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CORRIGENDUM TO "GENERALIZED MARTIN'S AXIOM AND SOUSLIN'S HYPOTHESIS FOR HIGHER CARDINALS"

BY

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ABSTRACT

Correct proofs are given for Theorem 3 and the Propositions of §§5, 6 of [4]. For the latter, we must modify the principle (S)″ in a technical way. We prove a weaker version of Theorem 2, where $\square$ is replaced by the stronger hypothesis $\mathrm{Pr}^b_{\aleph_1}$.

The burden of this note is to acknowledge and correct errors in [4] which were pointed out by Velleman in his review [7]. We are grateful to the referee for many helpful suggestions which spared all of us a Corrigendum[2].

§1. The most serious error affects "Lemma" 1, "Theorem" 2 and Theorem 3. The last is in fact true, but "Lemma" 1 of [4] is false (see (1.5) below) and thus the "proof" of "Theorem" 2 of [4] is irreparably false. However, Shelah has recently found a rather different approach to proving analogous results. See, below, the Corollary and Theorem 2′ of (1.1).

The difficulty in the "proof" of "Lemma" 1 in [4] is as follows. We cannot prove that CH $\Rightarrow$ (BA applies to the partial ordering $\mathbf{P}^C$); in fact (see below), we conjecture that CH $\Rightarrow$ (BA does not apply to any partial ordering $\mathbf{Q}$ whose regular open algebra is isomorphic to that of $\mathbf{P}^C$).

In the "Lemma" at the end of §3 it is claimed that $\mathbf{P}^C$ is κ-closed and (though this terminology is not used in the Lemma) well-met. Reference is made to [5] for proofs.

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$\mathbf{P}^C$ is a slightly simpler version of the partial ordering considered in [5], §1.4, where it is proved that the latter partial ordering is well-met (again, this terminology is not used) and $(<\kappa,\infty)$-distributive (the proof actually proves κ-strategic-closure). These proofs go over for $\mathbf{P}^C$, *but* $\mathbf{P}^C$ is not κ-closed.

Suppose $p(\alpha)=(s(\alpha),w(\alpha),u(\alpha))$ is an increasing sequence, where $\alpha<\theta<\kappa$, θ limit. Let $\gamma(\alpha)=\max s(\alpha)$, let $\gamma=\bigcup\{\gamma(\alpha):\alpha<\theta\}$, and suppose $\gamma>\gamma(\alpha)$ for all $\alpha<\theta$. Let $\bar{s}=\bigcup\{s(\alpha):\alpha<\theta\}$, $\bar{w}=\bigcup\{w(\alpha):\alpha<\theta\}$, $\bar{u}=\bigcup\{u(\alpha):\alpha<\theta\}$. Then, $(\forall\nu\in\text{range }\bar{u})\ \gamma\in B_\nu$.

However, it may be that $\{\bar{A}_{\gamma\nu}:\nu\in\text{range }\bar{u}\}$ is not linearly ordered by $\sqsubseteq_\gamma$, in which case $(\bar{s},\bar{w},\bar{u})$ cannot be extended to a condition. The problem arises if there is $\{\nu,\nu'\}\in[\text{range }\bar{u}]^2$ such that the following are both cofinal in γ:

$$Y_1=\{\alpha\in\bar{s}:\bar{A}_{\alpha\nu},\bar{A}_{\alpha\nu'}\in\bar{w}(\alpha)\wedge\bar{A}_{\alpha\nu}\sqsubsetneq_\alpha A_{\alpha\nu'}\},$$

$$Y_2=\{\alpha\in\bar{s}:\bar{A}_{\alpha\nu},\bar{A}_{\alpha\nu'}\in\bar{w}(\alpha)\wedge\bar{A}_{\alpha\nu'}\sqsubsetneq_\alpha\bar{A}_{\alpha\nu}\},$$

i.e., if the ν-branch and the ν'-branch intertwine cofinally beneath γ. In [5], this is prevented by guaranteeing, at even α, that if $\nu,\nu'\in\text{range }u(\alpha)$, $\nu<\nu'\Rightarrow\nu\in X_{\gamma(\alpha)\nu'}$.

In fact, this requirement can be built into $\mathbf{P}^C$, i.e., by requiring in the definition of $\mathbf{P}^C$:

(iv) let $\gamma=\max s$; then $(\nu,\nu'\in\text{range }u\wedge\nu<\nu')\Rightarrow\nu\in X_{\gamma\nu'}$.

With this change, $\mathbf{P}^C$ becomes κ-closed. Alas, it is no longer well-met (nor even neatly κ^+-normal), since now $(s,w,u\cup u')$ will not be a condition, since (iv) will fail. It can be extended to one, but not to a least one. Thus, the "proof" that, under CH, BA applies to $\mathbf{P}^C$ collapses and with it the "proof" of "Lemma" 1.

A situation similar to this obtains for the countable conditions $\mathbf{P}$ for forcing $\square_{\omega_1}$ (see [3], (4.1)-(4.7), or [6], I, §3). In this context Velleman proved the following (II.4.2) of [6]:

THEOREM (CH). R.O.($\mathbf{Q}$) $\cong$ R.O.($\mathbf{P}$) $\Rightarrow$ $\mathbf{Q}$ *countably closed* $\Rightarrow$ $\mathbf{Q}$ *ill met.*

CONJECTURE (CH). R.O.($\mathbf{Q}$) $\cong$ R.O.($\mathbf{P}^C$) $\Rightarrow$ $\mathbf{Q}$ *countably closed* $\Rightarrow$ $\mathbf{Q}$ *ill met.* See also (1.5), below.

