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Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I

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Abstract

The authors present a provably strongest form of Martin's axiom, called Martin's Maximum, and show its consistency. From it we derive the solutions to several classical problems in set theory, showing that $2^{\aleph_0} = \aleph_2$, the non-stationary ideal on ω_1 is $\aleph_2$-saturated, and several other results. We show as a consequence of our techniques that there can be no "nice" inner model of a supercompact cardinal. We generalize our results to cardinals above ω_1 to show, for example, the consistency of the statement "The non-stationary ideal on every regular cardinal κ is precipitous."

In this paper we present a provably maximal form of Martin's axiom ([M-So]) which we call Martin's Maximum. We show that it settles several classical questions in set theory, including the value of the continuum, Friedman's problem and the saturation of the non-stationary ideal on ω_1. We show that Martin's Maximum is consistent relative to the existence of a supercompact cardinal.

It is well-known ([So2]) that saturated ideals give rise to generic elementary embeddings. It was a widely held belief that the generic embedding had roughly the same consistency strength as the analogous non-generic embedding ([K1]). However the generic embedding associated with an $\aleph_2$-saturated ideal on ω_1 is analogous to an almost-huge embedding, which is much stronger than a supercompact cardinal. Thus, the results in this paper contradict the common ideology.

Using technology previously developed by Shelah, we were able to force over a model with a supercompact cardinal κ with a κ-c.c., (ω_1, ∞)-distributive partial ordering to make the non-stationary ideal on ω_1 restricted to a particular stationary set be $\aleph_2$-saturated.

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A major program in set theory initiated by Solovay, Mitchell and others was to construct canonical models of "ZFC + there is a supercompact cardinal". The models were supposed to have some of the crystalline structure of L. (This is the so-called inner-model problem.) The results in the previous paragraph drastically limit the possibilities of such an inner model. For example, they show that the canonical models cannot have the same $\aleph_1$, are generic extensions of one another, and so forth.

Similar techniques show that if there is a supercompact cardinal, then the theory of $L(\mathbf{R})$ does not change under set-generic forcing extensions. Woodin and Shelah have since strengthened this theorem a great deal by reducing the large cardinal hypothesis required.

We also show the consistency of "for all regular cardinals μ, the non-stationary ideal on μ is precipitous" from a supercompact cardinal. Further, we show that relative to a supercompact cardinal, Chang's conjecture is equivalent to a generic version of Chang's conjecture. From this we deduce the consistency of a generic huge embedding from a supercompact cardinal. (See [F2] for terminology.)

In Part II of this paper we will show that one can force over a model of "ZFC + there is a huge cardinal" to get fully non-regular ultrafilters on any succesor cardinal μ. We also construct ultrafilters giving rise to ultrapowers of small cardinality.

A summary of our results is as follows:

In Section 1, we present the axiom we call *Martin's Maximum* (MM) and show that it is a provably maximal version of Martin's axiom. We review the technology of semi-proper forcing developed by Shelah ([Sh1]) which is intimately connected with the work in this paper. We then show that Martin's Maximum is consistent with ZFC relative to a supercompact cardinal. Finally we show that MM implies various versions of Martin's Axiom discussed elsewhere in the literature. We also introduce the principle $(\dagger)$.

In Section 2 we deduce various consequences of MM. We first show Friedman's problem (every stationary subset of a regular cardinal $\kappa > \omega_2$ consisting of points of cofinality ω contains a closed set of order type ω_1). Using the same technique, we deduce $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and various other cardinal arithmetic consequences. We then show that under MM the non-stationary ideal on ω_1 is $\aleph_2$-saturated and that the saturation of the non-stationary ideal is preserved by c.c.c. forcing. Along the way we show the crucial combinatorial tool that MM implies: that every stationary subset of an $[H(\lambda)]^{\omega}$ reflects to a set of size ω_1. This implies the principle $(\dagger)$. We also obtain partial information about the quotient algebra $\mathscr{P}(\omega_1)/NS_{\omega_1}$. In particular we show that any new real in the forcing extension is (in a quite strong sense) a minimal degree.

Section 3 is a short section mostly devoted to a brief explication of versions of MA (Martin's Axiom) consistent with CH and results of Shelah and Woodin showing that there can be no nice inner model of a supercompact cardinal. We also show a joint result of the authors that weak versions of MA consistent with CH imply that the non-stationary ideal on ω_1 is presaturated.

In Section 4, we generalize our results to cardinals above ω_1 to show that for any regular μ, the non-stationary ideal on μ can be precipitous. Further, we get higher order ideals to be precipitous on sets such as $[\lambda]^{<\kappa}$ and $[\lambda]^{\kappa}$. These ideals are versions of the non-stationary ideal. We show that if we have Chang's conjecture at a regular cardinal κ, then by collapsing a supercompact cardinal to κ^+, we can make the Chang's-conjecture ideal precipitous. Finally, we show that if $2^\kappa = \kappa^+$ and $\kappa^{\underset{\smile}{\kappa}} = \kappa$, then there is a κ-closed, κ^+-c.c. forcing that makes any normal precipitous ideal on κ in V non-precipitous.

We now want to discuss the notion of closed and unbounded we use.

On $[\lambda]^{<\kappa}$ there are two different natural notions of closed and unbounded, one stronger than the other.

The weaker notion is the official one used in this paper, though all of the proofs work with the stronger notion. Recently, Woodin has exploited the stronger notion to great advantage; so we spell out the differences in the following definitions and lemma.

Definition. Let κ and λ be regular cardinals. Let $[\lambda]^{<\kappa} = \{x \subseteq \lambda: |x| < \kappa\}$ and $[\lambda]^{\kappa} = \{x \subseteq \lambda: |x| = \kappa\}$.

