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Universal forcing notions and ideals

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Abstract Our main result states that a finite iteration of Universal Meager forcing notions adds generic filters for many forcing notions determined by universality parameters. We also give some results concerning cardinal characteristics of the σ-ideals determined by those universality parameters.

Keywords ccc forcing notions · Sweet forcing notions · Universal forcing notions · Universality parameters · σ-ideals

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0 Introduction

One of the most striking differences between measure and category was discovered in Shelah [8] where it was proved that the Lebesgue measurability of $\mathbf{\Sigma}^1_3$ sets implies ω_1 is inaccessible in $\mathbf{L}$, while one can construct (in ZFC) a forcing notion $\mathbb{P}$ such that $\mathbf{V}^{\mathbb{P}} \models$ "projective subsets of $\mathbb{R}$ have the Baire property". For the latter result one builds a homogeneous ccc forcing notion adding a lot of Cohen reals. Homogeneity is obtained by multiple use of amalgamation (see [4] for a full explanation of how this works), the Cohen reals come from compositions with the Universal Meager forcing notion $\mathbb{UM}$ or with the Hechler forcing notion $\mathbb{D}$. The main point of that construction was isolating a strong version of ccc, so called *sweetness*, which is preserved in amalgamations. Later, Stern [11] introduced a weaker property, *topological sweetness*, which is also preserved in amalgamations. Sweet (i.e., strong ccc) properties of forcing notions were further investigated in [6], where we introduced a new property called *iterable sweetness* (see [6, Definition 4.2.1]) and we proved the following two results.

Theorem 1 1. (*See [6, Proposition 4.2.2]) If $\mathbb{P}$ is a sweet ccc forcing notion* (*in the sense of [8, Definition 7.2]*) *in which any two compatible elements have a least upper bound, then $\mathbb{P}$ is iterably sweet.*
2. (*See [6, Theorem* 4.2.4*]) If $\mathbb{P}$ is a topologically sweet forcing notion* (*in the sense of Stern [11, Definition* 1.2*]*) *and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name for an iterably sweet forcing, then the composition $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is topologically sweet.*

In [6, Sect. 2.3] we introduced a scheme of building forcing notions from so called *universality parameters* (see Definition 2 later). We proved that typically they are sweet (see [6, Proposition 4.2.5]) and in natural cases also iterably sweet. So the question arose if the use of those forcing notions in iterations gives us something really new. Specifically, we asked:

Problem 1 (See [6, Problem 5.5]) Is there a universality parameter $\mathfrak{p}$ satisfying the requirements of [6, Proposition 4.2.5(3)] such that no finite iteration of the Universal Meager forcing notion adds a $\mathbb{Q}^{\rm tree}(\mathfrak{p})$-generic real?

Bad news is that Problem 1 has a partially negative answer: if the universality parameter $\mathfrak{p}$ satisfies some mild conditions (i.e., is *regular*, see Definition 8), then finite iteration of $\mathbb{UM}$ will add a generic filter for the corresponding forcing notion, see Corollary 1.

Good news is that we have more examples of iterably sweet forcings and they are presented in a subsequent paper [7].

The structure of the present paper is as follows. In the first section we recall in a simplified form all the definitions and results we need from [6], and we define *regular universality parameters*. We also re-present the canonical examples we keep in mind in this context. In the second section we prove our main result: a sequence Cohen real–dominating real–Cohen real produces generic filters for forcing notions $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ determined by regular $\mathfrak{p}$ (see Theorem 2). In the following section we look at the σ-ideals $\mathcal{I}_{\mathfrak{p}}$ for regular $\mathfrak{p}$ and we prove a couple of inequalities concerning their cardinal characteristics.

Notation Our notation is rather standard and compatible with that of classical textbooks (like Jech [3] or Bartoszyński and Judah [1]). In forcing we keep the older convention that *a stronger condition is the larger one*. Our main conventions are listed below.

1. For a forcing notion $\mathbb{P}$, all $\mathbb{P}$-names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$). The complete Boolean algebra determined by $\mathbb{P}$ is denoted by $\mathrm{RO}(\mathbb{P})$.
2. For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.
3. A *tree* is a family T of finite sequences closed under initial segments. For a tree T, the family of all ω-branches through T is denoted by $[T]$, and we let
$$\max(T) \stackrel{\text{def}}{=} \{\nu \in T : \text{ there is no } \rho \in T \text{ such that } \nu \vartriangleleft \rho\}.$$
If η is a node in the tree T and $n < \omega$, then
$$\begin{aligned} \mathrm{succ}_T(\eta) &= \{\nu \in T : \eta \vartriangleleft \nu \ \&\ \mathrm{lh}(\nu) = \mathrm{lh}(\eta) + 1\} \text{ and} \\ T^{[\eta]} &= \{\nu \in T : \eta \trianglelefteq \nu \text{ or } \nu \trianglelefteq \eta\}, \\ T{\restriction}n &= \{\nu \in T : \mathrm{lh}(\nu) \le n\}. \end{aligned}$$
4. The Cantor space 2^ω and the Baire space ω^ω are the spaces of all functions from ω to 2, ω, respectively, equipped with the natural (Polish) topology.
5. The quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ are abbreviations for
$$(\exists m \in \omega)(\forall n > m) \quad \text{and} \quad (\forall m \in \omega)(\exists n > m),$$
respectively. For $f, g \in \omega^\omega$ we write $f <^* g$ ($f \le^* g$, respectively) whenever $(\forall^\infty n \in \omega)(f(n) < g(n))$ ($(\forall^\infty n \in \omega)(f(n) \le g(n))$, respectively).
6. $\mathbb{R}^{\ge 0}$ stands for the set of non-negative reals.

Basic convention In this paper, $\mathbf{H}$ is a function from ω to $\omega \setminus 2$ and
$$\mathcal{T} = \bigcup_{i<\omega} \prod_{j<i} \mathbf{H}(j) \quad \text{and} \quad \mathcal{X} = [\mathcal{T}]$$
and $\mathcal{T}{\restriction}n = \bigcup_{i \le n} \prod_{j<i} \mathbf{H}(j)$ (for $n<\omega$). The space $\mathcal{X}$ is equipped with the natural (Polish) product topology.

1 Regular universality parameters

Since our main result applies to a somewhat restricted class of universal parameters of [6, Sect. 2.3], we adopt here a simplified version of the definition of universality parameters (it fits better the case we cover). The main difference between our Definition 2 and [6, Definition 2.3.3] is that we work in the setting of complete tree creating pairs (so we may ignore (K, Σ) and just work with trees) and $\mathcal{F}^{\mathfrak{p}}$ is assumed to be a singleton (so we also ignore it incorporating its function into $\mathcal{G}^{\mathfrak{p}}$). This simplification should increase clarity, but we still include particular examples from [6, Sect. 2.4] (see Definitions 5, 6 at the end of this section).

Definition 1 1. *A finite* $\mathbf{H}$*-tree* is a tree $S \subseteq \mathcal{T}\restriction N$ for some $N < \omega$ such that $\max(S) \subseteq \prod_{i<N} \mathbf{H}(i)$. The integer N may be called *the level of the tree* S and it will be denoted by $\mathrm{lev}(S)$.
2. *An infinite* $\mathbf{H}$*-tree* is a tree $T \subseteq \mathcal{T}$ with $\max(T) = \emptyset$.
3. The family of all finite $\mathbf{H}$-trees will be denoted by $\mathrm{FT}[\mathbf{H}]$, and the set of all infinite $\mathbf{H}$-trees will be called $\mathrm{IFT}[\mathbf{H}]$.

Definition 2 (Compare [6, Definition 2.3.3]) *A simplified universality parameter* $\mathfrak{p}$ *for* $\mathbf{H}$ is a pair $(\mathcal{G}^{\mathfrak{p}}, F^{\mathfrak{p}}) = (\mathcal{G}, F)$ such that

(α) elements of $\mathcal{G}$ are triples $(S, n_{\mathrm{dn}}, n_{\mathrm{up}})$ such that S is a finite $\mathbf{H}$-tree and $n_{\mathrm{dn}} \le n_{\mathrm{up}} \le \mathrm{lev}(S)$, $(\{\langle\rangle\}, 0, 0) \in \mathcal{G}$;
(β) **if** $(S^0, n^0_{\mathrm{dn}}, n^0_{\mathrm{up}}) \in \mathcal{G}$, S^1 is a finite $\mathbf{H}$-tree, $\mathrm{lev}(S^0) \le \mathrm{lev}(S^1)$, and $n^1_{\mathrm{dn}} \le n^0_{\mathrm{dn}}$, $n^0_{\mathrm{up}} \le n^1_{\mathrm{up}} \le \mathrm{lev}(S^1)$, and $S^1\restriction\mathrm{lev}(S^0) \subseteq S^0$, **then** $(S^1, n^1_{\mathrm{dn}}, n^1_{\mathrm{up}}) \in \mathcal{G}$,
(γ) $F \in \omega^\omega$ is increasing,
(δ) **if**
- $(S^\ell, n^\ell_{\mathrm{dn}}, n^\ell_{\mathrm{up}}) \in \mathcal{G}$ (for $\ell < 2$), $\mathrm{lev}(S^0) = \mathrm{lev}(S^1)$,
- $S \in \mathrm{FT}[\mathbf{H}]$, $\mathrm{lev}(S) < \mathrm{lev}(S^\ell)$, and $S^\ell\restriction\mathrm{lev}(S) \subseteq S$ (for $\ell < 2$),
- $\mathrm{lev}(S) < n^0_{\mathrm{dn}}$, $n^0_{\mathrm{up}} < n^1_{\mathrm{dn}}$, $F(n^1_{\mathrm{up}}) < \mathrm{lev}(S^1)$,

then there is $(S^*, n^*_{\mathrm{dn}}, n^*_{\mathrm{up}}) \in \mathcal{G}$ such that
- $n^*_{\mathrm{dn}} = n^0_{\mathrm{dn}}$, $n^*_{\mathrm{up}} = F(n^1_{\mathrm{up}})$, $\mathrm{lev}(S^*) = \mathrm{lev}(S^0) = \mathrm{lev}(S^1)$, and
- $S^0 \cup S^1 \subseteq S^*$ and $S^*\restriction\mathrm{lev}(S) = S$.

