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THE KUNEN-MILLER CHART (LEBESGUE MEASURE, THE BAIRE PROPERTY, LAVER REALS AND PRESERVATION THEOREMS FOR FORCING)

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Abstract. In this work we give a complete answer as to the possible implications between some natural properties of Lebesgue measure and the Baire property. For this we prove general preservation theorems for forcing notions. Thus we answer a decade-old problem of J. Baumgartner and answer the last three open questions of the Kunen-Miller chart about measure and category. Explicitly, in §1: (i) We prove that if we add a Laver real, then the old reals have outer measure one. (ii) We prove a preservation theorem for countable-support forcing notions, and using this theorem we prove (iii) If we add ω_2 Laver reals, then the old reals have outer measure one. From this we obtain (iv) Cons(ZF) $\Rightarrow$ Cons(ZFC $+ \neg B(m) + \neg U(m) + U(c)$). In §2: (i) We prove a preservation theorem, for the finite support forcing notion, of the property "$F \subseteq {}^{\omega}\omega$ is an unbounded family." (ii) We introduce a new forcing notion making the old reals a meager set but the old members of ${}^{\omega}\omega$ remain an unbounded family. Using this we prove (iii) Cons(ZF) $\Rightarrow$ Cons(ZFC $+ U(m) + \neg B(c) + \neg U(c) + C(c)$). In §3: (i) We prove a preservation theorem, for the finite support forcing notion, of a property which implies "the union of the old measure zero sets is not a measure zero set," and using this theorem we prove (ii) Cons(ZF) $\Rightarrow$ Cons(ZFC $+ \neg U(m) + C(m) + \neg C(c)$).

§0. Introduction. First we give some easy definitions:

$A(m) \equiv$ The union of less than continuum many measure zero sets has measure zero.

$B(m) \equiv$ The real line is not the union of less than continuum many measure zero sets.

$U(m) \equiv$ Every set of reals of cardinality less than the continuum has measure zero.

$C(m) \equiv$ There does not exist a family F of measure zero sets, of cardinality less than the continuum, and such that every measure zero set is covered by some member of F.

$A(c)$, $B(c)$, $U(c)$ and $C(c)$ are defined similarly with "first category" (meager) replacing measure zero. These properties are studied by Rothberger [R], Martin

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and Solovay [MS], Kunen [K2], Miller [M1], [M2], Bartoszyński [B], and Raisonnier and Stern [RS]. The following facts are known (references can be found in [M2]):

$$
\begin{tikzcd}
 & & A(m) \arrow[ddll] \arrow[d] \arrow[ddrr] & & \\
 & & A(c) \arrow[dl] \arrow[dr] & & \\
B(m) \arrow[r] \arrow[ddrr] & U(c) \arrow[dr] & & B(c) \arrow[r] \arrow[dl] & U(m) \arrow[ddll] \\
 & & C(c) \arrow[d] & & \\
 & & C(m) & &
\end{tikzcd}
$$

(these implications are proved in ZFC). In [M1] a chart was given involving the possible relationships between these properties, and only three questions remained open. These questions will be answered in §§1, 2 and 3, respectively.

In §1 we will prove the following fact:

0.1. THEOREM. Cons(ZF) $\Rightarrow$ Cons(ZFC $+ \neg B(m) + \neg U(m) + U(c)$).

In order to prove this theorem, we first answer a well-known question of Baumgartner by proving

0.2. THEOREM. *If r is a Laver real over V, then $V[r] \models$ "$2^\omega \cap V$ has outer measure one".*

The natural approach, in order to build a model for Theorem 0.1, is to iterate with countable support ω_2 Laver reals, and to show that in the generic extension the old reals have outer measure one. For this we define when a forcing notion satisfies the property $*1$, and we prove that: (i) If a forcing notion satisfies $*1$, then in the generic extensions the old reals have outer measure one. (ii) The property $*1$ is preserved under countable-support iterated forcing. (iii) The Laver-real forcing satisfies $*1$. We conclude the section by showing that if the ground model is the constructible universe (or satisfies CH) then by adding ω_2 Laver reals we obtain a model for $\neg B(m) + \neg U(m) + \neg B(c) + U(c)$.

In the second section we will prove the following fact.

0.3. THEOREM. Cons(ZF) $\Rightarrow$ Cons(ZFC $+ U(m) + \neg B(c) + \neg U(c) + C(c)$).

In order to give a model for this theorem, we begin with a model satisfying $A(m) + \neg$CH. The final model is obtained by an ω_1-iterated forcing notion with finite support of σ-centered partially ordered sets as components. As every finite-support iterated forcing adds Cohen reals, it is not hard to see that in such a generic extension $U(c)$ fails. In order to show that $U(m)$ holds in the generic extension, we prove

0.4. THEOREM. *If P is σ-centered and $V \models A(m)$, then $0 \Vdash_P$ "$(\forall A \in [\mathbf{R}]^{<c^V})$ (A has measure zero)."*

So our problem was to show that $\neg B(c)$ and $C(c)$ hold in such a generic extension. In general this is not true, as one can verify by iterating Hechler reals. So the problem was to find a σ-centered partially ordered set P satisfying

(i) $\Vdash_P$ "$2^\omega \cap V$ is meager" and

(ii) $\Vdash_P$ "${}^\omega\omega \cap V$ is an unbounded family."

So we complete the proof of the theorem by showing

0.5. THEOREM. *There exists a σ-centered partially ordered set M* (*called the meager tree forcing*) *satisfying* (i) *and* (ii).

0.6. THEOREM. *The property "$F \subseteq {}^\omega\omega$ is an unbounded family" is preserved under finite-support iterated forcing.*

In fact we prove a more general preservation theorem which can be used in other contexts. The parallel of this theorem for countable-support iterated forcing was proved by Shelah [SH2].

In §3 we prove the following statement:

0.7. THEOREM. Cons(ZF) $\Rightarrow$ Cons(ZFC $+ \neg U(m) + C(m) + \neg C(c)$).

Once more we begin with a model satisfying $A(m) + \neg$CH, and we add with finite support ω_1-many random and Hechler reals alternately. Clearly in the generic extension $\neg U(m) + \neg C(c)$ holds and, in order to show that $C(m)$ holds, we prove the following theorem:

0.8. THEOREM. *If $M \subseteq N$ are models of* ZFC* (*see* [M2]) *and there exists $h \in N \cap \{{}^\omega([\omega]^{<\omega})\}$ and $f \in M \cap {}^\omega\omega$ such that, for every $m \in \omega$, $|h(m)| < f(m)$ and*

(*) *for every $g \in M \cap {}^\omega\omega$ there exists $n \in \omega$ such that for every $m \geq n$, $g(m) \in h(m)$,*

then there exists $h' \in N \cap {}^\omega([\omega]^{<\omega})$ such that for every $m \in \omega$, $|h'(m)| \leq m$ and h' satisfies (*).

Then, using a remark of [RS] and the proof of Lemma 1.1 of [RS], we obtain

0.9. THEOREM. *If $M \subseteq N$ then in N the union of all measure zero sets that belong to M is a measure zero set iff in N there exists $h \in {}^\omega([\omega]^{<\omega})$ such that, for every $n \in \omega$, $|h(n)| \leq n$ and h satisfies* (*) *of Theorem* 0.8.

Next we introduce the property of being "good" for partially ordered sets, and we prove that if P is good, then in the generic extension the union of all measure zero sets coded in the ground model is not a measure zero set. We prove that this property is preserved under finite-support iterated forcing notions and, finally, we prove that random-reals forcing is good and also that σ-centered partially ordered sets are good.

A general theorem about preservation under countable-support iterated forcing can be found in [SH3]. This theorem generalized the previous theorem, which appears in [SH1].

All our notation is standard, and can be found in [K1], [M2], and [SH1]. For $A \subseteq \mathbf{R}$ we denote

(1) by $\mu(A)$ the Lebesgue measure of A,

(2) by $\mu^*(A)$ the outer Lebesgue measure of A, and

(3) by $\mu_*(A)$ the inner Lebesgue measure of A.

We confuse $\mathbf{R}$ with the unit interval and with ${}^\omega 2$, in all our arguments. Theorems 0.7 and 0.3 were proved, independently, by Cichoń and Kamburelis, but they never published those results.

§1. Preserving "the old reals have outer measure 1".

1.1. Definition. Let $\langle P, \leq \rangle$ be a partially ordered set. We define the following properties:

(a) $\bigstar_1[P]$ iff for every sufficiently large cardinal χ, and for every countable $N \prec (H(\chi), \varepsilon, \leq_\chi)$, if $P \in N$ and $\langle \boldsymbol{I}_n : n < \omega \rangle \in N$ is a P-name of a sequence of rational intervals, and $\langle p_n : n < \omega \rangle \in N$, each $p_n \in P$, and $p_0 \vDash$ "$\Sigma|\boldsymbol{I}_n| = b \in \mathbf{Q}^+$" and for every $n \in \omega$, $p_n \Vdash$ "$\boldsymbol{I}_n = I_n$", then for every random real x over N, if $x \notin \bigcup_n I_n$ then there exists $p_0 \leq q \in P$, q is (N, P)-generic and

$$q \Vdash \text{“}x \text{ is random over } N[G_P]\text{”} \quad \text{and} \quad q \Vdash \text{“}x \notin \bigcup_n \boldsymbol{I}_n\text{”}.$$

(b) $\bigstar_2[P]$ iff for every P-name $\boldsymbol{A}$ of a subset of $\mathbf{R}$ and for every $p \in P$, if $p \Vdash$ "$\mu(\boldsymbol{A}) \leq c$", then

$$\mu_*\{x \in \mathbf{R} : (\exists q \in P)(p \leq q \wedge q \Vdash \text{“}x \notin A\text{”})\} \geq 1 - c.$$

(c) $\bigstar_3[P]$ iff for every $A \in V \cap \mathscr{P}({}^\omega 2)$ if $V \vDash$ "$\mu(A) > 0$" then $V^P \vDash$ "$\mu^*(\dot{A}) > 0$".