If $X \subseteq [\lambda]^{<\kappa}$ then X is strongly closed and unbounded if and only if there is a structure $\mathscr{A} = \langle \lambda, f_i \rangle_{i \in \omega}$ where $f_i: \lambda^{<\omega} \to \lambda$ and $X = \{N \prec \mathscr{A}: |N| < \kappa\}$. Note that any strongly closed and unbounded set contains countable subsets of λ.

X is closed and unbounded if and only if:

i) For all $y \in [\lambda]^{<\kappa}$ there is a $z \in X$ such that $y \subseteq z$.

ii) Whenever $\langle y_\alpha: \alpha < \beta \rangle \subseteq X$ where $\beta < \kappa$ and $\alpha < \alpha'$ implies $y_\alpha \subseteq y_{\alpha'}$ then $\bigcup_{\alpha<\beta} y_\alpha \in X$. ((ii) is equivalent to X being closed under unions of directed systems.)

Note that if $\kappa > \omega_1$ there are closed and unbounded sets containing no countable sets. Further any strongly closed and unbounded set is closed and unbounded.

The collection of strongly closed and unbounded sets generates a countably complete, normal and fine filter, $\mathscr{F}_s$ and the closed unbounded sets generate a $<\kappa$-complete normal and fine filter $\mathscr{F}$. The next lemma, essentially due to Kueker, [Ku], shows that $\mathscr{F}$ is the filter generated by adding $\{y \in [\lambda]^{<\kappa}: y \cap \kappa \in \kappa\}$ to $\mathscr{F}_s$.

LEMMA 0. *Let $\mu \leq \kappa < \lambda$ be regular cardinals, $\mathscr{F}_s(\lambda, \mu)$, $\mathscr{F}_s(\kappa, \mu)$ be the filters of strongly closed and unbounded sets on $[\lambda]^{<\mu}$ and $[\kappa]^{<\mu}$ respectively. Let $\mathscr{F}(\lambda, \mu)$ and $\mathscr{F}(\kappa, \mu)$ be the corresponding filters of closed and unbounded sets. Then:*

a) *$\mathscr{F}(\lambda, \mu)$ is the filter generated by*

$$\mathscr{F}_s(\lambda, \mu) \cup \{\{z \in [\lambda]^{<\mu}: z \cap \mu \in \mu\}\}.$$

b) *If $C \subseteq [\lambda]^{<\mu}$ is strongly closed and unbounded then $\{y \cap \kappa: y \in C\}$ is a strongly closed and unbounded subset of $[\kappa]^{<\mu}$.*

c) *If $C \subseteq [\lambda]^{<\mu}$ is closed and unbounded then $\{y \cap \kappa: y \in C\}$ contains a closed and unbounded set in $[\kappa]^{<\mu}$.*

d) *If $C \subseteq [\kappa]^{<\mu}$ is closed and unbounded (resp. strongly closed and unbounded) then $\{z \in [\lambda]^{<\mu}: z \cap \kappa \in C\}$ is closed and unbounded (resp. strongly closed and unbounded).*

Proof. a) Let $C \subseteq [\lambda]^{<\mu}$ be closed and unbounded. We must find $\langle f_i: [\lambda]^{<\omega} \to \lambda | i \in \omega\rangle$ such that $\{y \in [\lambda]^{<\mu}: y$ is closed under each f_i and $y \cap \mu \in \mu\} \subseteq C$. Let $\mathscr{L} = \langle H(\lambda), \varepsilon, C, \Delta, \{\mu\}\rangle$ where Δ is a well ordering of $H(\lambda)$. Let $N \prec \mathscr{L}$ be an elementary substructure of $\mathscr{L}$ of cardinality $< \mu$ such that $N \cap \mu \in \mu$.

For $\vec{\alpha} \in [N \cap \lambda]^{<\omega}$, we define by induction on $|\vec{\alpha}|$ an $M_{\vec{\alpha}} \in N \cap C$ so that $\vec{\alpha} \subseteq M_\alpha$ and if $\vec{\alpha} \supseteq \vec{\beta}$, $M_{\vec{\alpha}} \supseteq M_{\vec{\beta}}$. Then, since $M_{\vec{\alpha}} \in N$, $|M_{\vec{\alpha}}| \in N$ so $M_{\vec{\alpha}} \subseteq N$. The collection $\{M_{\vec{\alpha}}: \vec{\alpha} \in [N \cap \lambda]^{<\omega}\}$ is a directed system, hence $N \cap \lambda = \bigcup M_{\vec{\alpha}} \in C$.

Suppose we have defined $M_{\vec{\alpha}}$ for $|\vec{\alpha}| = n$. If $|\vec{\beta}| = n + 1$ then choose $M_{\vec{\beta}} \in N \cap C$ such that for all subsets $\vec{\alpha} \subseteq \vec{\beta}$, $|\vec{\alpha}| = n$, $\vec{\beta} \cup M_{\vec{\alpha}} \subseteq M_{\vec{\beta}}$. We can choose such an $M_{\vec{\beta}}$ since $\bigcup_{\vec{\alpha} \subset \vec{\beta}} M_{\vec{\alpha}} \in N$ and $N \models$ "C is unbounded". Clearly these $M_{\vec{\alpha}}$'s are as desired.

Let $\langle g_i: i \in \omega\rangle$ be Skolem functions for $\mathscr{L}$ that are closed under composition. Let $f_i: [\lambda]^{<\omega} \to \lambda$ be the restriction of g_i to domains and ranges in λ.

