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REMARKS ON SUPERATOMIC BOOLEAN ALGEBRAS

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1. Introduction and statement of results

The principal results of this paper are as follows: In Mitchell's model there are no thin-thick superatomic Boolean algebras (Theorem 3.2 and Corollary 3.3); if ZFC is consistent, then so is ZFC + "there is a thin-very tall superatomic Boolean algebra" (Theorems 7.1, 8.1 and 9.12); and (in ZFC) there exist zero-dimensional scattered topological spaces of arbitrary height below $\mathfrak{c}^+$ such that all levels are countable (Theorem 10.1).

A Boolean algebra is *superatomic* iff every homomorphic image is atomic. See [2] for a discussion of equivalent definitions of superatomic Boolean algebra (hereinafter abbreviated sBa). In particular, B is an sBa iff its Stone space $S(B)$ is scattered. A very useful tool for studying scattered spaces is the Cantor–Bendixson derivative $A^{(\alpha)}$ of a set $A \subseteq S(B)$, defined by induction on α as follows. Let $A^{(0)} = A$, $A^{(\alpha+1)} =$ the set of limit ponts of $A^{(\alpha)}$, and $A^{(\lambda)} = \bigcap \{A^{(\alpha)}: \alpha < \lambda\}$ if λ is a limit ordinal. Then $S(B)$ is scattered iff for some α, $S(B)^{(\alpha)} = 0$.

When this notion is transferred to the Boolean algebra B, we arrive at a sequence of ideals I_α, which we refer to as the Cantor–Bendixson ideals, defined by induction on α as follows. Let $I_0 = \{0\}$. Given I_α, let $I_{\alpha+1}$ be generated by I_α together with all $b \in B$ such that b/I_α is an atom in B/I_α. If λ is a limit ordinal, let $I_\lambda = \bigcup \{I_\alpha: \alpha < \lambda\}$. Then B is an sBa iff some $I_\alpha = B$.

The *height* of an sBa B, ht(B), is the least ordinal α such that B/I_α is finite (so then $B = I_{\alpha+1}$). For $\alpha <$ ht(B) let wd$_\alpha(B)$ be the cardinality of the set of atoms in B/I_α. The sequence $\langle \text{wd}_\alpha(B): \alpha < \text{ht}(B)\rangle$ is called the *cardinal sequence* of B. If B is given, then wd$_\alpha(B)$ may be abbreviated to wd$_\alpha$. For an infinite cardinal κ, B is called *κ-thin-thick* iff ht$(B) = \kappa + 1$ and wd$_\alpha = \kappa$ for $\alpha < \kappa$, wd$_\kappa = \kappa^+$. B is

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κ-thin-very thick if $\text{ht}(B)=\kappa+1$, $\text{wd}_\alpha=\kappa$ for $\alpha<\kappa$ and $\text{wd}_\kappa\geqslant\kappa^{++}$. B is *κ-thin-tall* if $\text{ht}(B)=\kappa^+$ and $\text{wd}_\alpha=\kappa$ for all $\alpha<\kappa^+$; B is *κ-thin-very tall* if $\text{ht}(B)=\kappa^{++}$ and $\text{wd}_\alpha=\kappa$ for all $\alpha<\kappa^{++}$. Following Roitman [8], we call B thin-thick if B is ω_1-thin-thick; B is thin-very tall if B is ω-thin-very tall.

Our set-theoretic terminology is fairly standard, and notions not defined here can be found in [4] or [6]. If λ is a (possibly finite) cardinal and A is a set, then $[A]^\lambda=\{B\subseteq A: |B|=\lambda\}$, $[A]^{<\lambda}=\{B\subseteq A: |B|<\lambda\}$ and $[A]^{\leqslant\lambda}=\{B\subseteq A: |B|\leqslant\lambda\}$. A *$\Delta$-system* is a family F of sets such that for some set Δ we have $A\cap B=\Delta$ for all $A, B\in F$, $A\neq B$. The set Δ is called the *kernel* of the Δ-system. It is well-known that every uncountable family of finite sets contains an uncountable Δ-system, and that if the continuum hypothesis (CH) is true, then every $F\subseteq[\omega_2]^\omega$ of cardinality ω_2 contains a Δ-system of cardinality ω_2 in which the kernel is an initial segment of every element.

We regard forcing as taking place over the universe V of set theory, and thus we speak of $V[G]$ where G is P-generic (over V), etc. While this is formally improper, it provides a convenient notation. The reader uncomfortable with this device may simply substitute for V a countable transitive model M of a sufficiently large fragment of the set theory in question.

The countable chain condition is always abbreviated c.c.c.

The rest of the paper is organized as follows.

In Section 2 we show that if one does c.c.c. forcing over a model of CH, then in the extension there are no ω_1-thin-very thick sBa's. This means that Martin's Axiom + "$2^{\aleph_0}$ large" is consistent with the non-existence of ω_1-thin-very thick sBa's, and hence that Roitman's theorem [8] that MA + $\neg$CH implies the existence of thin-thick sBa's is best possible.

Thin-thick sBa's are treated in Sections 3, 4 and 6. In Section 3 we introduce the notion of a graded almost-disjoint family and state a combinatorial proposition GR which implies that there are no thin-thick sBa's. GR has other consequences as well, for example: If $\langle A_\alpha: \alpha<\omega_2\rangle$ is an ordinary almost-disjoint family of elements of $[\omega_1]^{\omega_1}$, then $\{A_\alpha\cap A_\beta: \alpha<\beta<\omega_2\}$ has cardinality ω_2. In Section 6 GR is shown to hold in Mitchell's model. Section 4 contains a review of facts about Mitchell's model together with some simple new observations.

Section 5 is a 'dry run' for Section 6. It is shown that if many Cohen reals are adjoined to a model of CH, then in the extension there is no family $F\subseteq[\omega_1]^{\omega_1}$ such that $|F|=\omega_2$ and all pairwise intersections of elements of F are infinite. Such families, which we call *strongly almost-disjoint* families, were used by Roitman [8] to construct thin-thick sBa's. Previously, Weese [9] had used Canadian trees (also called weak Kurepa trees), i.e., trees of height ω_1 and cardinality ω_1 with at least ω_2 uncountable branches, to construct thin-thick sBa's. Mitchell [7] showed that relative to the existence of an inaccessible cardinal it is consistent that Canadian trees do not exist.

The next three sections of the paper are devoted to the construction by forcing of a thin-very tall sBa. Our method requires a double extension: the first by a countably closed notion of forcing and the second by c.c.c. forcing. We do not know whether c.c.c forcing alone will suffice.

Previously it was shown by Just and by Roitman (see [8]) that it is relatively consistent with ZFC that the continuum is large and thin-very tall sBa's do not exist. Juhasz and Weiss [5] constructed (in ZFC) sBa's of arbitrary height β below ω_2 such that $\mathrm{wd}_\alpha = \aleph_0$ for all $\alpha < \beta$. They asked the question whether it is consistent that thin-very tall sBa's exist.

The final section of the paper is devoted to the proof in ZFC that there are zero-dimensional scattered spaces of arbitrary height below $\mathfrak{c}^+$ with all levels countable. This answers a question of Juhasz.

The history of this paper is as follows. The first author originally announced the results in the first nine section of this paper, but the definition of property Δ and the attendant forcing construction in Section 9 had a serious error in it. The error was discovered by W. Fleissner, to whom the first author wishes to tender his thanks. Subsequently it was discovered that in fact there are no functions satisfying the original version of property Δ. A new version of property Δ and a completely new forcing construction to establish its consistency were supplied by the second author. The result in the last section is also due to him.

2. Martin's Axiom and ω_1-thin-very thick sBa's

In [8], Roitman showed that Martin's Axiom implies the existence of a strongly almost disjoint family of cardinality ω_2, and hence the existence of a thin-thick sBa. Her method was to begin with an ordinary almost disjoint family of size ω_2 and, using an observation of M. Wage, thin out the elements of the family one at a time until a strongly almost disjoint family is produced. As long as $2^{\aleph_0}$ is large enough, this method will work to produce strongly almost disjoint families of larger cardinality, provided one begins with a sufficiently large almost disjoint family. Unfortunately, as was shown in [1], the existence of such a family is not guaranteed.

In this section, we expand the observation in [1], with virtually the same proof, to ω_1-thin-very thick sBa's.

Theorem 2.1. *If* ZF *is consistent, then so is* ZFC + *Martin's Axiom* + $2^{\aleph_0}$ *large* + *there are no* ω_1*-thin-very thick sBa's.*

Thus Roitman's result is best possible.

Recall that if κ, λ, μ are cardinals, then the notation

$$\kappa \to (\lambda)^2_\mu$$

means that for any $f:[\kappa]^2 \to \mu$ there is $A \subseteq \kappa$ with $|A| = \lambda$ and f is constant on $[A]^2$ (we say A is *homogeneous* for f). One instance of the Erdös–Rado Theorem [3] asserts that if GCH holds, then

$$\kappa^{++} \to (\kappa^+)^2_\kappa \quad \text{for all } \kappa.$$

Proof of Theorem 2.1. Begin with a model of GCH and force Martin's Axiom + $2^{\aleph_0}$ as large as desired with a c.c.c. partial ordering P. Suppose that in the extension B is an ω_1-thin-very thick sBa. Let $\langle b_\alpha : \alpha < \omega_3 \rangle$ be representatives of distinct atoms in B/I_{ω_1}. Then if $\alpha < \beta$ we have $b_\alpha \wedge b_\beta \in I_{\omega_1}$ so $\exists \xi = \xi(\alpha, \beta) < \omega_1$ $b_\alpha \wedge b_\beta \in I_\xi$. But since P has the c.c.c. we may assume that the function $\xi(\alpha, \beta)$ lies in the ground model (there are only countably many possible values for $\xi(\alpha, \beta)$ so we could replace $\xi(\alpha, \beta)$ by the supremum of all such values).

By the Erdös–Rado Theorem with $\kappa = \omega_1$ there is $A \subseteq \omega_3$, $|A| = \omega_2$ and ξ is constant on all $(\alpha, \beta) \in A \times A$ with $\alpha < \beta$. Say $\xi(\alpha, \beta) = \xi_0$ for all $(\alpha, \beta) \in A \times A$, $\alpha < \beta$. But then since $b_\alpha \wedge b_\beta \in I_{\xi_0}$ whenever $\alpha, \beta \in A$, $\alpha \neq \beta$, it follows that B/I_{ξ_0} must contain at least ω_2 atoms, and this contradicts our hypothesis on B.