(d) $\bigstar_4[P]$ iff for every sufficiently large cardinal χ, and for every countable $N \prec (H(\chi), \in, \leq_\chi)$, if $P \in N$ and $\langle p_n, n < \omega \rangle \in N$, each $p_n \in P$, and $\langle \boldsymbol{A}_n < \omega \rangle \in N$, each $\boldsymbol{A}_n$ a P-name, and for every n, $p_n \Vdash$ "$\boldsymbol{A}_n \subseteq {}^\omega 2$ is a Borel set and $\mu(\boldsymbol{A}_n) < \varepsilon_n$" and $\lim_{n \to \infty} \varepsilon_n = 0$ and $x \in {}^\omega 2$ is random over N, then there exists $q \in P$ such that

(i) q is (N, P)-generic,

(ii) $q \Vdash$ "x is random over $N[\boldsymbol{G}_P]$", and

(iii) there exists n such that $q \geq p_n$ and $q \Vdash$ "$x \notin \boldsymbol{A}_n$".

1.2. Definition. P is *weakly homogeneous* iff $\Vdash_P$ "$(\forall p \in P)(\exists G_p \in V^P)(p \in G_p \subseteq P$ and G_P is a generic filter over $V)$".

1.3. *Fact. If P is weakly homogeneous, then $\bigstar_2[P]$ iff $\bigstar_3[P]$.*

Proof. (i) Suppose $\bigstar_2[P]$, and let $A \in V \cap \mathscr{P}({}^\omega 2)$ be such that there exists $p \in P$, $p \Vdash$ "$\mu(\dot{A}) = 0$". Therefore

$$\mu^*\{x \in {}^\omega 2; p \Vdash x \in \dot{A}\} = 0,$$

and this implies that $\mu(A) = 0$. So we have proved that $\bigstar_2[P] \Rightarrow \bigstar_3[P]$.

(ii) Suppose $\bigstar_3[P]$; without loss of generality, $\boldsymbol{A} = \bigcup_n \boldsymbol{I}_n$, each $\boldsymbol{I}_n$ a P-name of a rational interval. Let $\varepsilon > 0$ be sufficiently small, and define $X = \{x \in {}^\omega 2 : p \Vdash$ "$x \in \boldsymbol{A}$"$\}$.

In order to get a contradiction, suppose that $\mu^*(X) > c$; we pick $\langle p_m : m < \omega \rangle$, $p_m \leq p_{m+1}$, and $\langle I_n : n < \omega \rangle$ rational intervals and $h \in {}^\omega \omega$ increasing such that

$$p_m \Vdash \text{“}(\forall n \leq h(m))(\boldsymbol{I}_n = \dot{I}_n) \wedge \sum_{n > h(m)} |\boldsymbol{I}_n| < \varepsilon/4^m\text{”}.$$

Therefore $\sum_{n \leq h(m)} |I_n| \leq c$. Now we have that

$$\mu^*\left(X - \bigcup_n I_n\right) > 0.$$

Let $Y = X - \bigcup_n I_n$. Also, from the choice of p_m, p_m forces that Y is covered by $\bigcup_{l > h(m)} \boldsymbol{I}_l$. As P is weakly homogeneous, it is easy to prove that above p there exists a P-name $\langle \boldsymbol{I}_l^m : h(m) < l < \omega \rangle$ satisfying the above conclusion with p_m replaced by p and $\langle \boldsymbol{I}_n : h_m < n < \omega \rangle$ by $\langle \boldsymbol{I}_l^m : h(m) < l < \omega \rangle$. From this we have

$$p \Vdash \text{“} \sum_{l > h(m)} |\boldsymbol{I}_l^m| < \varepsilon/4^m\text{”} \quad \text{and} \quad p \Vdash \text{“} Y \subseteq \bigcup_{l > h(m)} \boldsymbol{I}_l^m\text{”};$$

therefore

$$p \Vdash \text{“} Y \subseteq \bigcap_m \bigcup_{l > h(m)} \boldsymbol{I}_l^m\text{”},$$

which implies that $p \Vdash$ “$\mu^*(Y) = 0$”, a contradiction. $\square$

1.4. *Fact.* $\bigstar_1[P]$ *iff* $\bigstar_4[P]$.

Proof. (i) $\bigstar_1[P] \Rightarrow \bigstar_4[P]$. Without loss of generality, working in N, we have $\boldsymbol{A}_n = \bigcup\{\boldsymbol{I}_{n,l}: l < \omega\}$, $\boldsymbol{I}_{n,l}$ a P-name of a rational interval. For every n we choose $\langle p_{n,l}: l < \omega\rangle$ and $\langle I_{n,l}: l < \omega\rangle$ such that $p_{n,l} \Vdash$ “$\boldsymbol{I}_{l,n} = I_{n,l}$”, $p_n = p_{n,0}$, and $p_{n,l} \le p_{n,l+1}$. Therefore $\mu(\bigcup_l I_{l,n}) \le \varepsilon_n$, and this implies that

$$\mu\left(\bigcap_n \bigcup_l I_{n,l}\right) = 0.$$

Also $\bigcap_n \bigcup_l I_{n,l} \in N$; therefore, as x is random over N, there exists $n \in \omega$, $x \notin \bigcup_l I_{n,l}$. Now using $\bigstar_1[P]$ there exists $q \ge p_n$, q is (N, P)-generic, and

$$q \Vdash \text{“}x \text{ is random over } N[\boldsymbol{G}_P]\text{”} \quad \text{and} \quad q \Vdash \text{“}x \notin \bigcup_l \boldsymbol{I}_{n,l} = \boldsymbol{A}_n\text{”}.$$

This concludes the proof of (i).

(ii) $\bigstar_4[P] \Rightarrow \bigstar_1[P]$ is clear. $\square$

1.5. THEOREM. *If* Lv *denotes the Laver real forcing, then* $\bigstar_3[\mathrm{Lv}]$.

1.6. REMARK. J. Baumgartner gives the following problem (see [M1, p. 113]): Show that if one adds a Laver real the ground reals have measure zero. Clearly, Theorem 1.5 gives a negative answer to this problem.

We will break the proof of Theorem 1.5 into a series of lemmas and definitions.

1.7. DEFINITION. (i) Lv is the set of trees $T \subseteq \omega^{<\omega}$ with the property that there exists $s \in T$ (called the *stem* of T) so that $\forall t \in T, t \subseteq s$ or $s \subseteq t$, and if $t \supseteq s$ and $t \in T$ then there are infinitely many $n \in \omega$ such that $t^\wedge\langle n\rangle \in T$.

(ii) $T_1 \le_{\mathrm{Lv}} T_2$ iff $T_1 \supseteq T_2$.

(iii) $p_\eta = \{\nu \in p: \eta \subseteq \nu \text{ or } \nu \subseteq \eta\}$, where $p \in \mathrm{Lv}$.

(iv) $T_1 \le^0_{\mathrm{Lv}} T_2$ iff $T_1 \le_{\mathrm{Lv}} T_2$ and they have the same stem.

(v) For $p \in \mathrm{Lv}$, let $s(p)$ be the stem of p.

This forcing notion was introduced in [L]. Without loss of generality, we write $\le$ instead of $\le_{\mathrm{Lv}}$.

1.8. LEMMA. *Let* $\langle \boldsymbol{I}_n: n < \omega\rangle$ *be a sequence of* Lv*-names of rational intervals such that* $\Vdash_{\mathrm{Lv}}$ “$\sum|\boldsymbol{I}_n| = q < 1$”, $q \in \mathbf{Q}^+$, *and let* $p \in \mathrm{Lv}$. *Then there exists* $p' \,{}^0\!\!\ge p$ *and there exists* $f: p' \to \{$*finite sequences of rational intervals*$\}$ *such that*

(i) $\eta \subseteq \nu \in p'$ *implies* $f(\eta)$ *is an initial segment of* $f(\nu)$,

(ii) $\eta \in p'$ *implies* $p' \vDash$ “$f(\eta)$ *is an initial segment of* $\langle \boldsymbol{I}_n: n < \omega\rangle$”, *and*

(iii) *for every* $\varepsilon \in \mathbf{Q}^+$ *and for every branch* x *of* p' *there exists* $n \in \omega$ *such that for every* $m \ge n$, $\mu(\bigcup f(x|m)) \ge q - \varepsilon$.

PROOF. We apply the following fact:

1.9. *Fact. If* $p_0 \in \mathrm{Lv}$ *and* $D \subseteq \mathrm{Lv}$ *is open dense, then there exists* $p_1 \,{}^0\!\!\ge p_0$ *such that* $p_1 \in D^{\mathrm{cl}}$, *where* $D^{\mathrm{cl}} = \{p \in \mathrm{Lv}: \{\eta \in p: p_\eta \in D\}$ *contains a front*$\}$ *and* $A \subseteq p$ *is a* *front if*

(i) $\eta \neq \nu \in A \to \eta \nsubseteq \nu$, *and*

(ii) *for every* $x \in \omega^\omega$ *if* $(\forall k)(x \mid k \in p)$ *then* $(\exists k)(x \mid k \in A)$.

Proof. By induction on the ordinal α, for each $\nu \in p_0$, when $\mathrm{rk}(\nu) = \alpha$ we define

(i) $\mathrm{rk}(\nu) = 0$ iff there exists $p_1 \in D$, $p_{0_\nu} \leq^0 p_1$,

(ii) $\mathrm{rk}(\nu) = \alpha > 0$ iff there exists n such that for every $m \geq n$ we have that if $\nu^\frown\langle m\rangle \in p_0$ then $\mathrm{rk}(\nu^\frown\langle m\rangle)$ is well defined and less than α, and

(iii) $\mathrm{rk}(\nu) = \infty$ if there is no α such that $\mathrm{rk}(\nu) = \alpha$.

Claim 1. *For every* $s(p_0) \subseteq \nu \in p_0$ *we have that* $\mathrm{rk}(\nu) < \infty$.

Proof of Claim 1. Let $\nu \in p_0$ be such that $\mathrm{rk}(\nu) = \infty$ and $s(p_0) \subseteq \nu$. We define $p^*_\nu = \{\rho \in p_0: \rho \subseteq \nu \text{ or } \nu \subseteq \rho$ and, for every $k \in [\lg(\nu), \lg(\rho)), \mathrm{rk}(\rho \restriction k) = \infty\}$. Clearly $p^*_\nu \subseteq p_{0_\nu}$, and by the definition of rk, $\nu \leq \rho \in p^*_\nu$. Then $(\exists^\infty n)(\rho^\frown\langle n\rangle \in p^*_\nu)$. Therefore $p^*_\nu \in \mathrm{Lv}$, and as D is dense there exists p^{**} such that $p^*_\nu \leq^0 p^{**} \in D$. By hypothesis $s(p^{**}) \in p^*_\nu$, and this implies that $\mathrm{rk}(s(p^{**})) = \infty$; but clearly $\mathrm{rk}(s(p^{**})) = 0$. □

Claim 2. *For every* $\nu \in p_0$ *there exists* p^1_ν *such that* $p_{0_\nu} \leq^0 p^1_\nu \in D^{\mathrm{cl}}$.

