{ if } \mathbf{t} = 0). \quad \boxplus_3$$

[Why? We have that $M_{\mathbf{a}} \models |\varphi_1(M) \cap \varphi(M)| \geq \frac{1}{2}|\varphi_1(M)|$ or $M_{\mathbf{a}} \models |\varphi_1(M) \backslash \varphi(M)| \geq \frac{1}{2}|\varphi_2(M)|$. As $a_* \notin \mathbb{N}$, clearly $\xi(\varphi_1(x) \wedge \varphi(x)) = \xi(\varphi_2(x))$ or $\xi(\varphi_2(x) \wedge \neg\varphi(x)) = \xi(\varphi_2(x))$. So clearly we are done proving $\boxplus_3$.]

Stage B. We arrived at a major point: how can we continue to omit members of $\Gamma_{\mathbf{a}}$? This stage of the proof is devoted to proving the following statement:

If $\bar{\varphi} \in \mathbb{P}$, $\sigma(x) \in \Sigma_{\mathbf{a}}$ and $p(x) \in \Gamma_{\mathbf{a}}$, then for some $\bar{\varphi}'$ and n we have

(a) $\bar{\varphi} \leq \bar{\varphi}' \in \mathbb{P}$ and $\boxplus_4$
(b) $\varphi'_0(x) \vdash \sigma(x) \notin (a_{p,n}, b_{p,n})$.

For the proof, we use what can be called a "wedge question":

Case 1. There is $d \in M_{\mathbf{a}}$ such that $\bar{\varphi}' := (\varphi_1(x) \wedge \sigma(x) = d, \varphi_2(x)) \in \mathbb{P}$.

In this case obviously $\bar{\varphi} \leq \bar{\varphi}' \in \mathbb{P}$. Also as $d \in M_{\mathbf{a}}$ and $M_{\mathbf{a}}$ omit $p(x)$ recalling $p(x) \in \Gamma_{\mathbf{a}}$, clearly d does not realize $p(x)$ hence for some n and $d \notin (a_{p,n}, b_{p,n})_{M_{\mathbf{a}}}$; so $\bar{\varphi}'$ and n are as promised.

Case 2. Not case 1. So

$$\xi(\varphi(x) \wedge \sigma(x) = d) \leq \xi(\varphi_2(x)) \text{ for every } d \text{ from } M_{\mathbf{a}}. \qquad (*)_{4.1}$$

Clearly there is a minimal $d_* \in M_{\mathbf{a}}$ satisfying

$$M_{\mathbf{a}} \models |\{c \in \varphi_1(M) : \sigma(c) \leq d_*\}| \geq \frac{1}{2}|\varphi_1(M_{\mathbf{a}})|. \qquad (*)_{4.2}$$

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So

$$M_{\mathbf{a}} \models |\{c \in \varphi_1(M) : \sigma(c) \geq d_*\}| \geq \frac{1}{2}|\varphi_2(M_{\mathbf{a}})|. \qquad (*)_{4.3}$$

But $M_{\mathbf{a}}$ omits the type $p(x)$ as $p(x) \in \Gamma_{\mathbf{a}}$ and $d_* \in M_{\mathbf{a}}$, so for some n and $d_* \notin (a_{p,n}, b_{p,n})$. So one of the following sub-cases occurs:

Sub-case 2a: $d_* \leq a_{p,n}$. Let $\varphi_1'(x) = \varphi_1(x) \wedge (\sigma(x) \leq a_{p,n+1})$ and $\varphi_2'(x) = \varphi_2(x)$. Now the pair $\bar{\varphi}' = (\varphi_1', \varphi_2')$ is as required, noting (by $(*)_{4.1}$ that

$$M_{\mathbf{a}} \models |\varphi_1'(M)| \geq |\{a \in \varphi_1(M) : \sigma(c) \leq d_*\}| \geq \frac{1}{2}|\varphi_1(M)|$$

hence

$$\xi(\varphi_1'(x)) = \xi(\varphi_1(x)) > \xi(\varphi_2(x)) = \xi(\varphi_2'(x)).$$

Sub-case 2b: $d_* \geq a_{p,n}$. Let $\varphi_1'(x) = \varphi_2(x) \wedge (\sigma(x) \geq b_{p,n+1})$ and $\varphi_2'(x) = \varphi_2(x)$. We note by $(*)_{4.2}$ that

$$M_a \models |\varphi_1'(M)| \geq |\{c \in \varphi_1(M) : \sigma(c) \geq d_*\}| \geq \frac{1}{2}|\varphi_1(M)|;$$

thus,

$$\xi(\varphi_1'(x)) = \xi(\varphi_1(x)) > \xi(\varphi_2(x)) = \xi(\varphi_2'(x)),$$

and so $\bar{\varphi}' = (\varphi_1', \varphi_2')$ is as required, and we are done proving $\boxplus_4$.

Stage C. How do we omit the new type? Recall that c_* will realize a type to which $\bar{\varphi} \in \mathbb{P}$ is an approximation and we have to omit the relevant type from clause (B)(c) of the claim.

This stage is dedicated to proving the following statement:

If $\bar{\varphi} \in \mathbb{P}$ and $\sigma(x) \in \Sigma_{\mathbf{a}}$, then for some $\bar{\varphi}'$,

(a) $\bar{\varphi} \leq \bar{\varphi}' \in \mathbb{P}$; $\boxplus_5$
(b) for some $a_1 < a_2 \leq a_*$, we have $\varphi_1'(x) \vdash a_1 \leq x < a_2$ and $\varphi_1'(x) \vdash \neg(F(a_1) \leq \sigma(x) < F(a_2))$.