Definition 3 (Compare [6, Definition 2.3.5]) Let $\mathfrak{p} = (\mathcal{G}, F)$ be a simplified universality parameter for $\mathbf{H}$.

1. We say that an infinite $\mathbf{H}$-tree T is $\mathfrak{p}$*-narrow* if for infinitely many $n < \omega$, for some $n = n_{\mathrm{dn}} < n_{\mathrm{up}}$ we have that $(T\restriction(n_{\mathrm{up}} + 1), n_{\mathrm{dn}}, n_{\mathrm{up}}) \in \mathcal{G}$.
2. We define a forcing notion $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$:
A condition in $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ is a pair $p = (N^p, T^p)$ such that $N^p < \omega$ and T^p is an infinite $\mathfrak{p}$-narrow $\mathbf{H}$-tree.
The order $\le$ on $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ is given by:
$(N^0, T^0) \le (N^1, T^1)$ if and only if

$$N^0 \le N^1,\ T^0 \subseteq T^1 \quad \text{and} \quad T^1\restriction N^0 = T^0\restriction N^0.$$

Proposition 1 (Compare [6, Proposition 2.3.6]) *Assume* $\mathfrak{p}$ *is a simplified universality parameter.*

1. *If* $T_0, T_1 \in \mathrm{IFT}[\mathbf{H}]$ *are* $\mathfrak{p}$*-narrow, then* $T_0 \cup T_1 \in \mathrm{IFT}[\mathbf{H}]$ *is* $\mathfrak{p}$*-narrow.*

2. $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ *is a Borel σ-centered forcing notion in which the compatibility relation is Borel as well.*

Definition 4 (Compare [6, Definition 3.2.1]) Let $\mathfrak{p} = (\mathcal{G}, F)$ be a simplified universality parameter for $\mathbf{H}$.

1. We say that $\mathfrak{p}$ *is suitable* whenever:
   (a) for every $n < \omega$, there is $N > n$ such that
   if $(S, n_{\mathrm{dn}}, n_{\mathrm{up}}) \in \mathcal{G}$, $N \le n_{\mathrm{dn}}$ and $\eta \in \prod_{i<n} \mathbf{H}(i)$,
   then $(\exists \nu \in \prod_{i<\mathrm{lev}(S)} \mathbf{H}(i))(\eta \vartriangleleft \nu \;\&\; \nu \notin S)$, and
   (b) for every $n < \omega$, there is $N > n$ such that
   if S is a finite $\mathbf{H}$-tree, $\mathrm{lev}(S) = n$, $\eta \in \prod_{i<N} \mathbf{H}(i)$ and $\eta{\restriction}n \in S$,
   then there is $(S^*, n_{\mathrm{dn}}, n_{\mathrm{up}}) \in \mathcal{G}$ such that $n < n_{\mathrm{dn}} \le n_{\mathrm{up}} < N$, $S \subseteq S^*$, $S^* \cap \prod_{i<\mathrm{lev}(S)} \mathbf{H}(i) = \max(S)$ and $\eta \in S^*$.
2. We say that a closed set $A \subseteq \mathcal{X}$ is $\mathfrak{p}$-narrow if the corresponding infinite $\mathbf{H}$-tree T (i.e., $A = [T]$) is $\mathfrak{p}$-narrow.
3. $\mathcal{I}^0_{\mathfrak{p}}$ is the ideal generated by $\mathfrak{p}$-narrow closed subsets of $\mathcal{X}$.
4. $\mathcal{I}_{\mathfrak{p}}$ is the σ-ideal generated by $\mathcal{I}^0_{\mathfrak{p}}$.
5. $\underset{\sim}{T}_{\mathfrak{p}}$ is a $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$-name such that $\Vdash_{\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})}$ "$\underset{\sim}{T}_{\mathfrak{p}} = \bigcup\{T^p{\restriction}N^p : p \in \underset{\sim}{G}_{\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})}\}$".

Proposition 2 (Compare [6, Proposition 3.2.3]) *Let $\mathfrak{p}$ be a suitable simplified universality parameter for* $\mathbf{H}$.

1. *Every set in $\mathcal{I}^0_{\mathfrak{p}}$ is nowhere dense in $\mathcal{X}$; all singletons belong to $\mathcal{I}^0_{\mathfrak{p}}$.*
2. *$\mathcal{I}_{\mathfrak{p}}$ is a proper Borel σ-ideal of subsets of $\mathcal{X}$.*
3. *In $\mathbf{V}^{\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})}$, $\underset{\sim}{T}_{\mathfrak{p}}$ is an infinite $\mathfrak{p}$-narrow $\mathbf{H}$-tree.*

Let us recall some of the examples of universality parameters from [6]. We represent them in a somewhat modified form to fit the simplified setting here.

Definition 5 (Compare [6, Example 2.4.9]) Let $g \in \omega^\omega$ and $\mathbf{F} : \mathrm{FT}[\mathbf{H}] \longrightarrow \mathbb{R}^{\ge 0}$ and $A \in [\omega]^\omega$. We define $\mathcal{G}^{g,A}_{\mathbf{F}}$ as the family consisting of $(\{\langle\rangle\}, 0, 0)$ and of all triples $(S, n_{\mathrm{dn}}, n_{\mathrm{up}})$ such that

(α) S is a finite $\mathbf{H}$-tree, $n_{\mathrm{dn}} < n_{\mathrm{up}} \le \mathrm{lev}(S)$, and
(β) $\big(\forall \nu \in S \cap \prod\limits_{i<n_{\mathrm{dn}}} \mathbf{H}(i)\big)\big(\exists \eta \in \prod\limits_{i<\mathrm{lev}(S)} \mathbf{H}(i)\big)\big(\nu \vartriangleleft \eta \;\&\; \eta \notin S\big)$,

and such that for some sequence $\langle Y_i : i \in A \cap [n_{\mathrm{dn}}, n_{\mathrm{up}})\rangle$ we have

(γ) $Y_i \in \mathrm{FT}[\mathbf{H}]$, $\mathrm{lev}(Y_i) = i+1$, $\mathbf{F}(Y_i) \le g(i)$ (for all $i \in A \cap [n_{\mathrm{dn}}, n_{\mathrm{up}})$), and
(δ) $\big(\forall \eta \in \max(S)\big)\big(\exists i \in A \cap [n_{\mathrm{dn}}, n_{\mathrm{up}})\big)\big(\eta{\restriction}(i+1) \in \max(Y_i)\big)$.

If $A = \omega$ then we may omit it and write $\mathcal{G}^g_{\mathbf{F}}$.

Proposition 3 *Let $A \in [\omega]^\omega$ and $F^A_{\mathbf{H}} \in \omega^\omega$ be an increasing function such that*

$$(\forall n < \omega)\big((n+1)^2 \cdot \prod_{i \le n} \mathbf{H}(i) < \big|A \cap (n, F^A_{\mathbf{H}}(n))\big|\big)$$

and let $g \in \omega^\omega$. If a function $\mathbf{F} : \mathrm{FT}[\mathbf{H}] \longrightarrow \mathbb{R}^{\geq 0}$ satisfies

$$\big(\forall S \in \mathrm{FT}[\mathbf{H}]\big)\big(|\max(S)| = 1 \;\Rightarrow\; \mathbf{F}(S) = 0\big),$$

then $(\mathcal{G}^{g,A}_{\mathbf{F}}, F_{\mathbf{H}})$ is a suitable simplified universality parameter.

Example 1 Let $g \in \omega^\omega$, $A \in [\omega]^\omega$ and let $F^A_{\mathbf{H}}$ be as in the assumptions of Proposition 3.