Proof of Claim 2. By induction on $\mathrm{rk}(\nu)$. If $\mathrm{rk}(\nu) = 0$, the proof is easy. If $\mathrm{rk}(\nu) = \alpha > 0$, it follows by the induction hypothesis. □

This concludes the proof of Fact 1.9 and the proof of Lemma 1.8. □

1.10. LEMMA. *If* $A \subseteq [0, 1]$ *and* $\mu^*(A) = 1$, *then* $\Vdash_{\mathrm{Lv}}$ "$\mu^*(A) = 1$".

PROOF. Suppose that there exist $\langle \boldsymbol{I}_n: n < \omega\rangle$, an Lv-name of a sequence of rational intervals, and $p \in \mathrm{Lv}$ such that

(i) $p \Vdash_{\mathrm{Lv}}$ "$\sum|\boldsymbol{I}_n| = q < 1$",

(ii) $p \Vdash$ "$(\forall x \in A)(\exists^\infty n)(x \in \boldsymbol{I}_n)$".

Therefore there exist f and p' satisfying the requirements of Lemma 1.8, and, without loss of generality, $p' = p$. Now let χ be a large enough regular cardinal, and let N be countable such that $N \prec \langle H(\chi), \in, \leq_\chi\rangle$ and f, p belong to N. Let $x \in A$ be such that x is random over N. $N[x]$ is a generic extension of N; therefore $N[x] \models \mathrm{ZFC}^*$. Working in $N[x]$, for every $\varepsilon \in \mathbf{Q}^+$ we define

(i) $h^\varepsilon(\eta) = \min\{l: \sum|f(\eta) \restriction l| \geq q - \varepsilon\} \cup \{\lg(f(\eta))\}$, and

(ii) $f_\varepsilon(\eta) = f(\eta) \restriction [h^\varepsilon(\eta), \lg(f(\eta)))$.

We need to find $p' \geq p$ and $\varepsilon \in \mathbf{Q}^+$ such that

$$p' \Vdash_{\mathrm{Lv}} \text{"}x \notin \bigcup_l \bigcup f_\varepsilon(\boldsymbol{\eta}_G \restriction l)\text{"},$$

where $\boldsymbol{\eta}_G$ is the generic branch. Clearly this is sufficient. Suppose that it is impossible. Then the following fact holds in $N[x]$.

1.11. *Fact. For every* $\varepsilon \in \mathbf{Q}^+$ *there exists* $T_\varepsilon \subseteq p$ *satisfying*

(a) $\eta \subseteq \nu \in T_\varepsilon \Rightarrow \eta \in T_\varepsilon$,

(b) $T_\varepsilon \neq \varnothing$,

(c) *for every branch* $\eta \in p$ *there exists* $m < \omega$ *such that* $\eta \restriction m \notin T_\varepsilon$, *and*

(d) *for every* $\eta \in T_\varepsilon$, *either* $x \in \bigcup f_\varepsilon(\eta)$ *or* $|\{\eta^\frown\langle n\rangle \in p: \eta^\frown\langle n\rangle \notin T_\varepsilon\}| < \aleph_0$.

Proof. We define when $D(\eta) \geq \alpha$, by induction on α:

(i) $D(\eta) \geq 0$ iff $x \notin f_\varepsilon(\eta)$, and

(ii) $D(\eta) \geq \alpha$ (>0) iff for every $\beta < \alpha$ there exist infinitely many $k \in \omega$ such that $D(\eta^\frown\langle k\rangle) \geq \beta$.

Then we define $D(\eta) = \alpha$ iff $D(\eta) \geq \alpha$ but $D(\eta) \ngeq \alpha + 1$. Otherwise $D(\eta) = \infty$.

Claim. $D(s(p)) = \infty$ *iff* $\exists p^1 \geq p$ *such that* $(\forall \eta \in p^1)(x \notin \bigcup f_\varepsilon(\eta))$.

Proof. ($\Leftarrow$) By induction on α it is easy to prove that, for every $\eta \in p^1$, if $s(p) \subseteq \eta$ then $D(\eta) \geq \alpha$.

($\Rightarrow$) We define $\alpha(\bigstar) = \sup\{D(\eta): \eta \in p \text{ and } D(\eta) < \infty\}$ and $p' = \{\eta \in p: \eta \subseteq s(p)$ or $s(p) \subseteq \eta$ and $D(\eta \restriction k) \geq \alpha(\bigstar) + 8$ or ∞ for every $k \in [\lg(s(p)), \lg(\eta))\}$. Clearly if $\nu \subseteq \eta \in p'$ then $\nu \in p'$. Suppose that $s(p) \subseteq \nu \in p'$; then $D(\nu) \geq \alpha(\bigstar) + 8$ or ∞, and therefore $A_\nu = \{k: D(\nu\char`^\langle k \rangle) \geq \alpha(\bigstar) + 7 \text{ or } \infty\}$ is infinite, and by the choice of $\alpha(\bigstar)$, we have

$$k \in A_\nu \Rightarrow D(\nu^\frown\langle k \rangle) \geq \alpha(\bigstar) + 8 \text{ or } \infty,$$

which says that $p' \in \mathrm{Lv}$. $\square$

So we have proved that $D(s(p)) < \infty$, and we define $T_\varepsilon = \{\eta \in p: \eta \subseteq s(p)$ or $\langle D(\eta \restriction k): \lg(s(p)) \leq k < \lg(\eta)\rangle$ is a decreasing sequence of ordinals$\}$. Clearly if $\eta \subseteq \nu \in T_\varepsilon$ then $\eta \in T_\varepsilon$. If $s(p) \subseteq \eta \in T_\varepsilon$ then $D(\eta) < \infty$, and if $x \notin \bigcup f_\varepsilon(\eta)$ then $\{k: \eta^\frown\langle k \rangle \in p \text{ and } D(\eta^\frown\langle k \rangle) \geq D(\eta)\}$ is finite, and this implies that there exists $n \in \omega$ such that for $k \in \omega - n$, if $\eta^\frown\langle k \rangle \in p$ then $\eta^\frown\langle k \rangle \in T_\varepsilon$. $\square$

Therefore, still working in $N[x]$, there exists $h_\varepsilon: T_\varepsilon \to \omega_1$ such that for every $\eta^\frown\langle n \rangle \in T_\varepsilon$, $h(\eta^\frown\langle n \rangle) < h(\eta)$. As x is random over N, there exists a Borel set $B \in N$, $\mu(B) > 0$, such that in N

$$(*) \qquad B \underset{\text{random}}{\Vdash} \text{“for every } \varepsilon > 0 \text{ there exist } h_\varepsilon \text{ and } T_\varepsilon \text{ as above”.}$$

Set $\varepsilon = \mu(B) \cdot 10^{-10}$, and let $\boldsymbol{h}_\varepsilon$ and $\boldsymbol{T}_\varepsilon$ be random names witnessing (*) for this ε; $\boldsymbol{h}_\varepsilon$ and $\boldsymbol{T}_\varepsilon$ belong to N. Using the ω^ω-bounding property of random forcing (see [SH1, p. 169]), we can find $B' \subseteq B$ with $\mu(B') \geq \frac{1}{2}\mu(B)$ and such that for each $\eta \in p$

(i) $\{n: (\exists B'' \subseteq B')(B'' \Vdash \text{“}\eta \in \boldsymbol{T}_\varepsilon \wedge \eta^\frown\langle n \rangle \notin \boldsymbol{T}_\varepsilon\text{”} \wedge \mu(B'' \cap \bigcup f_\varepsilon(\eta) = 0)\}$ is finite, and

(ii) $\{\alpha \in \omega_1: (\exists B'' \subseteq B')(B'' \Vdash h_\varepsilon(\eta) = \alpha)\}$ is finite.

Now we define

$$T^*_\varepsilon = \left\{\eta \in p: (\exists B'' \subseteq B')\left(B'' \underset{\text{random}}{\Vdash} \text{“}\eta \in \boldsymbol{T}_\varepsilon\text{”}\right)\right\},$$

$$H_\varepsilon(\eta) = \left\{\alpha \in \omega_1: (\exists B'' \subseteq B')\left(B'' \underset{\text{random}}{\Vdash} \text{“}\eta \in \boldsymbol{T}_\varepsilon \wedge \boldsymbol{h}_\varepsilon(\eta) = \alpha\text{”}\right)\right\}.$$

1.12. *Fact.* (a) $T^*_\varepsilon \subseteq p$, $T^*_\varepsilon \neq \emptyset$ *and, if* $\eta \subseteq \nu \in T^*_\varepsilon$, *then* $\eta \in T^*_\varepsilon$.

(b) *If* $\eta \in T^*_\varepsilon$ *and* $x \in B'$ *is random over* N, *and* $x \notin \bigcup f_\varepsilon(\eta)$ *and* $\eta \in \boldsymbol{T}_\varepsilon[x]$, *then* $\{n: \eta^\frown\langle n \rangle \in T_\varepsilon[x] \wedge \eta^\frown\langle n \rangle \notin T^*_\varepsilon\}$ *is finite.*

Proof. By the definition of T^*_ε and the choice of B'. $\square$

1.13. *Fact. If* $\nu = \eta^\frown\langle n \rangle \in T^*_\varepsilon$, *then* $\max H_\varepsilon(\nu) < \max H_\varepsilon(\eta)$.

Proof. Let $\alpha = \max(\nu)$ and $B'' \subseteq B'$ be such that

$$B'' \underset{\text{random}}{\Vdash} \text{“}\boldsymbol{h}_\varepsilon(\nu) = \alpha\text{”}.$$

Therefore

$$B'' \underset{\text{random}}{\Vdash} \text{“}\boldsymbol{h}_\varepsilon(\eta) \text{ is well defined and larger than } \alpha\text{”}.$$

This implies that $\alpha < \max H_\varepsilon(\eta)$. $\square$

1.14. COROLLARY. *For every $\eta \in {}^{\omega}\omega$ there exists $n \in \omega$ such that $\eta \restriction n \notin T^*_\varepsilon$.* $\square$

So there exists $h: T^*_\varepsilon \to \omega_1$ such that

$$\eta \subseteq \nu \in T^*_\varepsilon \Rightarrow h(\eta) > h(\nu).$$

Now, by induction on $h(\eta)$, for each $\eta \in T^*_\varepsilon$ we define a set $Y_\eta \subseteq [0,1]$ with $\mu(Y_\eta) \leq \varepsilon$ as follows:

(i) If η does not have an extension in T^*_ε, then $Y_\eta = \bigcup f_\varepsilon(\eta)$.