If $y \in [\lambda]^{<\mu}$, $y \cap \mu \in \mu$ and y is closed under each f_i then there is an $N \prec \mathscr{L}$ such that $N \cap \lambda = y$; hence $y \in C$.

b) Let $\langle f_i: i \in \omega\rangle$ be such that if $y \in [\lambda]^{<\mu}$ and y is closed under $\langle f_i: i \in \omega\rangle$, then $y \in C$. Without loss of generality we may assume that the f_i's are closed under composition. Let $\langle g_i: i \in \omega\rangle$ be the result of restricting the domains and ranges of each f_i to κ. If $z \in [\kappa]^{<\mu}$ and z is closed under the g_i's then there is a $y \in [\lambda]^{<\mu}$ such that y is closed under the f_i's and $y \cap \kappa = z$. Further, if y is closed under the f_i's then $y \cap \kappa$ is closed under the g_i's. Thus $\{z \in [\kappa]^{<\mu}$ there is a $y \in C$, $z = y \cap \kappa\}$ is exactly the set of $z \in [\kappa]^{<\mu}$ closed under $\{g_i: i \in \omega\}$.

c) By a) we may assume that C is of the form $\{y \in [\lambda]^{<\mu}: y \cap \mu \in \mu$ and y is closed under $\langle f_i: i \in \omega\rangle\}$ for some sequence of functions $\langle f_i: i \in \omega\rangle$. By b) there are functions $\langle g_i: i \in \omega\rangle$ such that for all $z \in [\kappa]^{<\mu}$, $z = y \cap \kappa$ for some y closed under the f_i's if and only if z is closed under the g_i's. For $y \in [\lambda]^{<\mu}$, $y \cap \mu \in \mu$ if and only if $y \cap \kappa \cap \mu \in \mu$. Hence $\{y \cap \kappa: y \in C\} = \{z: z \in [\kappa]^{<\mu}$ and $z \cap \mu \in \mu$, and z is closed under $\langle g_i: i \in \omega\rangle\}$.

d) is immediate. □

We note that if μ is ω_1, Lemma 0 implies that there is no difference between $\mathscr{F}_s(\lambda, \omega_1)$ and $\mathscr{F}(\lambda, \omega_1)$.

Notation. We now discuss the notation and conventions we shall use throughout this paper.

We will write $|X|$ for the cardinality of a set X and o.t. (X) for the order type of (X, ε).

Forcing will be used throughout this paper and we will frequently use both Boolean algebra and partial ordering notation. We will use $\| \ \|_{\mathscr{B}}$ for the Boolean value taken in a particular Boolean algebra $\mathscr{B}$ and drop the $\mathscr{B}$ if it is clear from context. When we use the symbol "$\geq$" it will be in the Boolean algebra convention; i.e. $p \leq q$ means that p is stronger than q. Similarly, when we write that p is below q we will mean that p is stronger than q.

We will write that $\|\phi\|_{\mathscr{B}} = 1$ if and only if ϕ is true in any forcing extension by $\mathscr{B}$. In an attempt to avoid culturally induced confusion of $p \geq q$ vs. $p \leq q$, in this paper we have followed the convention established by the New England Set Theory Seminar of using $p \Vdash q$ as an abbreviation for "p forces q to be in the canonical generic object." Solovay has pointed out that the relation "$p \Vdash q \in G$" is *not the same* as the partial order $\leq_{\mathbf{P}}$ for non-separative partial orderings $\mathbf{P}$. We hereby warn the reader that confusion may arise as a result of this.

In a similar abuse of notation we write $p\|q$ to mean that p decides the Boolean value $\|q \in G\|$ where G is the canonical term for a generic object. In general G will be the generic object. If ϕ is an n-ary formula and $\tau_1 \cdots \tau_n$ are terms we write $p\|\phi(\tau_1 \cdots \tau_n)$ to mean $p \Vdash \phi(\tau_1 \cdots \tau_n)$ or $p \Vdash \neg\phi(\tau_1 \cdots \tau_n)$. We will let $\mathscr{B}(\mathbf{P})$ be the complete Boolean algebra in which the separative quotient of $\mathbf{P}$ is dense.

We will also abuse notation by using $V^{\mathbf{P}}$ to stand both for the generic extension of V by a generic object $G \subseteq \mathbf{P}$ and for the Boolean-valued universe. Similarly we will write that $V^{\mathbf{P}} \models \phi$ for $\|\phi\|_{\mathscr{B}(\mathbf{P})} = 1$. A $\mathbf{P}$-term (or $\mathbf{P}$-name) will simply be an element of $V^{\mathbf{P}}$. If $Q \in V^{\mathbf{P}}$ is a $\mathbf{P}$-term for a partial ordering, a

Q-term in $V^{\mathbf{P}}$ is a $\mathbf{P}$-term τ such that $\|\tau^{V[G]}$ is a Q-term$\|_{\mathscr{B}}(\mathbf{P}) = 1$. We will occasionally explicitly work with terms, in which case we will attempt to use the system of dots and checks. For example $\dot{\alpha}$ might be a $\mathbf{P}$-term for an ordinal, whereas if α is an ordinal in V we will write $\check{\alpha}$ for its canonical term in $V^{\mathbf{P}}$.

We will use quotation marks around certain statements following $\models$ or $\Vdash$ when they occur in prose to delineate the extent of the symbol $\models$ or $\Vdash$. We will also use quotation marks to specify classes defined by the mathematical representation of the statement in quotes.

We will write $i: Q \hookrightarrow \mathbf{P}$ if i is a monomorphism of Q into $\mathbf{P}$ such that any maximal antichain in Q is sent to a maximal antichain in $\mathbf{P}$. Equivalently, i can be extended to a complete embedding $\bar{i}$: $\mathscr{B}(Q) \hookrightarrow \mathscr{B}(\mathbf{P})$. If i: $Q \hookrightarrow \mathbf{P}$ and $G \subseteq Q$ is generic we can form the Boolean algebra $\mathscr{B}(\mathbf{P})/G$ in $V[G]$ in the standard way. Then forcing with $\mathscr{B}(\mathbf{P})/G$ over $V[G]$ yields an ultrafilter $H \subseteq \mathscr{B}(\mathbf{P})$ which is generic over V. We will let $\mathbf{P}/Q$ be the Q-term for the Boolean algebra $\mathscr{B}(\mathbf{P})/G$. We will use $*$ for the two step iteration. Thus $\mathscr{B}(\mathbf{P}) \simeq \mathscr{B}(Q * \mathbf{P}/Q)$.