First note

if there is $\bar{\varphi}' \in \Phi$ such that $\bar{\varphi} \leq \bar{\varphi}'$ and $\xi(\varphi_1')/\xi(\varphi_2') > 2$, then the conclusion of $\boxplus_5$ holds. $(*)_{5.1}$

[Why? For pairwise disjoint sets, we have $|\bigcup_i A_i| = \sum_i |A_i|$. A version of this is provable in PA; we use this to find $\varphi_1''(x) \in \Phi$ such that $\varphi_1''(M_{\mathbf{a}}) \subseteq \varphi_1'(M_{\mathbf{a}})$, $M_{\mathbf{a}} \models |\varphi_1'(M_{\mathbf{a}})|/|\varphi_1''(M_{\mathbf{a}})| \leq |\varphi_2(M_{\mathbf{a}})|$ and $\sigma[\varphi_1'', \varphi_2]$, which was defined in Clause $\boxplus_1$(f), is constant, say, constantly e, hence $e \in \varphi_2(M_{\mathbf{a}})$. So $\xi(\varphi_1''(x)) \geq \xi(\varphi_1'(x)) - \xi(\varphi_2'(x)) > 2\xi(\varphi_2'(x)) - \xi(\varphi_2'(x)) = \xi(\varphi_2'(x))$ hence $(\varphi_1'', \varphi_2')$ belongs to $\mathbb{P}$ and $\bar{\varphi} \leq \bar{\varphi}' \leq (\varphi_1'', \varphi_2')$. As $e \in \varphi_2(M_{\mathbf{a}}) \subseteq [0, b_*)_{M_{\mathbf{a}}}$ and F is onto $M^{\mathbf{a}}_{<b_*}$ for some $d <_{M_{\mathbf{a}}} a_*$ we have $F(d) = e$. By $\boxplus_3$, without loss of generality, $\varphi''(x) \vdash x < d$ or $\varphi''(x) \vdash d \leq x$ so we can choose (a_1, a_2) as $(0, d)$ or as (d, a_*). So we are done.]

So we can assume $(*)_{5.1}$ does not apply. Without loss of generality, for no $\bar{\varphi}' \in \Phi$ do we have $\bar{\varphi} \leq \bar{\varphi}'$ and

$$\xi(\varphi_1')/\xi(\varphi_2') > (1 + \frac{1}{8})\xi(\varphi_1)/\xi(\varphi_2) \qquad (*)_{5.2}$$

(where $\frac{1}{8}$ is an arbitrary choice and can be replace by any fixed $\varepsilon > 0$).

[Why? We try to choose $\bar{\varphi}^n$ by induction on $n \in \mathbb{N}$ such that $\bar{\varphi}^n \in \mathbb{P}$, $\bar{\varphi}^0 = \bar{\varphi}$, $\bar{\varphi}^n \leq \bar{\varphi}^{n+1}$, and $\xi(\varphi_1^n)/\xi(\varphi_2') \geq (1 + \frac{1}{8})^n \xi(\varphi_1)/\xi(\varphi_2)$. So for some n we have $\xi(\varphi_1^n)/\xi(\varphi_2') > 2$ and we can apply $(*)_{5.1}$, contradiction. But $\bar{\varphi}^0$ is well defined, hence for some n, $\bar{\varphi}^n$ is well defined but we cannot choose $\bar{\varphi}^{n+1}$. Now $\bar{\varphi}^n$ is as required in $(*)_{5.2}$.]

Clearly,

if $a_1 < a_2$ are from $\varphi_1(M)$ $b_1 \leq F(a_1) < F(a_2) \leq b_2$ are from $\varphi_2(M)$ and they are such that $\xi(\varphi_1(x) \wedge a_1 \leq x < a_2 \wedge \sigma(x) \notin [b_1, b_2)) > \xi(\varphi_2(x) \wedge b_1 \leq x < b_2)$, $(*)_{5.3}$

then we are done. So from now on we assume that there are no a_1, a_2, b_1, and b_2 as in $(*)_{5.3}$.

(a) Let $k_* \in \mathbb{N}\backslash\{0\}$ be large enough such that $(\xi(\varphi_1) - \xi(\varphi_2))/\xi(\varphi_2) > 2/k_*$.
(b) Let $n(1) \in \mathbb{N}$ be large enough such that $\xi(\varphi_1) - \xi(\varphi_2) > 1/n(1)$ and $\xi(\varphi_2) > (k_* + 1)/n(1)$. $(*)_{5.4}$

(c) Let $n(2) > n(1)$.

We furthermore define:

(a) Let $\psi_1(x_1, x_2; y_1, y_2) := x_1 < x_2 \wedge \varphi_1(x_1) \wedge \varphi_1(x_2) \wedge y_1 < y_2 < b_*$.
(b) Let $\psi_2(x_1, x_2; y_1, y_2)$ be the conjunction of $\psi_1(x_1, x_2, y_1, y_2)$ and $|\{x : \varphi_1(x) \wedge x_1 \leq x < x_2\}|^{n(1)} \geq |\{y : \varphi_2(y) \wedge y_1 \leq y < y_2\}|^{n(1)} \times a_*$.
(c) Let $\vartheta_2(x_1, x_2) := (\exists y_1, y_2)(\psi_2(x_1, x_2; y_1, y_2)$. $(*)_{5.5}$
(d) Let $\psi_3(x_1, x_2; y_1, y_2)$ be the conjunction of $\psi_2(x_1, x_2; y_1, y_2)$ and $|\{x : \varphi_1(x) \wedge x_1 \leq x < x_2 \wedge \sigma(x) \notin [y_1, y_2)\}|^{n(2)} < |\{y : \varphi_2(a) \wedge y_1 \leq y < y_2\}|^{n(2)} \times a_*$.
(e) Finally, let $\vartheta_3(x_1, x_2) := (\exists y_1, y_2)\psi_3(x_1, x_2, y_1, y_2)$.