(ii) If η has extensions in T^*_ε then

$$Y_\eta = \bigcup_n \bigcap \{Y_{\eta^\wedge\langle l\rangle}: l \geq n \text{ and } \eta^\wedge\langle l\rangle \in T^*_\varepsilon\}.$$

Thus $Y_{s(p)}$ is well defined, $\mu(Y_{s(p)}) \leq \varepsilon$ and $Y_{s(p)} \in N$. Therefore there exists $x \in A \cap (B' - Y_{s(p)})$, x random over N. As $x \notin Y_{s(p)}$ it is not hard, using 1.12(b), to see that $\boldsymbol{T}_\varepsilon[x]$ has a branch. But this is a contradiction to the construction of the tree $\boldsymbol{T}_\varepsilon[x]$. This finishes the proof of Lemma 1.10. $\square$

Clearly Lemma 1.10 implies Theorem 1.5. $\square$

1.15. COROLLARY. $\bigstar_2[\mathrm{Lv}]$.

PROOF. Clearly Lv is weakly homogeneous; then Fact 1.3 gives the conclusion of this corollary. $\square$

1.16. THEOREM. $\bigstar_1[\mathrm{Lv}]$.

We will break the proof into a series of lemmas and definitions.

1.17. DEFINITION. (a) Let $\boldsymbol{A}$ be an Lv-name of a Borel subset of $\mathbf{R}$ such that $\Vdash_{\mathrm{Lv}}$ "$\boldsymbol{A} = \bigcup_n \boldsymbol{I}_n$ and $\sum|\boldsymbol{I}_n| = c$". For $p \in \mathrm{Lv}$, we say that $\langle \boldsymbol{I}_n: n < \omega\rangle$ *is interpreted over* p, if for every n there exists a front $A_n \subseteq p$, and (i) $(\forall \nu \in A_n)(p_\nu \Vdash \boldsymbol{I}_n = I^n_\nu)$; (ii) $(\forall \nu \in A_{n+1})(\exists \rho \in A_n)(\rho \subseteq \nu)$.

(b) Let χ be a regular cardinal, large enough, and let $N \prec \langle H(\chi, \in, \leq_\chi)\rangle$, $\|N\| = \aleph_0$, $p_0 \in N \cap \mathrm{Lv}$ and $\boldsymbol{A} \in N$ an Lv-name satisfying the condition of (a), interpreted over p_0. We define $Y \subseteq \mathbf{R}$ by putting x into Y iff there exists $q \in \mathrm{Lv}$ such that the following four conditions hold:

(i) $p_0 \leq q \in \mathrm{Lv}$.

(ii) For every open dense set $D \subseteq \mathrm{Lv}$, if $D \in N$ then there exists $r \in D^{\mathrm{cl}} \cap N$ such that $r \leq q$.

(iii) $q \Vdash$ "$x \notin \boldsymbol{A}$".

(iv) If $\models$ "$\langle \boldsymbol{J}_n: n < \omega\rangle$ is a sequence of rational intervals and $\sum|\boldsymbol{J}_n| < \infty$" and $\bar{J} = \langle \boldsymbol{J}_n: n < \omega\rangle \in N$, and $D_{\bar{J}} = \{r \in \mathrm{Lv}: \bar{J}$ is interpretable over r with front $\langle A^r_l: l < \omega\rangle\}$, then there exists $r \in D_{\bar{J}} \cap N$ and $k \in \omega$ such that

$$(\forall m \geq k\, \forall \eta \in q \cap A^r_m)(x \notin J^{r,m}_\eta).$$

1.18. LEMMA. *Y is a Σ^1_1 set of reals.*

PROOF. $x \in Y$ iff there exists $q \subseteq p_0$ such that the following four conditions hold:

(i) $q \in \mathrm{Lv}$ (Δ^1_0).

(ii) If $\langle E_l: l < \omega\rangle$ is an enumeration of $\{D^{\mathrm{cl}}: D \in N$ and $D \subseteq \mathrm{Lv}$ is an open dense set$\}$, $E_l = \langle r_{l,m}: m < \omega\rangle$, then for every l there exists m such that $q \subseteq r_{l,m}$ (Δ^1_0).

(iii) We know that $\boldsymbol{A}$ is interpreted over p_0. Therefore we can find $\langle A_n, I^n_\nu: n, \nu\rangle$ witnessing this and then, for every n, for every $\nu \in q \cap A_n$, $x \notin I^n_\nu$ (Δ^1_0).

(iv) We have an enumeration of $\{\langle r, \langle A^r_n, J^{r,n}_\nu: n, \nu\rangle\rangle: r \in D_{\bar{J}}\}$.

Clearly $D_{\bar J}^{\mathrm{cl}} = D_{\bar J}$ and then, for every $\bar J \in N$, there exist $r \in D_{\bar J} \cap N$ and $k \in \omega$ such that $(\forall m \geq k)(\forall \eta \in q \cap A_m^r)(x \notin J_\eta^{r,m})(\Delta_0^1)$. ☐

1.19. LEMMA. $\mu_*(Y) \geq 1 - c$.

PROOF. By Laver [L], there exists $q \in \mathrm{Lv}$ satisfying (i) and (ii) of Lemma 1.8. Also every $q' \geq q$ satisfies (i) and (ii) of Lemma 1.18. Now we define

$$Y_q = \{x \in \mathbf{R}\text{: there exists } q_x \in \mathrm{Lv},\ q_x \geq q \text{ and } q_x \Vdash \text{“}x \notin \boldsymbol{A} \cup \{\boldsymbol{B}\colon \boldsymbol{B} \in N \text{ and } \mu(\boldsymbol{B}) = 0\}\text{”}\}.$$

By $\bigstar_2$[Lv] we know that $\mu_*(Y_q) \geq 1 - c$. For every $x \in Y_q$, it is not hard to find $q' \geq q_x$ witnessing $x \in Y$ (remember that $\|N\| = \aleph_0$ and that if $q_x \Vdash$ “$x \notin \bigcup_n \bigcap_{m \geq n} J_m$” then there exists $q''^0 \geq q_x$ satisfying that there exists n such that for every $y \in [q'']$ and for every $m \geq n$, if $\eta \in A_m^{\bar J}$ and $\eta \subseteq y$ then $x \notin J_\eta^m$). ☐

Now working in N (as in 1.17(b)), we define $Q_0 = \mathrm{Levy}(\aleph_0, 2^{\aleph_0})$, and Q_1 is random real forcing over N^{Q_0}. Let $y \subseteq Q_0$ be generic over N, and x random over $N[y]$. Clearly the parameters of the definition of Y are in $N[y]$, and we can ask, in $N[y,x]$, if “$x \in Y$”. In $N[y]$ there exist $B_0, B_1 \in Q_1$ such that $\mu(B_0 \cup B_1) = 1$, $B_0 \cap B_1 = \varnothing$, and in $N[y]$ we have that $B_0 \Vdash$ “$x \in Y$” and $B_1 \Vdash$ “$x \notin Y$”. Also we know that $\mu(B_0) \geq 1 - c$. It is well known that x is random over N; therefore, working in N, we have

$$(*) \qquad Q_0 * Q_1 \cong R_0 * R_1$$

where R_0 is random real forcing. In N^{R_0} we can ask: “After R_1 does $x \in B_0$?”, and we obtain $B_0^* \in R_0$ such that

$$N \models B_0^* \Vdash_{R_0} \text{“}(\exists r \in R_1)(r \Vdash x \in B_0)\text{”}$$

and

$$N \models \sim B_0^* \Vdash_{R_0} \text{“}\phi \Vdash x \in B_1\text{”}.$$

Working in N^{Q_0}, it is not hard to prove that $B_0 \subseteq B_0^*$ (a.e.); therefore $\mu(B_0^*) \geq 1 - c$.

From this, if x is random real over N and $x \in B_0^*$, we can find $y \subseteq Q_0$ generic over N such that $(y, x) \subseteq Q_0 * Q_1$ is generic over N and $N[y,x] \models$ “$x \in Y$”; furthermore, Y is a Σ_1^1 set.

We can conclude that $V \models$ “$x \in Y$”. In other words, there exists a Borel set $B_0^* \in N$, $\mu(B_0^*) \geq 1 - c$, such that for every $x \in V \cap B_0^*$, if x is random over N then $x \in Y$. In this case we denote $Y = Y(N, p_0, \boldsymbol{A})$.

PROOF OF THEOREM 1.16. Given N, $\langle p_n\colon n < \omega\rangle$, $\langle \boldsymbol{I}_n\colon n < \omega\rangle$, $p_n \Vdash \boldsymbol{I}_n = I_n$, $\langle p_n\colon n < \omega\rangle \in N$, $p_n \leq p_{n+1}$, $x \notin \bigcup_n I_n$, and $\Vdash \sum |\boldsymbol{I}_n| = c$, we define, for every $k \in \omega$,

$$Y(N, p_n, \langle \boldsymbol{I}_{k+n}\colon k < \omega\rangle) = Y_n$$

and, by the above work, we can find a Borel set $B_n^* \in N$, $\mu(B_n^*) \geq 1 - (c - \sum_{l \leq n} |I_n|)$, such that for every $x \in V \cap B_n^*$, if x is random over N, then $x \in Y_n$. Without loss of generality $\langle B_n^*\colon n < \omega\rangle \in N$, and $\mu(\bigcup B_n^*) = 1$. Therefore if x is random real over N, then there exists $n \in \omega$ such that $x \in B_n^*$; and this implies that $x \in Y_n$. This concludes the proof of the theorem. ☐

1.20. THEOREM. *If P is a forcing notion and $\mathbf{Q}$ is a P-name of a forcing notion and $\bigstar_1[P]$ and $\Vdash_P$ "$\bigstar_1[\mathbf{Q}]$", then $\bigstar_1[P * \mathbf{Q}]$.*

PROOF. Let N, $P * \mathbf{Q} \in N$, $\langle \mathbf{I}_n: n \in \omega\rangle \in N$, $\langle (p_n, \mathbf{q}_n): n < \omega\rangle \in N$, $\langle I_n: n < \omega\rangle \in N$ and x random over N all satisfy the requirements of 1.1(a). For each n we define $\langle (p_n, \mathbf{q}_{n,l}): l < \omega\rangle \in N$ and $\langle \mathbf{I}_{n,l}: l < \omega\rangle \in N$, each $\mathbf{I}_{n,l}$ a P-name of a rational interval, as follows: for $0 \leq l \leq n$ let $\mathbf{I}_{n,l} = \dot{I}_l$ and $(p_n, \mathbf{q}_{n,l}) = (p_n, \mathbf{q}_n)$; for $l = n + m + 1$ let $\mathbf{q}_{n,l}$ be a P-name of a member of $\mathbf{Q}$ and $\mathbf{I}_{n,l}$ a P-name of a rational interval such that

$$p_n \Vdash_P \text{“}q_{n,n+m} \leq_{\mathbf{Q}} \mathbf{q}_{n,l} \quad \text{and} \quad \mathbf{q}_{n,l} \Vdash_{\mathbf{Q}} \text{“}\mathbf{I}_{n,l}[G_P] = \mathbf{I}_l\text{”}.$$

Now we define $\mathbf{A}_n = \bigcup_{l \geq n} \mathbf{I}_{n,l}$. Clearly for every $n \in \omega$, $\mathbf{A}_n \in N$ and $\mathbf{A}_n$ is such that $p_n \Vdash_P$ "$\mathbf{A}_n \subseteq 2^\omega$", and there exists $\langle \varepsilon_n: n < \omega\rangle$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $p_n \Vdash_P$ "$\mu(\mathbf{A}_n) \leq \varepsilon_n$".