(1.1) The correct proof of Theorem 3 and an analogue of Theorem 2 of [4] is as follows. For regular uncountable κ, we introduce three principles:

Pr^a_κ: $(\exists A\subseteq\kappa)\ (\kappa^+=(\kappa^+)^{L[A]})$,

Pr^b_κ: $(\exists A\subseteq\kappa^+)\ (\forall\delta)\ (\kappa<\delta<\kappa^+\Rightarrow L[A\cap\delta]\models\text{"card }\delta=\kappa\text{"})$,

Pr^c_κ: $(\exists A\subseteq\kappa^+)(\{\delta:\kappa<\delta<\kappa^+\wedge L[A\cap\delta]\models\text{"}\delta\text{ is regular"}\}$ is non-stationary.

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We easily prove:

PROPOSITION. (a) $\mathrm{Pr}^b_\kappa \Leftrightarrow \mathrm{Pr}^c_\kappa$;
(b) *If* κ^+ *is not* $(Mahlo)^L$ *then* Pr^c_κ.

PROOF. The left-to-right implication of (a) is obvious. For the other implication, use the obvious pressing-down function provided by Pr^c_κ to pack in well-orderings of κ of the appropriate order types. Taking $A = \varnothing$, (b) is clear.

We shall prove, below:

LEMMA. $\kappa^{<\kappa} = \kappa \wedge \mathrm{BA}_\kappa \wedge 2^\kappa > \kappa^+ \wedge \mathrm{Pr}^b_\kappa \Rightarrow \mathrm{Pr}^a_\kappa$.

This has the immediate corollary, taking $\kappa = \aleph_1$:

COROLLARY. $\mathrm{BACH} + 2^{\aleph_1} > \aleph_2 + \aleph_2$ *is not* $(Mahlo)^L \Rightarrow \exists(\aleph_1, 1)$ *morasses.*

PROOF OF COROLLARY. By the Proposition and the Lemma, the hypotheses yield $\mathrm{Pr}^a_{\aleph_1}$. As was argued in [3], $\mathrm{Pr}^a_{\aleph_1} \Rightarrow \exists(\aleph_1, 1)$ morasses, since clearly, in $\mathrm{Pr}^a_{\aleph_1}$, A can be chosen so that $\aleph_1^{L[A]} = \aleph_1$, as well.

Now the above Corollary replaces "Lemma" 1 of [4]. We have the following analogue of "Theorem" 2 of [4].

THEOREM 2′. $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{Pr}^b_{\aleph_1} + \mathrm{BA} \Rightarrow$ *there's an* $\aleph_2$*-super-Souslin tree and thus* (viz. (2.3) of [3]) $\neg \mathrm{SH}_{\aleph_2}$.

Then the proof of Theorem 3 of [4] goes through with the Corollary and Theorem 2′ in place of "Lemma" 1 and "Theorem" 2 of [4]. We prove the above Lemma in (1.2), (1.3).

(1.2) Let $A \subseteq \kappa^+$ witness Pr^b_κ. For $\delta < \kappa^+$, let $M_\delta = L[A \cap \delta]$. Note that $M_\delta \models \mathrm{card}\, \delta \leqq \kappa$. Now define $(A_\xi : \xi < \kappa^+)$ by recursion as follows: given $(A_\xi : \xi < \delta)$, let A_δ be the $<_{M_\delta}$-least element of $[\kappa]^\kappa \cap M_\delta$ which is almost disjoint from the A_ξ and non-stationary in the sense of M_δ. To see that A_δ is defined, first note that the definition of $(A_\xi : \xi < \delta)$ can be carried out in M_δ, so this sequence lies in M_δ, and in M_δ each A_ξ is non-stationary. But then, since $M_\delta \models \mathrm{card}\, \delta \leqq \kappa$, we can, in M_δ, take the diagonal intersection, C, of a sequence of club subsets, C_ξ (which avoids A_ξ), $\xi < \delta$, and we take $A_\delta =$ any non-stationary subset of C of power κ.

Having defined $\vec{A} = (A_\xi : \xi < \kappa^+)$, in (1.3), below, we shall define a partial ordering $\mathbf{P}(\vec{A})$, which will be κ-closed, well-met and, assuming $\kappa^{<\kappa} = \kappa$, κ-linked, so that BA_κ, the version of BA, for κ, applies to $\mathbf{P}(\vec{A})$. $\mathbf{P}(\vec{A})$ is a variant of the almost-disjoint set coding of A, using $\vec{A}$ as the almost disjoint family. A twist is that the "yes" part of the condition is required to have a co-initial segment of fewer than κ many A_i and to have additional information of size $< \kappa$, so that $\mathbf{P}(\vec{A})$ is really an almost-inclusion coding. It will then be fairly routine to show, in (1.4) below, that:

LEMMA. *There are κ^+ dense subsets of $\mathbf{P}(\vec{A})$ which, if met, yield $X \subseteq \kappa$ such that $\vec{A}$, $A \in L[X]$.*

The Lemma of (1.1) is then clear: the set X of the above Lemma witnesses $\mathrm{Pr}^a_{\aleph_1}$.

(1.3) We now define $\mathbf{P}(\vec{A})$.

DEFINITION. $f \in \mathbf{P}(\vec{A}) \Leftrightarrow f : \mathrm{dom}\, f \to 2$, $\mathrm{dom}\, f \subseteq \kappa$ and there's $W \in [\kappa^+]^{<\kappa}$ s.t.:
(a) $(\forall \alpha \in W)(\alpha \in A \wedge (\exists i \in \kappa)(\forall i < j < \kappa)(j \in A_\alpha \Rightarrow j \in \mathrm{dom}\, f \wedge f(j) = 1))$.
(b) $\mathrm{dom}\, f \setminus \bigcup \{A_\alpha : \alpha \in W\}$ has power $< \kappa$.
For $f, g \in P(\vec{A})$, set $f \leqq g$ iff $f \subseteq g$. $\mathbf{P}(\vec{A}) = (P(\vec{A}), \leqq)$.