It is straightforward to generalize Theorem 2.1 to larger cardinals. Details are left to the reader.

3. Graded almost disjoint families and GR

Let us say that a family $F \subseteq [\omega_1]^{\omega_1}$ is *graded almost disjoint* if there is a disjoint partition $\langle B_\alpha : \alpha < \omega_1 \rangle$ of ω_1 such that

(1) $\forall A \in F$ $\{\alpha : A \cap B_\alpha \neq 0\}$ is uncountable, and

(2) $\forall A_1, A_2 \in F$ if $A_1 \neq A_2$, then $\{\alpha : A_1 \cap A_2 \cap B_\alpha \neq 0\}$ is countable.

We will sometimes refer to a family as graded almost disjoint when the underlying set is not ω_1 but some other set of the same cardinality.

Note that if $B_\alpha = \{\alpha\}$, then (1) and (2) simply assert that F is an almost disjoint family in the usual sense. Thus graded almost disjointness generalizes almost disjointness.

Let GR denote the following hypothesis: Whenever $\langle A_\xi : \xi < \omega_2 \rangle$ is a graded almost disjoint family, then $\{A_\xi \cap A_\eta : \xi < \eta < \omega_2\}$ has cardinality ω_2.

It is easy to see that GR rules out both Canadian trees and strongly almost disjoint families of cardinality ω_2, but we can do even better.

Theorem 3.1. GR *implies that there are no thin-thick sBa's.*

Proof. Let B be a thin-thick sBa, and let $\langle I_\alpha : \alpha < \omega_1 \rangle$ be the ideals obtained from B by the Cantor–Bendixson process. Since B is ω_1-thin we have $|I_\alpha| \leq \omega_1$ for all $\alpha < \omega_1$. Let $\langle b_\xi : \xi < \omega_2 \rangle$ be representatives of distinct atoms in B/I_{ω_1}, and let $A_\xi = \{b \in I_{\omega_1} : b \leq b_\xi\}$. If $B_\alpha = I_{\alpha+1} - I_\alpha$, then we claim $\langle A_\xi : \xi < \omega_2 \rangle$ is graded almost disjoint relative to the B_α (here the underlying set is I_{ω_1}).

First let us check (1). Fix A_ξ. If $\alpha < \omega_1$, then there is an atom $b/I_\alpha \in B/I_\alpha$ such that $b/I_\alpha \leq b_\xi/I_\alpha$. But then $b - b_\xi \in I_\alpha$ so $b \wedge b_\xi \in I_{\alpha+1} - I_\alpha = B_\alpha$ and $b \wedge b_\xi \leq b_\xi$, so $A_\xi \cap B_\alpha \neq 0$.

Next check (2). Suppose $\xi < \eta < \omega_2$. Then $b_\xi \wedge b_\eta \in I_{\omega_1} = \bigcup \{I_\alpha : \alpha < \omega_1\}$. Say $b_\xi \wedge b_\eta \in I_\alpha$, $\alpha < \omega_1$. But then if $b \in A_\xi \cap A_\eta$ we would have $b \leq b_\xi, b_\eta$ so $b \leq b_\xi \wedge b_\eta$ and $b \in I_\alpha$ also. Hence $\{\beta : A_\xi \cap A_\eta \cap B_\beta \neq 0\} \subseteq \alpha$, and (2) is verified.

Finally, note that $A_\xi \cap A_\eta$ is completely determined by $b_\xi \wedge b_\eta \in I_{\omega_1}$, and since $|I_{\omega_1}| = \omega_1$ this violates GR.

The principal theorem of the next few sections is

Theorem 3.2. *In Mitchell's model* GR *holds so there are no thin-thick sBa's.*

By 'Mitchell's model' we mean the version in [7] in which an inaccessible cardinal is collapsed to become ω_2 (and $2^{\aleph_0}$). Thus we have

Corollary 3.3. *If it is consistent that an inaccessible cardinal exists, then it is consistent that there is no thin-thick sBa.*

4. Mitchell's model

In this section we review some facts about Mitchell's model from [7] and add some simple observations of our own.

With one or two exceptions, we use Mitchell's notation.

Let κ be a fixed strongly inaccessible cardinal from V. Let P be the set of finite functions mapping subsets of κ into 2, the standard partial ordering for adjoining κ Cohen subsets of ω. For $\alpha < \kappa$ let $P_\alpha = \{p \in P : \text{domain}(p) \subseteq \alpha + \omega\}$, where $\alpha + \omega$ is the ordinal sum, and let B_α be a complete Boolean algebra containing P_α as a dense subset. Let A be the set of all countable functions f with domain$(f) \subseteq \kappa$ and such that $\forall \alpha f(\alpha) \in B_\alpha$.

If G is P-generic over V, then in $V[G]$ we may define f_G for $f \in A$ so that f_G: domain$(f) \to 2$ and $\forall \alpha f_G(\alpha) = 1$ iff $\exists p \in G \cap P_\alpha\, p \leq f(\alpha)$. Let $Q = \{f_G : f \in A\}$, partially ordered by $\supseteq$. Then Mitchell's model is obtained by forcing first with P, then with Q, i.e., with $P * Q$.

There is a convenient alternate description of the forcing as well. Let $R = P \times A$, with $(p, f) \leq_R (q, g)$ iff $p \leq_P q$ and $p \Vdash_P f_G \supseteq g_G$. Then forcing with R is equivalent to forcing with $P \times Q$. We will usually drop subscripts on the orderings when it is unlikely to cause confusion.

In [7] Mitchell shows that R has the κ-chain condition and that if K is R-generic then in $V[K]$, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 = \kappa$.

Let us begin with two variations on Lemma 3.1 of [7].

Lemma 4.1. *If $(q, g) \leqslant (p, f)$, then $\exists h \in A\ f \subseteq h$ and $(q, g) \leqslant (q, h) \leqslant (q, g)$.*

Proof. Set domain(h) = domain(g) and $h(\alpha) = f(\alpha)$ if $\alpha \in$ domain(f), $h(\alpha) = g(\alpha)$ if $\alpha \in$ domain(g) − domain(f). It is easy to check that h works.

By Lemma 4.1, whenever $(q, g) \leqslant (p, f)$ we may always assume without loss of generality that $f \subseteq g$, and we shall frequently make that assumption.

The conclusion of Lemma 4.1 also illustrates that $\leqslant_R$ is not a partial ordering but a preordering; that, of course, causes no trouble.

Lemma 4.2. *Suppose $f, g_1, g_2 \in A$, $f \subseteq g_1, g_2$ and $f \restriction \xi = g_1 \restriction \xi = g_2 \restriction \xi$. Then $\exists h \in A\ f \subseteq h$, $h \restriction \xi = f \restriction \xi$, and $\{(\xi, 0)\} \Vdash h_G \supseteq (g_1)_G$ and $\{(\xi, 1)\} \Vdash h_G \supseteq (g_2)_G$.*

Proof. Here, of course, the forcing is with respect to the Cohen ordering. Let a be the Boolean algebra element $\{(\xi, 0)\}$. By extending g_1 and g_2 we may assume they have the same domain D. Define h by $h(\alpha) = f(\alpha)$ if $\alpha \in D$, $\alpha < \xi$ and $h(\alpha) = (a \wedge g_1(\alpha)) \vee (\bar{a} \wedge g_2(\alpha))$ if $\alpha \in D$, $\alpha \geqslant \xi$. It is straightforward to check that h works. Note that if $\alpha \in D$, $\alpha \geqslant \xi$ and $\alpha \in$ domain(f), then $h(\alpha) = (a \wedge f(\alpha)) \vee (\bar{a} \wedge f(\alpha)) = f(\alpha)$, so $h \supseteq f$.

The next lemma is well known as a tool for proving chain conditions.

Lemma 4.3. *Suppose $S \subseteq \{\xi < \kappa : \mathrm{cf}\, \xi \geqslant \omega_1\}$ is stationary and $\langle N_\xi : \xi \in S\rangle$ is a sequence of countable structures. Then there is stationary $T \subseteq S$ such that $\forall \xi, \eta \in T$ if $\xi < \eta$, then there is an isomorphism $\pi_{\xi\eta} : N_\xi \to N_\eta$ such that $\pi_{\xi\eta} : N_\xi \cap \kappa \to N_\eta \cap \kappa$ is an order-preserving bijection and $\pi_{\xi\eta} \restriction \xi$ is the identity. Moreover, if we assume that $\xi \in N_\xi$* always, then $\pi_{\xi\eta}(\xi) = \eta$.

Proof. This is routine. Without loss of generality we may assume $\kappa \cap N_\xi$ and $\in$ occur as predicates in N_ξ. Since κ is inaccessible and there are only $2^{\aleph_0}$ isomorphism types of these structures we may assume they all have the same isomorphism type. Finally, an easy application of Fodor's Theorem allows us to obtain stationary $T \subseteq S$ so that the mapping $\xi \mapsto N_\xi \cap \xi$ is constant on T. This suffices.

Lemma 4.4. *Suppose $\langle (p_\xi, f_\xi) : \xi < \kappa\rangle$ is a sequence of conditions from Mitchell's ordering R. Then there is $S \subseteq \kappa$ cofinal (even stationary) in κ such that $\{(p_\xi, f_\xi) : \xi \in S\}$ is pairwise compatible, and there is $(p, f) \in R$ such that*

$$(p, f) \Vdash \{\xi \in S : (p_\xi, f_\xi) \in K\} \text{ is cofinal in } \kappa,$$

where K denotes the R-generic set.

Proof. This can be proved directly or derived from Lemma 4.3 by building (p_ξ, f_ξ) into a structure N_ξ for each ξ. Details are left to the reader.

Let ν be a limit ordinal, $\nu<\kappa$. Let $P\restriction\nu=\{p\in P\colon \text{domain}(p)\subseteq\nu\}$ and let $P^\nu=\{p\in P\colon \text{domain}(p)\subseteq\kappa-\nu\}$. Let $A_\nu=\{f\in A\colon \text{domain}(f)\subseteq\nu\}$ and $A^\nu=\{f\in A\colon \text{domain}(f)\cap\nu=0\}$, and let $R_\nu=P\restriction\nu\times A_\nu$. This yields another way of decomposing R as an iteration of two orderings.