As x is random real over N and $\bigstar_4[P]$, we can find $n, p \in P$ such that p is (N, P)-generic and $p_n \leq_P p$ and $p \Vdash$ "$x \notin \mathbf{A}_n$". Fixing such n and p, let $G \subseteq P$ be generic over V containing p. Therefore we know the following:

(i) x is random real over $N[G] \prec \langle H(\chi)^{V[G]}, \varepsilon, \leq_\chi\rangle$.

(ii) $x \notin \bigcup_{l \geq n} I_{n,l}[G]$, and thus $x \notin \bigcup_{l \geq 0} I_{n,l}[G]$.

(iii) For each $l \in \omega$, $\mathbf{q}_{n,l}[G] \Vdash_{\mathbf{Q}[G]}$ "$\mathbf{I}_l = I_{n,l}[G]$".

Applying $\bigstar_1[\mathbf{Q}[G]]$, we find $q \in \mathbf{Q}[G]$ such that q is $(N[G], Q[G])$-generic, $q \geq \mathbf{q}_n[G]$,

$$q \Vdash_{Q[G]} \text{“}x \text{ is random over } N[G][G_{\mathbf{Q}[G]}]\text{”}, \quad \text{and} \quad q \Vdash_{Q[G]} \text{“}x \notin \bigcup_n \mathbf{I}_n\text{”}.$$

As G is arbitrary, we can find $\mathbf{q}$, a P-name of a member of $\mathbf{Q}$, satisfying all this. Now it is not hard to see that $(p, \mathbf{q})$ witnesses $\bigstar_1[P * \mathbf{Q}]$. This finishes the proof of the theorem. $\square$

1.21. THEOREM. *Let $\langle P_i; \mathbf{Q}_i: i < \delta\rangle$, $\delta = \bigcup \delta \neq 0$, be a countable-support iterated forcing system satisfying*

(i) *for every $i < j < \delta$, $i \neq \bigcup i$, P_j/P_i is a proper forcing notion, and*

(ii) *for every $i < \delta$, $\bigstar_1[P_i]$.*

Then $\bigstar_1[P_\delta]$.

PROOF. Let $N \prec \langle H(\chi), \in, \leq_\chi\rangle$, $\|N\| = \aleph_0$, $p_0 \leq p_1 \leq \cdots \in P_\delta$, $\langle p_l: l < \omega\rangle \in N$, $\langle \mathbf{I}_l: l < \omega\rangle \in N$, $p_n \Vdash_{P_\delta}$ "$\mathbf{I}_l = I_l$", $\mathbf{I}_l$ a P_δ-name of a rational interval, and

$$p_0 \Vdash_{P_\delta} \text{“}\sum_l |\mathbf{I}_l| = b \in \mathbf{Q}^+\text{”};$$

without loss of generality $b = \frac{1}{2}$, $x \in \mathbf{R}$ is a random real over N, and $x \notin \bigcup_l I_l$.

Let $\langle D_n: n < \omega\rangle$ be an enumeration of the open dense subsets of P_δ that belong to N. Let $\langle\langle \mathbf{I}_{n,l}: l < \omega\rangle: n < \omega\rangle$ be an enumeration of the sequences $\langle \mathbf{J}_l: l < \omega\rangle \in N$ such that $\mathbf{J}_l$ is a P_δ-name of a rational interval and $\Vdash_{P_\delta}$ "$\sum_l |\mathbf{I}_l| = \frac{1}{2}$" and $\mathbf{I}_{0,l} = I_l$. We fix $\alpha(0) < \alpha(1) < \cdots$ such that $\alpha(n) \in N$, $\alpha(0) = 0$, $\bigcup(\alpha(n)) = \delta$ and $\alpha(n) \neq \bigcup \alpha(n)$.

By induction on $k < \omega$, we will choose $q_k \in P_{\alpha(k)}$ and $P_{\alpha(k)}$-names

$$\langle \mathbf{p}_l^k: l < \omega\rangle; \quad \mathbf{I}_{\zeta,\nu}^k; \quad \mathbf{v}(k)$$

such that if $q_k \in G_k \subseteq P_{\alpha(k)}$, G_k generic over V, then

(a) $\mathbf{p}_l^k[G_k] \restriction \alpha(k) \in G_k$,

(b) $P_\delta/P_{\alpha(k)} \Vdash$ "$\mathbf{p}_l^k[G_k] \leq \mathbf{p}_{l+1}^k[G_k]$",

(c) $\mathbf{I}^k_{\zeta,\nu}$ is a $P_{\alpha(k)}$-name of a rational interval,
(d) $\mathbf{p}^k_l[G_k] \Vdash_{P_\delta/P_{\alpha(k)}}$ "$(\forall\zeta \le k, \forall\nu \le k+l)(\mathbf{I}_{\zeta,\nu} = \mathbf{I}^k_{\zeta,\nu}[G_k])$",
(e) $q_k \Vdash_{P_{\alpha(k)}}$ "$\mathbf{p}^k_l \in N[\mathbf{G}_k] \cap P_\delta/\mathbf{G}_k$",
(f) x is random over $N[\mathbf{G}_k]$,
(g) $(\forall\zeta \le k)(x \notin \bigcup\{\mathbf{I}^k_{\zeta,\nu}[\mathbf{G}_k]: \nu(\zeta) \le \nu < \omega\})$,
(h) $p^k_0 \vDash_{P_\delta/P_{\alpha(k)}}$ "if $\zeta < k$ and $\nu < \omega$ then $\mathbf{I}_{\zeta,\nu}$ has a $P_{\alpha(\nu)}$-name",
(i) $q_k \Vdash$ "$\boldsymbol{\nu}(k) < \omega$", $\nu(0) = 0$,
(j) $p^k_0 \le p^{k+1}_0 \in D_k$,
(k) $\langle \mathbf{I}^k_{\zeta,\nu}[G_k]: l \le k, \nu < \omega\rangle \in N[G_k]$ and $\langle \mathbf{p}^k_l[G_k]: l < \omega\rangle \in N[G_k]$, and
(l) $p_0 \restriction \alpha(k) \le q_k$, $q_{k+1} \restriction \alpha(k) = q_k$.

The induction. For $k = 0$ we set $p^0_l = p_0$, $q_0 = \varnothing \in P_0 = \{\varnothing\}$, $\nu(0) = 0$, and $I^0_{0,\nu} = I_\nu$.

For $k+1$ we will work in $N[G_k]$, $q_k \in G_k$. We fix $l \in \omega$, and by induction on $m \in \omega$ we define $p^k_{l,m}$ such that

$$\mathbf{p}^k_l[G_k] \le p^k_{l,0} \in P_\delta/G_k, \qquad p^k_{l,0} \in D_k, \qquad p^k_{l,m} \le p^k_{l,m+1};$$

for every $\zeta \le k+1$, $p^k_{l,0} \restriction [\nu,\delta)$ forces that $\mathbf{I}_{\zeta,\nu}$ has a $P_{\alpha(\nu)}$-name, and for every $\zeta \le k+1$ and $\nu < k+l+m$

$$p^k_{l,m} \Vdash \text{“}\mathbf{I}_{\zeta,\nu} = \mathbf{I}^{k,l}_{\zeta,\nu}[G_{k+1}]\text{”}.$$

Therefore the sequences $\langle p^k_{l,m}: l, m < \omega\rangle$ and $\langle \mathbf{I}^{k,l}_{\zeta,\nu}: \zeta \le k+1, l, \nu \le \omega\rangle$ belong to $N[G_k]$.

Now we define $\langle m_k(l): l < \omega\rangle$ such that $m_k(l) < \omega$ and

$$p^k_{l,m_k(l)} \Vdash \text{“}\textstyle\sum\{|\mathbf{I}_{k+1,\nu}|: m_k(l) \le \nu < \omega\} < 2^{-l}\text{”}$$

and we define

$$A^k_l = \text{“}\bigcup\{I^{k,l}_{\zeta,\nu}: \zeta \le k, l \le \nu < \omega\} \cup \bigcup\{\mathbf{I}_{k+1,\nu}: m_k(l) < \nu < \omega\}\text{”}.$$

It is not hard to find $\langle \varepsilon_l: l < \omega\rangle$ such that

$$p^k_{l,m_k(l)} \restriction \alpha(k+1) \underset{P_{\alpha(k+1)}}{\Vdash} \text{“}\mu(\mathbf{A}^k_l) \le \varepsilon_l\text{”}$$

and $\lim_{l\to\infty} \varepsilon_l = 0$.

Applying $\bigstar_4[P_{\alpha(k+1)}]$, we can find $q_{k+1} \in P_{\alpha(k+1)}/G_k$ and $l(k)$ such that

$$p^k_{l(k),m_k(l(k))} \restriction \alpha(k_1) \le q_{k+1}, \qquad q_{k+1} \underset{P_{\alpha(k+1)}}{\Vdash} x \notin \mathbf{A}_{l(k)},$$

$$q_{k+1} \vDash \text{“}x \text{ is random over } N[\mathbf{G}_{P_{\alpha(k+1)}}].$$

Now we define

$$p^{k+1}_l = p^k_{l(k),m_k(l(k))+l}, \qquad \mathbf{I}^k_{\zeta,\nu} = \mathbf{I}^{k,l(k)}_{\zeta,\nu}, \qquad \nu(k+1) = m_k(l(k)).$$

It is easy to check that this works. Now we define $q \in P_\delta$ by setting $q \restriction \alpha(k) = q_k$. Then:

(i) q is (N, P_δ)-generic [use (j) and (l)],
(ii) $p_0 \le q$, and
(iii) $q \vDash$ "x is random over $N[\mathbf{G}_{P_\delta}]$ and $x \notin \bigcup_n \mathbf{I}_n$.