REMARKS. (1) $\alpha \in W \Rightarrow A_\alpha \setminus \mathrm{dom}\, f$ has power $< \kappa$.
(2) $f^{-1}[\{0\}]$ has power $< \kappa$,
(3) W is uniquely determined by (a), (b),
(4) $(Q \in [P(\vec{A})]^{<\kappa} \wedge \bigcup Q$ is a function$) \Rightarrow \bigcup Q \in P(\vec{A})$.

Remark (4) immediately yields:

PROPOSITION. *$\mathbf{P}(\vec{A})$ is κ-closed and well-met; further, if $f^{-1}[\{0\}] = g^{-1}[\{0\}]$ then f, g are compatible.*

COROLLARY ($\kappa^{<\kappa} = \kappa$). *$\mathbf{P}(\vec{A})$ is κ-linked.*

PROOF. Enumerate $[\kappa]^{<\kappa}$ as $(a_i : i < \kappa)$; then set $P_i = P_{a_i} = \{f \in P : f^{-1}[\{0\}] = a_i\}$. Note that, in fact, $\mathbf{P}(\vec{A})$ is (κ, κ) centered.

We now complete the proof of the Lemma of (1.2). For $\alpha < \kappa^+$, $\xi < \kappa$, let $D_{\alpha,\xi} = \{f \in P(\vec{A}) : (\exists \zeta \in A_\alpha \setminus \xi) f(\zeta) = 0\}$, if $\alpha \notin A$; otherwise, let

$$D_{\alpha,\xi} = D_\alpha = \{f \in P(\vec{A}) : f^{-1}[\{1\}] \text{ includes a final segment of } A_\alpha\}.$$

Clearly, each $D_{\alpha,\xi}$ is dense, in either case.

Now suppose G is an ideal in $\mathbf{P}(\vec{A})$ meeting all the $D_{\alpha,\xi}$. Let $F = \bigcup G$, and let $X = F^{-1}[\{1\}]$ (so $X \subseteq \kappa$). Then clearly:
(!) $\alpha \in A \Leftrightarrow A_\alpha \subseteq^* X$ (i.e. $\mathrm{card}(A_\alpha \setminus X) < \kappa$).
Thus, $A \in L[X, \vec{A}]$.

We now claim that $\vec{A} \in L[X]$. To see this, simply note that the recursive definition of $\vec{A}$ can be carried out in $L[X]$: given $(A_\xi : \xi < \delta)$, use X to read off

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$A \cap \delta$, let $M_\delta = L[A \cap \delta]$, and then define A_δ from $(A_\xi : \xi < \delta)$ and M_δ as above in (1.2). Thus, $A \in L[X, \vec{A}] = L[X]$, so X witnesses $\mathrm{Pr}^{\mathrm{a}}_{\aleph_1}$. So, the proof of the Lemma of (1.2) and, therefore, the proof of the Lemma of (1.1), is complete.

(1.4) We should point out the limitations to the methods of (1.1)–(1.3).

LEMMA. *If $\omega < \kappa < \lambda$, λ is (strongly) Mahlo, κ regular, then there's κ-closed, λ-c.c.* $\mathbf{P}$ *s.t., in* $V^{\mathbf{P}}$, $2^\kappa = \lambda^+$, BA_κ *holds, but* $\mathrm{Pr}^{\mathrm{b}}_\kappa$ *fails.*

PROOF (Sketch). Let $\mathbf{P}_0 = \mathbf{P}^\lambda_0$ be the Lévy collapse of λ to become κ^+, let $\mathring{\mathbf{P}}_1 = \mathring{\mathbf{P}}^\lambda_1 \in V^{\mathbf{P}_0}$ be the natural length λ^+ iteration to make BA_κ true. It suffices then to show that in $V^{\mathbf{P}_0 * \mathring{\mathbf{P}}_1}$, $\mathrm{Pr}^{\mathrm{a}}_\kappa$ fails. By way of contradiction, suppose $A \in \mathcal{P}(\kappa) \cap V^{\mathbf{P}_0 * \mathring{\mathbf{P}}_1}$ witnesses $\mathrm{Pr}^{\mathrm{a}}_\kappa$. But then for some (strongly) inaccessible $\lambda' < \lambda$, the whole situation is reflected at λ'; i.e.

$$A \in V^{\mathbf{P}^{\lambda'}_0 \times \mathring{\mathbf{R}}}, \text{ and there witnesses } \mathrm{Pr}^{\mathrm{a}}_\kappa;$$

here $\mathring{\mathbf{R}}$ is an initial segment of the length $(\lambda')^+$ iteration, $\mathring{\mathbf{P}}^{\lambda'}_1$ for BA_κ in $V^{\mathbf{P}^{\lambda'}_0}$.

The main point is that the "evidence" for the chain condition of the $\mathring{\mathbf{Q}}_\alpha$'s (in the iteration for $\mathring{\mathbf{P}}^\lambda_1$) is very explicit and can therefore be reflected to suitable λ'; further, for suitable λ', $\mathbf{P}^\lambda_0 * \mathring{\mathbf{P}}^\lambda_1 \cong \mathbf{P}^{\lambda'}_0 * \mathring{\mathbf{P}}^{\lambda'}_1 * \mathring{\mathbf{Q}}$, for some $\mathbf{P}^{\lambda'}_0 * \mathring{\mathbf{P}}^{\lambda'}_1$-name $\mathring{\mathbf{Q}}$ for a partial ordering.