Suppose K is R-generic and $K_\nu=K\cap R_\nu$. Let $\bar A$ and $\bar R=P\times\bar A$ be the result of defining A and R in $V_1=V[K_\nu]$.

Lemma 4.5. *In the structure above, there is $\bar K$ which is $\bar R$-generic over V_1 and is such that $V_1[\bar K]=V[K]$.*

Proof. This is a variation on Lemma 3.7 of [7].

We know that $V[K]=V[G][H]$, where G is P-generic over V and H is Q-generic over $V[G]$. Let $G_\nu=G\cap P\restriction\nu$, $G^\nu=G\cap P^\nu$, $Q_\nu=\{f_G\colon f\in A_\nu\}$, $Q^\nu=\{f_G\colon f\in A^\nu\}$, $H_\nu=Q_\nu\cap H$ and $H^\nu=Q^\nu\cap H$. Then it is easy to see that $V_1=V[G_\nu][H_\nu]$. It is also clear that G^ν is P^ν-generic over V_1.

Let $\pi:\kappa\to\kappa-\nu$ be the order-preserving bijection. If $\bar G=\{p\pi\colon p\in G^\nu\}$, then $\bar G$ is P-generic over V_1, since the mapping $p\mapsto p\pi$ is an isomorphism between P^ν and P. Let $\bar Q=\{f_{\bar G}\colon f\in\bar A\}$.

Let $V_2=V_1[\bar G]$.. Then clearly $V_2=V_1[G^\nu]$ so $V[K]=V_2[H^\nu]$, and H^ν is Q^ν-generic over V_2. Thus it will suffice to show that in V_2 (even in $V[G]$) there is an injection carrying Q^ν onto a dense subset of $\bar Q$, for if $\bar H$ is determined by the image of H^ν then $V[K]=V_1[\bar G][\bar H]=V_1[\bar K]$, where $\bar K$ is $\bar R$-generic over V_1.

Thus we must define $\sigma\colon Q^\nu\to\bar Q$. Suppose $f\in A^\nu$. Define $f^*\in\bar A$ as follows. Let $\text{domain}(f^*)\ =\pi^{-1}\,\text{domain}(f)$ and for $\alpha\in\text{domain}(f^*)$ let

$$f^*(\alpha)=\sup\{p\in P\colon \exists q\in G_\nu\ q\cup p\pi^{-1}\leqslant f(\pi(\alpha))\}.$$

Then $(f^*)_{\bar G}(\alpha)=1$ iff $\exists p\in\bar G\, p\leqslant f^*(\alpha)$ iff $\exists p_1\in G^\nu\, p_1\pi\leqslant f^*(\alpha)$ iff $\exists p_1\in G^\nu\,\exists q\in G_\nu\ q\cup p_1\pi\pi^{-1}\leqslant f(\pi(\alpha))$ iff $\exists p_2\in G\, p_2\leqslant f(\pi(\alpha))$ iff $f_G(\pi(\alpha))=1$. So $(f^*)_{\bar G}=f_G\pi$. Let $\sigma(f_G)=f_G\pi$.

We need only check that the image of σ is dense in $\bar Q$. Suppose $g\in\bar A$. Since domain(g) is countable and $g(\alpha)$ is determined by a countable antichain in P for each $\alpha\in\text{domain}(g)$, it follows from Corollary 3.2 of [7] that $g\in V[G_\nu]$. Thus $\bar A\in V[G_\nu]$. By extending g if necessary we may assume domain$(g)\in V$. Let $\dot g$ be a name for g. Now define $f\in A^\nu$ so that $\text{domain}(f)=\pi''\,\text{domain}(g)$ and $f(\pi(\alpha))=\sup\{p\in P\colon p\restriction\nu\Vdash_{P_\nu}(p-p\restriction\nu)\pi\leqslant\dot g(\alpha)\}$ for $\alpha\in\text{domain}(g)$. But now $g_{\bar G}(\alpha)=1$ iff $\exists q\in\bar G\ q\leqslant g(\alpha)$ iff $\exists p\in G_\nu\ p\Vdash q\leqslant\dot g(\alpha)$ iff $\exists p\in Q_\nu\ \exists q_1\in G^\nu p\Vdash q_1\pi\leqslant\dot g(\alpha)$ iff $\exists p\in G_\nu\ \exists q_1\in G^\nu p\cup q_1\leqslant f(\pi(\alpha))$ iff $f_G(\pi(\alpha))=1$, so $\sigma(f_g)\leqslant g_{\bar G}$ in $\bar Q$ and we are done.

5. A dry run

Before entering into the proof of Theorem 3.1, it may be useful to prove a similar result in a simpler setting. In [1, pp. 427–428] it was remarked in particular that if the continuum is enlarged by adding Cohen reals, then in the extension there are no strongly almost disjoint families, but the proof was omitted. Since the proof has never appeared, and since it is closely related to the argument in the next section, we present it here.

For the purposes of this section, let P be the partial ordering for adjoining λ Cohen reals, where $\lambda \geqslant \omega_2$. Thus P consists of all finite functions mapping subsets of λ into 2.

Theorem 5.1. *Assume* CH. *Then in* V^P *there are no strongly almost disjoint families of cardinality* $\geqslant \omega_2$.

Proof. Suppose on the contrary that

$$\Vdash_P \langle \dot{A}_\xi : \xi < \omega_2 \rangle \text{ is strongly almost disjoint.}$$

Fix ξ, and define a sequence $\langle (p_\alpha, \beta_\alpha) : \alpha < \omega_1 \rangle$ so that $p_\alpha \in P$, the β_α are increasing elements of ω_1, and $p_\alpha \Vdash \beta_\alpha \in \dot{A}_\xi$. Without loss of generality we may assume that $\langle \text{domain}(p_\alpha) : \alpha < \omega_1 \rangle$ forms a Δ-system with kernel Δ, and that for some $p^\xi \in P$ we have $p_\alpha \restriction \Delta = p^\xi$ for all $\alpha < \omega_1$. We will write Δ^ξ, p^ξ_α, β^ξ_α to indicate the dependence on ξ.

Using CH and the methods of Lemma 4.3 above, it is straightforward to find $\xi < \eta < \omega_2$ so that $\langle \beta^\xi_n : n < \omega \rangle = \langle \beta^\eta_n : n < \omega \rangle$ and there is an order-preserving bijection

$$\pi_{\xi\eta} : \bigcup \{\text{domain}(p^\xi_n) : n < \omega\} \to \bigcup \{\text{domain}(p^\eta_n) : n < \omega\}$$

so that $p^\eta_n \pi_{\xi\eta} = p^\xi_n$ for all n, $\pi_{\xi\eta} \restriction \xi$ is the identity, and $\text{domain}(\pi_{\xi\eta}) \cap \text{range}(\pi_{\xi\eta}) \subseteq \xi$. Then in particular p^ξ and p^η are compatible, and p^ξ_n is compatible with p^η_n for all n.

Let $p = p^\xi \cup p^\eta$ and set $\beta_n = \beta^\xi_n = \beta^\eta_n$. We claim

$$p \Vdash \{n : \beta_n \in \dot{A}_\xi \cap \dot{A}_\nu\} \text{ is definite,}$$

and this contradiction will complete the proof.

Suppose $q \leqslant p$. Since the domain of q is finite, there must be n large enough that $\text{domain}(q) \cap \text{domain}(p^\xi_n) = \Delta^\xi$, $\text{domain}(q) \cap \text{domain}(p^\eta_n) = \Delta^\eta$. But then clearly $q' = q \cup p^\xi_n \cup p^\eta_n$ is a condition, and $q' \Vdash \beta_n \in \dot{A}_\xi \cap \dot{A}_\eta$. Since n could be arbitrarily large, this establishes the claim.

Once again there are easy generalizations of this result, as indicated in [1].

6. The proof of Theorem 3.2

Let us suppose now that K is R-generic over V, and in $V[K]$, $\langle A_\xi : \xi < \kappa \rangle$ is a counterexample to GR. Thus $\langle A_\xi : \xi < \kappa \rangle$ is a graded almost disjoint family with respect to some $\langle B_\alpha : \alpha < \omega_1 \rangle$ and $\{A_\xi \cap A_\eta : \xi < \eta < \kappa\}$ has cardinality $\leqslant \omega_1$. Then by the κ-chain condition for R and by Lemma 4.5, we may assume without loss of generality that $\langle B_\alpha : \alpha < \omega_1 \rangle$ and $\langle A_\xi \cap A_\eta : \xi < \eta < \kappa \rangle$ lie in V, and that the information above is all forced with respect to R, using $\langle \dot{A}_\xi : \xi < \kappa \rangle$ as a name for $\langle A_\xi : \xi < \kappa \rangle$.

For convenience, if $\alpha \in B_\beta$, then we say level$(\alpha) = \beta$.

Let $\xi < \kappa$. We say ξ is of *type* A if $\exists (p_\xi, f_\xi) \in R \, \exists \alpha_\xi < \omega_1 \, \exists$ countable $Z \subseteq \kappa \, \forall (q, g) \in R \, \forall \beta$ if $(q, g) \leqslant (p_\xi, f_\xi)$, $g \supseteq f_\xi$, level$(\beta) \geqslant \alpha_\xi$ and $(q, g) \Vdash \beta \in \dot{A}_\xi$, then $(q \restriction Z, g) \Vdash \beta \in \dot{A}_\xi$; otherwise ξ is of *type* B.

We consider two cases, each of which will lead to a contradiction.

Case 1: $S_1 = \{\xi : \xi$ is of type A$\}$ *has cardinality* κ.

Fix $\xi \in S_1$ and p_ξ, f_ξ, α_ξ, Z as in the definition of type A. By enlarging Z, if necessary, we may assume domain$(p_\xi) \subseteq Z$.

Lemma 6.1. *The definition of type* A *holds for* ξ *if* "$\beta \in \dot{A}_\xi$" *is replaced by* "$\beta \notin \dot{A}_\xi$".