[It is sufficient to show that for every $n \in \omega$, $q \Vdash$ "$x \notin \bigcup_{v(l) \leq l} \boldsymbol{I}_{n,l}$"; and this follows from (g) and (h).]

This concludes the proof of the theorem. $\square$

1.22. THEOREM. con(ZF) $\Rightarrow$ cons(ZFC $+ \neg B(m) + \neg U(m) + \neg B(c) + U(c)$).

PROOF. Let $V = L$ be the constructible universe, let $P \in L$ be the ω_2-iteration of Laver reals with countable support, and let $G \subseteq P$ be generic over V. Then $V(G) \models$ "$\neg B(m) + \neg U(m) + \neg B(C) + U(C)$".

(i) $\neg B(m) + \neg B(C)$. By [SH1, pp. 206–207], P has the Laver property; and this implies that in $V[G]$ the real numbers are included in the union of the meager measure zero sets coded in V.

(ii) $\neg U(m)$. By $\bigstar_1[P]$ we have that $\mu^*(2^\omega \cap V) = 1$.

(iii) $U(c)$. For every $f \in {}^\omega\omega$,

$$A_f = \{g \in {}^\omega\omega: (\forall^\infty n)(g(n) < f(n))\}$$

is a meager set. And in $V[G]$ for every $A \in [{}^\omega\omega]^{<c}$ there exists $f \in {}^\omega\omega$ such that $A \subseteq A_f$. $\square$

§2. Preserving "the old reals are unbounded".

2.1 DEFINITION: We say $\oplus(\bar{f}, R, S)$ iff the following seven conditions hold:

(a) $\bar{f} = \langle f_a : a \in I \rangle$.

(b) I is a directed set of indices.

(c) For every $J \subseteq I$, if $|J| \leq \aleph_0$ then there exists $a \in I$ such that, for every $b \in J$, $b \leq_I a$.

(d) For every $a \in I$, $f_a \in {}^\omega H(\omega)$.

(e) $S \subseteq H(\omega) \times H(\omega)$ and $R \subseteq H(\omega) \times H(\omega)$ and $xSy \wedge yRz \Rightarrow xSz$.

(f) For every $a, b \in I$, $a \leq_I b \Rightarrow (\forall^\infty n)(f_a(n) R f_b(n))$.

(g) For every $f \in {}^\omega H(\omega)$ there exists $a \in I$ such that $(\exists^\infty n)(f(n) S f_a(n))$.

2.2. THEOREM. *Let $\bar{f}$, R, S, $\bar{Q} = \langle P_i; \boldsymbol{Q}_i : i < \delta \rangle$ be in V, satisfying*

(i) *$\bar{Q}$ is a finite-support iterated forcing system such that, for every $i < \delta$, $\Vdash_{P_i}$ "$\boldsymbol{Q}_i \models$ c.c.c.",*

(ii) *for every $i < \delta$, $\Vdash_{P_i}$ "$\oplus(\bar{f}, R, S)$, and*

(iii) *if $\delta = \gamma + 1$ then $\Vdash_{P_\gamma * Q_\gamma}$ "$\oplus(\bar{f}, R, S)$".*

Then, if $P_\delta = \varinjlim \bar{Q}$, then $\Vdash_{P_\delta}$ "$\oplus(f, R, S)$".

PROOF. If $\delta = \gamma + 1$, then the conclusion follows from (iii). If $\delta = 0$, it is clear. Therefore we will prove the theorem when $\delta = \bigcup \delta \neq 0$. Conditions (a), (b), (d), (e) and (f) of Definition 2.1 are clear. As $P_\delta \models$ "c.c.c.", we know that

$$\models_{P_\delta} \text{"}(\forall B \subseteq V)(|B| = \aleph_0 \Rightarrow (\exists A \in V)(B \subseteq A \wedge |A| = \aleph_0)).$$

This implies that (c) of Definition 2.1 holds after forcing with P_δ.

So we need to check 2.1(g). Let $\boldsymbol{g} \in V^{P_\delta}$ be such that $\Vdash_{P_\delta}$ "$\boldsymbol{g}: \omega \to H(\omega)$",

(i) If $\operatorname{cof}(\delta) > \aleph_0$ then 2.1(g) follows from the c.c.c. of P_δ.

(ii) If $\operatorname{cof}(\delta) = \aleph_0$, then we fix a well-order $\leq_\omega$ of $H(\omega)$ and a sequence $\langle \alpha_n : n < \omega \rangle$ of ordinals such that $\alpha_n < \alpha_{n+1}$ and $\alpha_n \to \delta$.

For each n we define $\boldsymbol{g}^n \in V^{P_{\alpha_n}}$ as follows:

$$\boldsymbol{g}^n(i) = \min_{\leq_\omega} \{a \in H(\omega): (\exists p \in P_\delta / \boldsymbol{G}_{P_{\alpha_n}})(p \Vdash \boldsymbol{g}(i) = a)\}.$$

Clearly $\Vdash_{P_{\alpha_n}}$ "$\boldsymbol{g}^n \in {}^\omega H(\omega)$". For each n there exists $a_n \in I$ such that

$$\Vdash_{P_{\alpha_n}} \text{“}(\exists^\infty i)(\boldsymbol{g}^n(i) S f_{a_n}(i))\text{”}.$$

Using the c.c.c. of P_δ, we can find $b \in I$ such that, for every $n \in \omega$, $a_n \leq_I b$. Therefore, for every $n \in \omega$, $\Vdash_{P_{\alpha_n}}$ "$(\exists^\infty i)(\boldsymbol{g}^n(i) S f_b(i))$".

2.3. *Claim.* $\Vdash_{P_\delta}$ "$(\exists^\infty i)(\boldsymbol{g}(i) S f_b(i))$".

Proof. If this does not hold, then there exist $p \in P_\delta$ and $k \in \omega$ with

$$p \Vdash_{P_\delta} \text{“}(\forall i > k)(\neg \boldsymbol{g}(i) S f_b(i))\text{”}.$$

There exists $n \in \omega$ such that $p \in P_{\alpha_n}$, and this implies that there exist $m \in \omega - (k+1)$ and $p \leq q \in P_{\alpha_n}$ such that

$$q \Vdash_{P_{\alpha_n}} \text{“}\boldsymbol{g}^n(m) S f_b(m)\text{”}.$$

Then, by the definition of $\boldsymbol{g}^n$, there exists $r \in P_\delta$, $q \leq r$, with $r \Vdash_{P_\delta}$ "$\boldsymbol{g}^n(m) = \boldsymbol{g}(m)$"; and this implies

$$r \Vdash_{P_\delta} \text{“}\boldsymbol{g}(m) S f_b(m)\text{”}.$$

As $p \leq r$ and $k < m$, we have found a contradiction. This concludes the proof of the claim. □

Clearly the claim implies Definition 2.1(g), and this finishes the proof of the theorem. □

2.4. Definition. The *meager forcing* M is the partially ordered set defined by setting $(t, w) \in M$ iff there exists $n(t) \in \omega$ such that

(a) if $t \in {}^{n(t) \geq}2$ and $\eta \subseteq \nu \in t$ then $\eta \in t$;

(b) if $\eta \in t$ and $\lg(\eta) < n(t)$ then $n\char`^\langle 0 \rangle \in t$ or $\eta\char`^\langle 1 \rangle \in t$, $w \subseteq {}^\omega 2$ and $|w| < \aleph_0$; and

(c) if $x \in w$ then $x \restriction n(t) \in t$.

The order for M is given by setting $(t_1, w_1) \leq (t_2, w_2)$ iff $t_1 = t_2 \cap {}^{n(t_1)}2$ and $w_1 \subseteq w_2$.

2.5. *Fact.* (i) *M is σ-centered.*

(ii) $\Vdash_M$ "*$V \cap 2^\omega$ is a meager set*".

Proof. (i) Clearly $M = \bigcup_{t \in \omega > 2} M_t$, where $M_t = \{(t, w) \in M\}$.

(ii) Let $\boldsymbol{T} = \bigcup\{t \colon \exists (t, w) \in \boldsymbol{G}_M\}$; then $\vDash_M$ "$\boldsymbol{T}$ is a meager perfect tree" and $\Vdash_M$ "$(\forall x \in 2^\omega \cap V)(\exists n \in \omega)(\exists t \in {}^n 2)(\forall k \geq n)(t\char`^x \restriction [n, k) \in T)$". □

If $p \in M$, then $t(p)$, $w(p)$ and $n(p)$ are defined satisfying $p = (t(p), w(p))$ and $n(p) = n(t(p))$.

Remark. M adds Cohen reals.

2.6. Lemma. *If $p \in M$, $k \in \omega$ and τ is an M-name of an ordinal, then there exists $m(k, p, \tau) = m < \omega$ such that if $q \in M$ and $p \leq q$ and $t(p) = t(q)$ and $|w(q) - w(p)| \leq k$, then there exists $r \in M$ such that $q \leq r$ and $n(r) \leq m$ and r decides the value of τ.*

Proof. If this does not hold, then for every m there exists $q_m \in M$ such that $p \leq q_m$ and $t(p) = t(q_m)$ and $|w(q_m) - w(p)| \leq k$ and for every $r \geq q_m$ if r decides the value of τ then $n(r) > m$.

Set $w(q_m) - w(p) = \{x_1^m, \ldots, x_{k(m)}^m\}$, $k(m) \leq k$. Thinning $\langle q_m \colon m < \omega \rangle$ if necessary, we can assume that $k(m) = k(\bigstar)$. If $l \in [1, k(\bigstar)]$ then $x_l^m \restriction m = x_l^{m+1} \restriction m$. Let $\langle y_1, \ldots, y_{k(\bigstar)} \rangle$ be such that for every $m \in \omega$ and $l \in [1, k(\bigstar)]$ we have $y_l \restriction m = x_l^m \restriction m$. Therefore $p \leq (t(p), w(p) \cup \{y_1, \ldots, y_{k(\bigstar)}\})$, and (let $r \in M$ and $\sigma \in \mathrm{ord}$)

$$r \Vdash \text{“}\tau = \sigma\text{”}, \qquad (t(p), w(p) \cup \{y_1, \ldots, y_{k(\bigstar)}\}) \leq r.$$

Let $n(*) = n(r) + 8$ and $r^* = (t(r), w(q_{n(*)}) \cup w(r))$.