But then $L[A] \subseteq V^{\mathbf{P}^{\lambda'}_0 \times \mathring{\mathbf{R}}}$ and in $V^{\mathbf{P}^{\lambda'}_0 * \mathring{\mathbf{R}}}$, λ is still Mahlo, so A cannot witness $\mathrm{Pr}^{\mathrm{a}}_\kappa$ in $V^{\mathbf{P}_0 * \mathring{\mathbf{P}}_1}$.

In a similar fashion, we obtain:

LEMMA. *If $\omega < \kappa < \lambda$, λ weakly compact, κ regular, then there's κ-closed λ-c.c.* $\mathbf{P}$ *s.t. in* $V^{\mathbf{P}}$, $2^\kappa = \lambda^+$, $(\mathrm{S})_\kappa$ *(the version for κ of the principle* (S) *of* [4]) *holds but* $\mathrm{Pr}^{\mathrm{b}}_\kappa$ *fails.*

(1.5) Of course, as was known quite early to Jensen, [1], $\mathrm{Pr}^{\mathrm{b}}_\kappa \Rightarrow \Box_\kappa$. However, as we shall now show:

PROPOSITION. $\mathrm{BACH} + 2^{\aleph_1} > \aleph_2 + \Box_{\aleph_1} \not\Rightarrow \mathrm{Pr}^{\mathrm{b}}_{\aleph_1}$.

Before proving the Proposition, we note that this shows that "Lemma" 1 of [4] is in fact false. In an early version of this paper, we had claimed a result analogous to the Proposition, for arbitrary regular κ in place of $\aleph_1$, and without assuming $\kappa^{<\kappa} = \kappa$, but with $2^\kappa = \kappa^+$ in place of $\mathrm{BA}_\kappa + 2^\kappa > \kappa^+$. The referee found problems with the proof we gave, but suggested another for $\kappa = \aleph_1$ which, as he pointed out, yielded the Proposition as stated. The obstacle to generalizing this proof to higher κ is that the $<\kappa$-size conditions for forcing $\Box_\kappa$ are not

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κ-closed; this affects the strategic-closure property of the tail of the iteration, $\mathbf{P}_0 * \mathring{\mathbf{P}}_1$, even though appropriate initial segments and the whole iteration are strategically-closed.

PROOF OF PROPOSITION. Let κ be a Mahlo cardinal in the ground model, V. Let $\mathbf{P}_0 \in V$ be the analogue of the countable conditions for forcing $\square_{\aleph_1}$, viz. [3], §4, except that if $p = (a, \vec{c}) \in P_0$, then a is allowed to be a countable subset of κ, and not required to be in $[\omega_2]^{\aleph_0}$. It is easy to see that forcing with $\mathbf{P}_0$ simultaneously collapses κ to become $\aleph_2$ and adds a $\square_{\aleph_1}$-sequence $\vec{C}$; thus $\vec{C}$ is defined on limit ordinals $< \kappa = \omega_2$ of the extension, V_1. Over V_1, make a further extension, by countably-closed conditions, $\mathring{\mathbf{P}}_1$, to make $\mathrm{BA} + 2^{\aleph_1} > \aleph_2$ hold in V_2. Now by exactly the same arguments as in the first Lemma of (1.4) $\mathrm{Pr}^{\mathrm{b}}_{\aleph_1}$ fails in V_2.

§2. The second error discovered by Velleman is less serious. It affects the Propositions of §§5, 6. However, a reformulation of (S)$''$, given below, yields correct proofs of these Propositions. It should be noted that, in the absence of "Lemma" 1, the Proposition of §5 is needed to conclude that (the reformulation of) (S)$''$, together with CH and $2^{\aleph_1} > \aleph_2$ implies that there's an $\aleph_2$-super-Souslin tree.

Let us first examine the error in the "proof" of the Proposition of §6, in order to motivate our reformulation of (S)$''$. The difficulty occurs in the (extremely sketchy) argument for (c) of (S)$''$, where it was claimed that $R_{ij}(\bigcup_n p^n, \bigcup_n q^n)$. As Velleman pointed out, this would require that $\overline{d(\bigcup_n p^n)}$ and $\overline{d(\bigcup_n q^n)}$ have the same order type, which is supposed to follow, but doesn't, from the hypothesis that for each n, $\overline{d(p^n)}$ and $\overline{d(q^n)}$ have the same order type. However, the desired conclusion does follow if the hypotheses are strengthened in the following way:

(*) for $p \in P$ and limit ordinals $\lambda < \omega_2$, let $(\theta(\lambda)_\alpha : \alpha < \mathrm{cf}\,\lambda)$ be a sequence of ordinals cofinal in λ and let $\alpha(p, \lambda)$ be the least α such that $d(p) \cap \lambda \subseteq \theta(\lambda)_\alpha$, if there is such, and $\alpha(p, \lambda) = \mathrm{cf}\,\lambda$ if not. If $p, q \in P$ and $p \cong q$, let $p \cong^s q$ (p is strongly isomorphic to q) iff whenever $\lambda \in d(p)$ is a limit ordinal, $\mathrm{cf}\,\lambda = \mathrm{cf}(f_{pq}(\lambda))$ and $\alpha(p, \lambda) = \alpha(q, f_{pq}(\lambda))$. We then require:

(a) $\forall n\, R_{ij}(p^n, q^n)$; further, letting $\pi_n : \overline{d(p^n)} \to \overline{d(q^n)}$ be the order isomorphisms, for all n,

(b) $\pi_n \subseteq \pi_{n+1}$,

(c) $p^n \cong^s q^n$.