Proof. Suppose $(q, g) \Vdash \beta \notin \dot{A}_\xi$. If $(q \restriction Z, g) \nVdash \beta \notin \dot{A}_\xi$, then $\exists (r, h) \leqslant (q \restriction Z, g)$, $h \supseteq g$, $(r, h) \Vdash \beta \in \dot{A}_\xi$, and since ξ is of type A we may assume domain$(r) \subseteq Z$. But since r is compatible with $q \restriction Z$, it is compatible with q, and thus

$$(r \cup q, h) \Vdash \beta \in \dot{A}_\xi \quad \text{and} \quad \beta \notin \dot{A}_\xi,$$

which is impossible.

Now let $\xi_0 = \min(\kappa - Z)$. Then $\xi_0 < \omega_1$ so without loss of generality we may assume $\xi_0 \subseteq$ domain(f_ξ).

Lemma 6.2. *If* $(q, g) \leqslant (p_\xi, f_\xi)$, $g \supseteq f$, level$(\beta) \geqslant \alpha_\xi$ *and* $(q, g) \Vdash \beta \in \dot{A}_\xi$, *then* $(q \restriction Z, f_\xi) \Vdash \beta \in \dot{A}_\xi$.

Proof. We already know $(q \restriction Z, g) \Vdash \beta \in \dot{A}_\xi$ so we may as well assume that $q = q \restriction Z$. If $(q, f_\xi) \nVdash \beta \in A_\xi$, then $\exists (q', g') \leqslant (q, f_\xi)$, $g' \supseteq f_\xi$ with $(q', g') \Vdash \beta \notin \dot{A}_\xi$, and by Lemma 6.1 we may assume domain$(q') \subseteq Z$ also. But now by Lemma 4.2 since $f_\xi \restriction \xi_0 = g \restriction \xi_0 = g' \restriction \xi_0$, there is $h \supseteq f_\xi$ such that $\{(\xi_0, 0)\} \Vdash h_G \supseteq g_G$ and $\{(\xi_0, 1)\} \Vdash h_G \supseteq g'_G$. But since $\xi_0 \notin Z$ and domain$(q') \subseteq Z$ we have

$$(q' \cup \{(\xi_0, 0)\}, h) \Vdash \beta \in \dot{A}_\xi$$

while

$$(q' \cup \{(\xi_0, 1)\}, h) \Vdash \beta \notin \dot{A}_\xi.$$

But now since $h \supseteq f_\xi$ and $\xi_0 \notin Z$ we have since ξ is of type A that $(q', h) \Vdash \beta \in \dot{A}_\xi \wedge \beta \notin \dot{A}_\xi$, which is absurd.

Thus if $(p_\xi, f_\xi) \in K$ and level$(\beta) \geqslant \alpha_\xi$, then if $\beta \in A_\xi$ there must be $q < p_\xi$, domain$(q) \subseteq Z$, with $(q, f_\xi) \Vdash \beta \in \dot{A}_\xi$. Since there are only countably many q with domain$(q) \subseteq Z$ it follows that for some q, $\{\text{level}(\beta): (q, f_\xi) \Vdash \beta \in \dot{A}_\xi\}$ is cofinal in ω_1. Thus A_ξ must have a subset in the ground model with levels cofinal in ω_1. By Lemma 4.4 and the assumption for this case there is $(p, f) \in R$ which will force this to happen κ times. Since the cardinality of $\mathcal{P}(\omega_1) \cap V$ is only ω_1, two of the A_ξ must contain the same subset with levels cofinal in ω_1, and this contradicts our assumption that the family was graded almost disjoint. This completes Case 1.

Case 2: $S_2 = \{\xi: \text{cf}\, \xi \geq \omega_1$ and ξ is of type $B\}$ *is stationary.*
Suppose $\xi \in S_2$.

Lemma 6.3. *Let $Z \subseteq \kappa$ be countable, $(p, f) \in R$ and $\alpha < \omega_1$. Then $\exists g, q_1, q_2, \beta$ level$(\beta) \geq \alpha$, $g \supseteq f$, $q_1, q_2 \leq p$, $q_1 \restriction Z = q_2 \restriction Z$ and $(q_1, g) \Vdash \beta \in \dot{A}_\xi$, $(q_2, g) \Vdash \beta \notin \dot{A}_\xi$.*

Proof. Suppose not. We claim that ξ is of type A with $p_\xi = p$, $f_\xi = f$, $\alpha_\xi = \alpha$. Suppose q, g, β are as in the definition of type A with $(q, g) \Vdash \beta \in \dot{A}_\xi$. If $(q \restriction Z, g) \nVdash \beta \in \dot{A}_\xi$, then there is $(r, h) \leq (q \restriction Z, g)$, $h \supseteq g$ with $(r, h) \Vdash \beta \notin \dot{A}_\xi$. But now if we let $q_2 = r$, $q_1 = q \cup (r \restriction Z)$, then $q_1 \restriction Z = q_2 \restriction Z$ and $(q_1, h) \Vdash \beta \in \dot{A}_\xi$, $(q_2, h) \Vdash \beta \notin \dot{A}_\xi$.

Now, using Lemma 6.3 repeatedly, we may build sequences $\langle h_\tau: \tau < \omega_1\rangle$ and $\langle q^1_\tau, q^2_\tau: \tau < \omega_1\rangle$ so that $(q^1_\tau, h_\tau) \Vdash \beta_\tau \in \dot{A}_\xi$, $(q^2_\tau, h_\tau) \Vdash \beta_\tau \notin \dot{A}_\xi$, and if $Z_\tau = \bigcup\{\text{domain}(q^1_\delta) \cup \text{domain}(q^2_\delta: \delta < \tau\}$ then $q^1_\tau \restriction Z_\tau = q^2_\tau \restriction Z_\tau$. We also assume the h_τ are increasing, and that level(β_τ) is increasing as a function of τ (although this is not strictly necessary).

Without loss of generality we may assume that $\{\text{domain}(q^1_\tau) \cup \text{domain}(q^2_\tau): \tau < \omega_1\}$ forms a Δ-system with kernel Δ, and that for some fixed q_ξ we have $q^1_\tau \restriction \Delta = q^2_\tau \restriction \Delta = q_\xi$ for all τ. (This uses the fact that $q^1_\tau \restriction Z_\tau = q^2_\tau \restriction Z_\tau$ and that $\Delta \subseteq Z_\tau$ eventually.)

Let $f_\xi = h_\omega$.

Fix a regular cardinal λ so large that $R \in H(\lambda)$, the collection of all sets hereditarily of cardinality $<\lambda$ ($\lambda = \kappa^+$ will do), and for each $\xi < \kappa$ let N_ξ be a countable elementary substructure of $H(\lambda)$ containing ξ, q_ξ, f_ξ, $\langle q^i_{n,\xi}: n < \omega\rangle$, $\langle Z^\xi_n: N < \omega\rangle$ and $\langle \beta^\xi_n: n < \omega\rangle$ as predicates, where $q^i_{n,\xi}$, Z^ξ_n, β^ξ_n denote q^i_n, Z_n, β_n as defined from ξ.

Now apply Lemma 4.3 to get $\xi < \eta$ and an isomorphism $\pi_{\xi\eta}: N_\xi \to N_\eta$ such that $\pi_{\xi\eta}: N_\xi \cap \kappa \to N_\eta \cap \kappa$ is order-preserving and the identity on ξ, and $\pi_{\xi\eta}(\xi) = \eta$. By taking η large enough we may also ensure that $N_\xi \cap \kappa \subseteq \eta$.

Then (q_ξ, f_ξ) and (q_η, f_η) are compatible, for $\pi_{\xi\eta}(q_\xi \restriction \xi) = q_\xi \restriction \xi = q_\eta \restriction \eta$ and domain$(q_\xi) \subseteq N_\xi \cap \kappa \subseteq \eta$, so q_ξ and q_η are compatible, and a similar argument may be applied to f_ξ and f_η.

Suppose now $(p, f) \leq (q_\xi \cup q_\eta, f_\xi \cup f_\eta)$ and $X \in V$ is arbitrary. Then we may find n so large that domain$(p) \cap \bigcup\{Z^\xi_i: i < \omega\} \subseteq Z^\xi_n$ and domain$(p) \cap \bigcup\{Z^\eta_i: i < \omega\} \subseteq Z^\eta_n$. But that means that $q^i_{n,\xi}$, $q^i_{n,\eta}$ are all compatible with p. If

$\beta_n \in X$, then consider $q = q^2_{n,\xi} \cup q^2_{n,\eta} \cup p$, which is a condition because $\pi_{\xi\eta}$ is the identity on ξ and domain$(q^1_{n,\xi}) \subseteq N_\xi \subseteq n$. But clearly

$$(q, f) \Vdash \beta_n \notin \dot{A}_\xi \wedge \beta_n \notin \dot{A}_\eta \text{ so } \beta_n \notin \dot{A}_\xi \cap \dot{A}_\eta.$$

If $\beta_n \notin X$, then consider $q = q^1_{n,\xi} \cup q^1_{n,\eta} \cup p$ and note that

$$(q, f) \Vdash \beta_n \in \dot{A}_\xi \text{ and } \beta_n \in \dot{A}_\eta \text{ so } \beta_n \in \dot{A}_\xi \cap \dot{A}_\eta.$$

In either case it is clear that $(q, f) \Vdash X \neq \dot{A}_\xi \cap \dot{A}_\eta$. Since X was arbitrary this means

$$(q_\xi \cup q_\eta, f_\xi \cup f_\eta) \Vdash \dot{A}_\xi \cap \dot{A}_\eta \notin V,$$

which completes Case 2 and the proof.

7. Forcing a thin-very tall sBa

The remainder of this paper is devoted to the proof that by forcing one can produce a thin-very tall sBa. The proof breaks naturally into three parts, and it will occupy the next three sections. In an effort to make the proof more understandable, we shall present it backwards.

In this section we define a partial ordering P depending on a special function f^*, and we show that if P has the c.c.c., then forcing with P adjoins a thin-very tall sBa. In the next section we show that if f^* satisfies a certain property, then P will indeed have the c.c.c. Finally, in Section 9 we show how to force the existence of a function f^* having the desired property.

Let $T = \omega_2 \times \omega$ and for $\alpha < \omega_2$ let $T_\alpha = \{\alpha\} \times \omega$. The partial ordering P is designed to impose a partial Boolean-algebra structure on T. More specifically, P will adjoint a partial ordering $\leqslant$ of T and a function i on $[T]^2$ such that the meet $s \wedge t$ of $s, t \in T$ is represented as the supremum of the elements of $i\{s, t\}$ in the strong sense that if $v \leqslant s, t$ then $\exists u \in i\{s, t\}\ v \leqslant u$. Conditions will consist of finite bits of information about $\leqslant$ and i, and $i\{s, t\}$ will always be a finite subset of T.