So, by our assumptions (for clause (b), use "$(*)_{5.3}$ does not apply") we have

$$\begin{aligned}
&\text{(a)} \quad \text{if } a_1 < a_2 \text{ are from } \varphi_1(M_{\mathbf{a}}) \text{ then } M_{\mathbf{a}} \models \psi_1[a_1, a_2; F^{[\varphi_2]}(a_1), F^{[\varphi_2]}(a_2)]\\
&\text{(b)} \quad \text{if } M_{\mathbf{a}} \models \psi_2[a_1, a_2; F^{[\varphi_2]}(a_1), F^{[\varphi_2]}(a_2)] \text{ then}\\
&\qquad M_{\mathbf{a}} \models \psi_3[a_1, a_2; F^{[\varphi_2]}(a_1), F^{[\varphi_2]}(a_2)]\\
&\qquad M_{\mathbf{a}} \models \vartheta_3[a_1, a_2].
\end{aligned} \qquad (*)_{5.6}$$

Clearly

$$\text{if } M_{\mathbf{a}} \models \psi_3[a_1, a_2; b_1^\iota, b_2^\iota] \text{ for } \iota = 1, 2 \text{ then } [b_1^1, b_2^1)_{M_{\mathbf{a}}} \cap [b_1^2, b_2^2)_{M_{\mathbf{a}}} \neq \emptyset. \qquad (*)_{5.7}$$

It is well known that in linear orders, the intersection of any finite family of intervals is non-empty if and only if the intersection of any two is non-empty. A version of this statement can be proved in PA; therefore, for some $\sigma(x_1, x_2) \in \Sigma^2_{\mathbf{a}}$, we have

$$\begin{aligned}
&\text{if } M_{\mathbf{a}} \models \vartheta_3[a_1, a_2], \text{ then } \sigma(a_1, a_2) \in \varphi_2(M) \text{ and for every } b_1, b_2 \in \varphi_2(M),\\
&\quad \text{we have that } M_{\mathbf{a}} \models \psi_3[a_1, a_2; b_1, b_2] \text{ implies that } \sigma(a_1, a_2) \in [b_1, b_2)_{M_{\mathbf{a}}}.
\end{aligned} \qquad (*)_{5.8}$$

Now

(a) let $\varepsilon \in \mathbb{Q}_M \subseteq \mathbb{R}_M$ be a true rational such that $\xi(\varphi_2) > \varepsilon > \xi(\varphi_2)k_*/(k_* + 1) + 1/n(1)$ and $(*)_{5.9}$
(b) let $d_* = \lfloor (a_*)^\varepsilon \rfloor \in M_{\mathbf{a}}$ computed in $\mathbb{R}_{\mathbf{a}}$ and $c_* = \lfloor (a_*)^{\varepsilon - 1/n(1)} \rfloor$.

In $M_{\mathbf{a}}$, we can define an increasing sequence $\langle a_{1,i} : i < i(*)\rangle$ with $i(*) \in M_{\mathbf{a}}$ such that

$$\begin{aligned}
&a_{1,0} = 0,\\
&a_{1,i(*)} = a_*, \text{ and}\\
&a_{1,i+1} = \min\{a : \varphi_1(a) \text{ and } a_{1,i} < a \text{ and } |\varphi_1(M_{\mathbf{a}}) \cap [a_{1,i}, a)| \text{ is } d_*\}.
\end{aligned} \qquad (*)_{5.10}$$

Then $d_* \leq |\{a : \varphi_1(a) \text{ and } a_{1,i(*)-1} \leq a < a_*\}| \leq 2d_*$. In $M_{\mathbf{a}}$, we can define

(a) $u = \{i < i(*) : M_{\mathbf{a}} \models \vartheta_3[a_{1,i}, a_{1,i+1}]\}$ and $v = \{i < i(*) : i \notin u\}$. $(*)_{5.11}$
(b) Furthermore, let $\varphi_{1,i}(x) := \varphi_1(x) \wedge a_{1,i} \leq x < a_{1,i+1}$.

The following statement follows and will be used in *Case 1* below:

(a) For $i < i(*)$, we have $\xi(\varphi_{1,i}(x)) = \varepsilon$.
(b) If $i < i(*)$ and $i \in v$, then

$$\begin{aligned}
M_{\mathbf{a}} \models\ &|\varphi_2(M_{\mathbf{a}}) \cap (F(a_{1,i}), F(a_{1,i+1}))_{M_{\mathbf{a}}}|\\
&\geq |\varphi_{1,i}(M_{\mathbf{a}})| \times a_*^{-1/n(1)} = b_* \times a_*^{-1/n(1)}.
\end{aligned} \qquad (*)_{5.12}$$

[Why? Clause (a) is obvious by the definition of $\xi(-)$ and $a_{1,i+1}$. For clause (b) note that by the definition of ϑ_3 in $(*)_{5.5}$(e) we have $M_{\mathbf{a}} \models \neg\psi_3[a_{1,i}, a_{2,i}; F^{[\varphi_2]}(a_{1,i}), F^{[\varphi_2]}(a_{2,i})]$, but by $(*)_{5.6}$(a) we have $M_{\mathbf{a}} \models \psi_3[a_{1,i}, a_{2,i}; F(a_{1,i}), F(a_{1,i})]$. By the definition of ψ_3 in $(*)_{5.5}$(d) we are done.]