In doing Boolean algebra computations in $\mathscr{B}$ we will use Σ and $\vee$ for the sum or join of elements of $\mathscr{B}$; similarly we will use $\prod$ or $\wedge$ for the meet of elements of $\mathscr{B}$.

We will use the notation $\mathbf{P}_\alpha$ for an α-stage iteration. If we have defined an iteration $\langle \mathbf{P}_\beta \colon \beta < \alpha \rangle$ we will write $\xrightarrow{\lim} \langle \mathbf{P}_\beta \colon \beta < \alpha \rangle$ and $\xrightarrow{\lim} \langle \mathbf{P}_\beta \colon \beta < \alpha \rangle$ for the direct and inverse limits of $\langle \mathbf{P}_\beta \colon \beta < \alpha \rangle$ respectively. An iteration is determined by its "factors" and the type of supports allowed in the iteration. If p is a condition in an iteration, then the support of p, which we write $\operatorname{supp}(p)$, is the set of β in which p gives non-trivial information in the βth factor. We can represent a condition p by $p = \langle p(\beta) \colon \beta \in \operatorname{supp}(p) \rangle$. (See [B1] for a very good exposition of iterated forcing.)

We will say that a partial ordering $\mathbf{P}$ is (κ, ∞)-distributive whenever $\langle D_\alpha \colon \alpha < \beta \rangle$ is a collection of $< \kappa$-many dense open sets in $\mathbf{P}$, $\bigcap_{\alpha < \beta} D_\alpha$ is dense and open. (This is equivalent to $\mathbf{P}$ not adding new $< \kappa$-sequences.) An exception to this is that we may write (ω, ∞)-distributive to mean (ω_1, ∞)-distributive.

There are several partial orderings we will use quite frequently. We will write $\operatorname{Col}(\kappa, \lambda), \operatorname{Col}(\kappa, \leq \lambda), \operatorname{Col}(\kappa, < \lambda)$ for the Levy collapses of λ, everything less than or equal to λ and everything less than λ to have cardinality κ respectively. If X is an arbitrary set, we will write $\operatorname{Col}(\kappa, X)$ for the Levy collapse of X to have cardinality κ.

We will typically use λ for a large enough generic regular cardinal. We will write $\lambda \gg \kappa$ for a regular λ at least two power set operations greater than κ, i.e. $\lambda \geq 2^{2^\kappa}$. In contexts where we use it, it will not matter exactly what λ is as long as it is sufficiently large and regular. We will write $H(\lambda)$ for the collection of sets hereditarily of power less than λ. We will write $[H(\lambda)]^{<\kappa}$ and $[H(\lambda)]^{\kappa}$ to mean all subsets of $H(\lambda)$ of power $<\kappa$ and power κ respectively. We will use Δ as an arbitrary well-ordering of $H(\lambda)$ in order type $|H(\lambda)|$. The sets of set-theoretical rank less than κ will be called R_κ. We will use OR for the class of ordinals and $\mathrm{cof}(\gamma)$ for the class of ordinals of cofinality γ. Thus $\kappa \cap \mathrm{cof}(\gamma)$ is the set of ordinals less than κ of cofinality γ.

Cardinal exponentiation will be denoted in the usual way; i.e. $\kappa^\lambda = |\{f | f: \lambda \to \kappa\}|$.

We will often be interested in ideals. All ideals will be proper and countably complete and contain all finite ordinals. If $\mathscr{I}$ is an ideal on a set z, then $\mathscr{P}(z)/\mathscr{I}$ is the Boolean algebra constructed by taking $\mathscr{P}(z)$ modulo $\mathscr{I}$. If $A \in \mathscr{P}(z)$, A is $\mathscr{I}$-positive if and only if $A \notin \mathscr{I}$. We let $\mathscr{I} \restriction A$ be the ideal generated by $\mathscr{I} \cup \{\tilde{A}\}$. The set of $\mathscr{I}$-positive sets will be written $\mathscr{I}^+$. The filter dual to $\mathscr{I}$ will be called $\breve{\mathscr{I}}$. If A is positive then $[A]_{\mathscr{I}}$ is the equivalence class of A modulo $\mathscr{I}$.

If $z = \kappa$ for some set κ and $\langle A_\alpha: \alpha < \kappa \rangle \subseteq \mathscr{P}(z)$ then $\Delta_{\alpha<\kappa} A_\alpha = \{\beta:$ for all $\alpha < \beta$, $\beta \in A_\alpha\}$ and $\nabla_{\alpha<\kappa} A_\alpha = \{\beta:$ there is an $\alpha < \beta$, $\beta \in A_\alpha\}$. If $z = [\lambda]^\kappa$ or $[\lambda]^{<\kappa}$ and $\langle A_\alpha: \alpha < \lambda \rangle \subseteq \mathscr{P}(z)$ then $\Delta_{\alpha<\lambda} A_\alpha = \{x \in z:$ for all $\alpha \in x$, $x \in A_\alpha\}$ and $\nabla_{\alpha<\lambda} A_\alpha = \{x \in z:$ there is an $\alpha \in x$, $x \in A_\alpha\}$.

We will be particularly interested in the non-stationary ideals on various sets z. A set $x \subseteq z$ is non-stationary if and only if it is in the dual to the closed unbounded filter. We refer the reader to earlier remarks about the closed unbounded filter on various sets. We will write NS_Z for the non-stationary ideal on Z.