Given $\leqslant$ and i, we may define $a_t \subseteq \omega$ for $t \in T$ by $a_t = \{n\colon (0, n) \leqslant t\}$. If B is the subalgebra of $\mathcal{P}(\omega)$ generated by the a_t, then we will show that for each $\alpha < \omega_2$, $\{a_t\colon t \in T_\alpha\}$ form a set of representatives of the atoms in B/I_α. It will follow that B is a thin-very tall sBa.

Let $f^*\colon [\omega_2]^2 \to [\omega_2]^{\leqslant\omega}$ be fixed. Define P as the set of all $p = (x_p, \leqslant_p, i_p)$ satisfying the following conditions:

(1) $x_p \in [T]^{<\omega}$.

(2) $\leqslant_p$ is a partial ordering of x_p with the property that if $s \in T_\alpha$, $t \in T_\beta$ and $s <_p t$, then $\alpha < \beta$.

(3) $i_p\colon [x_p]^2 \to [x_p]^{<\omega}$ is such that

(3.1) if $s \in T_\alpha$, $t \in T_\beta$, $s \neq t$ and $\alpha \leqslant \beta$, then

(3.1.1) if $\alpha = \beta$, then $i_p\{s, t\} = 0$,

(3.1.2) if $s <_p t$, then $i_p\{s, t\} = \{s\}$,
(3.1.3) if $\alpha < \beta$ and $s \not<_p t$, then

$$i_p\{s, t\} \subseteq x_p \cap \bigcup \{T_\tau : \tau \in f^*\{\alpha, \beta\}, \tau < \alpha\};$$

(3.2) if $\{s, t\} \in [x_p]^2$, then $\forall u \in i_p\{s, t\}\ u \leqslant_p s, t$, and if $v \leqslant_p s,t$, then $\exists u \in i_p\{s, t\}\ v \leqslant_p u$.

Set $p \leqslant q$ iff $x_p \supseteq x_q$, $\leqslant_p \restriction x_q = \leqslant_q$ and $i_p \restriction [x_q]^2 = i_q$.

Theorem 7.1. *Assume P has the c.c.c. Then forcing with P adjoins a thin-very tall superatomic Boolean algebra.*

Proof. We need the c.c.c. only to verify that cardinals are preserved.

Let G be P-generic, and let $\leqslant = \bigcup \{\leqslant_p : p \in G\}$, $i = \bigcup \{i_p : p \in G\}$. It is clear that $\leqslant$ is a partial ordering of T. For $t \in T$ let $a_t = \{n \in \omega : (0, n) \leqslant t\}$. Then clearly if $s \leqslant t$ we have $a_s \subseteq a_t$. Also, if $s, t \in T_\alpha$ and $s \neq t$, then by (3.1.1) and (3.2) we must have $a_s \cap a_t = 0$. Thus $\{a_t : t \in T_\alpha\}$ is a disjoint family. More generally, by (3.2) we have $a_s \cap a_t = \bigcup \{a_u : u \in i\{s, t\}\}$ for all s, t.

Lemma 7.2. *Suppose $\alpha < \beta < \omega_2$ and $t \in T_\beta$. Then $\{s \in T_\alpha : s \leqslant t\}$ is infinite.*

Proof. Fix $p \in P$ with $t \in x_p$. Let $s \in T_\alpha$, $s \notin x_p$. We will find $q \leqslant p$ with $s \leqslant_q t$. By an elementary density argument, this will suffice. Let $x_q = x_p \cup \{s\}$. Put $t_1 \leqslant_q t_2$ iff $t_1 \leqslant_p t_2$ or else $t_1 = s$ and $t \leqslant_p t_2$. Then clearly $\leqslant_q$ is a partial ordering and $s \leqslant_q t$. Let $i_q\{t_1, t_2\} = i_p\{t_1, t_2\}$ unless one of t_1, t_2, say t_1, is s. Let $i_q\{s, t_2\} = \{s\}$ if $s \leqslant_q t_2$ and $i_q\{s, t_2\} = 0$ otherwise. Since no element of x_q lies strictly below s in $<_q$ it is easy to verify (3.2).

With $\alpha = 0$, Lemma 7.2 implies that a_t is infinite whenever $t \in T_\beta$, $\beta > 0$. Of course, if $t = (0, n)$, then $a_t = \{n\}$.

Let B be the subalgebra of $\mathcal{P}(\omega)$ generated by the a_t for $t \in T$.

Lemma 7.3. *B is superatomic. Moreover, if I_α is the Cantor–Bendixson ideal produced at the αth step, then for all $\alpha < \omega_2$, I_α is generated by $\{a_t : t \in \bigcup \{T_\beta : \beta < \alpha\}\}$, and $\{a_t : t \in T_\alpha\}$ is a set of representatives of the atoms of B/I_α. Thus B is thin-very tall.*

Proof. We proceed by induction on α.

It is clear that if $\{a_t : t \in T_\alpha\}$ is a set of representatives of atoms, then $I_{\alpha+1}$ is generated by $I_\alpha \cup \{a_t : t \in T_\alpha\}$, so we need only show that B/I_α is atomic and $\{a_t : t \in T_\alpha\}$ is a set of representatives of atoms assuming the characterization of I_α in the lemma.

First we had better check that if $t \in T_\alpha$ then $a_t \notin I_\alpha$. If this is not true, then $\exists t_1, \ldots, t_n \in \bigcup \{T_\beta : \beta < \alpha\}$ such that $a_t \subseteq a_{t_1} \cup \cdots \cup a_{t_n}$. Let β be maximal with some $t_i \in T_\beta$. Then $\beta < \alpha$ so by Lemma 7.2 we can find $s \leqslant t$, $s \in T_\beta$, $s \neq$

$t_1, \ldots, t_n$. But then $a_s \subseteq a_t$ and by inductive hypothesis a_s is not covered by the a_{t_i} for $t_i \in T_\tau$, $\tau < \beta$. Since $a_s \cap a_{t_i} = 0$ for $t_i \in T_\beta$, we must have $a_t \notin I_\alpha$.

Next we must show that if $b \in B$, $b \notin I_\alpha$, then $\exists t \in T_\alpha\ a_t \subseteq b$ modulo I_α, i.e., $a_t - b \in I_\alpha$. We may put b in disjunctive normal form relative to the a_t's, and write $b = b_1 \cup b_2 \cup \cdots \cup b_n$, where each b_i is the intersection of elements of the form a_t and their complements. If $b \notin I_\alpha$, then some $b_i \notin I_\alpha$, so without loss of generality we may assume $b = b_i$. Thus we may set $b = (a_1 \cap \cdots \cap a_n) - (b_1 \cup \cdots \cup b_m)$, where the a's and b's are of the form a_t. Since we know $a_s \cap a_t = \bigcup \{a_u : u \in i\{s, t\}\}$, it is easy to see by induction that (if $n > 0$) $a_1 \cap \cdots \cap a_n$ may be written as a union of elements of the form a_t. Thus we may set

$$\begin{aligned} b &= (c_1 \cup \cdots \cup c_k) - (b_1 \cup \cdots \cup b_m) \\ &= (c_1 - (b_1 \cup \cdots \cup b_m)) \cup \cdots \cup (c_k - (b_1 \cup \cdots \cup b_m)), \end{aligned}$$

and, as above, assume without loss of generality that $b = c - (b_1 \cup \cdots \cup b_m)$ where, to allow for the case $n = 0$, we permit the possibility both that $c = a_t$ and $c = \omega$.

We treat the case $c = a_t$; the case $c = \omega$ may be reduced to this one in a way we will describe later. Suppose $t \in T_{\bar{\alpha}}$. Fix i with $1 \leq i \leq m$. Then $b_i = a_s$ for $s \in T_\beta$, some β.

If $\bar{\alpha} = \beta$, then either $s = t$, in which case $b = 0$ so $b \in I_\alpha$, a contradiction, or else $s \neq t$ in which case $a_t - a_s = a_t$ so b_i is redundant, and may be eliminated.

If $\bar{\alpha} < \beta$, then we cannot have $a_t \subseteq a_s$ lest $b = 0$, so clearly $i\{s, t\} \subseteq \{u : u \in \bigcup \{T_\tau : \tau < \bar{\alpha}\}\}$. Hence we may write $a_t - a_s = a_t - (a_{u_1} \cup \cdots \cup a_{u_l})$, where $u_1, \ldots, u_l \in \bigcup \{T_\tau : \tau < \bar{\alpha}\}$.

Thus, again without loss of generality, we may assume

$$b = a_t - (a_{u_1} \cup \cdots \cup a_{u_l}),$$

where $u_1, \ldots, u_l \in \bigcup \{T_\tau : \tau < \bar{\alpha}\}$. Let $\tau_1 > \cdots > \tau_j$ enumerate all τ such that some $u_i \in T_\tau$. We may assume $\tau_j \geq \alpha$ since we are working modulo I_α. Now choose a sequence $a_t \supseteq a_{t_1} \supseteq \cdots \supseteq a_{t_j}$ inductively as follows. Using Lemma 7.2, choose $t_1 \in T_{\tau_1}$ such that $t_1 \leq t$ and $t_1 \neq u_i$, all i. Then, again using Lemma 7.2, choose $t_2 \leq t_1$ such that $t_2 \in T_{\tau_2}$ and $t_2 \neq u_i$, all i, etc. If $\tau_j > \alpha$, then we may choose $\bar{t} \in T_\alpha$, $\bar{t} \leq t_{\tau_j}$; otherwise let $\bar{t} = t_{\tau_j}$. But now note that $a_{\bar{t}} \supseteq b$ and $\bar{t} \in T_\alpha$, as desired.

Finally, if $c = \omega$, then choose $t \in T_{\bar{\alpha}}$, where $\bar{\alpha}$ is so large that for all b_i we have $b_i \in \{a_s : s \in \bigcap \{T_\tau : \tau < \bar{\alpha}\}\}$, replace b by $a_t - (b_1 \cup \cdots \cup b_m)$, and proceed as above.

This completes the proof of Lemma 7.3 and Theorem 7.1.