The following observation will be used in *Case 2* below:

$$\text{if } i_1 < i_2 \text{ are from } u \text{ then } F(a_{1,i_1}) < \sigma^M(a_{1,i_2}, a_{1,i_2+1}). \qquad (*)_{5.13}$$

[Why? Obvious by $(*)_{5.8}$.]

We define terms $\sigma_1(x_1, x_2), \sigma_2(x_1, x_2) \in \Sigma^2_{\mathbf{a}}$ such that if $i < i(*)$ then

(a) $M_{\mathbf{a}} \models \sigma_1(a_{1,i}, a_{1,i+1}) < \sigma(a_{1,i}, a_{1,i+1}) < \sigma_2(a_{1,i}, a_{1,i+1})$,
(b) $\varphi_2(M_{\mathbf{a}}) \cap [\sigma^{M_{\mathbf{a}}}(a_{1,i}, a_{1,i+1}), \sigma_2^{M_{\mathbf{a}}}(a_{1,i}, a_{1,i+1}))_{M_{\mathbf{a}}}$ has $\leq c_*$ elements in the sense of $M_{\mathbf{a}}$,
(c) $\varphi_2(M_{\mathbf{a}}) \cap [\sigma_1^{M_{\mathbf{a}}}(a_{1,i}, a_{1,i+1}), \sigma^{M_{\mathbf{a}}}((a_{1,i}, a_{1,i+1}))_{M_{\mathbf{a}}}$ has $\leq c_*$ in the sense of $M_{\mathbf{a}}$, $(*)_{5.14}$
(d) if $i < j$ are from u, then $M_{\mathbf{a}} \models \sigma_2(a_{1,i}, a_{1,i+1}) < \sigma(a_{1,j}, a_{1,j+1})$,
(e) $M_{\mathbf{a}} \models \sigma_1(a_{1,i}, a_{1,i+1}) < \sigma(a_{1,i}, a_{1,i+1}) < \sigma_2(a_{1,i}, a_{1,i+1})$, and
(f) if $i \in u$, then $M_{\mathbf{a}} \models \sigma_1(a_{1,i}, a_{1,i+1}) < F(a_{1,i}) < \sigma(a_{1,i}, a_{1,i+1}) < F(a_{1,i+1}) < \sigma_2(a_{1,i}, a_{1,i+1})$.

[Why? Let $\sigma_2(a_{1,i}, a_{1,i+1})$ be maximal such that the relevant part of (a) and (b) and (d) hold and $\sigma_1(a_{1,i}, a_{1,i+1})$ be minimal such that the other part of (a) and (c) and (e) hold. By $(*)_{5.8}$, we can finish.]

Now the proof splits into cases:

Case 1: $M_{\mathbf{a}} \models |v| \geq i(*)/3$. Here we shall use $\boxplus_2$(c). Let $v_1 = \{i \in v : M_{\mathbf{a}} \models \text{“}|\{j \in v : j < i\}| \text{ is even”}\}$, so $M_{\mathbf{a}} \models |v_1| \geq i(*)/6$. Let $\varphi_1'(x) := \varphi_1(x) \wedge (\exists z)[z \in v_1 \wedge x \in [a_{1,z}, a_{1,z+1}) \wedge \neg(\exists y)(\varphi_2(y) \wedge y \in [a_{1,z}, a_{1,z+1}) \wedge y < x)]$.

Let $\varphi_2'(x) := \varphi_2(x) \wedge (\text{the number } |\{y : \varphi_2(a) \wedge y < x\}| \text{ is divisible by } c_*)$.

Now

(a) $\bar{\varphi}' := (\varphi_1'(x), \varphi_2'(x)) \in \mathbb{P}$,
(b) $\xi(\varphi_1'(x)) = \xi(\varphi_1(x)) - \varepsilon$, and $(*)_{5.15}$
(c) $\xi(\varphi_2'(x)) = \xi(\varphi_2(x)) - \varepsilon + 1/n(1)$.

So

$$\begin{aligned}\xi(\varphi_1'(x))/\xi(\varphi_2'(x)) &= (\xi(\varphi_1(x)) - \varepsilon)/(\xi(\varphi_2(x) - \varepsilon + 1/n(1))\\ &\geq (\xi(\varphi_1(x) - \xi(\varphi_2(x))/(\xi(\varphi_2) - \xi(\varphi_2)k_*/(k_* + 1))\\ &= (k_* + 1)(\xi(\varphi_1(x)) - \xi(\varphi_2(x)))/\xi(\varphi_2) > 2,\end{aligned}$$

and we fall under $(*)_{5.1}$, thus finishing the proof of $\boxplus_5$.