1. Let $\mathbf{F}_0, \mathbf{F}_1 : \mathrm{FT}[\mathbf{H}] \longrightarrow \mathbb{R}^{\geq 0}$ be defined by

$$\mathbf{F}_0(S) = \max\big(|\mathrm{succ}_S(s)| : s \in S\big) - 1 \quad \text{and} \quad \mathbf{F}_1(S) = |\max(S)| - 1$$

   (for $S \in \mathrm{FT}[\mathbf{H}]$). Then both $(\mathcal{G}^{g,A}_{\mathbf{F}_0}, F^A_{\mathbf{H}})$ and $(\mathcal{G}^{g,A}_{\mathbf{F}_1}, F^A_{\mathbf{H}})$ are suitable simplified universality parameters.
2. Let $\mathbf{F}_2 : \mathrm{FT}[\mathbf{H}] \longrightarrow \mathbb{R}^{\geq 0}$ be defined by $\mathbf{F}_2(\{\langle\rangle\}) = 0$ and

$$\mathbf{F}_2(S) = \Big|\big\{\eta\big(\mathrm{lev}(S) - 1\big) : \eta \in \max(S)\big\}\Big| - 1$$

   when $\mathrm{lev}(S) > 0$. Then $(\mathcal{G}^{g,A}_{\mathbf{F}_2}, F^A_{\mathbf{H}})$ is s suitable simplified universality parameter.
3. Suppose that (K, Σ) is a local tree creating pair for $\mathbf{H}$ (see [5, Sect. 1.3, Definition 1.4.3]) such that
   - for each $n < \omega$, $\eta \in \prod_{i<n} \mathbf{H}(i)$ and a non-empty set $X \subseteq \mathbf{H}(n)$, there is a unique tree creature $t_{\eta,X} \in K$ satisfying $\mathrm{pos}(t_{\eta,X}) = \{\eta^\frown\langle k\rangle : k \in X\}$,
   - if $n < \omega$, $\eta \in \prod_{i<n} \mathbf{H}(i)$, $X \subseteq \mathbf{H}(n)$ and $|X| = 1$, then $\mathbf{nor}[t_{\eta,X}] = 0$.

   For $S \in \mathrm{FT}[\mathbf{H}]$ let

$$\mathbf{F}_3(S) = \mathbf{F}^{K,\Sigma}_3(S) \stackrel{\mathrm{def}}{=} \max(\mathbf{nor}[t_\eta] : \eta \in \hat{S}),$$

   where $\langle t_\eta : \eta \in \hat{S}\rangle$ is the unique finite tree–candidate such that $\mathrm{pos}(t_\eta) = \mathrm{succ}_S(\eta)$ for $\eta \in \hat{S} = S \setminus \max(S)$. Then $(\mathcal{G}^{g,A}_{\mathbf{F}_3}, F^A_{\mathbf{H}})$ is a suitable simplified universality parameter.

Remark 1 The universality parameters from Example 1 are related to the PP-property and the strong PP–property (see [10, Chap. VI, 2.12*], compare also with [5, Sect. 7.2]). Note that if $A \in [\omega]^\omega$ and $g \in \omega^\omega$ then an infinite $\mathbf{H}$-tree T is $(\mathcal{G}^{g,A}_{\mathbf{F}_2}, F^A_{\mathbf{H}})$-narrow if and only if there exist sequences $\bar{w} = \langle w_i : i \in A\rangle$ and $\bar{n} = \langle n_k : k < \omega\rangle$ such that

- $\big(\forall i \in A\big)\big(w_i \subseteq \mathbf{H}(i)\ \&\ |w_i| \leq g(i) + 1\big)$, and
- $n_k < n_{k+1} < \omega$ for each $k < \omega$, and
- $\big(\forall \eta \in [T]\big)\big(\forall k < \omega\big)\big(\exists i \in A \cap [n_k, n_{k+1})\big)\big(\eta(i) \in w_i\big)$.

Definition 6 (Compare [6, Example 2.4.10]) Let $\mathcal{G}^{\rm cmz}_{\mathbf{H}}$ consist of $(\{\langle\rangle\},0,0)$ and of all triples $(S,n_{\rm dn},n_{\rm up})$ such that $S\in {\rm FT}[\mathbf{H}]$, $n_{\rm dn}\leq n_{\rm up}\leq {\rm lev}(S)$ and

$$\frac{|S\cap\prod_{i<n_{\rm up}}\mathbf{H}(i)|}{|\prod_{i<n_{\rm up}}\mathbf{H}(i)|}\leq\sum_{i=n_{\rm dn}}^{n_{\rm up}}\frac{1}{(i+1)^2}.$$

Proposition 4 1. *Let $F_{\mathbf{H}}$ be as $F^{\omega}_{\mathbf{H}}$ in Proposition 3. Then $\mathfrak{p}^{\rm cmz}_{\mathbf{H}}=(\mathcal{G}^{\rm cmz}_{\mathbf{H}},F_{\mathbf{H}})$ is a suitable simplified universality parameter.*
2. *An infinite $\mathbf{H}$-tree T is $\mathfrak{p}^{\rm cmz}_{\mathbf{H}}$-narrow - if and only if $[T]$ is of measure zero (with respect to the product measure on $\mathcal{X}$).*
3. *$\mathcal{I}_{\mathfrak{p}^{\rm cmz}_{\mathbf{H}}}$ is the σ-ideal of subsets of $\mathcal{X}$ generated by closed measure zero sets.*

Definition 7 1. *A coordinate-wise permutation for* $\mathbf{H}$ is a sequence $\bar\pi=\langle\pi_n: n<\omega\rangle$ such that (for each $n<\omega$) $\pi_n:\mathbf{H}(n)\longrightarrow\mathbf{H}(n)$ is a bijection. We say that such $\bar\pi$ is *an n-coordinate-wise permutation* if π_i is the identity for all $i>n$.
2. *A rational permutation for* $\mathbf{H}$ is an n-coordinate-wise permutation for $\mathbf{H}$ (for some $n<\omega$). The set of all n-coordinate-wise permutations for $\mathbf{H}$ will be called ${\rm rp}^n_{\mathbf{H}}$ and the set of all rational permutation will be denoted by ${\rm rp}_{\mathbf{H}}$ (so ${\rm rp}_{\mathbf{H}}=\bigcup_{n\in\omega}{\rm rp}^n_{\mathbf{H}}$).
3. Let $\bar\pi$ be a coordinate-wise permutation for $\mathbf{H}$. We will treat $\bar\pi$ as a bijection from $\bigcup_{n\leq\omega}\prod_{i<n}\mathbf{H}(i)$ onto $\bigcup_{n\leq\omega}\prod_{i<n}\mathbf{H}(i)$ such that for $\eta\in\prod_{i<n}\mathbf{H}(i)$ $(n\leq\omega)$ and $i<n$ we have $\bar\pi(\eta)(i)=\pi_i(\eta(i))$.

Definition 8 A simplified universality parameter $\mathfrak{p}=(\mathcal{G},F)$ for $\mathbf{H}$ is called *a regular universality parameter* whenever

(a) $\mathfrak{p}$ is suitable (see Definition 4(1)), and
(b) $\mathcal{G}$ is invariant under rational permutations, that is
if $\bar\pi\in{\rm rp}_{\mathbf{H}}$ and $(S,n_{\rm dn},n_{\rm up})\in\mathcal{G}$, then $(\bar\pi[S],n_{\rm dn},n_{\rm up})\in\mathcal{G}$.

Proposition 5 1. *Suppose that $A\in[\omega]^{\omega}$, $F^A_{\mathbf{H}}\in\omega^{\omega}$, $g\in\omega^{\omega}$ and a function $\mathbf{F}:{\rm FT}[\mathbf{H}]\longrightarrow\mathbb{R}^{\geq 0}$ are as in Proposition 3. Assume also that*

$$\big(\forall S\in{\rm FT}[\mathbf{H}]\big)\big(\forall\bar\pi\in{\rm rp}_{\mathbf{H}}\big)\big(\mathbf{F}(S)=\mathbf{F}(\bar\pi[S])\big).$$

Then $(\mathcal{G}^{g,A}_{\mathbf{F}},F^A_{\mathbf{H}})$ (see Definition 5) is a regular universality parameter.
2. *For $i=0,1,2$, $(\mathcal{G}^{g,A}_{\mathbf{F}_i},F^A_{\mathbf{H}})$ (defined in Example 1(1,2)) is a regular universality parameter. Also $\mathfrak{p}^{\rm cmz}_{\mathbf{H}}$ of Proposition 4(1) (see also Definition 6) is regular.*

From now on we will assume that all universality parameters under considerations are regular. The ideals associated with regular parameters are much nicer than those in the general case, and they are more directly connected with the respective universal forcing notions.

Lemma 1 *Let $\mathfrak{p}$ be a regular universality parameter. Suppose that $T\in{\rm IFT}[\mathbf{H}]$ is a $\mathfrak{p}$-narrow tree. Then there are a $\mathfrak{p}$-narrow tree $T^*\in{\rm IFT}[\mathbf{H}]$ and a strictly increasing sequence $\bar n=\langle n_k:k<\omega\rangle\subseteq\omega$ such that* *(a)* $T \subseteq T^*$ *and for every* $k < \omega$:

(b)$_k$ *if* $\nu_0, \nu_1 \in T^* \cap \prod_{i \le n_k} \mathbf{H}(i)$ *and* $\bar{\pi} \in \mathrm{rp}^{n_k}_{\mathbf{H}}$ *are such that* $\bar{\pi}(\nu_0) = \nu_1$, *then* $\bar{\pi}[(T^*)^{[\nu_0]}] = (T^*)^{[\nu_1]}$, *and*

(c)$_k$ *if a finite* $\mathbf{H}$*-tree* $S \in \mathrm{FT}[\mathbf{H}]$ *is such that*
- $\mathrm{lev}(S) = n_{k+1} + 1$, *and*
- *for all* $\nu_0 \in S \cap \prod_{i \le n_k} \mathbf{H}(i)$ *and* $\nu_1 \in T^* \cap \prod_{i \le n_k} \mathbf{H}(i)$ *and* $\bar{\pi} \in \mathrm{rp}^{n_k}_{\mathbf{H}}$ *such that* $\bar{\pi}(\nu_0) = \nu_1$ *we have:* $\bar{\pi}[S^{[\nu_0]}] \subseteq (T^*)^{[\nu_1]}$,

then $(S, n_k + 1, n_{k+1}) \in \mathcal{G}$.