8. The countable chain condition

Next we turn to the construction of a function f^* which will make the partial ordering P of Theorem 7.1 satisfy the countable chain condition.

Let us say that $f:[\omega_2]^2\to[\omega_2]^{\leqslant\omega}$ has *property* Δ iff $f\{\alpha,\beta\}\subseteq\min\{\alpha,\beta\}$ for all $\alpha,\beta<\omega_2$ and for any uncountable set D of finite subsets of ω_2, $\exists a,b\in D\ a\neq b$ and $\forall\alpha\in a-b\ \forall\beta\in b-a\ \forall\tau\in a\cap b$

(a) if $\alpha,\beta>\tau$, then $\tau\in f\{\alpha,\beta\}$,
(b) if $\beta>\tau$, then $f\{\alpha,\tau\}\subseteq f\{\alpha,\beta\}$,
(c) if $\alpha>\tau$, then $f\{\beta,\tau\}\subseteq f\{\alpha,\beta\}$.

Theorem 8.1. *If f^* has property Δ, then the partial ordering P of Section 7 has the countable chain condition.*

Proof. Suppose on the contrary that A is an uncountable antichain. If $y_p=\{\alpha: T_\alpha\cap x_p\neq 0\}$ for $p\in A$, then by thinning out A if necessary we may assume that the y_p form a Δ-system with kernel Δ. Moreover, we may assume that the y_p all have the same cardinality and that for $p,q\in A$ the unique order-preserving bijection $\pi_{pq}:y_p\to y_q$ is the identity on Δ. We may also assume that π_{pq} lifts to an isomorphism of x_p with x_q given by $\pi_{pq}(\alpha,n)=(\pi_{pq}(\alpha),n)$. Of course, this may require a further thinning out of the antichain A. We may assume in addition that $s\leqslant_p t$ iff $\pi_{pq}(s)\leqslant_q\pi_{pq}(t)$. And finally and most crucially, because of the condition (3.1.3) in the definition of P and the countability of each $f^*\{\alpha,\beta\}$, we may assume that whenever $s,t\in x_p\cap x_q$ then $i_p\{s,t\}=i_q\{s,t\}$ (so in particular π_{pq} is the identity on $i_p\{s,t\}$).

Now since f^* has property Δ we may find $p,q\in A$ such that $\forall\alpha\in y_p-y_q\ \forall\beta\in y_q-y_p\ \forall\tau\in y_p\cap y_q$

(a) if $\alpha,\beta>\tau$, then $\tau\in f^*\{\alpha,\beta\}$,
(b) if $\beta>\tau$, then $f^*\{\alpha,\tau\}\subseteq f^*\{\alpha,\beta\}$,
(c) if $\alpha>\tau$, then $f^*\{\beta,\tau\}\subseteq f^*\{\alpha,\beta\}$.

We claim p and q are compatible, and this will complete the proof of Theorem 8.1.

We must determine $r\leqslant p,q$. Let $x_r=x_p\cup x_q$. Put $s\leqslant_r t$ iff $s\leqslant_p t$ or $s\leqslant_q t$ or $\exists u\in x_p\cap x_q$ either $s\leqslant_p u\leqslant_q t$ or $s\leqslant_q u\leqslant_p t$. Note in particular that $\leqslant_r\restriction x_p=\leqslant_p$, $\leqslant_r\restriction x_q=\leqslant_q$.

Lemma 8.2. *$\leqslant_r$ is a partial ordering.*

Proof. We need only check transitivity. Let $s\leqslant_r t\leqslant_r u$. If $s,t,u\in x_p$ or $s,t,u\in x_q$ we are done. Suppose $s\leqslant_p s'\leqslant_q t\leqslant_q u$. Then $s\leqslant_p s'\leqslant_q u$ so $s\leqslant_r u$. If $s\leqslant_p s'\leqslant_q t\leqslant_q t'\leqslant_p u$, then $s\leqslant_p s'\leqslant_q t'\leqslant_p u$. But $s'\leqslant_p t'$ since $s',t'\in x_p\cap x_q$, so $s\leqslant_p u$. All other cases are similar to one of these.

Finally, let $i_r\{s,t\}=i_p\{s,t\}$ if $s,t\in x_p$ and $i_r\{s,t\}=i_q\{s,t\}$ if $s,t\in x_q$. Suppose $s\in x_p-x_q$, $t\in x_q-x_p$. Then let $i_r\{s,t\}=\{u\in x_r: u\leqslant_r s,t$ and if $u\in T_\alpha$, then $\alpha\in f^*\{\beta,\tau\}\}$, where $s\in T_\beta$, $t\in T_\tau$, unless $s\leqslant_r t$ or $t\leqslant_r s$, in which case we define $i_r\{s,t\}$ to make (3.1.2) true. We must check condition (3).

Let us begin with (3.1). It is clear that (3.1.1) and (3.1.2) are satisfied, and (3.1.3) follows from the definition of $i_r\{s, t\}$ above. Furthermore, the first part of (3.2), namely that $\forall u \in i_r\{s, t\}\ u \leqslant_r s, t$ is clear.

We must check the second part of (3.2). Let $\{s, t\} \in [x_r]^2$, $v \leqslant_r s, t$. If $\{v, s, t\} \subseteq x_p$ or $\{v, s, t\} \subseteq x_q$, then it is clear that $\exists u \in i_r\{s, t\}\ v \leqslant_r u$. For the rest, we consider two cases.

Case 1: $s, t \in x_q$, $v \in x_p - x_q$.

Case 1.1: $s, t \in x_q - x_p$. Then $\exists s', t' \in x_p \cap x_q\ v \leqslant_p s' \leqslant_q s$, $v \leqslant_p t' \leqslant_q t$. By (3.2) for p, there is $u' \in i_p\{s', t'\}$ such that $u' \leqslant_p s', t'$ and $v \leqslant_p u'$. But $u' \in x_p \cap x_q$ since $i_p\{s', t'\} = i_q\{s', t'\}$, so by (3.2) for q there is $u \in i_q\{s, t\}$ such that $u' \leqslant_q u$ (since $u' \leqslant_q s', t'$ we have $u' \leqslant_q s, t$). But now $v \leqslant_p u' \leqslant_q u$ so $v \leqslant_r u$ and we are done since $i_r\{s, t\} = i_q\{s, t\}$.

Case 1.2: $s \in x_q \cap x_p$, $t \in x_q - x_p$. Then we proceed as in Case 1.1, except that we take $s' = s$.

The case $s, t \in x_p$, $v \in x_q - x_p$ is exactly similar to Case 1.

Case 2: $s \in x_p - x_q$, $t \in x_q - x_p$. If $s \leqslant_r t$ or $t \leqslant_r s$, then (3.2) is clear since $\leqslant_r$ is a partial ordering. So suppose othewise. Say $v \in x_p$. The case $v \in x_q$ is similar. Then $v \leqslant_p s$, $v \leqslant_p t' \leqslant_q t$ for some $t' \in x_p \cap x_q$. Let $s \in T_\alpha$, $t \in T_\beta$, $t' \in T_\tau$. Note that $\tau \in y_p \cap y_q$. By (3.2) for p there is $u \in i_p\{s, t'\}$ with $v \leqslant_p u$. It will suffice to show $u \in i_r\{s, t\}$. Say $u \in T_\delta$. If $\delta = \tau$, then $u = t'$, and since $\tau < \alpha, \beta$, we have $\tau \in f^*\{\alpha, \beta\}$. Hence $u \in i_r\{s, t\}$. If $\delta < \tau$, then $\delta \in f^*\{\alpha, \tau\} \subseteq f^*\{\alpha, \beta\}$ (since $\tau < \beta$) so again $u \in i_r\{s, t\}$.

This completes the verification of (3). Thus P has the countable chain condition.

9. A function with property Δ

Finally, we must establish that the existence of a function with property Δ is consistent with the axioms of ZFC. This we do by forcing.

Recall that $f: [\omega_2]^2 \to [\omega_2]^{<\omega}$ has property Δ iff $f\{\alpha, \beta\} \subseteq \min\{\alpha, \beta\}$ for all $\alpha, \beta < \omega_2$ and for any uncountable set D of finite subsets of ω_2, $\exists a, b \in D\ a \neq b$ and $\forall \alpha \in a - b\ \forall \beta \in b - a\ \forall \tau \in a \cap b$

(a) if $\alpha, \beta > \tau$, then $\tau \in f\{\alpha, \beta\}$,

(b) if $\beta > \tau$, then $f\{\alpha, \tau\} \subseteq f\{\alpha, \beta\}$,

(c) if $\alpha > \tau$, then $f\{\beta, \tau\} \subseteq f\{\alpha, \beta\}$.

In constructing f^* it is natural to try to force with countable approximations to f^*. Let H be the family of all functions h such that for some $a \in [\omega_2]^{\leqslant\omega}$, $h: [a]^2 \to [a]^{<\omega}$ and $h\{\alpha, \beta\} \subseteq \min\{\alpha, \beta\}$ for all $\alpha, \beta \in a$. We may try forcing with H, ordered by reverse inclusion. It is easy to see that H is countably closed and, assuming CH, has the ω_2-chain condition. Unfortunately, this ordering does not work. Roughly speaking, the problem is that if f^* is the function adjoined by forcing with H, then $f^*\{\alpha, \beta\}$ is too often small.

To see this a little more clearly, consider the proof of the ω_2-chain condition for H. Given any $A \subseteq H$ of cardinality ω_2, it is easy by a Δ-system argument to find $g, h \in A$ with $\text{domain}(g) = [a]^2$, $\text{domain}(h) = [b]^2$, $g \neq h$ and there are sets x, y, z with $a = x \cup y$, $b = x \cup z$, $\forall \alpha \in x\, \forall \beta \geq y\, \forall \tau \in z\ \alpha < \beta < \tau$, and there is a (unique) order-preserving mapping $\pi: a \to b$ which lifts to an isomorphism of g with h. (Note that π is the identity on $a \cap b = x$ since x is an initial segment of both a and b.) Now we may amalgamate g and h by defining $f \supseteq g \cup h$ on $[a \cup b]^2$ so that if $\alpha \in a - b$ and $\beta \in b - a$, then $f\{\alpha, \beta\}$ is an arbitrary subset of $(a \cup b) \cap \min\{\alpha, \beta\}$. For our purposes it is best to make $f\{\alpha, \beta\}$ as large as possible, namely $f\{\alpha, \beta\} = (a \cup b) \cap \min\{\alpha, \beta\}$. We refer to f as the *maximal amalgamation* of g and h. This kind of amalgamation is called a *head-tail-tail* amalgamation because of the order relation between x, y and z. We refer to x as the *head*, y as the *first tail* and z as the *second tail* of the amalgamation.