Case 2: $M_{\mathbf{a}} \models |u| \geq i(*)/3$. In $M_{\mathbf{a}}$, define $u_2 = \{i \in u : |\{j \in \varphi_1(M) : j < i\}| \text{ is even}\}$. So $M_{\mathbf{a}} \models |u_1| \geq i(*)/6$. Now $M_{\mathbf{a}}$ satisfies

$$\begin{aligned}|\varphi_1(M)| \leq (i(*) + 1)d_* &\leq 7\Big|\bigcup\{\varphi_1(M_{\mathbf{a}}) \cap [a_{1,i}, a_{1,i+1}) : i \in u_1\}\Big|\\ &= 7\sum\{|\varphi_1(M_{\mathbf{a}}) \cap [a_{1,i}, a_{1,i+1})| : i \in u_1\}|\\ &\leq 7\sum\{|\varphi_2(M_{\mathbf{a}}) \cap [\sigma_1(a_{1,i}, a_{1,i+1}), \sigma_2(a_{1,i}, a_{1,i+1}))| \times a_*^{1/n(1)} : i \in u_1\}|\\ &\leq 7\Big|\bigcup\{\varphi_2(M_{\mathbf{a}}) \cap [\sigma_1(a_{1,i}, a_{2,i+1}), \sigma_2(a_{1,i}, a_{1,i+1})) : i \in u_2\}\Big| \times a_*^{1/n(1)}\\ &< 7 \times |\varphi_2(M_{\mathbf{a}})| \times a_*^{1/n(1)}.\end{aligned}$$

Together $|\varphi_2(M)| \leq 7 \times |\varphi_2(M)| \times a_{a_*}^{1/n(1)}$. But as $n(1)$ was chosen large enough, i.e., $1/n(1) < \xi(\varphi_2(M_{\mathbf{a}}) - \xi(\varphi_2(M_{\mathbf{a}}))$, contradiction. So we are done proving $\boxplus_5$.

Stage D. We define the sets $\mathbf{T} = \mathbf{T}_1 \cup \mathbf{T}_2 \cup \mathbf{T}_3$ of tasks where

$$\mathbf{T}_1 = \{(1, \varphi(x)) : \varphi(x) \in \Phi\}, \text{ towards completeness;}$$
$$\mathbf{T}_2 = \{(2, \sigma(x), \varphi(x)) : \sigma(x) \in \Sigma_{\mathbf{a}} \text{ and } \varphi(x) \in \Gamma_{\mathbf{a}}\}, \text{ towards preserving } \varphi(x) \text{ is omitted; and} \qquad \boxplus_6$$
$$\mathbf{T}_3 = \{(3, \sigma(x)) : \sigma(x) \in \Sigma_{\mathbf{a}}\}, \text{ toward "stopping } F\text{" and "omitting the new type".}$$

Clearly $\mathbf{T}$ is countable, let $\langle \mathbf{s}_n : n < \omega\rangle$ be an enumeration of $\mathbf{T}$. We now choose $\bar{\varphi}^n$ by recursion on n such that:

(a) $\bar{\varphi}^n \in \mathbb{P}$,
(b) $\bar{\varphi}^m \le \bar{\varphi}^n$ for $m < n$,
(c) if $n = m+1$ and $\mathbf{s}_m = (1, \varphi(x))$, then $\varphi_n(x) \vdash \varphi(x)$ or $\varphi_n(x) \vdash \neg\varphi(x)$, $\boxplus_7$
(d) if $n = m+1$ and $\mathbf{s}_m = (2, \sigma(x), p(x))$, then for some k, we have that $\varphi_n(x) \vdash \sigma(x) \notin (a_{p,k}, b_{p,k})$, and
(e) if $n = m+1$ and $\mathbf{s}_m = (3, \sigma(x))$, then for some $a_{1,m} < a_{2,m} \le a_*$, we have that $\varphi'_n(x) \vdash a_{1,m} \le x < a_{2,m}$ and $\neg(F(a_{1,n}) \le \sigma(x) < a_{2,m})$.

Why does the recursion work? Choosing $\bar{\varphi}^0$ is trivial; for the successor step, if $\mathbf{s}_m \in \mathbf{T}_1$, we use $\boxplus_3$; if $\mathbf{s}_m \in \mathbf{T}_2$, we use $\boxplus_4$; and if $\mathbf{s}_m \in \mathbf{T}_3$, we use $\boxplus_5$. In more detail, if $\mathbf{s}_m = (3, \sigma(x))$, let $\sigma'(x)$ be defined by $\sigma'(x) = \min\{y : y = b_* \text{ or } \sigma(x) \le y \wedge \varphi(y)\}$; now apply $\boxplus_5$ to $(\bar{\varphi}^m, \sigma'(-))$.

We note that:

$$\{\varphi_n(x) : n < \omega\} \text{ is a complete type over } M_{\mathbf{a}}. \qquad (*)_{7.1}$$

[Why? By clause (c) of $\boxplus_7$ and the choice of $\mathbf{T}_1$.]

By compactness, there are N and c_* such that

(a) $M_{\mathbf{a}} \prec N$ and
(b) c_* realizes $\{\varphi_n(x) : n < \omega\}$. $(*)_{7.2}$

Since T is a completion of PA, it has definable Skolem functions, and therefore, without loss of generality,

$$N \text{ is the Skolem hull of } M_{\mathbf{a}} \cup \{c_*\}. \qquad (*)_{7.3}$$

Now, by $\boxplus_7$(d),

$$N \text{ omits every } p \in \Gamma_{\mathbf{a}}. \qquad (*)_{7.4}$$

Also by $\boxplus_7$(c)

$$N \text{ omits } \{F(a) < x < F(a_1) : a_0 < c_* < a_1 \le a_* \text{ and } \{a_0, a_1\} \subseteq M_{\mathbf{a}}\}. \qquad (*)_{7.5}$$

By renaming, without loss of generality, the set of elements of N is a countable ordinal, so we can finish the proof of the claim. □

As announced at the beginning of the proof of Main Claim 4.5, in the following, we shall give an informal discussion of the proof that will help the reader:

(1) Note that a natural approach is to approximate the type of c_* by formulas $\varphi(x)$ with parameters from M such that $\varphi(x) \vdash x < a_*$ and $M_{\mathbf{a}}$ thinks that $|\varphi(M_{\mathbf{a}})|$ is large enough than b_*. Then for $\sigma(x) \in \Sigma_{\mathbf{a}}$ (i.e., a term with parameters from M), which maps $\{c : c <_M a_*\}$ into $\{d : d <_M b_*\}$ we have to ensure $\sigma(c_*)$ will not realize the undesirable type, so it is natural to "shrink" $\varphi(x)$ to $\varphi'(x)$ such that "$|\varphi'(M)|$ is large enough then $|\varphi(M)|/b_*$" in the sense of M and $\sigma(-)$ is constant on $\varphi'(M)$. This requires that also $|\varphi(M)|/b_*$ is large. So a natural choice to make the statement "$\varphi(x)$ is large" precise would be, e.g., "for every $n \in \mathbb{N}$, $M \models |\varphi(M)| \ge b_*^n$". This is fine if $\neg(a_* E^4_M b_*)$, but not if we know $\neg(a_* E^3_M b_*)$ but possibly $a_* E^4_M b_*$. This issue motivates the main definition of $\mathbb{P}$ above.

(2) We shall say that $(\varphi_1, \varphi_2) \in \mathbb{P}$ is a "poor man's substitute" to the original problem when (a) $[0, a_*)_{M_{\mathbf{a}}}$ is replaced by $\varphi_1(M)$, (b) $[0, b_*)_{M_{\mathbf{a}}}$ is replaced by $\varphi_2(M)$, (c) $F\restriction[0, a_*)_{M_{\mathbf{a}}}$ is replaced by $F\restriction\varphi_1(M_{\mathbf{a}})$, really rounded to $\varphi_2(M)$, and (d) $a_* > b_* \wedge \neg(a_* E^3_{M_{\mathbf{a}}}, b_*)$ is replaced by $\xi(\varphi_1) > \xi(\varphi_2)$. (Cf. $\boxplus_1$(e) in the proof of Main Claim 4.5.)

(3) Why do we demand condition $\boxplus_2$(d) in the proof of Main Claim 4.5?

Assume $M_{\mathbf{a}} \prec M$ and $a \in M \backslash M_{\mathbf{a}}$, $a <_M a_*$ and F^+ is an automorphism of $M{\restriction}\{<\}$ extending F then $\{f(a_1) : a_1 <_M a_*, a_1 <_M a\}$, $\{F(a_2) : a_2 \in M_{\mathbf{a}}, a_* < a_2 < a_*\})$ is a cut of $M_{\mathbf{a}}$, which $F^+(a)$ realizes in M. If M "thinks" $|\varphi_2(M)|$ is $\ll b_*$, F may be one-to-one from $\varphi_2(M)$ onto some definable subset $\varphi_2'(M) \subseteq [0, b_*)_M$. A reasonable suggestion is to demand $|\varphi_2(M)| \gg b_*$. To make the discussion more transparent, consider the case $M_{\mathbf{a}} \models a_* < b_* b_*$.

But then let E be the definable convex equivalent relation on $[0, a_*)$ such that each equivalence class is of size b_{**}, then the cut the new element realizes is really a cut of $[0, a_*)/E$. Now F^+ maps every E-equivalence class to some E'-equivalence class, E' a definable convex equivalence relation on $[0, b_{**})$ and F as a map from $[0, a_*)/E$ into $[0, b_*)/E'$ is defined, possible if $|[0, a_*)/E| = |[0, b_*)/E'|$.

The solution is via clause $\boxplus_2$(d), which tells us that in parts (1) and (2) of this discussion, $\xi(\varphi_1) > \xi(\varphi_2)$ is a real substitute, cf. clause (d) in part (2).

(4) Why do we have clause $\boxplus_2$(c) as part of the definition of $\mathbb{P}$? If not, $\varphi_2(-)$ might be irrelevant to the type we like to omit. Therefore $\boxplus_2$(c) is necessary in the definition.

(5) We ask and answer some natural questions about the construction. Let $\bar{\phi} = (\phi_1, \phi_2) \in \mathbb{P}$ be such an approximation.

Question: Why do we approximate a complete type?
Answer: If we divide φ_1 to two sets, at least one has the same $\xi(-)$.

Question: Why can we continue to omit $p(x) \in \Gamma_{\mathbf{a}}$?
Answer: If $\sigma(-)$ is a function with domain φ_1 definable in $M_{\mathbf{a}}$, let d_* be maximal such that $|\{a \in \varphi_1(M) : \sigma(a) < d_*\}| \leq \frac{1}{2}|\varphi_2(M)|$, i.e., is in the middle in the right sense. If $\sigma^{-1}\{d_*\}$ is large enough, it is easy to see that everything is fine; otherwise, for some n we have $d_* \notin (a_{p,n}, b_{p,n})$, so $\varphi_1(M) \wedge \sigma(x) \notin (a_{p,n}, b_{p,n})$ has $\geq \frac{1}{2}(\varphi_2(M))$ elements.

Question: Why can we guarantee that such $\sigma(x)$ does not realize the forbidden new type?
Answer: This is a major point. If $\xi(\varphi_1) > 2\xi(\varphi_2)$, this is easy (as in the case we use $\neg a_* E^4_M b_*$). If for some $a_1 < a_2$, we have $\xi(\varphi_2') > \xi(\varphi_2')$, we let

$$\varphi_2'(x) = (\varphi_2(x) \wedge a_1 \leq x < a_2 \wedge \sigma(x) \notin (F(a_1), F(a_2))$$

and

$$\varphi_2''(x) := (\varphi_2 \wedge F(a_1) \leq x < F(a_2))$$

and we are done, so assume there are no such a_1 and a_2.

