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A CLOSED $(n+1)$-CONVEX SET IN R^2 IS A UNION OF n^6 CONVEX SETS

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ABSTRACT

It is shown that if a closed set S in the plane is $(n+1)$-convex, then it has no more than n^4 holes. As a consequence, it can be covered by $\leqq n^6$ convex subsets. This is an improvement on the known bound of $2^n \cdot n^3$.

§0. Introduction

0.1. DEFINITIONS.

S — a subset (usually closed) of $\mathbf{R}^d$.

ker S — the convex kernel of S, is the subset of all points of S in relation to which S is star-like. ker S is always convex.

When S is closed, ker S is also closed.

$G(S)$ — the graph of non-seeing associated with S, i.e. the graph whose vertices are the points of S, and whose edges connect exactly the pairs of points $a, b \in S$ s.t. $[a, b] \not\subset S$.

Visual Independency — a subset F of S is called visually independent if no two of its points see each other in S (that is, the interval connecting them is not contained in S).

$\alpha(S)$ — the degree of visual independence of S, is the supremum of cardinalities of the visually independent subsets of S. Caution: attention should be paid to whether the supremum is obtained or not in the case $\alpha(S)$ is an infinite cardinal. When $\alpha(S) = n < \aleph_0$ we say that S is

$(n+1)$*-convex*, as in any subset of S of size $n+1$ there are two points that see each other.

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$\beta(S)$ — the chromatic number of $G(S)$.

$\gamma(S)$ — the minimal cardinal k such that S is the union of k convex sets.

FACT. For any S, $\alpha(S) \leqq \beta(S) \leqq \gamma(S)$.

PROOF. Suppose there is a visually independent subset F of S of size m. Then its points are the vertices of a complete subgraph of size m of $G(S)$. Therefore the chromatic number of $G(S)$ is at least m. This establishes the first inequality. For the second, suppose that $\langle C_i \mid i < \lambda \rangle$, λ a cardinal, is a sequence of convex sets, such that $\bigcup_{i<\lambda} C_i = S$. By induction define $B_i = \bigcup_{j<i} C_j$. Now colour each $B_{i+1} - B_i$ with f_i, $\langle f_i \mid i < \lambda \rangle$ being a sequence of distinct colours. As each C_i is convex, this is a valid colouring.

We concern ourselves here with the question: how do restrictions on $\alpha(S)$ and $\beta(S)$ influence $\gamma(S)$, namely, what upper bounds in terms of $\alpha(S)$ and $\beta(S)$ can be put on $\gamma(S)$?

Early work in this direction is found in McKinney [3], who proved that for closed S, $\beta(S) = 2$ implies $\gamma(S) = 2$ and $\beta(S) = 3$ implies $\gamma(S) = 3$. The pentagonal star shows that $\alpha(S) = 2$ does *not* imply $\gamma(S) = 2$.

From now on, S is always a closed subset of the plane. Valentine proved in [4] that for such S, $\alpha(S) = 2$ implies $\gamma(S) \leqq 3$. Further progress was made by Eggleston, who proved in [2] that for compact S a finite $\alpha(S)$ implies a finite $\gamma(S)$. The next progress was made by Breen and Kay in [1]. They set a bound of order $m^3 \cdot 2^m$ on $\gamma(S)$ — for S having a finite $\alpha(S) = m$. This bound was reduced to $k \cdot 2^m$ (k a constant) by M. Perles.

The objective of this paper is to establish the bound of $\alpha(S)^6$ for closed S in $\mathbf{R}^2$ with finite $\alpha(S)$.

Breen and Kay have managed to prove that $\gamma(S) = \alpha(S)$ for S which is supported by a line H, namely, is contained in the closed half-plane defined by H, and which is star-like in relation to a point p on H. So a bound of $2 \cdot \alpha(S)$ follows immediately for star-like S by slicing S through the point in relation to which it is star-like. By the proof of Breen and Kay, for a simply connected set S with $\alpha(S) = m$, $\gamma(S) \leqq m^2$. Further, when a set S has a finite number of holes, it can be sliced by a finite number of lines, one line through each hole, into a finite number of simply connected sets, the degree of visual independence of each being not greater than the original $\alpha(S)$. The exponential factor in the bound of Breen and Kay comes in when trying to estimate the number of holes in a set S with a finite $\alpha(S)$ in terms of $\alpha(S)$.