Proof We will define n_k and $T^* \restriction (n_k + 1)$ inductively. We let $n_0 = 0$ and $T^* \restriction 1 = T \restriction 1$. Now suppose that n_k and $T^* \restriction (n_k + 1)$ have been already chosen. Let

$$T^+ = \bigcup \{\bar{\pi}[T] : \bar{\pi} \in \mathrm{rp}^{n_k}_{\mathbf{H}}\}.$$

It follows from Proposition 1(1) that T^+ is a $\mathfrak{p}$-narrow tree, so we may pick $n_{k+1} > n_k$ such that $\left(T^+ \restriction n_{k+1}, n_k + 1, n_{k+1}\right) \in \mathcal{G}$. We choose $T^* \restriction (n_{k+1} + 1)$ so that

$$T^* \cap \prod_{i \le n_{k+1}} \mathbf{H}(i) = \left\{\eta \in T^+ : \mathrm{lh}(\eta) = n_{k+1} + 1 \ \&\ \eta \restriction (n_k + 1) \in T^* \cap \prod_{i \le n_k} \mathbf{H}(i)\right\},$$

completing the inductive definition. Now it should be clear that $\bar{n}$ and T^* are as required.

Proposition 6 *Let* $\mathfrak{p}$ *be a regular universality parameter.*

1. *The ideal* $\mathcal{I}^0_{\mathfrak{p}}$ *is invariant under coordinate-wise permutations.*
2. *For every* $A \in \mathcal{I}_{\mathfrak{p}}$ *there is* $A^* \in \mathcal{I}^0_{\mathfrak{p}}$ *such that*

$$A \subseteq \bigcup \{\bar{\pi}[A^*] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}\}.$$

3. $\Vdash_{\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})}$ " $\underset{\sim}{T}_{\mathfrak{p}}$ *is a* $\mathfrak{p}$*-narrow tree such that for every closed set* $A \in \mathcal{I}^0_{\mathfrak{p}}$ *coded in* $\mathbf{V}$, *there is* $n < \omega$ *with* $A \subseteq \bigcup \{[\bar{\pi}[\underset{\sim}{T}_{\mathfrak{p}}]] : \bar{\pi} \in \mathrm{rp}^n_{\mathbf{H}}\}$".

Proof (3) It follows from Proposition 2(3) that $\Vdash_{\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})}$ " $\underset{\sim}{T}_{\mathfrak{p}}$ is $\mathfrak{p}$-narrow ".

Suppose now that $p = (N, T) \in \mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ and $A \subseteq \mathcal{X}$ is a closed set from $\mathcal{I}^0_{\mathfrak{p}}$. Pick $S \in \mathrm{IFT}[\mathbf{H}]$ such that

- $|S \cap \prod\limits_{i \le N} \mathbf{H}(i)| = 1$, say $S \cap \prod\limits_{i \le N} \mathbf{H}(i) = \{\nu_0\}$, and
- S is $\mathfrak{p}$-narrow, and
- $A \subseteq \bigcup \{[\bar{\pi}[S]] : \bar{\pi} \in \mathrm{rp}^N_{\mathbf{H}}\}$.

Now we may pick a condition $q \in \mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ stronger than p and such that $N^q = N$ and

$$\bigcup \{\bar{\pi}[S] : \bar{\pi} \in \mathrm{rp}^N_{\mathbf{H}} \ \&\ \bar{\pi}(\nu_0) \in T\} \subseteq T^q.$$

Then $q \Vdash_{\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})}$ " $A \subseteq \bigcup \{[\bar{\pi}[\underset{\sim}{T}_{\mathfrak{p}}]] : \bar{\pi} \in \mathrm{rp}^N_{\mathbf{H}}\}$ ".

2 Generic objects for regular universal forcing notions

In this section we present our main result: a sequence

Cohen real – dominating real – Cohen real

produces generic filters for forcing notions $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ determined by regular universality parameters $\mathfrak{p}$.

Theorem 2 *Let $\mathfrak{p} = (\mathcal{G}, F)$ be a regular universality parameter for* $\mathbf{H}$.

1. *Suppose that $\mathbf{V} \subseteq \mathbf{V}^* \subseteq \mathbf{V}^{**}$ are universes of set theory, $\mathfrak{p} \in \mathbf{V}$, $T \in \mathbf{V}^*$ and $c \in \omega^\omega \cap \mathbf{V}^{**}$ are such that*
   (a) *$T \in \mathrm{IFT}[\mathbf{H}]$ is a $\mathfrak{p}$-narrow tree such that for every closed set $A \in \mathcal{I}^0_{\mathfrak{p}}$ coded in $\mathbf{V}$, there is $n < \omega$ with $A \subseteq \bigcup\{[\bar\pi[T]] : \bar\pi \in \mathrm{rp}^n_{\mathbf{H}}\}$, and*
   (b) *c is a Cohen real over $\mathbf{V}^*$.*
   Then, in $\mathbf{V}^{**}$, there is a generic filter $G \subseteq \left(\mathbb{Q}^{\rm tree}(\mathfrak{p})\right)^{\mathbf{V}}$ over $\mathbf{V}$.
2. *Suppose that $\mathbf{V} \subseteq \mathbf{V}^* \subseteq \mathbf{V}^{**}$ are universes of set theory, $\mathfrak{p} \in \mathbf{V}$, $c \in \omega^\omega \cap \mathbf{V}^*$ and $d \in \omega^\omega \cap \mathbf{V}^{**}$ are such that*
   (a) *c is a Cohen real over $\mathbf{V}$, and*
   (b) *d is dominating over $\mathbf{V}^*$.*
   Then, in $\mathbf{V}^{**}$, there is a $\mathfrak{p}$-narrow tree $T \in \mathrm{IFT}[\mathbf{H}]$ such that for every closed set $A \in \mathcal{I}^0_{\mathfrak{p}}$ coded in $\mathbf{V}$, there is $n < \omega$ with $A \subseteq \bigcup\{[\bar\pi[T]] : \bar\pi \in \mathrm{rp}^n_{\mathbf{H}}\}$.

Proof (1) The proof essentially follows the lines of the proof of a similar result for the Universal Meager forcing notion, see Truss [12, Lemma 6.4]. So suppose that T, c are as in the assumptions. Let $\bar n = \langle n_k : k < \omega\rangle, T^* \in \mathbf{V}^*$ be as given by Lemma 1 for T (so they satisfy Lemma 1(a–c)).

Consider the following forcing notion $\mathbb{C}^* = \mathbb{C}^*(\bar n, T^*)$:
A condition in $\mathbb{C}^*$ is a finite $\mathbf{H}$-tree S such that $\mathrm{lev}(S) = n_k + 1$ for some $k < \omega$.
The order relation $\leq_{\mathbb{C}^*}$ on $\mathbb{C}^*$ is given by:
$S_0 \leq_{\mathbb{C}^*} S_1$ if and only if $S_0 \subseteq S_1$ and $S_1{\restriction}\mathrm{lev}(S_0) = S_0$, and

($\otimes$) *if* $\mathrm{lev}(S_0) = n_k + 1$, $\mathrm{lev}(S_1) = n_\ell + 1$, $\nu_0 \in \max(S_0)$, $\nu_1 \in T^* \cap \prod_{i \leq n_k} \mathbf{H}(i)$ and $\bar\pi \in \mathrm{rp}^{n_k}_{\mathbf{H}}$ are such that $\bar\pi(\nu_1) = \nu_0$,
then $\bar\pi[(T^*)^{[\nu_1]}{\restriction}(n_\ell + 1)] \subseteq (S_1)^{[\nu_0]}$.

Plainly, $\mathbb{C}^*$ is a countable atomless forcing notion, so it is equivalent to the Cohen forcing $\mathbb{C}$. Therefore the Cohen real c determines a generic filter $G^c \subseteq \mathbb{C}^*$ over $\mathbf{V}^*$. Letting $T^c = \bigcup G^c$ we get an infinite $\mathbf{H}$-tree, $T^c \in \mathbf{V}^{**}$. By an easy density argument, for infinitely many $k < \omega$, for each $\nu_0 \in T^c \cap \prod_{i \leq n_k} \mathbf{H}(i)$, $\nu_1 \in T^* \cap \prod_{i \leq n_k} \mathbf{H}(i)$ and $\bar\pi \in \mathrm{rp}^{n_k}_{\mathbf{H}}$ such that $\bar\pi(\nu_0) = \nu_1$ we have

$$(T^*)^{[\nu_1]}{\restriction}(n_{k+1} + 1) = \bar\pi[(T^c)^{[\nu_0]}]{\restriction}(n_{k+1} + 1).$$

Hence T^c is $\mathfrak{p}$-narrow (remember Lemma 1(c)). Also, because of the definition of the order,

($\circledast$) if $S \in G^c$, $\mathrm{lev}(S) = n_k + 1$, then for every $\nu_0 \in \max(S)$, $\nu_1 \in T^* \cap \prod_{i \leq n_k} \mathbf{H}(i)$ and $\bar\pi \in \mathrm{rp}^{n_k}_{\mathbf{H}}$ such that $\bar\pi(\nu_1) = \nu_0$ we have $\bar\pi[(T^*)^{[\nu_1]}] \subseteq T^c$.