An ideal $\mathscr{I}$ on $[\kappa]^\lambda$ will be said to concentrate on $[\kappa']^{\lambda'}$ if and only if $\{x \in [\kappa]^\lambda: x \cap \kappa'$ has cardinality $\lambda'\} \in \breve{\mathscr{I}}$.

If $\mathscr{A}$ and $\mathscr{L}$ are structures we will write $\mathscr{A} \prec \mathscr{L}$ if $\mathscr{A}$ is an elementary substructure of $\mathscr{L}$. We will write $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$ if and only if whenever $\mathscr{L} = (\kappa; \lambda, f_i)_{i \in \omega}$ is a structure there is an elementary substructure $\mathscr{A} \prec \mathscr{L}$ such that $|\mathscr{A}| = \kappa'$ and $|\mathscr{A} \cap \lambda| = \lambda'$. (This is Chang's conjecture.) If $\mathscr{A}$ is a structure with Skolem functions (or a well-ordering) and $X \subseteq \mathscr{A}$ then $\mathrm{Sk}^{\mathscr{A}}(X)$ is the Skolem hull of X in $\mathscr{A}$.

If $\eta \in (\kappa)^{<\omega}$ then $l(\eta)$ is the length of η. We will use $\frown$ for concatenation, so that $\eta^\frown\alpha$ will be η concantenated with α.

If $x \subseteq \mathrm{OR}$ then $\sup x$ will be the proper supremum of X (i.e. $\sup X = \bigcup\{y + 1: y \in X\}$).

We will use the notation *Proposition* (T), where T is a theory, to mean that the proposition is proved in the theory T.

We will write $j: V \to M$ for an elementary embedding j from V into a transitive class M. We will write $\mathrm{crit}(j)$ for the critical point of j, i.e. the first ordinal moved by j.

1. The consistency proof

In this section we work towards proving the consistency of *Martin's Maximum*, a maximal strengthening of Martin's Axiom [M-So].

Definition. If Q is a partial ordering, Q *preserves stationary subsets of* ω_1 if and only if there is a $q \in Q$ such that whenever $S \subseteq \omega_1$, $S \in V$ is stationary, then $\|S \text{ is stationary}\|_{\mathbf{P}} \geq q$.

If $\mathscr{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense sets in Q and $G \subseteq Q$ is a filter, we say that G is generic for $\mathscr{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ if and only if for each α, $G \cap D_\alpha \neq \varnothing$.

Martin's Maximum is the following statement:

If $\mathbf{P}$ is a partial ordering that preserves stationary subsets of ω_1 and $\mathscr{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense sets in $\mathbf{P}$ then there is a filter $G \subseteq \mathbf{P}$, such that G is generic for $\mathscr{D}$.

In general, if Γ is a class of partial orderings we will say that MA holds for Γ if and only if:

For all $Q \in \Gamma$ and all sequences $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense sets in Q, there is a filter $G \subseteq Q$ such that G is generic for $\mathscr{D}$.

We point out that $\Gamma =$ "the class of Q such that Q preserves stationary subsets of ω_1" is a maximal class for which MA can hold.

PROPOSITION 1. *Suppose that Q does not preserve stationary subsets of ω_1, then there is a sequence of sets $\mathscr{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ such that there is no $\mathscr{D}$-generic filter $G \subseteq Q$.*

Proof. Since Q does not preserve stationary subsets of ω_1 there is a term $S \in V^Q$ such that $\|\dot{S} \in V \text{ and } S \text{ is stationary in } V\| = 1$ and a term $\dot{C} \in V^Q$ such that $\|\dot{C} \text{ is club in } \omega_1 \text{ and } C \cap S = \varnothing\| = 1$.

Let $D_0 = \{q \in Q\colon \text{for some } S \in V,\ S \text{ stationary},\ q \Vdash \dot{S} = \check{S}\}$. Let $D_\alpha = \{q \in Q\colon\ q\|\text{"}\alpha \in C\text{"}$ and if $q \Vdash \text{"}\alpha \notin C\text{"}$ then there is a $\gamma < \alpha$, $q \Vdash C \cap (\gamma, \alpha) = \varnothing$ and for some $\beta \geq \alpha$, $q \Vdash \text{"}\beta \in C\text{"}\}$.

For each α, choose a term for an ω-sequence of ordinals $\langle \alpha_n : n \in \omega \rangle$ such that

$$\|\text{if } \alpha \in C \text{ then } \langle \alpha_n : \eta \in \omega \rangle \subseteq C \quad \text{and } \sup\langle \alpha_n : n \in \omega \rangle = \alpha\| = 1.$$

Let $D_{\alpha, n} = \{q \in Q\colon \text{either } q \Vdash \alpha \notin C \text{ or } q \Vdash \alpha \in C \text{ and for some } \beta \in \omega_1,\ q \Vdash \dot{\alpha}_n = \beta\}$.

Suppose $G \subseteq Q$ is generic for $\langle D_\alpha : \alpha < \omega_1 \rangle \cup \langle D_{\alpha, n} : \alpha < \omega_1,\ n \in \omega \rangle$. Let $C = \{\alpha\colon \text{there is a } q \in G \text{ such that } q \Vdash \alpha \in C\}$. Then C is closed since: if $\langle \alpha_n : n \in \omega \rangle \subseteq C$ is an increasing sequence with supremum α and $\alpha \notin C$ then for some $q \in D_\alpha \cap G$, $q \Vdash \alpha \notin C$. Hence there is a $\gamma < \alpha$ such that $q \Vdash C \cap (\gamma, \alpha) = \varnothing$. Take some $\alpha_n > \gamma$ and a $\gamma \in G$ such that $r \Vdash \alpha_n \in C$. Then r and q are incompatible. This is a contradiction. Hence C is closed.