2.7. *Claim.* (i) $r^* \in M$.

(ii) $r \leq r^*$ *and* $q_{n(\star)} \leq r^*$.

(iii) $r^* \models \tau = \sigma$ *and* $n(r^*) = n(r) < n(*)$.

PROOF. Remember that $\{y_1, \ldots, y_{k(\star)}\} \subseteq w(r)$. □

Now 3.7(iii) contradicts the choice of $q_{n(*)}$, and this finishes the proof of the lemma. □

2.8. DEFINITION. We say that $F \subseteq \omega^\omega$ is *unbounded* if

$$(\forall g \in \omega^\omega)(\exists f \in F)(\exists^\infty n)(g(n) \leq f(n)).$$

2.9. THEOREM. *If* $F \in V$ *is unbounded, then* $\Vdash_M$ "*F is unbounded*".

PROOF. Suppose that there exists an M-name $\boldsymbol{g}$ of a member of ω^ω and $p \in M$ such that

$$p \Vdash_M \text{“}(\forall f \in F)(\forall^\infty n)(f(n) < \boldsymbol{g}(n)).$$

Let $N \prec \langle H(\chi), \in, \leq_\chi \rangle$ be such that $\|N\| = \aleph_0$, and $P, p, \boldsymbol{g}$ are in N. Pick $f \in F$ such that, for every $h \in N \cap \omega^\omega$, $(\exists^\infty n)(h(n) \leq f(n))$. Working in N, for each $p^0 \in M$, for every $k \in \omega$ we define

$$h_{p^0}(k) = \max\{i \in \omega \colon (\exists q, r \in M)(t(q) = t(p_0) \text{ and } |w(p_0) - w(q)| \leq k, \\ n(r) \leq m(k, p_0, \boldsymbol{g}(k)) \text{ and } r \Vdash \text{“}\boldsymbol{g}(h) = i\text{”})\}.$$

Using the above lemma, it is not hard to show that for every $k \in \omega$, $h_{p^0}(k) \in \omega$.

By assumption there exist $q \geq p$ and $k_0 \in \omega$ such that $q \Vdash (\forall k \geq k_0)(f(k) < \boldsymbol{g}(k))$. Set $p_0 = (t(q), w(q) \cap N)$. Clearly $p_0 \in N$. Choose $k_1 = |w(q) - w(p_0)|$ and k_2 such that $h_{p^0}(k_2) < f(k_2)$ and $k_1 < k_2$.

By Lemma 2.6, there exist $r \geq q$, p_0 such that $n(r) \leq m(k_2, p_0, \boldsymbol{g}(k_2))$, and i such that $r \Vdash \text{“}\boldsymbol{g}(k_2) = i\text{”}$; and this implies that

$$r \Vdash \text{“}\boldsymbol{g}(k_2) \leq h_{p^0}(k_2) < f(k_2)\text{”},$$

which is a contradiction to the choice of q. This concludes the proof of the theorem. □

2.10. THEOREM. cons(ZF) $\Rightarrow$ cons(ZFC $+ \neg B(m) + U(m) + \neg B(c) + \neg U(c) + C(c)$).

PROOF. Let $V \models \text{“}A(m) + 2^{\aleph_0} > \aleph_1\text{”}$, and let $\bar{Q} = \langle P_i; \boldsymbol{Q}_i \colon i < \omega_1 \rangle$ be a finite-support iterated forcing system satisfying, for every $i < \omega_i$, $\Vdash_{P_i}$ "$\boldsymbol{Q}_i$ is the meager forcing M" and if i is a limit ordinal, then $P_i = \varinjlim \bar{Q} \restriction i$. Let $P_{\omega_1} = \varinjlim \bar{Q}$. Then

$$\Vdash_{P_{\omega_1}} \text{“}\neg B(m) + U(m) + \neg B(c) + \neg U(c) + C(c)\text{”}.$$

(a) $\neg B(m)$. As Cohen reals are added in every limit stage of cofinality ω, it is well known that

$$\Vdash_{P_{i+\omega}} \text{“}2^\omega \cap V[\boldsymbol{G} \restriction i] \text{ has measure zero”}$$

and, by c.c.c.,

$$(*) \qquad \Vdash_{P_{\omega_1}} \text{“}2^\omega = \bigcup_{i<\omega_1} 2^\omega \cap V[\boldsymbol{G} \restriction \omega \cdot i]\text{”}.$$

(b) $U(m)$. It is not hard to show that for every $i < \omega_1$, P_i is σ-centered. Therefore, for every P_i-name $\boldsymbol{\tau}$ for a real number, there exists $A_{\boldsymbol{\tau}} \in V$ such that $\mu(A_{\boldsymbol{\tau}}) = 0$ and

$\Vdash_{P_\iota}$ "$\tau \in A_\tau$". Now using (*) and $A(m)$, we can prove that for every $X \in V^{P_{\omega_1}}$, if $\Vdash_{P_{\omega_1}}$ "$X \in [2^\omega]^{<c}$", then there exist $AX \in V$ with $\mu(AX) = 0$ and $\Vdash_{P_{\omega_1}}$ "$X \subseteq AX$".

(c) $\neg B(c)$. The ω_1-meager trees of the generic sequence witness this.

(d) $\neg U(c)$. The ω_1-Cohen reals given by the support of the iteration witness this.

(e) $C(c)$. If $\Vdash_{P_{\omega_1}}$ "$\neg C(c)$", then

$$\Vdash_{P_{\omega_1}} \text{“}\bigcup\{B: B \in V \text{ and } B \text{ meager}\} \text{ is meager”}.$$

(Remember that $V \models A(m)$. Therefore $V \models A(c)$.) And this implies that

$$\Vdash_{P_{\omega_1}} \text{“}\omega^\omega \cap V \text{ is bounded”}.$$

But using Theorems 2.9 and 2.2, we can prove

$$\Vdash_{P_{\omega_1}} \text{“}\omega^\omega \cap V \text{ is unbounded”}.$$

(In order to see this, in V we define $\langle f_i: i < c\rangle$ such that $i < j < c$ implies $(\forall^\infty n)(f_i(n) < f_j(n))$, and we define aRb iff $|a| < |b|$ iff aSb.) This concludes the proof of the theorem. □

§3. Preserving "the union of the old measure zero sets is not a measure zero set".

3.1. THEOREM. *Let $M \subseteq N$ be models of* ZFC*. *Then the following statements are equivalent*:

(i) *There exists $h \in {}^\omega([\omega]^{<\omega}) \cap N$ such that, for every $n \in \omega$, $|h(n)| \leq n$ and for every $f \in {}^\omega\omega \cap M$ there exists $n \in \omega$ such that, for every $m \geq n$, $f(m) \in h(m)$.*

(ii) *There exist $h \in {}^\omega([\omega]^{<\omega}) \cap N$ and $g \in {}^\omega\omega \cap M$ such that, for every $n \in \omega$, $|h(n)| \leq g(n)$ and for every $f \in \omega^\omega \cap M$ there exists $n \in \omega$ such that, for every $m \geq n$, $f(m) \in h(m)$.*

PROOF. (i) ⇒ (ii) is clear.

(ii) ⇒ (i). Suppose we have h and g satisfying the requirements of (ii). Then we set $G_l: \omega^l \to \omega$, the canonical one-to-one and onto function from ω^l to ω, and for each $i < l$ we define $G_{l,i}: \omega \to \omega$ by setting $G_{l,i}(k) = \pi_i(G_l^{-1}(k))$, where π_i is the projection function over the ith coordinate.

In M we pick $\langle n_i: i < \omega\rangle$ such that $n_i < n_{i+1}$ and $g(n_i) < n_i$, and in N we define the function $h': \omega \to \omega$. If $i \in [n_l, n_{l+1})$, then

$$h'(i) = G''_{n_{l+1}-n_l} h(l),$$

where $j + l = i$. Clearly h' is well defined and satisfies $|h'(i)| \leq |h(l)|$ for $n_l \leq i < n_{l+1}$, and in this case $|h(l)| \leq g(l) < n_l \leq i$. Therefore $|h'(i)| \leq i$.

Now we will show that for every $f \in {}^\omega\omega \cap M$ there exists $n \in \omega$ such that, for every $m \geq n$, $f(m) \in h'(m)$. We define

$$f'(l) = G_{n_{l+1}-n_l}(f(n_l), \ldots, f(n_{l+1} - 1)).$$

Then clearly $f' \in {}^\omega\omega \cap M$, and thus there exists $k \in \omega$ such that, for every $l \geq k$, $f'(l) \subseteq h(l)$. Therefore,

$$G_{n_{l+1}-n_l, j}(f'(l)) = f(n_l + j),$$

where $j \in n_{l+1} - n_l$, and this implies that $f(n_l + j) \in h'(i)$, where $i = n_l + j$. Hence, for every $i \geq n_k$, $f(i) \in h'(i)$. □

3.2. COROLLARY. *Let* $M \subseteq N$ *be models of* ZFC*. *Then the following are equivalent*:

(i) *In* N *the union of all measure zero sets coded in* M *is a measure zero set.*

(ii) *Theorem* 3.1(ii).

(iii) *Theorem* 3.1(i).

PROOF. (i) $\Rightarrow$ (ii). Some little changes in the proof of [RS, 1.1] give that by (i) there exists $h \in {}^{\omega}([\omega]^{<\omega})$ such that, for every $n \in \omega$, $|h(n)| \leq n^2$ and for every $f \in {}^{\omega}\omega \cap M$ there exists $n \in \omega$ such that, for every $m \geq n$, $f(m) \in h(m)$.

(ii) $\Rightarrow$ (iii) is proved in 3.1.

(iii) $\Rightarrow$ (i) was proved by Bartoszyński; see [RS]. $\square$

3.3 DEFINITION. Let P be a forcing notion satisfying the countable chain condition.

(a) We say that $x \in {}^{\omega}\omega$ is *N-big* iff, for every $h \in {}^{\omega}([\omega]^{<\omega})$, if there exists $k \in \omega$ such that $|h(n)| \leq n^k$ for every $n \in \omega$, then there exist infinitely many $n \in \omega$ such that $x(n) \notin h(n)$.