The bound of $\alpha(S)^6$ will follow from the proof of Breen and Kay once a bound of $\alpha(S)^4$ is obtained for the number of holes in S: Slice the set to m^4

simply connected subsets, and cover each by m^2 convex subsets. A bound of m^8 on the number of holes was obtained by S. Shelah. It was further reduced to m^4 by M. Perles.

§1. A bound on the number of holes for a set S with $\alpha(S)$ finite

The fixed assumptions in this section are that S is a closed subset of the plane, and that $\alpha(S) = m$ is a natural number.

1.1. Definition. A hole in S is a bounded connected component of the complement of S.

1.2. Theorem. *For S as above, namely a closed subset of the plane with $\alpha(S) = m$, the number of holes in S does not exceed m^4.*

Proof. We begin by analysing the structure of $K := \mathrm{cl}(\mathrm{conv}(H))$, where H is a hole of S. We first note that $K = \mathrm{cov}(\mathrm{cl}(H))$.

Next we have:

(i) $H \subset \mathrm{Int}(K)$,

(ii) every extremal point of K belongs to the boundary of H,

(iii) if a, b are extremal points of K such that $(a, b) \subset \mathrm{Int}(K)$, then (a, b) is not included in S.

1.3. Fact. (i) is obvious, and (ii) follows from what is noted above. To see (iii) assume that (a, b) is in $\mathrm{Int}(K)$, so dividing $\mathrm{Int}(K)$ into two open sets. As K is the convex hull of its extremal points, there must be an extremal point of K on each side of $[a, b]$ (otherwise K would be limited to one side). As from (ii) these points are on the boundary of H, there are two points of H "near" them, namely, also on different sides of $[a, b]$. As H is open and connected, it is arcwise connected, so an arc connecting these points must intersect (a, b) — necessarily at a point of H.

An immediate corollary from (iii) is that there are at most $2 \cdot m + 1$ extremal points of K (where $m = \alpha(S)$), for otherwise arrange $2 \cdot m + 2$ extremal points clockwise on $\mathrm{bd}(H)$ and get a contradiction to $(m + 1)$-convexity of S by considering the set of points with even index: they do not see each other because of (iii). This corollary implies that K is a polygon.

Let $H_1, H_2, \ldots, H_N$ be a list of $N = m^4 + 1$ different holes of S. We intend to derive a contradiction to $(m + 1)$-convexity. Our next step is dedicated to showing that we may assume that for two different holes from the list, H_i and H_j, the different respective polygons K_i and K_j have no vertex in common. So for each i choose $z_i \in H_i$, and let ε be such that for all i the closed disk $B[z_i, 2\cdot\varepsilon]$ is included in H_i. Consider $S^* = S + B[0, \varepsilon]$:

(i) S^* is closed.

(ii) $\alpha(S^*) \leqq \alpha(S)$.

(iii) S^* has among its holes N holes $H_1^*, \ldots, H_N^*$ of S^* with $z_i \in H_i^*$.

(iv) When $i < j$, K_i^* and K_j^* have no common vertex.

(i) follows from S^*, being the addition of a closed and a compact set. To prove (ii) suppose $a_i^* = a_i + u_i$, $0 < i \leqq m+1$ are $m+1$ distinct points of S^* with $a_i \in S$ and $u_i \in B[0, \varepsilon]$. From (m + 1)-convexity of S, without loss of generality assume a_1 and a_2 see each other in S. Since $[a_{1,a2}] \subset S$, $[a_1, a_2] + B[0, \varepsilon] \subset S^*$. But $[a_1, a_2] + B[0, \varepsilon]$ is a convex set with a_1, a_2 in it, so $[a_1^*, a_2^*] \subset S^*$. Now for (iii): from the choice of ε it is clear that z_i is not covered by S^*, and that the connected component of z_i in the complement of S^* is bounded. Let H_i^* be the connected component of z_i in the complement of S^*. Lastly, (iv) follows by combining the facts that the vertices of K_i^* are on the boundary of H_i^*, and that boundaries of different H_i^*'s are disjoint.