Let T be such that for some $q \in G$, $q \Vdash \dot{S} = \check{T}$. By assumption T is stationary in V. But $T \cap C = \varnothing$ and C is closed and unbounded. This is a contradiction. □

We mention a slight strengthening of MA for Γ.

MA^+ for Γ is the statement: Whenever $Q \in \Gamma$ is a partial ordering, $\mathscr{D} = \langle D_\alpha \colon \alpha < \omega_1 \rangle$ is a sequence of dense sets in Q and $S \in V^Q$ is a term for a stationary subset of ω_1 in V^Q, then there is a $\mathscr{D}$-generic filter $G \subseteq Q$ such that $S^G = \{\alpha \colon \text{there is a } p \in G,\ p \Vdash \alpha \in S\}$ is stationary in V. Baumgartner has shown that for $\Gamma =$ "the class of c.c.c. partial orderings" (ordinary MA), MA^+ is equivalent to MA.

We now develop the tools to show the consistency of MA for Γ for various Γ's. We need the notion of a semi-proper partial ordering, which is due to Shelah. (See [Sh1].)

Definition. A partial ordering $\mathbf{P}$ is $\aleph_1$*-semi-proper* if and only if there is a club set $C \subseteq [H(2^{2^{(\mathbf{P})+}})]^\omega$ such that for all $N \in C$ and all $p \in N \cap \mathbf{P}$ there is a $q \Vdash p$ $q \Vdash$ (for all $\tau \in N$) (if τ is a $\mathbf{P}$-term for an element of ω_1 then $\tau^{V[G]} \in N$). Here $\tau^{V[G]}$ is the realization of τ in $V[G]$ where G is any generic object with $q \in G$.

Definition. A q as above will be called a *semi-master condition* for N and $\mathbf{P}$.

Note. A small amount of reflection will show that $2^{2^{|\mathbf{P}|^+}}$ can be replaced by any sufficiently large regular cardinal λ and yield an equivalent definition.

For the readers' edification we reproduce a theorem of Shelah [Sh1] that motivates $\aleph_1$-semi-properness.

PROPOSITION 2. *Suppose* $\mathbf{P}$ *is* $\aleph_1$-semi-proper; *then* $\mathbf{P}$ *preserves stationary subsets of* ω_1 (*in particular* $\omega_1^{V^{\mathbf{P}}} = \omega_1^V$).

Proof. Let S be a stationary subset of ω_1, $S \in V$ and $\dot{C} \in V^{\mathbf{P}}$ be a term for a club subset of ω_1 and $p \in \mathbf{P}$. Let $N \prec \langle H(2^{2^{|\mathbf{P}|^+}}), \varepsilon, \Delta, \dot{C}, S, \{P\} \rangle$ be a countable elementary substructure of $H(2^{2^{|\mathbf{P}|^+}})$ (where Δ is a well-ordering of $H(2^{2^{|\mathbf{P}|^+}})$) such that N has a semi-master condition, $q \Vdash p$, and $N \cap \omega_1 \in S$.

Let $\delta = N \cap \omega_1$. For each $\beta < \delta$, there is a term $\tau\beta \in N$ such that $\|\tau\beta \in C \text{ and } \tau\beta > \beta\| = 1$. For each such term $\tau\beta$, $q \Vdash$ "$\tau\beta \in \delta$". Hence $q \Vdash$ "$\dot{C}$ is unbounded in δ" and hence $q \Vdash$ "$\delta \in \dot{C}$". However, $\delta \in S$ and thus

$q \Vdash$ "$\delta \in C \cap S$, so $C \cap S \neq \varnothing$". We have shown that every closed unbounded set in V^Q has non-empty intersection with S; hence $V^Q \vDash S$ is stationary. $\square$

This argument is the prototype for showing that a particular partial ordering **P** preserves stationary subsets of ω_1.

A natural question arises: Can "preserving stationary substes of ω_1" be equivalent to "$\aleph_1$-semi-proper"?

As we shall see, the two properties are inequivalent in general (e.g. in L). However, the main advance in this paper is the following lemma:

LEMMA 3. *Suppose κ is a supercompact cardinal and* **P** *is an $\aleph_1$-semi-proper partial ordering such that*

a) $V^{\mathbf{P}} \vDash$ "$\kappa = \aleph_2$" *and* **P** *is* κ c.c.

b) *For each $\gamma \in$ OR there is a γ^+-supercompact embedding $j\colon V \to M$ such that $j(\mathbf{P}) = \mathbf{P} * \mathrm{Col}(\omega_1, \leq \gamma) * \mathbf{R}$ and* **R** *is $\aleph_1$-semi-proper in $M^{\mathbf{P} * \mathrm{Col}(\omega_1, \leq \gamma)}$.*

Then in $V^{\mathbf{P}}$: For all partial orders Q,

(†) *Q is $\aleph_1$-semi-proper if and only if Q preserves stationary subsets of ω_1.*

We postpone the proof of this lemma to prove:

PROPOSITION 4. *Suppose $A \subseteq [H(\lambda)]^\omega$ is stationary; then in $V^{\mathrm{Col}(\omega_1, |H(\lambda)|)}$, A is stationary in $[H(\lambda)^V]^\omega$. (In fact A is preserved by any countably closed forcing.)*

Proof. Let $\lambda' \geq 2^{2^{|H(\lambda)|^+}}$ be a regular cardinal, $p \in \mathrm{Col}(\omega_1, |H(\lambda)|)$ and $\dot{C}$ be a term for a closed unbounded set in $[H(\lambda)^V]^\omega$.

Let $N \prec \langle H(\lambda'), \varepsilon, \Delta, \dot{C}, \{P\}, A, \mathrm{Col}(\omega_1, |H(\lambda)|)\rangle$ be a countable elementary substructure of $H(\lambda')$ such that $N \cap H(\lambda) \in A$. Let $\delta = N \cap \omega_1$.