(b) We say that P is *good* iff, for every N-big $x \in {}^{\omega}\omega$, if $P \in N$ then

$$\Vdash_P \text{“}x \text{ is } N[G_P]\text{-big”}.$$

3.4. LEMMA. *If* P *is good and* $\Vdash_P$ "Q *is good*", *then* $P * Q$ *is good.*

PROOF. Easy. $\square$

3.5. LEMMA. *If* $\bar{Q} = \langle P_i; Q_i\colon i < \delta\rangle$ *is a finite-support iterated forcing system and, for every* i, $\Vdash_{P_i}$"Q_i *is good*", *then* $P_\delta = \varinjlim \bar{Q}$ *is good.*

PROOF (induction on δ). If $\delta = \gamma + 1$, then use the induction hypothesis and Lemma 3.4. If $\delta = \bigcup \delta \neq \varnothing$, then let $N \prec \langle H(\chi), \in, \leq_\chi\rangle$ be such that $P_\delta \in N$ and $\|N\| = \aleph_0$, and let $x \in \omega^\omega$ be N-big. Let $p \in P_\delta$, $\boldsymbol{h} \in N^{P_\delta}$, and $k \in \omega$ be such that

$$\Vdash_{P_\delta} \text{“}\boldsymbol{h} \in {}^{\omega}([\omega]^{<\omega}) \text{ and } (\forall n)(|\boldsymbol{h}(n)| \leq n^k)\text{”},$$

$$p \Vdash_P \text{“}(\forall n \geq l)(x(n) \in \boldsymbol{h}(n))\text{”}.$$

Let $\delta(*) = \sup(\delta \cap N)$ and $p_1 = p \restriction \delta(*)$, and let $\alpha < \delta(*)$ be such that $p_1 \in P_\alpha$. Let $G_\alpha \subseteq P_\alpha$ be generic over V, $p_1 \in G_\alpha$. By the induction hypothesis, x is $N[G_\alpha]$-big. Working in $N[G_\alpha]$, we can find $\langle r_n\colon n < \omega\rangle$ with $r_n \in P_\delta/G_\alpha$, $r_n \leq r_{n+1}$ and

$$r_n \Vdash \text{“}\boldsymbol{h}(n) = a_n\text{”}, \qquad p_1 \leq r_0$$

when $\langle a_n\colon n < \omega\rangle \in N[G_\alpha]$. The function $n \to a_n$ belongs to $N[G_\alpha]$ and, for every n, $|a_n| \leq n^k$. Therefore there exist infinitely many $n \in \omega$ such that $x(n) \notin a_n$. So let $n > l$ satisfy this, and thus $r_n \Vdash$"$x(n) \notin \boldsymbol{h}(n)$". But r_n and p are compatible, and this is a contradiction to the choice of p and l. $\square$

3.6. THEOREM. *If* P *is a* σ*-centered partially ordered set, then* P *is good.*

PROOF. (a) $P \models$ "c.c.c." clearly.

(b) Suppose $N \prec \langle H(\chi), \in, \leq_\chi\rangle$, $P \in N$, $\|N\| = \aleph_0$, and let $x \in {}^{\omega}\omega$ be N-big. Let $\boldsymbol{h} \in N^P$ be such that, for some fixed $k \in \omega$,

$$\Vdash_P \text{“}\boldsymbol{h} \in {}^{\omega}([\omega]^{<\omega}) \text{ and } (\forall n)(|\boldsymbol{h}(n)| \leq n^k)\text{”}.$$

By hypothesis there exists $\langle D_n\colon n < \omega\rangle$ such that $P = \bigcup_n D_n$ and each D_n is directed.

Now we define $t^n(r)$ and $T^n(r)$ by

$$t^n(r) = \{a \in [\omega]^{<\omega} : (\exists q \geq r)(q \Vdash \boldsymbol{h}(n) = a)\}, \qquad T^n(r) = \bigcap t^n(r).$$

Therefore $|T^n(r)| \leq n^k$. Also,

$$r_1 \leq r_2 \Rightarrow t^n(t_1) \supseteq t^n(r_2) \wedge T^n(r_1) \subseteq T^n(r_2).$$

We know that D_l is directed. Therefore there exists $r^{n,l} \in D_l$ satisfying

$$(\forall r \in D_l)(r^{n,l} \leq r \Rightarrow T^n(r) = T^m(r^{n,l})).$$

Now we define $h^l(n) = T^n(r^{n,l})$. Clearly $h^l \in N \cap {}^{\omega}([\omega]^{<\omega})$ and, for every $n \in \omega$, $|h^l(n)| \leq n^k$. Therefore there exist infinitely many $n \in \omega$ such that $x(n) \notin h^l(n)$. Let $G \subseteq P$ be generic over V. In $V[G]$ we need to prove that there exist infinitely many $n \in \omega$ such that $x(n) \notin \boldsymbol{h}[G](n)$. If this fails, there exists $r \in P$ such that

$$r \Vdash \text{“}(\neg \exists^\infty n)(x(n) \notin \boldsymbol{h}(n))\text{”}.$$

There exists $m \in \omega$ such that

$$r \Vdash \text{“}(\forall n > m)(x(n) \in \boldsymbol{h}(n))\text{”}.$$

There exists $l \in \omega$ such that $r \in D_l$. Let $n > m$ be such that $x(n) \notin h^l(n)$. This implies that there exist $r'' \in P, r'' \geq r'$, such that $r'' \Vdash$ “$x(n) \notin \boldsymbol{h}(n)$”; and this is a contradiction to the choice of r. □

3.7. Theorem. *If P is random real forcing, then P is good.*

Proof. Suppose $\boldsymbol{h}$, N, $x \in \omega^\omega$ are as in the definition of good. We define

$$B_{n,i} = \|i \in \boldsymbol{h}(n)\|.$$

Clearly $B_{n,i} \in P$, and $a_n = \{i : \mu(B_{n,i}) \geq 1/n\}$. Clearly

$$|a_n| \leq \frac{|\boldsymbol{h}(n)|^2}{1/n} = n^{2k+1}.$$

The function $n \to a_n$ belongs to N, and therefore $(\exists^\infty n)(x(n) \notin a_n)$. Let $G \subseteq P$ be generic over V, and let $p \in P$ and $l \in \omega$ be such that

$$p \Vdash \text{“}(\forall n > l)(x(n) \in \boldsymbol{h}(n))\text{”}.$$

There exists $m \in \omega$, $l < m$, such that $\mu(p) > 1/m$ and $x(m) \notin a_m$. Therefore $\mu(B_{m,x(m)}) < 1/m$, and this implies that

$$p^* = p - B_{m,x(m)} \in P \quad \text{and} \quad p \leq p^* \Vdash \text{“}x(m) \notin \boldsymbol{h}(m)\text{”}.$$

This is a contradiction. This finishes the proof of the theorem. □

3.8. Theorem. *Let P be a good forcing notion, and let $G \subseteq P$ be generic over V. Then $V[G] \models$ “the union of all measure zero sets added in V is not a measure zero set”.*

Proof. Suppose the conclusion of the theorem does not hold. Then by Corollary 3.2 there exists $h: \omega \to [\omega]^{<\omega}$ such that

$$(*) \qquad |h(n)| \leq n \quad \text{for every } n \in \omega,$$

$$(**)\qquad \text{for every } x \in \omega^\omega \cap V \text{ there exists } k \in \omega \text{ such that } x(n) \in h(n) \text{ for every } n \geq k.$$

Let $\mathbf{h} \in V^P$ be a name for such an h, and $p \in G$ forcing this. Let $N \prec \langle (H(\chi), \in, \leq_\chi \rangle$ be such that $P \in N$, $\|N\| = \aleph_0$, $\mathbf{h} \in N$, $p \in N$, and x an N-big member of ω^ω. Therefore x is $N[G]$-big, and this implies that $(**)$ fails for this x. $\square$

3.9. THEOREM. cons(ZF) $\Rightarrow$ cons(ZFC $+ \neg B(m) + \neg U(m) + C(m) + \neg C(c)$).

PROOF. Let $V \models$ "$A(m) + \neg$CH", and let $\bar{Q} = \langle P_\alpha; \mathbf{Q}_\alpha : \alpha < \omega_1 \rangle$ be a finite-support iterated forcing such that

(i) if α is odd, then $\models_{P_\alpha}$ "$\mathbf{Q}_\alpha$ is random real forcing", and

(ii) if α is even, then $\models_{P_\alpha}$ "$\mathbf{Q}_\alpha$ is Hechler real forcing".

Let $P_{\omega_1} = \varinjlim \bar{Q}$. Then P_{ω_1} is good and if $G \subseteq P_{\omega_1}$ is generic over V, then

$$(*)\qquad V[G] \models \text{"the union of every measure zero set coded in } V \text{ is not a measure zero set".}$$

3.10. *Claim.* $V[G] \models \neg B(m) + \neg U(m) + C(m) + \neg C(c)$.

Proof. (a) $\neg U(m)$. The ω_1-random reals of the generic sequence witness this fact.

(b) $\neg B(m)$. As Cohen reals are added in every even stage, it is possible to show that $\mu(2^\omega \cap V[G \restriction \alpha]) = 0$ for every $\alpha < \omega_1$; and by c.c.c. of P_{ω_1} we can prove that

$$2^\omega = \bigcup_{\alpha \in \omega_1} 2^\omega \cap V[G \restriction \alpha].$$

(c) $\neg C(c)$. Each pair of Hechler reals add a meager set which contains the union of all meager sets coded in the ground model. We use the c.c.c. and the fact that every meager set is contained in a Borel meager set in order to show that the ω_1-sequence of meager sets obtained from the Hechler reals witnesses $\neg C(c)$.

(d) $C(m)$. As in $V \models A(m)$, we can build $\langle A_i : i < 2^{\aleph_0} \rangle \in V$ such that for every $i < 2^{\aleph_0}$ we have $\mu(A_i) = 0$, and for every measure zero set $A \in V$ there exists $i < 2^{\aleph_0}$ with $A \subseteq A_i$, and if $i < j < 2^{\aleph_0}$ then $A_i \subseteq A_j$. As $P_{\omega_1} \models$ c.c.c., if $V[G] \models \neg C(m)$ then there exists a measure zero set $A \in V[G]$ such that, for every $i < 2^{\aleph_0}$, $A_i \subseteq A$. But this implies that P_{ω_1} is not good, a contradiction. $\square$

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