By replacing S with S^*, if necessary, we may assume that our N holes H_i are such that their respective K_i's are polygons with no vertices in common, and so we do from now on.

Next, assume (by rotating the axes, if necessary) that for any two different vertices v, v' from the set of all vertices of all K_i's the segment (v, v') is not parallel to the y-axis. So a single leftmost vertex of each K_i is well defined. Rearrange the holes by increasing the order of the x-coordinate of the leftmost vertex.

We summarize our assumptions:

(i) $H_0, \ldots, H_N$ are $m^4 + 1$ holes of S;

(ii) $K_i = \mathrm{con}(\mathrm{cl}(H_i))$ is a polygon;

(iii) $i < j$ implies K_i and K_j have no vertex in common;

(iv) each K_i has a leftmost vertex $z_i = \langle x_i, y_i\rangle$, and $i < j$ implies $x_i < x_j$;

(v) z_i belongs to the boundary of H_i.

We intend to associate with each hole H_i a ray R_i and two segments J_i^+, J_i^-. To do so we must first recall

1.4. Fact. If S is closed in the plane and $\alpha(S) < \infty$, then S is locally star-like in relation to every point of S.

Proof. Suppose the property fails for point w in S. That is, for each neighbourhood B of w, $S \cap B$ fails to be star-like in relation to w. Define by

Sh:354

induction a sequence $\langle x_n \mid n < \infty \rangle$ of visually independent points of S monotonically converging to w and such that no x_n see w, as follows. Let $x_0 \in S$ be a point which does not see w. It must exist, or S would be star-like in relation to w. Suppose $x_0, \ldots, x_{n-1}$ are defined. Because S is closed, there is an open neighbourhood B of w which all of the defined x's do not see. Utilize the fact that $S \cap B$ is not star-like to choose $x_n \in S \cap B$ such that x_n does not see w. The sequence we have constructed contradicts the finiteness of $\alpha(S)$, thus proving our fact.

To simplify notation, let us describe the definition of the ray R and the segments J^+, J^- for one hole H from our list, where H has convex closure K and where $z_0 = \langle x_0, y_0 \rangle$ is a leftmost vertex of H. The definition is identical for all H_i's.

Let $h > 0$ be a small real number. We define the box

$$P(h) = \{\langle x, y \rangle \mid x_0 < x < x_0 + h,\ |y - y_0| < h\}.$$

Choose h_0 so small that $S \cap P(h)$ is star-like in relation to z_0. This is possible because of 1.4. For each $0 < h < h_0$ define

$$\Sigma(h) = \{\sigma \mid \langle x, y_0 \rangle + h \cdot \langle 1, \sigma \rangle \in H\},$$

where $\Sigma(h)$ is an open set in the line $L(h) = \{\langle x, y \rangle \mid x = x_0 + h\}$. For $0 < h \leqq h_0$, $\Sigma(h)$ is not empty, for if it were empty for some such h, the line $L(h)$ would divide H into two connected components, contrary to its definition. We also prove now

1.5. Claim. If $0 < h' < h \leqq h_0$ then $\Sigma(h') \subseteq \Sigma(h)$.

Proof. Let $\sigma \in \Sigma(h')$. This is to say that $z_0 + h' \cdot \langle 1, \sigma \rangle \in H$. We have to show that $\sigma \in \Sigma(h)$. If this were not the case, we would have the point $z_0 + h \cdot \langle 1, \sigma \rangle \in S \cap P(h_0)$. But then from our choice of h_0 (which assures that the $S \cap P(h_0)$ is star-like in relation to z_0) also $z_0 + h' \cdot \langle 1, \sigma \rangle$ would be in S — a contradiction. This proves the claim.

From 1.5 and the fact that each $\Sigma(h)$ is not empty, it follows by compactness that the following is not empty:

$$\bigcap_{0 < h \leqq h_0} \mathrm{cl}(\Sigma(h)).$$

So let us choose a point σ_0 from this non-vacuous intersection, and define the ray $R = \{z_0 + t \cdot \langle 1, \sigma_0 \rangle \mid t > 0\}$. We also define $\sigma_0 = \inf\{\Sigma(h_0)\}$ and $\sigma^+ =$

$\sup\{\Sigma(h_0)\}$ and define the segments $J_- = \{z_0 + h \cdot \langle 1, \sigma_- \rangle \mid 0 < h \leqq h_0\}$ and $J^+ = \{z_0 + h \cdot \langle 1, \sigma^+ \rangle \mid 0 < h \leqq h_0\}$. Note that one of the segments J^+, J_- may lie on R.