Suppose now that $D \in \mathbf{V}$ is an open dense subset of $\left(\mathbb{Q}^{\text{tree}}(\mathfrak{p})\right)^{\mathbf{V}}$. In $\mathbf{V}^*$ we define $C_D \subseteq \mathbb{C}^*$ as the collection of all $S \in \mathbb{C}^*$ such that for some $k < \omega$ and $T' \in \text{IFT}[\mathbf{H}] \cap \mathbf{V}$ we have

- $(n_k, T') \in D$, and $T' \restriction (n_k + 1) = S$, and
- $T' \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \text{rp}^{n_k}_{\mathbf{H}}\}$.

Claim 1 $C_D \in \mathbf{V}^*$ is an open dense subset of $\mathbb{C}^*$.

Proof Working in $\mathbf{V}^*$, let $S_0 \in \mathbb{C}^*$, $n_{k_0} = \text{lev}(S_0) - 1$. Pick an infinite $\mathbf{H}$-tree $S^+ \in \text{IFT}[\mathbf{H}]$ such that $S^+ \restriction \text{lev}(S_0) = S_0$ and

if $\eta \in S^+$, $\nu \in T^*$ and $\text{lh}(\eta) = \text{lh}(\nu) = n_{k_0} + 1$ and $\bar{\pi} \in \text{rp}^{n_{k_0}}_{\mathbf{H}}$ are such that $\bar{\pi}(\nu) = \eta$,
then $(S^+)^{[\eta]} = \bar{\pi}[(T^*)^{[\nu]}]$.

Plainly, S^+ is $\mathfrak{p}$-narrow. In $\mathbf{V}$, take a maximal antichain $\mathcal{A} \subseteq D$ of $\mathbb{Q}^{\text{tree}}(\mathfrak{p})^{\mathbf{V}}$ such that $N^p > n_{k_0}$ for each $p \in \mathcal{A}$. It follows from Proposition 1 that then also $\mathcal{A}$ is a maximal antichain of $\mathbb{Q}^{\text{tree}}(\mathfrak{p})^{\mathbf{V}^*}$ in $\mathbf{V}^*$. Therefore some condition $p = (N^p, T^p) \in \mathcal{A}$ is compatible with $(n_{k_0} + 1, S^+) \in \left(\mathbb{Q}^{\text{tree}}(\mathfrak{p})\right)^{\mathbf{V}^*}$. Note that then ($N^p > n_{k_0}$ and)

$$T^p \restriction (n_{k_0} + 1) = S^+ \restriction (n_{k_0} + 1) = S_0$$

and $S^+ \cap \prod_{i<N^p} \mathbf{H}(i) \subseteq T^p \cap \prod_{i<N^p} \mathbf{H}(i)$. Take $k < \omega$ such that $n_k > N^p > n_{k_0}$ and $T^p \subseteq \bigcup\{\bar{\pi}[T] : \bar{\pi} \in \text{rp}^{n_k}_{\mathbf{H}}\}$ (remember the assumption Theorem 2(1)(a) on T). Let

$$S_1 \stackrel{\text{def}}{=} (S^+ \cup T^p) \restriction (n_k + 1) \in \text{FT}[\mathbf{H}].$$

Then $S_1 \in \mathbb{C}^*$ is a condition stronger than S_0.

Note that $T^p, S_1 \in \mathbf{V}$, so we may find $T' \in \text{IFT}[\mathbf{H}] \cap \mathbf{V}$ such that:

- $T' \cap \prod_{i \leq n_k} \mathbf{H}(i) = \max(S_1)$, and
- if $\eta \in T^p \cap \prod_{i \leq n_k} \mathbf{H}(i)$, then $(T')^{[\eta]} = (T^p)^{[\eta]}$, and
- if $\eta \in (T' \setminus T^p) \cap \prod_{i \leq n_k} \mathbf{H}(i)$, then $(T')^{[\eta]} = \bar{\pi}[(T^p)^{[\nu]}]$ for some $\nu \in T^p \cap \prod_{i \leq n_k} \mathbf{H}(i)$ and $\bar{\pi} \in \text{rp}^{n_k}_{\mathbf{H}}$ such that $\bar{\pi}(\nu) = \eta$.

It follows from the choice of k and from the choice of T^* (remember Lemma 1(a,b)) that for each $\nu_0 \in T^p, \nu_1 \in T^*, \bar{\pi} \in \text{rp}^{n_k}_{\mathbf{H}}$ such that $\text{lh}(\nu_0) = \text{lh}(\nu_1) = n_k + 1$ and $\bar{\pi}(\nu_1) = \nu_0$ we have $(T^p)^{[\nu_0]} \subseteq \bar{\pi}[(T^*)^{[\nu_1]}]$. Therefore,

$$T' \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \text{pr}^{n_k}_{\mathbf{H}}\}.$$

It should also be clear that $(n_k, T') \in \left(\mathbb{Q}^{\text{tree}}(\mathfrak{p})\right)^{\mathbf{V}}$ is stronger than $p \in D$, and therefore it also belongs to D. Consequently, (n_k, T') witnesses that $S_1 \in C_D$, proving the density of C_D.

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To show that C_D is open suppose that $S_0 \in C_D$, $S_0 \leq_{\mathbb{C}^*} S_1 \in \mathbb{C}^*$. Let $\mathrm{lev}(S_0) = n_k + 1$, $\mathrm{lev}(S_1) = n_\ell + 1$ and let (n_k, T') witness that $S_0 \in C_D$. By the definition of the order of $\mathbb{C}^*$, $\bar{\pi}\big[(T^*)^{[\nu_1]}\restriction(n_\ell + 1)\big] \subseteq (S_1)^{[\nu_0]}$ whenever $\nu_0 \in \max(S_0)$, $\nu_1 \in T^* \cap \prod_{i\leq n_k} \mathbf{H}(i)$ and $\bar{\pi} \in \mathrm{rp}^{n_k}_{\mathbf{H}}$ is such that $\bar{\pi}(\nu_1) = \nu_0$. Since $T' \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \mathrm{pr}^{n_k}_{\mathbf{H}}\}$ and $T' \cap \prod_{i\leq n_k} \mathbf{H}(i) = \max(S_0)$, we may conclude that $T' \cap \prod_{i\leq n_\ell} \mathbf{H}(i) \subseteq \max(S_1)$. Consequently we may find $T'' \in \mathrm{IFT}[\mathbf{H}] \cap \mathbf{V}$ such that

- $T' \subseteq T''$ and $T'' \cap \prod\limits_{i\leq n_\ell} \mathbf{H}(i) = \max(S_1)$, and
- $T''\restriction(n_k + 1) = T'\restriction(n_k + 1)$, and
- $T'' \subseteq \bigcup\{\bar{\pi}[T'] : \bar{\pi} \in \mathrm{rp}^{n_\ell}_{\mathbf{H}}\} \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \mathrm{rp}^{n_\ell}_{\mathbf{H}}\}$.

Then easily $(n_\ell, T'') \in \big(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})\big)^{\mathbf{V}}$ is stronger than (n_k, T'), so it belongs to D and thus it witnesses that $S_1 \in C_D$.

Claim 2 Let

$$G \stackrel{\mathrm{def}}{=} \Big\{p \in \big(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})\big)^{\mathbf{V}} : T^p \subseteq T^c \ \&\ T^p\restriction(N^p + 1) = T^c\restriction(N^p + 1)\Big\} \in \mathbf{V}^{**}.$$

Then G is a generic filer in $\big(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})\big)^{\mathbf{V}}$ over $\mathbf{V}$.

Proof By Proposition 1(1), G is a directed subset of $\big(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})\big)^{\mathbf{V}}$. We need that $G \cap D \neq \emptyset$ for every open dense subset $D \in \mathbf{V}$ of $\big(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})\big)^{\mathbf{V}}$. So let $D \in \mathbf{V}$ be an open dense subset of $\big(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})\big)^{\mathbf{V}}$ and let C_D be as defined before Claim 1. It follows from Claim 1 that $G^c \cap C_D \neq \emptyset$, say $S \in G^c \cap C_D$. Then for some $k < \omega$ and $T' \in \mathrm{IFT}[\mathbf{H}] \cap \mathbf{V}$ we have

- $(n_k, T') \in D$, and
- $T' \cap \prod_{i\leq n_k} \mathbf{H}(i) = \max(S)$, and $T' \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \mathrm{pr}^{n_k}_{\mathbf{H}}\}$.

Now, by ($\circledast$), we may conclude that $T' \subseteq T^c$ getting $(n_k, T') \in G$.

(2) Suppose that c, d and $\mathbf{V} \subseteq \mathbf{V}^* \subseteq \mathbf{V}^{**}$ are as in the assumptions. In $\mathbf{V}$, consider the following forcing notion $\mathbb{C}^{**}$:
A condition in $\mathbb{C}^{**}$ is a pair $(\bar{n}, S)$ such that

(α) $\bar{n} = \langle n_i : i \leq k\rangle \subseteq \omega$ is a strictly increasing finite sequence (so $k < \omega$),
(β) $S \in \mathrm{FT}[\mathbf{H}]$ is a finite $\mathbf{H}$-tree such that $\mathrm{lev}(S) = n_k + 1$, and for $\ell < k$:
$(\gamma)_\ell$ if $\nu_0, \nu_1 \in S$, $\mathrm{lh}(\nu_0) = \mathrm{lh}(\nu_1) = n_\ell + 1$, and $\bar{\pi} \in \mathrm{rp}^{n_\ell}_{\mathbf{H}}$ is such that $\bar{\pi}(\nu_0) = \nu_1$, then $\bar{\pi}[S^{[\nu_0]}] = S^{[\nu_1]}$, and
$(\delta)_\ell$ **if**

- $T \in \mathrm{FT}[\mathbf{H}]$, $\mathrm{lev}(T) = n_{\ell+1} + 1$ and
- for each $\nu_0 \in S$, $\nu_1 \in T$, $\mathrm{lh}(\nu_0) = \mathrm{lh}(\nu_1) = n_\ell + 1$ and $\bar{\pi} \in \mathrm{rp}^{n_\ell}_{\mathbf{H}}$ such that $\bar{\pi}(\nu_0) = \nu_1$ we have $T^{[\nu_1]} \subseteq \bar{\pi}[S^{[\nu_0]}]$,

then there is $n < \omega$ such that $n_\ell + 1 < n \leq F(n) < n_{\ell+1}$ and $(T, n_\ell + 1, n) \in \mathcal{G}$.