This is the paradigm for our strengthening of (S)$''$; we are grateful to the referee for formulating the notion of $\cong^s$ and pointing out the necessity of something like (c).

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We shall define, for regular $\kappa > \omega$, classes of partial orderings called κ-elegant. GMA_κ (κ-elegant) is that form of GMA which applies to κ-elegant partial orderings $\mathbf{P}$ and collections $\{D_\alpha : \alpha < \theta\}$ of dense subsets of $\mathbf{P}$, where $\theta < 2^\kappa$. The definition will be in terms of the existence of a system of *four*-place relations on P, $(\mathbf{T}_{ij} : i \leqq j,\ i, j \in S^\kappa_{\kappa^+})$, where $S^\kappa_{\kappa^+} = \{\alpha \in \mathrm{Lim} \cap \kappa^+ : \mathrm{cf}\, \alpha = \kappa\}$. In order to explicate the meaning of the $\mathbf{T}_{ij}$ in terms of the relations R_{ij} of (S)″, let us accept the paradigm that $R_{ij}(r, s)$ is witnessed by an order-preserving (on ordinals) isomorphism π from the $< \kappa$-many ordinals mentioned by r, all of which are smaller than j, to the $< \kappa$ many ordinals mentioned by s, and such that $\pi \restriction i = \mathrm{id} \restriction i$, $\pi(i) = j$ and π is an "isomorphism" of r and s (as "structures" on the sets of ordinals they mention). Then we can think of $\mathbf{T}_{ij}$ as meaning:

$$(*) \qquad \mathbf{T}_{ij}(p, p', q, q') \Leftrightarrow p \leqq p', q \leqq q', R_{ij}(p, q),$$

$R_{ij}(p', q')$, and if π, π' witness $R_{ij}(p, q)$, $R_{ij}(p', q')$ then $\pi' \supset \pi$; further $p \cong^s q$ and $p' \cong^s q'$.

This will make it clear that the Propositions of §§5, 6 hold if (S)″ is replaced by $\mathrm{GMA}_{\aleph_1}$ ($\aleph_1$-elegant). We turn, then, to defining the κ-elegant partial orderings.

Let $\kappa > \omega$ be regular.

Definition 1. $S^\kappa_{\kappa^+} = \{\alpha \in \mathrm{Lim} \cap \kappa^+ : \mathrm{cf}\, \alpha = \kappa\}$.

Definition 2. $\mathbf{P} = (P, \geqq)$ is *κ-fine* if there exists $(\mathbf{T}_{ij} : i \leqq j,\ i, j \in S^\kappa_{\kappa^+})$, each $\mathbf{T}_{ij} \subseteq P^4$ such that:

(i) $\mathbf{T}_{ij}(p, p', q, q') \Rightarrow p \leqq p' \wedge q \leqq q' \wedge \mathbf{T}_{ij}(p, p, q, q) \wedge \mathbf{T}_{ij}(p', p', q', q')$,

(ii) $p \leqq p' \Rightarrow (\exists i_0 \in S^\kappa_{\kappa^+})(\forall i, j \in S^\kappa_{\kappa^+} \backslash i_0)(i \leqq j \Rightarrow \mathbf{T}_{ij}(p, p', p, p')$,

(iii) $\mathbf{T}_{ij}(p, p, q, q) \Rightarrow p, q$ are compatible.

(iv) if $(p_\alpha : \alpha < \theta)$, $(p'_\alpha : \alpha < \theta)$ are increasing where $\theta < \kappa$ and $(\forall \alpha \leqq \beta < \theta)$ $\mathbf{T}_{ij}(p_\alpha, p_\beta, p'_\alpha, p'_\beta)$ then:

$$(\exists p, p' \in P)(\forall \alpha < \theta) \mathbf{T}_{ij}(p_\alpha, p, p'_\alpha, p').$$

Remark 3. It is natural, but not necessary, to also require:

(a) $(\mathbf{T}_{ij}(p, p', q, q') \wedge \mathbf{T}_{ij}(p', p'', q', q'')) \Rightarrow \mathbf{T}_{ij}(p, p'', q, q'')$,

(b) $(\mathbf{T}_{ij}(p, p'', q, q'') \wedge \mathbf{T}_{ij}(p', p'', q', q'') \wedge p \leqq p' \wedge q \leqq q') \Rightarrow \mathbf{T}_{ij}(p, p', q, q')$.

Remark 4. $\mathbf{P}$ is κ-fine $\Rightarrow \mathbf{P}$ is κ-closed.

Proof. This is like the remark, following the definition of (S)″, that (a), (c) of (S)″ imply that $\mathbf{P}$ is $\aleph_1$-closed.

Definition 5. $\mathbf{P}$ is *κ-elegant* if $\mathbf{P}$ is κ-fine and the $\mathbf{T}_{ij}$ also satisfy:

(v) if $(p_\alpha : \alpha < \kappa^+)$, $(p'_\alpha : \alpha < \kappa^+)$ are sequences from P and $(\forall \alpha < \kappa^+) p_\alpha \leqq p'_\alpha$, then $(\exists \text{ club } C \subseteq \kappa^+)(\exists \text{ regressive } g : \kappa^+ \to \kappa^+)(\forall i, j \in C \cap S^{\kappa}_{\kappa^+})((i \leqq j \wedge g(i) = g(j)) \Rightarrow \mathbf{T}_{ij}(p_i, p'_i, p_j, p'_j))$.