We would always like to work with head-tail-tail maximal amalgamations, but there is no way to express this in a forcing condition unless we include somehow the fact that f was obtained by amalgamating g and h. Thus the condition involving f would also involve its 'history', namely g and h. But when two of these more elaborate conditions are amalgamated the resulting condition will be still more complicated. In order to treat this process uniformly we make the following definitions.

Our forcing conditions will be certain countable (or finite) subsets p of H with the property that $\bigcup p \in p$. We refer to $\bigcup p$ as the *base* of p and write $\bigcup p = \text{base}(p)$. If p and q are such sets we put $p \leq q$ iff $\text{base}(q) \in p$ and $q = \{h \in p: h \subseteq \text{base}(q)\}$. It is clear that $\leq$ is a partial ordering.

Now, by induction on $\alpha < \omega_1$, we define sets $P_\alpha \subseteq [H]^{\leq\omega}$ which we refer to as sets of *level* α. At the end we will set $P = \bigcup\{P_\alpha: \alpha < \omega_1\}$, and we will force with P ordered by $\leq$ above.

P_0 consists of sets of the form $\{h\}$ where $\text{domain}(h) = \{\{\alpha, \beta\}\}$ for some $\{\alpha, \beta\} \in [\omega_2]^2$ and $h\{\alpha, \beta\} = 0$.

Let $\alpha = \beta + 1$. Then $p \in P_\alpha$ iff $\exists q, r \in P_\beta\ q \neq r$ and if $g = \text{base}(q)$, $h = \text{base}(r)$, $\text{domain}(g) = [a]^2$, $\text{domain}(h) = [b]^2$, then there are x, y, z with $a = x \cup y$, $b = x \cup z$, $\forall \alpha \in x\, \forall \beta \in y\, \forall \tau \in z\ \alpha < \beta < \tau$ and there is an order-preserving bijection $\pi: a \to b$ which lifts to an isomorphism of q with r, and $p = q \cup r \cup \{f\}$ where f is the maximal amalgamation of g and h. We say p is obtained by amalgamating q and r.

Finally, if α is a limit ordinal, then we put $p \in P_\alpha$ iff $p = \bigcup\{p_n: n < \omega\} \cup \{\bigcup p\}$, where $p_0 \geq p_1 \geq p_2 \geq \cdots$, each $p_n \in P_{\alpha_n}$ and $\langle \alpha_n: n < \omega\rangle$ is an increasing sequence cofinal in α.

Let $P = \bigcup\{P_\alpha: \alpha < \omega_1\}$, ordered by $\leq$.

If $h \in H$ let us define the *support* of h, support(h), to be the unique set a such that $\text{domain}(h) = [a]^2$. If $p \in P$, let $\text{support}(p) = \text{support}(\text{base}(p))$.

Lemma 9.1. *If* $p \in P_{\alpha+1}$ *and* p *is obtained by amalgamating* $q, r \in P_\alpha$, *then* $p \leqslant q, r$.

Proof. By symmetry we need only check $p \leqslant q$. Clearly base$(q) \in p$, and $q \subseteq \{h \in p: h \subseteq \text{base}(q)\}$. Suppose $h \in p$, $h \subseteq \text{base}(q)$. If $h \notin r$, then $h \in q$ and we are done. But if $h \in r$, then support$(h) \subseteq$ support$(q) \cap$ support(r), so h is fixed by the isomorphism between q and r and it follows again that $h \in q$. Hence $p \leqslant q$.

Lemma 9.2. *Assume* CH. *Then* $(P, \leqslant)$ *has the* ω_2*-chain condition.*

Proof. If $A \subseteq P$ has cardinality ω_2, then an easy Δ-system argument will produce distinct $q, r \in A$ which can be amalgamated into a condition $p \in P$. But then by Lemma 9.1 p extends both q and r.

Lemma 9.3. *If* $p, q, r \in P$, $p \leqslant q, r$ *and* base$(r) \in q$, *then* $q \leqslant r$.

Proof. Easy.

Lemma 9.4. *If* $p, q \in P$, $p \leqslant q$, $p \neq q$, $p \in P_\alpha$ *and* $q \in P_\beta$, *then* $\beta < \alpha$.

Proof. By induction on α. If $\alpha = 0$, this is vacuously true. If $\alpha = \beta + 1$, then p is obtained by amalgamating p_1 and p_2, say, so base$(q) \in p_1$ or p_2. But then by Lemma 9.3, $p_1 \leqslant q$ or $p_2 \leqslant q$ and we may apply the inductive hypothesis. If α is limit, then we have $p = \bigcup \{p_n: n < \omega\} \cup \{\bigcup p\}$, so base$(q) \in p_n$ for some n. But again by Lemma 9.3 we have $p_n \leqslant q$ and we are done by inductive hypothesis.

Lemma 9.5. $(P, \leqslant)$ *is countably closed.*

Proof. This is easy from Lemma 9.4 and the definition of P_α when α is limit.

It follows from Lemmas 9.2 and 9.5 that if CH holds then forcing with P preserves cardinals and cofinalities. If G is P-generic, then it is clear that for some set $A \subseteq \omega_2$ we have $\bigcup\bigcup G: [A]^2 \to [A]^{<\omega}$. *We do not claim that* $A = \omega_2$. This could be arranged, but it would complicate the definition of P.

Lemma 9.6. $|A| = \omega_2$.

Proof. Given any $p \in P$ it is easy to find an isomorphic condition q such that support(q) lies entirely above support(p). But then p and q can be amalgamated. Thus A will be cofinal in ω_2. Since cardinals are preserved, $|A| = \omega_2$.

Of course as long as $\bigcup\bigcup G$ satisfies all the other requirements of propety Δ, it is sufficient for its support, namely A, to be cofinal in ω_2. Now we concentrate on checking the other requirements of property Δ.

Lemma 9.7. *Suppose $p \in P$ and $h \in p$. Then $q = \{g \in p: g \subseteq h\} \in P$ (and of course $p \leq q$).*

Proof. By induction on α we must check the lemma for all $p \in P_\alpha$. Details are left to the reader.

Suppose $p \in P$. A sequence $\langle h_\xi: \xi < \zeta + 1\rangle$ of elements of p is called a *path through p* if

(1) $\{h_0\} \in P_0$,
(2) $h_{\xi+1}$ is an immediate $\subseteq$-successor of h_ξ,
(3) if ξ is limit, then $\bigcup\{h_\eta: \eta < \xi\} = h_\xi$,
(4) $h_\zeta = \text{base}(p)$.

If $\nu_1 < \nu_2 \leq \zeta$, then we also refer to $\langle h_\xi: \nu_1 \leq \xi \leq \nu_2\rangle$ as a path *from* h_{ν_1} *to* h_{ν_2} (in p).

Lemma 9.8. *Suppose $p \in P$ and $h \in p$. Then there is a path through p which contains h.*

Proof. By induction on α we check this for all $p \in P_\alpha$. All cases are easy. (If α is limit one must paste together paths between base(p_n) and base(p_{n+1}) from some point on, but the details are quite simple.)

Now suppose $p \in P$ and $s = \langle h_\xi: \xi < \zeta + 1\rangle$ is a path through p. If $\alpha \in$ support(p), then let $t(\alpha)$ be the least ordinal ξ such that $\alpha \in$ support(h_ξ). We think of $t(\alpha)$ as the *time* at which α appears. Naturally $t(\alpha)$ depends on both p and s.

By condition (3) in the definition of a path it is clear that $t(\alpha)$ is always a successor ordinal (or 0). If $t(\alpha) = \xi + 1$, then by Lemma 9.7 and condition (2) in the definition of a path it is clear that $h_{\xi+1}$ is obtained as the maximal amalgamation of h_ξ with some other function g. Since $t(\alpha) = \xi + 1$ we must have $\alpha \in \text{support}(g) - \text{support}(h_\xi)$. If $\pi: \text{support}(g) \to \text{support}(h_\xi)$ is the isomorphism, then we let $a(\alpha)$, the *ancestor* of α, be $\pi(\alpha) \in \text{support}(h_\xi)$. If $t(\alpha) = 0$, then $a(\alpha)$ is undefined. Of course $t(a(\alpha)) < t(\alpha)$ so the sequence $\alpha, a(\alpha), a^2(\alpha), \ldots$ is finite. Let $t^*(\alpha) = \{t(a^n(\alpha)): n \geq 0\}$.

Following are the two main lemmas for checking property Δ.

Lemma 9.9. *Suppose $p \in P$, $h = \text{base}(p)$ and $s = \langle h_\xi: \xi < \zeta + 1\rangle$ is a path through p. Suppose $\alpha, \beta \in \text{support}(p)$ with $t(\alpha) < t(\beta)$ and $t(\alpha) \notin t^*(\beta)$. Then $h\{\alpha, \beta\} \supseteq \min\{\alpha, \beta\} \cap \text{support}(h_{t(\alpha)})$.*

Proof. By induction on $t(\beta)$. Let $t(\beta)=\xi+1$. Then $h_{\xi+1}$ was produced by a head-tail-tail maximal amalgamation and β was in one of the tails. If α is also in one of the tails, then it cannot be in the same tail as β since $t(\alpha)\neq t(\beta)$. Hence since $h_{\xi+1}$ is maximal we have

$$h\{\alpha,\beta\}=h_{\xi+1}\{\alpha,\beta\}=\min\{\alpha,\beta\}\cap \text{support}(h_{\xi+1})$$
$$\supseteq \min\{\alpha,\beta\}\cap \text{support}(h_{t(\alpha)}).$$

So we may assume that α belongs to the head of the amalgamation producing $h_{\xi+1}$. By the isomorphism in the amalgamation we have $h_{\xi+1}\{\alpha,\beta\}=h_{\xi+1}\{\alpha,a(\beta)\}$ (recall α is fixed since it lies in the head). Also $\alpha<a(\beta)$ since $a(\beta)$ is in one of the tails.