The order relation $\leq_{\mathbb{C}^{**}}$ on $\mathbb{C}^{**}$ is the end-extension relation: $(\bar{n}^0, S^0) \leq_{\mathbb{C}^{**}} (\bar{n}^1, S^1)$ if and only if $\bar{n}^0 \trianglelefteq \bar{n}^1$, $S_0 \subseteq S_1$ and $S_1 \restriction \mathrm{lev}(S_0) = S_0$.

Since $\mathbb{C}^{**}$ is a countable atomless forcing notion, the Cohen real $c \in \mathbf{V}^*$ determines a generic filter $G^c \subseteq \mathbb{C}^{**}$ over $\mathbf{V}$, $G^c \in \mathbf{V}^*$. Put

$$\bar{n}^c = \bigcup\{\bar{n} \colon (\exists S)((\bar{n}, S) \in G^c)\} \in \mathbf{V}^* \quad \text{and} \quad T^c = \bigcup\{S \colon (\exists \bar{n})((\bar{n}, S) \in G^c)\} \in \mathbf{V}^*.$$

Then $\bar{n}^c = \langle n_i^c : i < \omega\rangle \subseteq \omega$ is strictly increasing and $T^c \in \mathrm{IFT}[\mathbf{H}]$, and

$$\Big(\forall \ell < \omega\Big)\Big(\big(T^c \restriction n_{\ell+1}, n_\ell + 1, n_{\ell+1}\big) \in \mathcal{G}\Big)$$

Note that if $\nu_0, \nu_1 \in T^c$, $\mathrm{lh}(\nu_0) = \mathrm{lh}(\nu_1) = n_\ell + 1$ and $\bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_\ell}$ is such that $\bar{\pi}(\nu_0) = \nu_1$, then $\bar{\pi}[(T^c)^{[\nu_0]}] = (T^c)^{[\nu_1]}$.

Since, in $\mathbf{V}^{**}$, there is a dominating real over $\mathbf{V}^*$, we may find $K^* = \{k_i^* : i < \omega\} \in [\omega]^\omega \cap \mathbf{V}^{**}$ (the enumeration is increasing) such that

$$(\forall K \in [\omega]^\omega \cap \mathbf{V}^*)(\forall^\infty i)(K \cap [k_i^*, k_{i+1}^*) \neq \emptyset).$$

Let A be the set of all $\eta \in \mathcal{X}$ such that

$$\big(\forall i < \omega\big)\big(\exists \ell \in [k_i^*, k_{i+1}^*)\big)\big(\eta \restriction (n_{\ell+1} + 1) \in \bigcup\{\bar{\pi}[T^c] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_\ell}\}\big).$$

Clearly, A is a closed subset of $\mathcal{X}$ (coded in $\mathbf{V}^{**}$). Let $T^* \in \mathrm{IFT}[\mathbf{H}] \cap \mathbf{V}^{**}$ be an infinite $\mathbf{H}$-tree such that $[T^*] = A$.

Claim 3 The tree T^* is $\mathfrak{p}$-narrow.

Proof Let $i < \omega$. For $\ell \in [k_i^*, k_{i+1}^*)$ let

$$T^\ell \stackrel{\text{def}}{=} \big\{\nu \in \mathcal{T} \restriction (n_{k_{i+1}^*} + 1) : \nu \restriction (n_{\ell+1} + 1) \in \bigcup\{\bar{\pi}[T^c] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_\ell}\}\big\}$$

and then let

$$S^i \stackrel{\text{def}}{=} \bigcup\{T^\ell : k_i^* \leq \ell < k_{i+1}^*\}$$

(so $T^\ell \in \mathrm{FT}[\mathbf{H}]$ and also $S^i \in \mathrm{FT}[\mathbf{H}]$). Note that for each ℓ as above, by $(\delta)_\ell$, there is n^ℓ such that $n_\ell + 1 < n^\ell \leq F(n^\ell) < n_{\ell+1}$ and $(T^\ell, n_\ell + 1, n^\ell) \in \mathcal{G}$. Thus we may use repeatedly Definition 2(δ) to conclude that $(S^i, n_{k_i^*} + 1, n_{k_{i+1}^*}) \in \mathcal{G}$. Since

$$T^* = \bigcap_{i<\omega}\{\eta \in \mathcal{T} : \eta \restriction (n_{k_{i+1}^*} + 1) \in S^i\}$$

we may easily finish the proof of the Claim.

Claim 4 For every $\mathfrak{p}$-narrow tree $T' \in \mathrm{IFT}[\mathbf{H}] \cap \mathbf{V}$ there is $k < \omega$ such that $T' \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_k}\}$.

Proof Let $T' \in \mathbf{V}$ be $\mathfrak{p}$-narrow.

Suppose $(\bar{n}^0, S^0) \in \mathbb{C}^{**}$, $\bar{n}^0 = \langle n_i^0 : i \leq k\rangle$. Since $T^+ \stackrel{\text{def}}{=} \bigcup\{\bar{\pi}[T'] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_k^0}\}$ is also a $\mathfrak{p}$-narrow tree, we may find $m > n_k^0 + 1$ such that $(T^+ \restriction (F(m)+3), n_k^0 + 1, m) \in \mathcal{G}$. Let $\bar{n}^1 = \bar{n}^0 {}^\frown \langle F(m)+2\rangle$, and let a finite $\mathbf{H}$-tree S^1 be such that

$$\max(S^1) = \left\{ \eta \in T^+ \cap \prod_{i<F(m)+3} \mathbf{H}(i) : \eta \restriction (n_k^0 + 1) \in S^0 \right\}.$$

It should be clear that $(\bar{n}^1, S^1) \in \mathbb{C}^{**}$ is a condition stronger than $(\bar{n}^0, S^0)$.

Using the above considerations we may employ standard density arguments to conclude that the set

$$K_{T'} \stackrel{\text{def}}{=} \Big\{ \ell < \omega : \text{for all } \nu_0 \in T^c, \nu_1 \in T' \text{ such that } \mathrm{lh}(\nu_0) = \mathrm{lh}(\nu_1) = n_\ell + 1, \\ \bigcup\{\bar{\pi}[(T')^{[\nu_1]} \cap \prod_{i \leq n_{\ell+1}} \mathbf{H}(i)] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_\ell} \;\&\; \bar{\pi}(\nu_1) = \nu_0\} \subseteq T^c \Big\}$$

is infinite (and, of course, $K_{T'} \in \mathbf{V}^*$). Therefore, by the choice of the set $K^* \in \mathbf{V}^{**}$, for some $N < \omega$ we have $(\forall i \geq N)(K_{T'} \cap [k_i^*, k_{i+1}^*) \neq \emptyset)$. Thus, for each $i \geq N$ we may find $\ell \in [k_i^*, k_{i+1}^*)$ such that

if $\nu_0 \lhd \eta \in T'$, $\nu_1 \in T^c$, $\mathrm{lh}(\nu_0) = \mathrm{lh}(\nu_1) = n_\ell + 1$, $\mathrm{lh}(\eta) = n_{\ell+1} + 1$, and $\bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_\ell}$ is such that $\bar{\pi}(\nu_0) = \nu_1$,
then $\bar{\pi}(\eta) \in T^c$.

Hence we may conclude that

$$(\forall i \geq N)(\exists \ell \in [k_i^*, k_{i+1}^*)) \left(T' \cap \prod_{i \leq n_{\ell+1}} \mathbf{H}(i) \subseteq \bigcup\{\bar{\pi}[T^c] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_\ell}\} \right)$$

and therefore $T' \subseteq \bigcup\{\bar{\pi}[T^*] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n_N}\}$ (just look at the choice of T^*).

The proof of Theorem 2 is complete.

Corollary 1 *Suppose that $\mathfrak{p}$ is a regular universality parameter for $\mathbf{H}$ and $\mathbb{P}$ is either the Hechler forcing (i.e., standard dominating real forcing) or the Universal Meager forcing (i.e., the amoeba for category forcing). Then $\mathbb{Q}^{\text{tree}}(\mathfrak{p})$ can be completely embedded into* $\mathrm{RO}(\mathbb{P} * \underset{\sim}{\mathbb{P}})$.

Remark 2 Let $\mathfrak{p}$ be a regular universality parameter. By the argument used in the proof of Proposition 7 one may show the following observation.