Starting below p we build a sequence of conditions $\langle p_n\colon n \in \omega\rangle \subseteq N$, such that $p_{n+1} \Vdash p_n$ and for each dense open set $D \subseteq \mathrm{Col}(\omega_1, |H(\lambda)|)$ if $D \in N$ then there is an n, $p_n \in D$. This is easy since N is countable. Since $\mathrm{Col}(\omega_1, |H(\lambda)|)$ is countably closed there is a $q \Vdash p_n$ for each $n \in \omega$. Clearly $q \in \bigcap\{D\colon D \subseteq \mathrm{Col}(\omega_1, |H(\lambda)|)$ and D is dense and open and $D \in N\}$. Hence $q\colon \delta \to N \cap H(\lambda)$ is surjective. Further, since $\|\dot{C}$ is club in $[H(\lambda)^V]^\omega\| = 1$,

$$q \Vdash \text{"}\cup(C \cap N) \supseteq N \cap H(\lambda)^V\text{"}$$

and so $q \Vdash N \cap H(\lambda)^V \in \dot{C}$. But $N \cap H(\lambda)^V \in A$, so that

$$q \Vdash \dot{C} \cap A \neq \varnothing.$$

Thus given any p and any term $\dot{C}$ for a club set in $[H(\lambda)^V]^\omega$ there is a $q \Vdash p$ such that $q \Vdash \dot{C} \cap A \neq \varnothing$. Hence A is stationary in $V^{\mathrm{Col}(\omega_1, |H(\lambda)|)}$. $\square$

We will have many arguments involving sequences such as the $\langle p_n: n \in \omega\rangle$ as in the last proof. We give such a sequence a name: Let $\mathbf{P}$ be a partial ordering, and $N \prec \langle H(\lambda), \varepsilon, \mathbf{P}, \Delta \dots\rangle$. A sequence of conditions, $\langle p_n: n \in \omega\rangle \subseteq N \cap \mathbf{P}$ such that $p_{n+1} \Vdash p_n$ and for each dense open set $D \subseteq \mathbf{P}$, $D \in N$ there is an n, such that $p_n \in D$, will be called a *generic sequence* for N.

Note that generic sequences always exist (although $\inf_{n \in \omega} p_n$ may be zero in $B(\mathbf{P})$).

We sum up the arguments in Proposition 2 and Proposition 4 in the following definition and lemma.

Definition. Suppose $N \prec H(\lambda)$ is countable and $\langle p_n: n \in \omega\rangle$ is a generic sequence for N. Then p is a *strong master condition* for N if for all n, $p \Vdash p_n$.

LEMMA $*$. *Let* $\mathbf{P}$ *be a partial ordering and* $\lambda = 2^{2^{|\mathbf{P}|^+}}$, $N \prec H(\lambda)$, $|N| = \omega$. *Then*

a) *If p is a strong master condition for N then p is a semi-master condition for N.*

b) *If $\dot{C} \in V^{\mathbf{P}}$ is a term for a club subset of ω_1 and $\dot{C} \in N$ and p is a semi-master condition for N then $p \Vdash N \cap \omega_1 \in \dot{C}$.*

c) *If S is a stationary subset of ω_1 such that for all $q \in \mathbf{P}$ and all $C \in H(\lambda)$ there is $p \Vdash q$ and an $N \prec H(\lambda)$, $|N| = \omega$, $C \in N$, and $N \cap \omega_1 \in S$ such that p is a semi-master condition for N then S is stationary in $V^{\mathbf{P}}$.*

Proof. This is as in Propositions 2 and 4.

We now return to the proof of Lemma 3: Suppose that $Q \in V^{\mathbf{P}}$ is a partial ordering such that Q preserves stationary subsets of ω_1 and Q is not $\aleph_1$-semi-proper. Let $\lambda = 2^{2^{|Q|^+}}$. Since Q is not $\aleph$-semi-proper there is a stationary set $A \subseteq [H(\lambda)]^{\omega}$ such that for all $N \in A$ there is a $p \in N \cap Q$ such that there is no semi-master condition q for N with $q \Vdash p$. By the normality of the non-stationary ideal on $[H(\lambda)]^{\omega}$ there is a fixed p such that on a stationary set $A \subseteq [H(\lambda)]^{\omega}$, for all $N \in A$ there is no semi-master-condition q for N such that $q \Vdash p$. By modifying Q we can assume that p is the trivial condition. Let $\gamma = |H(\lambda)|$.

Consider j: $V \to M$ such that j is a γ^+-supercompact embedding and $j(\mathbf{P}) = \mathbf{P} * \mathrm{Col}(\omega_1, \leq \gamma) * \mathbf{R}$ and $\mathbf{R}$ is $\aleph_1$-semi-proper in $M^{\mathbf{P} * \mathrm{Col}(\omega_1, \leq \gamma)}$. Then by standard large cardinal theory, since $\mathbf{P}$ is κ-c.c., j can be extended to an elementary embedding $\hat{j}$: $V^{\mathbf{P}} \to M^{j(\mathbf{P})}$. We confuse j and $\hat{j}$. (See [B1].)

In $M^{\mathbf{P} * \mathrm{Col}(\omega_1, \leq \gamma) * \mathbf{R}}$, $A \subseteq [H(\lambda)^{\mathbf{P}}]^{\omega}$ is stationary, since $\mathrm{Col}(\omega_1, \leq \gamma)$ keeps A stationary and in $M^{\mathbf{P} * \mathrm{Col}(\omega_1, \leq \gamma)}$, A can be coded as a stationary subset of ω_1. (A is a stationary subset of some $[X]^{\omega}$ with $|X| = \omega_1$.) Since $\mathbf{R}$ is $\aleph_1$-semi-proper, $\mathbf{R}$ preserves stationary subsets of ω_1 and hence preserves the stationariness of A.