REMARK 6. $\mathbf{P}$ is κ-elegant $\Rightarrow \mathbf{P}$ is κ^+-normal.

PROOF. This is clear from (iii), (v).

Correct proofs can now be given for the Propositions of §§5, 6. In what follows, "elegant" means "$\aleph_1$-elegant", "GMA (elegant)" means $\mathrm{GMA}_{\aleph_1}$($\aleph_1$-elegant)". Also, $S^1_2 = S^{\aleph_1}_{\aleph_2}$.

LEMMA 7 (CH). *Let $\mathbf{P}$ be the Velleman partial order for adjoining a morass. Then $\mathbf{P}$ is elegant, and there's a collection of $\aleph_2$ dense sets which, if met, guarantee the existence of an $(\aleph_1, 1)$-morass.*

PROOF. Let $i, j \in S^1_2$, $i \leqq j$. Set $\mathbf{T}_{ij}(p, p', q, q') \Leftrightarrow p \leqq p'$, $q \leqq q'$, $p \mid \aleph_1 = q \mid \aleph_1$, $p' \mid \aleph_1 = q' \mid \aleph_1$, $S^{p'}_{\aleph_1} \subset j$, and there's order-preserving $\sigma : S^{p'}_{\aleph_1} \to_{\text{onto}} S^{q'}_{\aleph_1}$ such that $\sigma \mid i = \mathrm{id} \mid i$, $S^{q'}_{\aleph_1} \cap j \subseteq i$, $\sigma(0) = 0$, $\sigma(\eta + 1) = \sigma(\eta) + 1$, $\lambda \in \mathrm{Lim} \Rightarrow \mathrm{cf}\, \lambda = \mathrm{cf}\, \sigma(\lambda)$, $\sigma'' S^p_{\aleph_1} = S^q_{\aleph_1}$. Then Definition 2(i) is clear. For (ii), we take i_0 such that $S^{p'}_{\aleph_1} \subseteq i_0$, so for $i_0 \leqq i \leqq j$, we take $g = \mathrm{id} \mid S^{p'}_{\aleph_1}$. (5.7) of [3] gives (iii).

For (iv), let $g_n : S^{p_{n+1}}_{\aleph_1} \to_{\text{onto}} S^{p'_{n+1}}_{\aleph_1}$ *be order-preserving. Then* $n \leqq m \Rightarrow g_n \subseteq g_m$. Let $\bar{S} = \bigcup_n S^{p_{n+1}}_{\aleph_1}$, $\bar{S}' = \bigcup_n S^{p'_{n+1}}_{\aleph_1}$, $\bar{g} = \bigcup_n g_n$, so $\bar{g} : \bar{S} \to_{\text{onto}} \bar{S}'$ is order-preserving. Let $\sigma = \sup \bar{S}$, $\sigma' = \sup \bar{S}'$; for $\eta \in S^1_2 \cap \bar{S}$, let $\lambda_\eta = \sup \bar{S} \cap \eta$; similarly, for $\eta \in S^1_2 \cap \bar{S}'$, let $\lambda'_\eta = \sup \bar{S}' \cap \eta$. Let

$$S = \bar{S} \cup \{\sigma\} \cup \{\lambda_\eta : \eta \in S^1_2 \cap \bar{S}\}, \qquad S' = \bar{S}' \cup \{\sigma'\} \cup \{\lambda'_\eta : \eta \in S^1_2 \cap \bar{S}'\};$$

then define order-preserving $g : S \to_{\text{onto}} S'$, by $g \mid \bar{S} = \bar{g}$; $g(\sigma) = \sigma'$, $g(\lambda_\eta) = \lambda'_{\bar{g}(\eta)}$. Let p, p' be the upper bounds for $\{p_n : n < \omega\}$, $\{p'_n : n < \omega\}$ respectively, constructed in (5.9) of [3]. Then for all n, g witnesses that $\mathbf{T}_{ij}(p_n, p, p'_n, p')$.

For Definition 5(v) we let $C = \{j < \omega_2 : i < j \Rightarrow S^{p'_i}_{\aleph_1} \subseteq j\}$, and let $g : \omega_2 \to \omega_2$ be regressive such that $(i < j, \quad i, j \in S^1_2 \cap C \wedge g(i) = g(j)) \Rightarrow S^{p_i}_{\aleph_1} \cap i = S^{p_j}_{\aleph_1} \cap j$, $S^{p'_i}_{\aleph_1} \cap i = S^{p'_j}_{\aleph_1} \cap j$, o.t. $S^{p_i}_{\aleph_1} =$ o.t. $S^{p_j}_{\aleph_1}$, o.t.$S^{p'_i}_{\aleph_1} =$ o.t. $S^{p'_j}_{\aleph_1}$, $p_i \mid \aleph_1 = p_j \mid \aleph_1$, $p'_i \mid \aleph_1 = p'_j \mid \aleph_1$, and for all $\xi <$ o.t. $S^{p_i}_{\aleph_1}$ all $\zeta <$ o.t. $S^{p'_i}_{\aleph_1}$, " the ξth element $S^{p_i}_{\aleph_1} =$ the ζth-element of $S^{p'_i}_{\aleph_1}$" iff "the ξth element of $S^{p_j}_{\aleph_1} =$ the ζth element of $S^{p'_j}_{\aleph_1}$". Then g is as required.

COROLLARY 8. $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{GMA}$ (*elegant*) $\Rightarrow$ *there exist $\aleph_2$-Souslin trees.*

PROOF. By Lemma 7, and (2.16) of [3].