We make the same definitions as above for each hole in our list, but, still for simplicity of notation, let us concentrate on a single hole H. We observe that since for every $0 < h < h_0$, $\sigma_0 \in \mathrm{cl}(\Sigma(h))$, either

(a) $(\sigma, \sigma + \varepsilon) \subseteq \Sigma(h)$ for some $\varepsilon > 0$

or

(b) $(\sigma, \sigma - \varepsilon) \subseteq \Sigma(h)$ for some $\varepsilon > 0$.

An argument similar to the argument we used in 1.5 shows

1.6. FACT. If (a) holds for ε and h and $h < h' \leqq h_0$, then (a) holds for ε and h'; the same for (b).

Now either (a) or (b) must hold for arbitrarily small h; therefore 1.6 gives

1.7. FACT. Either

(a′) for all $0 < h \leqq h_0$, (a) holds

or

(b′) for all $0 < h \leqq h_0$, (b) holds.

Let us denote by R^+ (R^-) the open upper (lower) half-plane determined by R. A straightforward corollary of 1.7 is

1.8. COROLLARY. *If* (a′) *holds, then whenever v is a point on R such that $x(v) < h_0$, there is an open neighborhood B of v such that $B \cap R^+ \subseteq H$; the same with R^- if* (b′) *holds.*

1.8 intuitively says that there is an open stripe of H either right above or right below R near z_0.

We now define Q^+ to be the upper right quarter-plane determined by R and $\{\langle x, y \rangle \mid x = x_0\}$ and Q^- to be the lower right quarter-plane, and we add R to the first in case (b′) holds, and to the latter otherwise. More precisely, if (b′) holds then

$$Q^+ = \{z_0 + t \cdot \langle 1, \sigma \rangle \mid t > 0, \sigma > \sigma_0\}$$

and

$$Q^- = \{z_0 + t \cdot \langle 1, \sigma \rangle \mid t > 0, \sigma \leqq \sigma_0\}$$

and analogously when (a′) holds. Now we can finally prove our main result.

1.9. LEMMA. *If $w \in S \cap Q^+$ then there is an h_w such that the sub-segment $\{z_0 + t \cdot \langle 1, \sigma_- \rangle \mid 0 < t < h_w\} \subset J_-$ is not seen by w, and if $w \in S \cup Q^-$ then*

Sh:354

there is an h_w *such that the sub-segment* $\{z - 0 + t \cdot \langle 1, \sigma^+ \rangle \mid 0 < t < h_w\}$ *of* J_+ *is not seen by* w.