If $\mathbf{V} \subseteq \mathbf{V}^*$ *are universes of set theory,* $\mathfrak{p} \in \mathbf{V}$*, and* $T \in \mathbf{V}^*$ *is a* $\mathfrak{p}$*-narrow tree such that for every closed set* $A \in \mathcal{I}^0_{\mathfrak{p}}$ *coded in* $\mathbf{V}$*, there is* $n < \omega$ *with*

$$A \subseteq \bigcup \{[\bar{\pi}[T]] : \bar{\pi} \in \mathrm{rp}^n_{\mathbf{H}}\},$$

then, in $\mathbf{V}^*$*, there is a dominating real over* $\mathbf{V}$*.*

Consequently we may use Proposition 6 to conclude that $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ adds a dominating real. Since $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ is Borel ccc (see Proposition 1) we may use [9] to conclude that $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ adds also a Cohen real.

3 Ideals $\mathcal{I}_{\mathfrak{p}}$

Let us recall that for an ideal $\mathcal{I}$ of subsets of the space $\mathcal{X}$ we define cardinal coefficients of $\mathcal{I}$ as follows:

the *additivity of* $\mathcal{I}$ is $\mathbf{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \;\&\; \bigcup \mathcal{A} \notin \mathcal{I}\}$;
the *covering of* $\mathcal{I}$ is $\mathbf{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \;\&\; \bigcup \mathcal{A} = \mathcal{X}\}$;
the *cofinality of* $\mathcal{I}$ is $\mathbf{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \;\&\; (\forall B \in \mathcal{I})(\exists A \in \mathcal{A})(B \subseteq A)\}$;
the *uniformity of* $\mathcal{I}$ is $\mathbf{non}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{X} \;\&\; A \notin \mathcal{I}\}$.

The *dominating* and *unbounded* numbers are, respectively,

$$\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \;\&\; (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})(g \leq^* f)\}$$
$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \;\&\; (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})(f \not\leq^* g)\}.$$

Below, $\mathcal{M}$ denotes the σ-ideal of meager subsets of $\mathcal{X}$ (or of any other Polish perfect space).

For the rest of this section let us fix a regular universality parameter $\mathfrak{p} = (\mathcal{G}, F)$ for $\mathbf{H}$.

Corollary 2 $\mathbf{add}(\mathcal{M}) \leq \mathbf{add}(\mathcal{I}_{\mathfrak{p}})$.

Proof It follows from Theorem 2(2) that

$$\min\big(\mathfrak{b}, \mathbf{cov}(\mathcal{M})\big) \leq \mathbf{add}(\mathcal{I}_{\mathfrak{p}}).$$

By well known results of Miller and Truss we have $\min\big(\mathfrak{b}, \mathbf{cov}(\mathcal{M})\big) = \mathbf{add}(\mathcal{M})$ (see [1, Corollary 2.2.9]), so the corollary follows.

Corollary 3 $\mathbf{cof}(\mathcal{I}_{\mathfrak{p}}) \leq \mathbf{cof}(\mathcal{M})$.

Proof By a well known result of Fremlin we have $\mathbf{cof}(\mathcal{M}) = \max\big(\mathfrak{d}, \mathbf{non}(\mathcal{M})\big)$ (see [1, Theorem 2.2.11]). Thus it is enough to show that

$$\mathbf{cof}(\mathcal{I}_{\mathfrak{p}}) \leq \max\big(\mathfrak{d}, \mathbf{non}(\mathcal{M})\big).$$

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The above inequality follows by a standard "dualization argument" applied to Theorem 2(2). Let us, however, describe the main steps of the same proof presented in a combinatorial fashion. Let $\mathbb{C}^{**}$ be the forcing notion defined at the beginning of the proof of Theorem 2(2). Let

$$\mathcal{Y} \stackrel{\text{def}}{=} \Big\{(\bar{n}, T) \in \omega^\omega \times \mathrm{IFT}[\mathbf{H}] : \big(\forall k < \omega\big)\big((\bar{n}\restriction(k+1), T\restriction(n_k+1)) \in \mathbb{C}^{**}\big)\Big\}$$

be equipped with the natural Polish topology. Let $\kappa = \max\big(\mathfrak{d}, \mathbf{non}(\mathcal{M})\big)$ and choose sequences $\langle K_\alpha : \alpha < \kappa\rangle$ and $\langle(\bar{n}^\alpha, T^\alpha) : \alpha < \kappa\rangle$ so that

(i) $K_\alpha = \{k_i^\alpha : i \in \omega\} \in [\omega]^\omega$ (the enumeration is increasing),
(ii) $(\forall K \in [\omega]^\omega)(\exists \alpha < \kappa)(\forall^\infty i \in \omega)(K \cap (k_i^\alpha, k_{i+1}^\alpha) \neq \emptyset)$,
(iii) the set $\{(\bar{n}^\alpha, T^\alpha) : \alpha < \kappa\}$ is not meager (in $\mathcal{Y}$).

For $\alpha, \beta < \kappa$ and $N < \omega$ let

$$A^N_{\alpha,\beta} \stackrel{\text{def}}{=} \big\{\eta \in \mathcal{X} : \big(\forall i \geq N\big)\big(\exists \ell \in [k_i^\alpha, k_{i+1}^\alpha)\big)\big(\eta\restriction(n^\beta_{\ell+1}+1) \in \bigcup\{\bar{\pi}[T^\beta] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n^\beta_\ell}\}\big)\big\}.$$

Then:

$(*)_1$ Each $A^N_{\alpha,\beta}$ is a closed $\mathfrak{p}$-narrow subset of $\mathcal{X}$.

[Why? See the proof of Claim 3.]

$(*)_2$ For each $\mathfrak{p}$-narrow tree $T' \in \mathrm{IFT}[\mathbf{H}]$, there is $\beta < \kappa$ such that the set

$$K^\beta_{T'} \stackrel{\text{def}}{=} \Big\{\ell < \omega : \text{for all } \nu_0 \in T^\beta, \nu_1 \in T' \text{ such that } \mathrm{lh}(\nu_0) = \mathrm{lh}(\nu_1) = n^\beta_\ell + 1,$$
$$\bigcup\{\bar{\pi}[(T')^{[\nu_1]} \cap \prod_{i \leq n^\beta_{\ell+1}} \mathbf{H}(i)] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}^{n^\beta_\ell} \ \&\ \bar{\pi}(\nu_1) = \nu_0\} \subseteq T^\beta\Big\}$$

is infinite.

[Why? By (iii) and an argument similar to the one in the proof of Claim 4.]

$(*)_3$ For each $\mathfrak{p}$-narrow tree $T' \in \mathrm{IFT}[\mathbf{H}]$ there are $\alpha, \beta < \kappa$ and $N < \omega$ such that $[T'] \subseteq A^N_{\alpha,\beta}$.

[Why? By $(*)_2$ + (ii) and an argument as in the proof of Claim 4.]

Consequently, $\{A^N_{\alpha,\beta} : \alpha, \beta < \kappa \ \&\ N < \omega\}$ is a cofinal family in $\mathcal{I}^0_{\mathfrak{p}}$. Hence, by Proposition 6(2),

$$\Big\{\bigcup\{\bar{\pi}[A^0_{\alpha,\beta}] : \bar{\pi} \in \mathrm{rp}_{\mathbf{H}}\} : \alpha, \beta < \kappa\Big\}$$

is a basis of $\mathcal{I}_{\mathfrak{p}}$.

Proposition 7 $\mathbf{add}(\mathcal{I}_{\mathfrak{p}}) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \mathbf{cof}(\mathcal{I}_{\mathfrak{p}})$.

Proof Recall that $\mathcal{M} = \mathcal{M}(\mathcal{X})$ is the σ-ideal of meager subsets of $\mathcal{X}$. We are going define two functions

$$\phi^* : \mathcal{M} \longrightarrow \omega^\omega \quad \text{and} \quad \phi : \omega^\omega \longrightarrow \mathcal{I}^0_{\mathfrak{p}}.$$

First, for each $n < \omega$, pick a finite $\mathbf{H}$-tree $S_n \in \mathrm{FT}[\mathbf{H}]$ such that

(a) $\mathrm{lev}(S_n) > n$, $S_n \restriction n = \mathcal{T} \restriction n$,
(b) $(S_n, n, \mathrm{lev}(S_n)) \in \mathcal{G}$,

and let $m_n = \mathrm{lev}(S_n)$. Put $M_n = \max\{m_j : j \le n\}$ (for $n < \omega$). Now, for $f \in \omega^\omega$ let $f^* \in \omega^\omega$ be defined by $f^*(0) = f(0)$, $f^*(n+1) = M_{f^*(n)+f(n+1)}$, and let

$$\phi(f) = \big\{\eta \in \mathcal{X} : (\forall n < \omega)(\eta \restriction m_{f^*(2n)} \in S_{f^*(2n)})\big\}.$$

Note that, for any $f \in \omega^\omega$, f^* is strictly increasing and $\phi(f) \in \mathcal{I}^0_{\mathfrak{p}}$.

Now suppose that $B \subseteq \mathcal{X}$ is meager and let $T_n \in \mathrm{IFT}[\mathbf{H}]$ be such that $T_n \subseteq T_{n+1}$(for $n < \omega$) and each $[T_n]$ is nowhere dense (in $\mathcal{X}$) and $B \subseteq \bigcup\limits_{n<\omega} [T_n]$. Let $\phi^*(B) \in \omega^\omega$ be defined by letting $\phi^*(B)(0) = 0$ and

$$\phi^*(B)(n{+}1) = \min\Big\{k < \omega : k > M_{\phi^*(B)(n)} \ \& \ (\forall \eta \in \prod_{i \le M_{\phi^*(B)(n)}} \mathbf{H}(i))(\exists \nu \in \prod_{i<k} \mathbf{H}(i))(\eta \lhd \nu \notin T_{n+1})\Big\}.$$

Claim 5 If $f \in \omega^\omega$ and $B \subseteq \mathcal{X}$ is meager, and $(\exists^\infty n < \omega)(\phi^*(B)(n) < f(n))$, then $\phi(f) \setminus B \neq \emptyset$.