LEMMA 9. *Let $\mathbf{P}$ be admissible (viz §6 of [4]). Then, assuming* CH, $\mathbf{P}$ *is elegant. Thus* $\mathrm{CH} + \mathrm{GMA}$ (*elegant*) $\Rightarrow \mathrm{GMA}_{\aleph_1}$ (*admissible*).

PROOF. We shall content ourselves with giving the definition of the $\mathbf{T}_{ij}$; with these changes the proofs of the Propositions of §6 yield the Lemma. So let $\mathbf{T}_{ij}(p,q,p',q')$ iff $p \leqq p'$, $q \leqq q'$, $\overline{d(p')} \cap \omega_2 \subseteq j$, $\overline{d(q')} \cap j \subseteq i$, $p' \cong_{i \cup (d(p') \cap d(q'))} q'$, $p \cong_{i \cup (d(p) \cap d(q))} q$, $f_{pq} = f_{p'q'} \restriction \overline{d(p)}$, and $p \cong^s q$, $p' \cong^s q'$. In fact, the formulation of κ-elegant and GMA_κ (κ-elegant) were known to us, except for the requirement of $\cong^s$, but we were "tempted" by the somewhat simpler formulation of (S)$''$.

To conclude, we sketch a relative consistency proof for $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{GMA}$ (elegant), following (1.1)–(1.3) of [2]. The proof generalizes to arbitrary regular $\kappa > \omega$ (as do the proofs of Lemmas 7 and 9, and, if κ is a successor cardinal, Corollary 8).

LEMMA 10. *For all δ, if $(\mathbf{P}_\alpha : \alpha \leqq \delta)$, $(\mathring{\mathbf{Q}}_\alpha : \alpha < \delta)$ is a countable support iteration where $P_0 = \{\emptyset\}$ and for $\alpha < \delta$, $\Vdash_{\mathbf{P}_\alpha}$ "$\mathring{\mathbf{Q}}_\alpha$ is elegant", then $\mathbf{P}'_\delta$ is countably closed and $\aleph_2$-normal.*

PROOF. By induction on δ, the case $\delta = 0$ being clear. By Remark 4, $(\forall \alpha < \delta)(\Vdash_{\mathbf{P}_\alpha}$ "$\mathring{\mathbf{Q}}_\alpha$ is countably closed"), so $\mathbf{P}_\delta$ is countably-closed. By the induction hypothesis, for all $\alpha < \delta$, $\omega_i = \omega_i^{V^{\mathbf{P}_\alpha}}$, $i = 1,2$, and $S^1_2 = (S^1_2)^{V^{\mathbf{P}_\alpha}}$.

For $\alpha < \delta$, let $(\mathring{\mathbf{T}}^\alpha_{ij} : i,j \in S^1_2,\ i \leqq j)$ witness that $\mathring{\mathbf{Q}}_\alpha$ is elegant in $V^{\mathbf{P}_\alpha}$. For $f \in P_\delta$, let $\mathrm{sp}(f) =$ the support of $f = \{\alpha < \delta : f(\alpha) \neq 1_{\mathring{\mathbf{Q}}_\alpha}\}$. We now show that $\mathbf{P}_\delta$ is $\aleph_2$-normal. Let $(f_i : i < \omega_2) \in {}^{\omega_2}P_\delta$. We define by recursion increasing sequences $f_i = (f^n_i : n < \omega)$. We set $f^1_i = f^0_i = f_i$. If $n \geqq 1$, having defined the f^n_i, for each $\xi < \delta$, we have:

$$\Vdash_{\mathbf{P}_\xi} (f^n_i(\xi) : i < \omega_2) \in {}^{\omega_2}\mathring{\mathbf{Q}}_\xi.$$

Therefore, there's $\mathring{C}^n_\xi$, $\mathring{g}^n_\xi$ such that:

$$\Vdash_{\mathbf{P}_\xi} \text{"}\mathring{g}^n_\xi : \omega_2 \to \omega_2 \text{ is regressive} \wedge \mathring{C}^n_\xi \subseteq \omega_2 \text{ is club} \wedge$$

$$(\forall i,j \in \mathring{C}^n_\xi \cap S^1_2)((i \leqq j \wedge \mathring{g}^n_\xi(i) = \mathring{g}^n_\xi(j)) \Rightarrow \mathring{\mathbf{T}}^\xi_{ij}(f^{n-1}_i(\xi), f^n_i(\xi), f^{n-1}_j(\xi), f^n_j(\xi))).\text{"}$$

By the induction hypothesis, we may assume each $\mathring{C}^n_\xi = C^n_\xi \in V$ is club $\subseteq \omega_2$. We easily find $f^{n+1}_i \geqq_\delta f^n_i$ such that

$$(\forall \xi \in \mathrm{sp}(f^n_i))(\exists \alpha^n_\xi(i) < i)(f^{n+1}_i \restriction \xi \Vdash_\xi \mathring{g}^n_\xi(i) = \alpha^n_\xi(i)).$$

If $\xi \notin \mathrm{sp}(f^n_i)$, let $\alpha^n_\xi(i) = 0$. Let $C_\xi = \bigcap_n C^n_\xi$, let $\{\xi_\alpha : \alpha < \omega_2\}$ enumerate $\bigcup\{\mathrm{sp}(f^n_i) : i < \omega_2,\ n < \omega\}$, and let $C = \Delta_{\alpha < \omega_2} C_{\xi_\alpha} = \{i < \omega_2 : (\forall \alpha < i) i \in C_{\xi_\alpha}\}$. We easily find club $E \subseteq C$, and regressive $g : \omega_2 \to \omega_2$ such that if $i,j \in E \cap S^1_2$, $g(i) = g(j)$ and $i \leqq j$, then:

(1) $\bigcup_{n<\omega} \mathrm{sp} f^n_i \cap \{\xi_\gamma : \gamma < i\} = \bigcup_{n<\omega} \mathrm{sp} f^n_j \cap \{\xi_\gamma : \gamma < j\}$,

(2) $\bigcup_{n<\omega} \mathrm{sp} f^n_i \subseteq \{\xi_\gamma : \gamma < j\}$,

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(3) $\{\langle\gamma, n, \alpha^n_{\xi_\gamma}(i)\rangle : n < \omega, \gamma < i\} \subseteq \{\langle\gamma, n, \alpha^n_{\xi_\gamma}(j)\rangle : n < \omega, \gamma < j\}$.