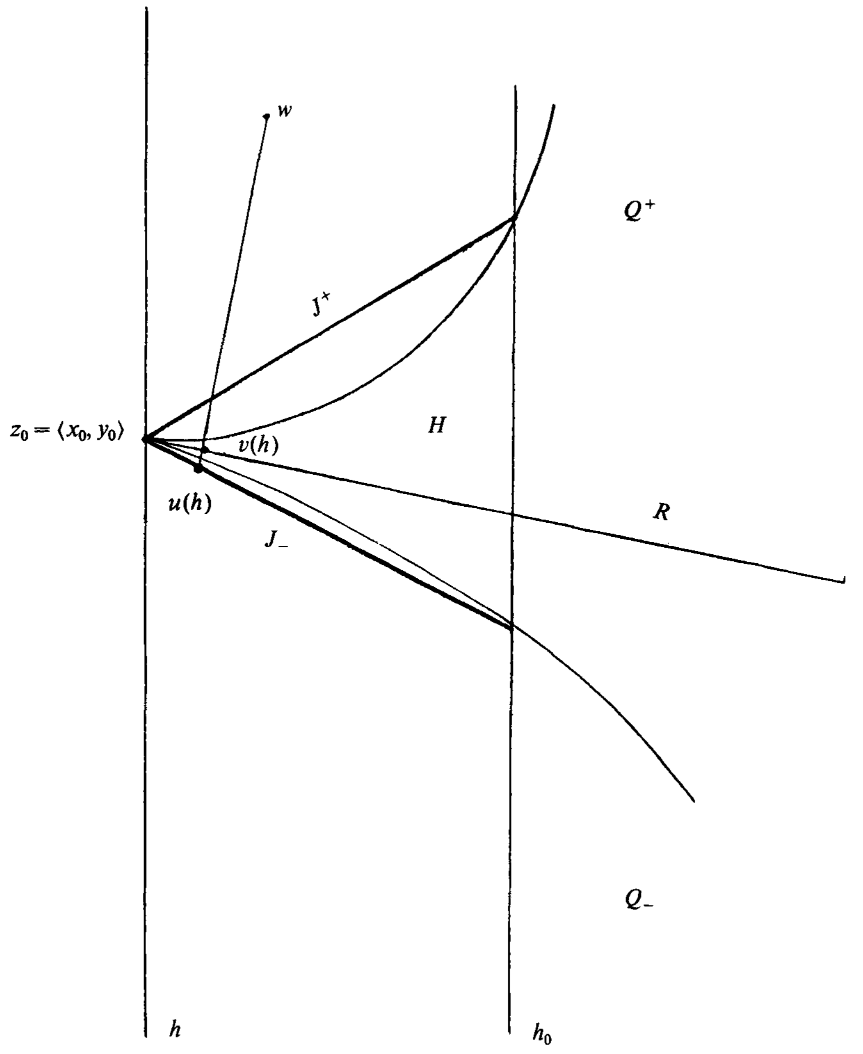

Fig. 1.

PROOF. This is easy if one looks at Fig. 1. Suppose (a′) holds. Let $w \in S \cap Q^+$. Denote by $u(h)$, for small enough h, the point on J_- with $x(u) = h$. By connecting w to $u(h)$ we intersect R at the unique point $v(h)$. Clearly, as h tends to 0, also $x(v(h))$ tends to 0. So set h small enough that

$x(v(h))$ is smaller than $h(0)$. By 1.8 the segment $\langle w, u(h)\rangle$ passes in an open neighbourhood of $v(h)$ which is in H, and, furthermore, the same is true for all $h' < h$. Now let $w \in Q_-$. If w is not on R, the same argument works. If w is on R, find h so small that there is a point $v(h)$ on R with a neighborhood in H containing the segment $\langle v(h), v(h) + \langle 0, n\rangle\rangle$. Take the part of J^+ which this segment in H shades from w. The case when (b′) holds is symmetric.

Recall now that we have the definitions and Lemma 1.9 for all the holes in our list. Look at the sequence of $m^4 + 1$ slopes $\langle \sigma_i \mid i < N\rangle$ of the rays $\langle R_i \mid i < N\rangle$. There is either a weakly increasing subsequence of length $m^2 + 1$ or a weakly decreasing such subsequence. Without loss of generality assume that $\langle \sigma_0, \dots, \sigma_M\rangle$ is a weakly increasing sequence, where $M = m^2 + 1$. Observe now the set $\{Q_i^+ \mid i < M\}$ partially ordered by inclusion. By Dillworth's theorem it has either an antichain of size $m + 1$ or a chain of size $m + 1$.

So let us first suppose that $Q_0^+, \dots, Q_m^+$ forms a chain with respect to inclusion. This means that if $i < j < m + 1$ then $z_j \in Q_i^+$. We define now by induction on i a set of $m + 1$ visually independent points of S such that each $x_i \in J_{-i}$ and $i < j$ implies x_i does not see z_j. By repeated use of the lemma choose $x_0 \in J_{-0}$ which does not see any of $\{z_j \mid j > 1\}$. Suppose that x_i were chosen for $i < n \leqq m$. There is a neighborhood of z_n which all x_i, $i < n$ do not see. Inside this neighourhood find a point $x_n \in J_{-n}$. So x_0 satisfies both requirements. We may choose x_{m+1} to be z_{m+1}. The set we have obtained is the desired contradiction to $\alpha(S) = m$.

Next suppose that $Q_0^+, \dots, Q_m^+$ forms an antichain with respect to inclusion. This means evidently that $i < j$ implies $z_j \in Q_{-j}$. The same inductive process as above works also here. This completes our proof of the bound m^4 on the number of holes of S.

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