We consider two possible reasons for the "failure" of a suggested pair (a_1, a_2). One reason is that maybe the length of the interval $[F(a_1), F(a_2))$ of $\varphi_2(M_1)$ is too large. The second is that it is small enough but $\sigma(-)$ maps the large majority of $\varphi_1(M) \cap [a_1, a_2)$ into $[F(a_1), F(a_2))$. In the second version, we can define a version of its property satisfied by $(a_1, a_2, F(a_1), F(a_2))$. So we have enough intervals of a "pseudo second kind" (where "pseudo" means using the definable version of the property). So dividing $\varphi_1(M)$ into convex subsets of equal (suitable) size (essentially this means that $a_*^{\xi(\varphi_2)}$, $\zeta \in \mathbb{R}_{>0}$ is small enough) by $\langle a_i : i < i(*)\rangle$ we have: for some such interval $[a_i, a_{i+1})$ there are b_1 and b_2 as above. For those for which we cannot define $(F(a_1), F(a_2))$ we can define it up to a good approximation. If there are enough, (this may include "pseudo cases" in respect to F), we can replace $\varphi_1(M)$ by $\varphi_1'(M) = \{a_i : i < i(*)\}$ and $\varphi_2'(-)$ defined by the function above.

So $|\varphi'(M)|$ is significantly smaller than $|\varphi_1(M)|$, essentially $\xi(\varphi_1') = \xi(\varphi_1) - \xi(a_{i+1} - a_i) \sim \xi(\varphi_1) - \xi(\varphi_2) + \zeta$ where ζ is quite small. But we are over-compensating, so we decrease $\varphi_2(x)$ to $\varphi_2'(x)$ which is quite close to $\{F(a_i) : [a_i, a_{i+1})$ is of the pseudo second kind$\}$ and $\xi(\varphi_2')$ is essentially $\xi(\varphi_2) - \xi(\varphi_2) + \zeta \sim \zeta$. So both lose similarly in the $\xi(-)$ measure, but now, if we have arranged the numbers correctly $\xi(\varphi_2') > 2\xi(\varphi_2')$, a case we know how to solve.

If there are not enough is of the pseudo second kind, the function essentially inflates the image getting a finite cardinality arithmetic contradiction.

Theorem 4.6 *Assume* $\diamondsuit_{\aleph_1}$. *If* M *is a countable model of* PA, *then* M *has an elementary extension* N *of cardinality* $\aleph_1$ *such that* $E^5_N = E^3_N$, *i.e., is 3-order rigid.*

Proof. Without loss of generality, M has universe a countable ordinal. As we are assuming $\diamondsuit_{\aleph_1}$, we choose a partial function F_α from α to α for $\alpha < \aleph_1$, i.e., $\bar{F} = \langle F_\alpha, \alpha < \aleph_1 \rangle$ such that for every partial function $F : \aleph_1 \to \aleph_1$, for stationarily many countable limit ordinals δ we have $F_\delta = F \restriction \delta$.

We now choose $\mathbf{a}_\alpha \in \mathrm{AP}_{\aleph_0}$ by recursion on $\alpha < \aleph_1$ such that

(a) $M_{\mathbf{a}_0} = M$ and $\Gamma_{\mathbf{a}_0} = \varnothing$,
(b) $\langle \mathbf{a}_\beta : \beta \le \alpha \rangle$ is $\le_{\mathrm{AP}}$-increasing continuous, and
(c) if $\alpha = \delta + 1$ where δ is a countable limit ordinal, $M_{\mathbf{a}_\delta}$ has universe δ and for some a_δ, b_δ the tuple $(\mathbf{a}_\delta, a_\delta, b_\delta, F_\delta)$ satisfies the assumptions of Main Claim 4.5 on $(\mathbf{a}, a_*, b_*, F)$, they are necessarily unique (cf. Main Claim 4.5(A)(c)), then $\mathbf{a}_{\delta+1}$ satisfies its conclusion (for some c_δ).

Why does the recursion work? For $\alpha = 0$, recall clause (a). For $\alpha = 1$, as $\Gamma_{\mathbf{a}_0} = \varnothing$ let $M_{\mathbf{a}_1}$ be a countable model such that $M = M_{\mathbf{a}_0} \prec M_{\mathbf{a}_1}$, $M \neq M_{\mathbf{a}_1}$ and withour loss of generality, the universe of $M_{\mathbf{a}_1}$ is a countable ordinal.

Finally, let $\Gamma_{\mathbf{a}_1} = \varnothing$. If α is a limit ordinal, use Main Claim 4.4(2), i.e., choose the union. This obviously works. For $\alpha = \beta + 1$, if clause (c) applies, then use Claim 4.5. For $\alpha = \beta + 1 > 1$, if clause (c) does not apply, this is easier than 4.5 (or choose (a_*, h_*, F) such that $(\mathbf{a}_\beta, a_*, b_*, F)$ are as in the assumption 4.5, this is possible because $M_{\mathbf{a}_\beta}$ is non-standard, cf. the case $\alpha = 1$, and note that $a, b \in M_{\mathbf{a}_\beta} \setminus \mathbb{N}$ implies $a E^5_{M_\mathbf{a}} b$ because $M_\mathbf{a}$ is countable; so we can use 4.5).