Proof Assume that $(\exists^\infty n < \omega)(\phi^*(B)(n) < f(n))$. Then the set

$$K = \big\{n < \omega : (\exists k < \omega)(f^*(2n) \le \phi^*(B)(k) < \phi^*(B)(k+1) < f^*(2n+2))\big\}$$

is infinite. [Why? Assume towards contradiction that K is finite. Then for some N we have $f^*(2n) \le \phi^*(B)(n+N)$ for all $n < \omega$. Take $n \ge N$ such that $\phi^*(B)(n+N) < f(n+N)$. Then $f^*(2n) < f(n+N) < f^*(n+N) \le f^*(2n)$, a contradiction.]

Now we may pick $\eta \in \mathcal{X}$ such that for each $n < \omega$ we have:

(i) $\eta \restriction m_{f^*(2n)} \in S_{f^*(2n)}$, and
(ii) if $n \in K$, then for some k such that

$$f^*(2n) \le \phi^*(B)(k) < \phi^*(B)(k+1) < f^*(2n+2)$$

we have $\eta \restriction \phi^*(B)(k+1) \notin T_{k+1}$.

It should be clear that the choice is possible; note that for n, k as in (ii) we have

$$f^*(2n) < \mathrm{lev}(S_{f^*(2n)}) = m_{f^*(2n)} \le M_{f^*(2n)} \le M_{\phi^*(B)(k)}.$$

The proposition follows from Claim 5: if $\mathcal{F} \subseteq \omega^\omega$ is an unbounded family, then $\bigcup\{\phi(f) : f \in \mathcal{F}\} \notin \mathcal{I}_\mathfrak{p}$, and if $\mathcal{B} \subseteq \mathcal{I}_\mathfrak{p}$ is a basis of $\mathcal{I}_\mathfrak{p}$, then $\{\phi^*(B) : B \in \mathcal{B}\}$ is a dominating family in ω^ω. (Remember $\mathcal{I}_\mathfrak{p} \subseteq \mathcal{M}$.)

It was shown in [2] that the additivity of the σ-ideal generated by closed measure zero sets (i.e., the one corresponding to $\mathfrak{p}^{\mathrm{cmz}}_{\mathbf{H}}$ of Definition 6) is $\mathbf{add}(\mathcal{M})$. We have a similar result for another specific case of $\mathcal{I}_\mathfrak{p}$:

Proposition 8 *Suppose that* $\mathbf{H} : \omega \longrightarrow \omega \setminus 2$ *is increasing and* $g : \omega \longrightarrow \omega \setminus 2$ *is such that* $g(n) + 1 < \mathbf{H}(n)$ *for all* $n < \omega$. *Let* $A \in [\omega]^\omega$ *and* $\mathfrak{p} = (\mathcal{G}^{g,A}_{\mathbf{F}_2}, F^A_{\mathbf{H}})$ *(see Example* 1(2)*). Then* $\mathbf{add}(\mathcal{I}_\mathfrak{p}) = \mathbf{add}(\mathcal{M})$.

Proof Since $\mathfrak{p}$ is a regular universality parameter (by Proposition 5), we know that $\mathbf{add}(\mathcal{M}) \le \mathbf{add}(\mathcal{I}_\mathfrak{p}) \le \mathfrak{b}$ (by Corollary 2 and Proposition 7). So for our assertion it is enough to show that $\mathbf{add}(\mathcal{I}_\mathfrak{p}) \le \mathbf{cov}(\mathcal{M})$.

Let us start with analyzing sets in $\mathcal{I}_\mathfrak{p}$. Suppose that $\bar{n}, \bar{w}$ are such that

$(\otimes)_0$ $\bar{n} = \langle n_k : k < \omega\rangle$ is a strictly increasing sequence of integers such that $A \cap [n_k, n_{k+1}) \neq \emptyset$ for each $k < \omega$,

$(\otimes)_1$ $\bar{w} = \langle w_i : i \in A\rangle$, $w_i \in [\mathbf{H}(i)]^{g(i)+1}$ for each $i \in A$.

Put

$$Z(\bar{n}, \bar{w}) \stackrel{\text{def}}{=} \left\{\eta \in \prod_{i<\omega} \mathbf{H}(i) : (\forall^\infty k < \omega)(\exists i \in A \cap [n_k, n_{k+1}))(\eta(i) \in w_i)\right\}.$$

It follows from Remark 1 that $Z(\bar{n}, \bar{w}) \in \mathcal{I}_\mathfrak{p}$. Moreover, for every $Z \in \mathcal{I}_\mathfrak{p}$ there are $\bar{n}, \bar{w}$ satisfying $(\otimes)_0 + (\otimes)_1$ and such that $Z \subseteq Z(\bar{n}, \bar{w})$ (by Remark 1 and Proposition 6(2)).

Claim 6 Suppose that $\bar{n}^\ell, \bar{w}^\ell$ satisfy $(\otimes)_0 + (\otimes)_1$ above (for $\ell = 0, 1$). Assume that $Z(\bar{n}^0, \bar{w}^0) \subseteq Z(\bar{n}^1, \bar{w}^1)$. Then $(\exists^\infty k < \omega)(\forall i \in A \cap [n^0_k, n^0_{k+1}))(w^0_i = w^1_i)$.

Proof If the assertion fails, then (as $|w^0_i| = |w^1_i| = g(i) + 1 < \mathbf{H}(i)$) we have $(\forall^\infty k < \omega)(\exists i \in A \cap [n^0_k, n^0_{k+1}))(w^0_i \setminus w^1_i \neq \emptyset)$. Consequently we may pick $\eta \in Z(\bar{n}^0, \bar{w}^0)$ such that $(\forall i \in A)(\eta(i) \notin w^1_i)$. Then $\eta \notin Z(\bar{n}^1, \bar{w}^1)$, contradicting $Z(\bar{n}^0, \bar{w}^0) \subseteq Z(\bar{n}^1, \bar{w}^1)$.

Claim 7 Suppose that $f : \omega \longrightarrow \omega \setminus 2$, $\kappa < \mathbf{add}(\mathcal{I}_\mathfrak{p})$ and $\{f_\alpha : \alpha < \kappa\} \subseteq \prod\limits_{i<\omega} f(i)$. Then there is a function $f^* \in \prod_{i<\omega} f(i)$ such that

$$\big(\forall \alpha < \kappa\big)\big(\exists^\infty i < \omega\big)\big(f^*(i) = f_\alpha(i)\big).$$

Proof Pick an increasing sequence $\langle a_i : i < \omega\rangle$ of members of A so that $f(i) < \left|[\mathbf{H}(a_i)]^{g(a_i)+1}\right|$ for all $i < \omega$). For each i fix a one-to-one mapping $\psi_i : f(i) \longrightarrow [\mathbf{H}(a_i)]^{g(a_i)+1}$. Now, for $\alpha < \kappa$, $k < \omega$ and $j \in A$ let

$$n^\alpha_k = a_k \quad \text{and} \quad w^\alpha_j = \begin{cases} \psi_i\big(f_\alpha(i)\big) & \text{if } j = a_i,\ i < \omega, \\ g(j) + 1 & \text{if } j \notin \{a_i : i < \omega\}. \end{cases}$$

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Then $\bar{n}^\alpha, \bar{w}^\alpha$ satisfy $(\otimes)_0 + (\otimes)_1$ above and thus $Z(\bar{n}^\alpha, \bar{w}^\alpha) \in \mathcal{I}_{\mathfrak{p}}$ (for all $\alpha < \kappa$). Since $\kappa < \mathbf{add}(\mathcal{I}_{\mathfrak{p}})$ we know that $\bigcup\limits_{\alpha<\kappa} Z(\bar{n}^\alpha, \bar{w}^\alpha) \in \mathcal{I}_{\mathfrak{p}}$ and therefore we may find $\bar{n}, \bar{w}$ such that they satisfy $(\otimes)_0 + (\otimes)_1$ and $(\forall \alpha < \kappa)\big(Z(\bar{n}^\alpha, \bar{w}^\alpha) \subseteq Z(\bar{n}, \bar{w})\big)$. It follows from Claim 6 that

$$\big(\forall \alpha < \kappa\big)\big(\exists^\infty k < \omega\big)\big(\psi_k(f_\alpha(k)) = w^\alpha_{a_k} = w_{a_k}\big).$$

Let $f^* \in \prod_{i<\omega} f(i)$ be such that if $k < \omega$ and $w_{a_k} \in \mathrm{Range}(\psi_k)$, then $\psi_k(f^*(k)) = w_k$. It should be clear that then f^* is as required.

The proposition follows now from Claim 7 and the inequality $\mathbf{add}(\mathcal{I}_{\mathfrak{p}}) \leq \mathfrak{b}$.

To generalize the above result to the ideals $\mathcal{I}_{\mathfrak{p}}$ (for a regular universality parameter $\mathfrak{p}$) one would like to know the answer to the following question.

Problem 2 Suppose that $\mathfrak{p}$ is a regular universality parameter for **H**. Does this imply $\mathbf{add}(\mathcal{I}_{\mathfrak{p}}) \leq \mathbf{cov}(\mathcal{M})$?

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