We show that for such i, j, $\{f^n_i : n < \omega\} \cup \{f^n_j : n < \omega\}$ has an upper bound, so that, in particular, f_i, f_j are compatible.

We define by recursion $h \restriction \xi$ for $\xi \leqq \delta$ such that $h \restriction \xi$ is an upper bound for $\{f^n_i \restriction \xi : n < \omega\} \cup \{f^n_j \restriction \xi : n < \omega\}$. This is trivial if $\xi = \bigcup \xi$, or $\xi = \zeta + 1$, where $\zeta \notin (\bigcup\{\mathrm{sp} f^n_i : n < \omega\} \cap \bigcup\{\mathrm{sp} f^n_j : n < \omega\})$, so, suppose $\xi = \zeta + 1$, where $\zeta \in \bigcup\{\mathrm{sp} f^n_i : n < \omega\} \cap \bigcup\{\mathrm{sp} f^n_j : n < \omega\}$ and that $h \restriction \zeta$ is an upper bound for $\{f^n_i \restriction \zeta : n < \omega\} \cup \{f^n_j \restriction \zeta : n < \omega\}$.

For such ζ, $(\exists \gamma < i)(\zeta = \xi_\gamma)$ (by (1), (2)), and $(\forall n < \omega)$ $(\alpha^n_{\xi_\gamma}(i) = \alpha^n_{\xi_\gamma}(j), i, j \in C_{\xi_\gamma})$, $(\alpha^n_{\xi_\gamma}(i) = \alpha^n_{\xi_\gamma}(j), i, j \in C_{\xi_\gamma})$, by properties of C, g. Also,

$$(\forall n < \omega)(\zeta \in \mathrm{sp} f^{n+1}_i \cap \mathrm{sp} f^{n+1}_j \Rightarrow h \restriction \zeta \Vdash_\zeta \mathring{\mathbf{T}}^\zeta_{ij}(f^n_i(\zeta), f^{n+1}_i(\zeta), f^n_j(\zeta), f^{n+1}_j(\zeta)))$$

and so by Definition 2(iv)

$$(\exists \mathring{p}, \mathring{p}')h \restriction \zeta \Vdash_\zeta (\forall n < \omega)(\mathring{\mathbf{T}}^\zeta_{ij}(f^n_i(\zeta), \mathring{p}, f^n_j(\zeta), \mathring{p}')).$$

But then, by Definition 2(i), (iii), $(\exists \mathring{r})(\Vdash_\zeta \mathring{r} \in \mathring{Q}_\zeta) \wedge h \restriction \zeta \Vdash_\zeta$ "$\mathring{r} \geqq_\zeta \mathring{p}, \mathring{p}'$, so, by Definition 2(i), $h \restriction \zeta \Vdash_\zeta$ "$\mathring{r}$ is an upper bound for $\{f^n_i(\zeta) : n < \omega\} \cup \{f^n_j(\zeta) : n < \omega\}$". Set $h(\zeta) = \mathring{r}$, and the induction hypotheses are preserved.

THEOREM 11. $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{GMA}_{\aleph_1}(\textit{elegant}))$.

PROOF. Iterate, in length κ, forcing with elegant p.o.'s of power $< \kappa$, where κ regular $> \aleph_2$, $\kappa^{<\kappa} = \kappa$, and every elegant p.o. of power $< \kappa$ is treated κ many times. This last is possible by the fact that $\mathbf{P}_\kappa$ is $\aleph_2$-c.c. by Lemma 10, and observing that if $(\mathring{\mathbf{T}}_{ij} : i, j \in S^1_2, i \leqq j)$ witnesses, in $V^{\mathbf{P}_\kappa}$, that $\mathring{\mathbf{Q}}$ is elegant, then for some $\xi < \kappa, \mathring{\mathbf{Q}}, f \in V^{\mathbf{P}_\xi}$, where, in $V^{\mathbf{P}_\kappa}, f : S^1_2 \times S^1_2 \times Q^4 \to 2$, $f(i, j, p, p', q, q') = 1$ iff $\mathring{\mathbf{T}}_{ij}(p, p', q, q')$ (so that $(\mathring{\mathbf{T}}_{ij} : i, j \in S^1_2, i \leqq j) \in V^{\mathbf{P}_\xi}$). Finally, by an argument totally analogous to that of Lemma 1.2 of [2], if $(\mathring{\mathbf{T}}_{ij} : i, j \in S^1_2, i \leqq j) \in V^{\mathbf{P}_\xi}$ and in $V^{\mathbf{P}_\kappa}$, $(\mathring{\mathbf{T}}_{ij} : i, j \in S^1_2, i \leqq j)$ witness that $\mathring{\mathbf{Q}}$ is elegant, then they do in $V^{\mathbf{P}_\xi}$ also, since (again by Lemma 10, in $V^{\mathbf{P}_\xi}$) the tail of the iteration, $\mathring{\mathbf{P}}_{\xi,\kappa}$ is countably closed and $\aleph_2$-normal.

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