Now that we have seen that the recursion works, let $N = \bigcup \{M_{\mathbf{a}_\alpha} : \alpha < \aleph_1\}$. Clearly N is a model of T of cardinality $\aleph_1$. We know that $E^3_N \subseteq E^5_N$ by Claim 3.3. Towards a contradiction, assume $a_* E^5_N b_*$ but $\neg(a_* E^3_N b_*)$ where $a_*, b_* \in N \setminus M$. Without loss of generality, $b_* < a_*$ and let F be an order-isomorphism from $N_{<a_*}$ onto $N_{<b_*}$. So $S = \{\delta : F \restriction \delta = F_\delta\}$ is stationary and $E = \{\delta : a_*, b_* \in M_{\mathbf{a}_\delta}, M_{\mathbf{a}_\delta}$ has universe δ and F maps $M^{\mathbf{a}_\delta}_{<a_*}$ onto $M^{\mathbf{a}_\delta}_{<b_*}\}$ is a club of $\aleph_1$.

Choose $\delta \in S \cap E$ and use the choice of $\mathbf{a}_{\delta+1}$, i.e., clause (c) to get a contradiction. □

Theorem 4.7 *Assume $\lambda = \lambda^{<\lambda}$ and $\diamondsuit_S$ where $S = S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$. Then for any model M of* PA *there is a λ-saturated model N of $\mathrm{Th}(M)$ of cardinality λ^+ such that $E^5_N \subseteq E^4_N$.*

Proof. The proof is similar to that of Theorem 4.6, but the analogue of Main Claim 4.5 is much easier. □

Conjecture 4.8 *(1) Assume λ is strong limit singular of cofinality $\aleph_0$ and $\diamondsuit_S$ where $S = S^{\lambda^+}_{\aleph_0} = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \aleph_0\}$ and $\square_\lambda$. If M is a model of* PA*, then $\mathrm{Th}(M)$ has a λ-universal model N of cardinality λ^+ which is 3-order rigid.*

(2) Any model M of PA *has a 3-order rigid elementary extension.*

5 Weaker version of PA

We may wonder what is the weakest version of PA needed in the results proved in this paper. In the following, we define some weak versions of PA and comment on which of them are sufficient for which results of Sections 1 and 2. When we discuss the exponentiation function $x \mapsto 2^x$, we prefer to add a new function symbol to the vocabulary and the relevant axioms, rather than an axiom stating that some definition has those properties. In the following, all models are models of PA_{-4} (cf. Definition 5.1), having the signature of τ_{PA} if not said otherwise.

Definition 5.1 We define the first order theories PA_ℓ for $\ell \in \{-1, \dots, -4\}$.

(a) For $\ell = -4$, we use the obvious axioms of addition and product and order, that is axioms describing the non-negative part of a discrete ordered ring.
(b) If $\ell = -3$, we also add division with remainder by any $n \in \mathbb{N}$.
(c) If $\ell = -2$, we use the axioms of PA_{-3} and also add division with remainder.
(d) If $\ell = -1$, we use the axioms of PA_{-2} and add a unary function F_2 written 2^x with the obvious axioms for $x \mapsto 2^x$, including

$$(\forall x)(\exists y)(2^y \le x < 2^{y+1}).$$

Claim 5.2 If M and N are models of PA_{-4}; then Claim 2.4(1), (1A), (1B), (3), (4), Claim 2.5, Claim 3.1(1), (2), (3), (5), and Observation 2.8(1), (2), (3) hold.

Claim 5.3 If M is a model of PA_{-3}, then Claim 2.4(2) holds.

Proof. The only difference to the original proof is: Why can we choose c_1 and c_2?

Now if we assume $M \models \mathsf{PA}$ this is obvious, but we are assuming $M \models \mathsf{PA}_{-3}$, still we can divide $b - a$ by $n - 1$ and then get c_1 and $m < n - 1$ such that $b - a = (n - 1) \times c_2 + m$. Let $c_2 = a - c_1$, so $b = a + (n - 1) \times c_2 + m = c_1 + n \times c_2 + m$. We still have to justify using $a - c_2$, i.e., showing $c_2 \leq a$, but otherwise $b - a = (n - 1) \times c_2 + m \geq (n - 1) \times a + m$, i.e., $b \geq n \times a + m$, contradiction. □

Theorem 5.4 *If M_1 and M_2 are models of PA_{-1}, then Theorem 2.6 holds, i.e., if M_2 is 2-order-rigid and M_1 and M_2 are order-isomorphic, then M_1 and M_2 are almost $\{<, +\}$-isomorphic.*

Proof. The proof proceeds as for the proof of Theorem 2.6 with the following minor additions: in the proof of $(*)_3$ we use $M_2 \models \mathsf{PA}_{-4}$; in the proof of $(*)_5$ we use $M_\ell \models \mathsf{PA}_{-2}$; in the proof of $(*)_0$ we use $M_2 \models \mathsf{PA}_{-2}$; in the proof of $(*)_7$ we use $M_1 \models \mathsf{PA}_{-4}$; and in the proof of $(*)_8$, $\circledast_2$ we use $M_\ell \models \mathsf{PA}_{-1}$. □

Claim 5.5

(a) If $M \models \mathsf{PA}_{-3}$ then Observation 2.8*(4)* holds.
(b) If $M \models \mathsf{PA}_{-1}$ then Observation 2.8*(5)* holds.

Proof. Straightforward. □

We close the paper with an open question (for more context, cf. Definition 2.3(2)):

Question 5.6 Is there a 2-order rigid model of PA_{-1}?

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