If $t(\alpha)<t(a(\beta))$, then we are done by inductive hypothesis. Thus we may assume $t(a(\beta))<t(\alpha)$. But then α is in a tail of the amalgamation at stage $t(\alpha)$ and $a(\beta)$ must be in the other tail since $\alpha<a(\beta)$ and $t(a(\beta))\neq t(\alpha)$. Hence $h\{\alpha,\beta\}=h\{\alpha,a(\beta)\}=\min\{\alpha,\beta\}\cap\text{support}(h_{t(\alpha)})$ and we are done.

Lemma 9.10. *Suppose $p\in P$, $h=\text{base}(p)$ and $s=\langle h_\xi:\xi<\zeta+1\rangle$ is a path through p. Suppose also $\alpha,\beta,\tau\in\text{support}(p)$, $t(\tau)<t(\alpha),t(\beta)$, $t(\alpha)\notin t^*(\beta)$ and $t(\beta)\notin t^*(\alpha)$. Then*

(a) *if $\tau<\alpha,\beta$, then $\tau\in h\{\alpha,\beta\}$, and*

(b) *if $\tau<\beta$, then $h\{\alpha,\tau\}\subseteq h\{\alpha,\beta\}$.*

Proof. (a) By symmetry we may assume $t(\alpha)<t(\beta)$. By Lemma 9.9, $h\{\alpha,\beta\}\supseteq\min\{\alpha,\beta\}\cap\text{support}(h_{t(\alpha)})$. But since $t(\tau)<t(\alpha)$ we have $\tau\in\min\{\alpha,\beta\}\cap\text{support}(h_{t(\alpha)})$.

(b) Suppose $t(\alpha)<t(\beta)$. Then since $\alpha,\tau\in\text{support}(h_{t(\alpha)})$ we have $h\{\alpha,\tau\}=h_{t(\alpha)}\{\alpha,\tau\}\subseteq\min\{\alpha,\tau\}\cap\text{support}(h_{t(\alpha)})$. But now the conclusion follows from Lemma 9.9 and the fact that $\min\{\alpha,\tau\}\leqslant\min\{\alpha,\beta\}$.

Suppose $t(\beta)<t(\alpha)$. Let us attack this case by induction on $t(\alpha)$. If β and α lie in different tails of the amalgamation at stage $t(\alpha)$, then we have $h\{\alpha,\tau\}=h_{t(\alpha)}\{\alpha,\tau\}\subseteq h_{t(\alpha)}\{\alpha,\beta\}=h\{\alpha,\beta\}$ by maximality of $h_{t(\alpha)}$. Thus we may assume β (and hence also τ, since $\tau<\beta$) lie in the head. But then $\beta<a(\alpha)$ and by the isomorphism we have $h\{\alpha,\beta\}=h\{a(\alpha),\beta\}$ and $h\{\alpha,\tau\}=h\{a(\alpha),\tau\}$. If $t(\beta)<t(a(\alpha))$, we are done by inductive hypothesis, so we must handle the case $t(a(\alpha))<t(\beta)$. But since $\beta<a(\alpha)$ it must be the case that β and $a(\alpha)$ lie in different tails of the amalgamation at stage $t(\beta)$. Hence by maximality of $h_{t(\beta)}$ we have $h\{\alpha,\tau\}=h_{t(\beta)}\{a(\alpha),\tau\}\subseteq h_{t(\beta)}\{a(\alpha),\beta\}=h\{\alpha,\beta\}$, and the proof is complete.

Lemma 9.11. *Suppose G is P-generic and $h=\bigcup\bigcup G$. Let A be the support of h, i.e.,* $\text{domain}(h)=[A]^2$. *Then for any uncountable set D of finite subsets of A, $\exists a,b\in D$ $a\neq b$ and $\forall\alpha\in a-b$ $\forall\beta\in b-a$ $\forall\tau\in a\cap b$*

(a) *if* $\alpha, \beta > \tau$, *then* $\tau \in h\{\alpha, \beta\}$,
(b) *if* $\beta > \tau$, *then* $h\{\alpha, \tau\} \subseteq h\{\alpha, \beta\}$,
(c) *if* $\alpha > \tau$, *then* $h\{\beta, \tau\} \subseteq h\{\alpha, \beta\}$.

Proof. Without loss of generality we may assume that D is a Δ-system. Thus we suppose that $\Vdash$"$\langle \dot{a}_\xi : \xi < \omega_1 \rangle$ is a Δ-system of subsets of $\dot{A}$ with kernel $\dot{\Delta}$", where $\dot{A}$ is a term for A. Now define a descending sequence $\{p_\xi : \xi < \omega_1\rangle$ of elements of P as follows: Choose p_0 so that for some (real) $\Delta \subseteq \text{support}(p_0)$ we have $p_0 \Vdash \dot{\Delta} = \Delta$. Given p_ξ, find $p_{\xi+1} \leqslant p_\xi$ so that for some $a_\xi \subseteq \text{support}(p_{\xi+1})$ we have $p_{\xi+1} \Vdash \dot{a}_\xi = a_\xi$. Finally, if ξ is limit choose p_ξ so that $p_\xi = \bigcup \{p_\eta : \eta < \xi\} \cup \{\bigcup p_\xi\}$.

Next let $\langle h_\xi : \xi < \omega_1 \rangle$ be a 'path' through $\bigcup \{p_\xi : \xi < \omega_1\}$ obtained by taking a path through p_0 and concatenating it with paths from base(p_ξ) to base$(p_{\xi+1})$ for all ξ. It is easy to see that for any p_η some initial segment of $\langle h_\xi : \xi < \omega_1 \rangle$ is a path through p_η.

Hence for $\alpha \in \bigcup \{\text{support}(p_\xi) : \xi < \omega_1\}$ we may define $t(\alpha)$ as the least ξ with $\alpha \in \text{support}(h_\xi)$ as before. Define $t^*(\alpha)$ as before. But now since each $t^*(\alpha)$ is finite, it is easy to find $\xi, \eta < \omega_1$ with $\xi < \eta$ and

(1) $\forall \tau \in \Delta \, \forall \alpha \in a_\xi \cap a_\eta - \Delta \; t(\tau) < t(\alpha)$,
(2) $\forall \alpha \in a_\xi - \Delta \, \forall \beta \in a_\eta - \Delta \; t(\alpha) \notin t^*(\beta)$ and $t(\beta) \notin t^*(\alpha)$.

Of course $p_{\eta+1}$ now satisfies the hypothesis of Lemma 9.10 whenever $\alpha \in a_\xi - \Delta$, $\beta \in a_\eta - \Delta$ and $\tau \in \Delta$ as well as whenever $\alpha \in a_\eta - \Delta$, $\beta \in a_\xi - \Delta$ and $\tau \in \Delta$. The conclusion of this lemma is now forced by $p_{\eta+1}$ as one sees immediately from Lemma 9.10.

By mapping A onto ω_2 in an order-preserving fashion we obtain finally

Theorem 9.12. *Forcing with* P *produces a function* f^* *with property* Δ.

10. Thin-tall topological spaces

Let us turn our attention from Boolean algebras to topological spaces.

Recall that if X is a topological space, then by induction on α we define $X^{(\alpha)}$ by: $X^{(0)} = X$, $X^{(\alpha+1)}$ is the set of limit points of $X^{(\alpha)}$, and for limit α, $X^{(\alpha)} = \bigcap \{X^{(\beta)} : \beta < \alpha\}$. X is *scattered* if some $X^{(\alpha)} = 0$; the least such α is the *height* of X. We refer to $X^{(\alpha)} - X^{(\alpha+1)}$ as the αth *level* of X.

Our purpose in this section is to prove in ZFC the following theorem, which anwers a question of I. Juhasz.

Theorem 10.1. *There are regular (even zero-dimensional) scattered spaces of arbitrary height below* $\mathfrak{c}^+$, *where* $\mathfrak{c}$ *denotes the cardinality of the continuum, such that all levels are countably infinite.*

Proof. A family F of subsets of ω is *independent* provided that whenever $a_1, \ldots, a_n, b_1, \ldots, b_m$ are distinct members of F, then $(a_1 \cap \cdots \cap a_n) - (b_1 \cup \cdots \cup b_m)$ is infinite. It is well-known that independent families of cardinality $\mathfrak{c}$ exist. Let F denote such a family.

Fix $\xi < \mathfrak{c}^+$ and for convenience assume $\mathfrak{c} \leqslant \xi$. Let $X = \xi \times \omega$ and let $f: X \to F$ be a bijection. For each $x = (\alpha, n) \in X$ let $A_x = \{(\beta, m): \beta < \alpha \text{ and } m \in f(x)\} \cup \{x\}$. We give X the topology determined by declaring each A_x to be clopen; thus X is clearly zero-dimensional.

It will suffice to verify by induction on α that $X^{(\alpha)} = (\xi - \alpha) \times \omega$. This is clear for $\alpha = 0$ and α limit. Suppose $\alpha = \beta + 1$ and $X^{(\beta)} = (\xi - \beta) \times \omega$. It is clear that if $x = (\beta, n)$, then x is isolated in $X^{(\beta)}$ since $A_x \cap X^{(\beta)} = \{x\}$. We must check that these are the only isolated points in $X^{(\beta)}$.

Let $y = (\alpha, m)$ with $\alpha > \beta$, and let $y_1, \ldots, y_k, z_1, \ldots, z_l$ be distinct points of X with $y \in U = (A_{y_1} \cap \cdots \cap A_{y_k}) - (A_{z_1} \cup \cdots \cup A_{z_l})$, a basic neighborhood of y. But if $n \in (f(y_1) \cap \cdots \cap f(y_k)) - (f(z_1) \cup \cdots \cup f(z_l))$, then clearly $(\beta, n) \in U$, and since there are infinitely many such n, y cannot be isolated. This completes the proof.

Note added in proof (19 November 1986)

(1) Using methods developed by S. Todorčević, Boban Veličković has shown that the existence of a function with property Δ follows from Jensen's principle $\Box_{\omega_1}$.

(2) Those familiar with Velleman's simplified morasses will recognize that the forcing to produce a function with property Δ is really just adjoining a certain kind of simplified morass.

(3) We are grateful to Petr Simon for calling our attention to the paper "Long chains in Rudin–Frolik order" by Eva Butkovičová in *Comment. Math. Univ. Carol.* 24 (1983) 563–569, in which she obtains implicitly Theorem 10.1 of this paper.

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