Since $M^{j(\mathbf{P})} \models$ "$j(Q)$ preserves stationary subsets of ω_1", A is stationary in $M^{j(\mathbf{P}) * j(Q)}$. In $M^{j(\mathbf{P}) * j(Q)}$,

$$C = \left\{X \prec H(j(\lambda))^{M^{j(\mathbf{P}) * j(Q)}} : \text{if } \tau \in X \cap H(\lambda)^{\mathbf{P}} \text{ is a } Q\text{-term for an element of } \omega_1 \text{ then } j(\tau)^{M^{j(\mathbf{P}) * j(Q)}} \in X\right\}$$

is a club set in $([H(j(\lambda))]^{\omega})^{M^{j(\mathbf{P}) * j(Q)}}$. Since A is a stationary set there is an $X \in C$ such that $X' = X \cap H(\lambda)^{V^{\mathbf{P}}} \in A$. In $M^{j^{(\mathbf{P})}}$, let $q \in j(Q)$, $x' \in A$ be such that $q \Vdash$ (there is an $X \in C$) $(X \cap H(\lambda)^{V^{\mathbf{P}}} = X')$. Then $q \Vdash$ if $\tau \in X'$ is a Q-term for an element of ω_1 and then $j(\tau)^{M^{j(\mathbf{P} * Q)}} \in X'$.

Since X' is countable, $j(X') = j''(X')$. Hence,

$$q \Vdash \text{"If } \sigma \in j(X') \text{ is a term for an element of } \omega_1, \text{ then } \sigma^{M^{j(\mathbf{P} * Q)}} \in j(X')."$$

So,

$$M^{j^{(\mathbf{P})}} \models \text{there is a } q, \text{ and } q \text{ is a semi-master condition for } j(X').$$

Thus

$$V^{\mathbf{P}} \models (\text{there is a } q)\ (q \text{ is a semi-master condition for } X').$$

But $X' \in A$ and so X' has *no* semi-master condition, a contradiction. □

We note that an example of such a $\mathbf{P}$ is $\mathbf{P} = \mathrm{Col}(\omega_1, < \kappa)$. So if κ is supercompact $V^{\mathbf{P}} \models (\dagger)$.

THEOREM 5. *If* "ZFC + *there is a supercompact cardinal* κ" *is consistent, then so is* "ZFC + *Martin's Maximum*". (*In fact we get the* "+" *version of Martin's Maximum.*)

We use the following theorem of Laver:

THEOREM (Laver, ([L1]). *Let κ be a supercompact cardinal. Then there is a function L: $\kappa \to R_\kappa$ such that for every set $Q \in V$ and every cardinal λ there are a $\lambda' > \lambda$ and a λ'-supercompact embedding j: $V \to M$ such that $j(L)(\kappa) = Q$.*

The paradigm for our proof is the proof by Baumgartner of the consistency of the proper forcing axiom.

We will use technology developed by Shelah in [Sh1], [Sh2] to do our iteration. A central notion in [Sh2] is iterating with "revised countable supports". Rather than redevelop these notions we will treat them axiomatically.

We will use the following properties of revised countable support (RCS) iterations:

a) To specify an iteration of length γ, $\mathbf{P}_\gamma$, it is enough to specify for $\alpha < \gamma$ the factor iterations Q_α such that $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * Q_\alpha$.

b) If β is a limit ordinal and $\langle \mathbf{P}_\alpha \colon \alpha < \beta \rangle$ have been defined then the revised countable support limit, $\mathrm{RCS}\lim\langle \mathbf{P}_\beta \colon \beta < \alpha \rangle \subseteq \underleftarrow{\lim}\langle \mathbf{P}_\beta \colon \beta < \alpha \rangle$.

c) If κ is inaccessible and for all $\alpha < \kappa$, $|\mathbf{P}_\alpha| < \kappa$ then

$$\mathrm{RCS}\lim\langle \mathbf{P}_\alpha \colon \alpha < \kappa \rangle = \underrightarrow{\lim}\langle \mathbf{P}_\alpha \colon \alpha < \kappa \rangle.$$

d) If, for all $\alpha < \beta$, $V^{\mathbf{P}_\alpha} \models Q_\alpha$ is $\aleph_1$-semi-proper and $V^{\mathbf{P}_\alpha * Q_\alpha} \models |\mathbf{P}_\alpha * Q_\alpha| = \aleph_1$, and $\mathbf{P}_\alpha$ is an RCS iteration then $\mathrm{RCS}\lim\langle \mathbf{P}_\alpha \colon \alpha < \beta \rangle$ is $\aleph_1$-semi-proper.

e) If for all $\alpha < \beta$, $V^{\mathbf{P}_\alpha} \models Q_\alpha$ is $\aleph_1$-semi-proper and $V^{\mathbf{P}_\alpha * Q_\alpha} \models |\mathbf{P}_\alpha * Q_\alpha| = \aleph_1$ and $\mathbf{P}$ is an RCS iteration then for all $\alpha < \beta$, $V^{\mathbf{P}\alpha} \models$ "$\mathbf{P}_\beta/\mathbf{P}_\alpha$ is an RCS iteration with $\aleph_1$-semi-proper factors".

f) If $\mathbf{P}_\beta$ is an RCS iteration and $\alpha < \beta$ then

$$\mathbf{P}_\beta \sim \mathbf{P}_\alpha * Q_\alpha * \mathbf{P}_\beta/\mathbf{P}_{\alpha+1}.$$

We are now in a position to define our partial ordering for forcing Martin's Maximum. Let L be a Laver function. Our iteration will be an RCS iteration. Hence we need only specify the factors $\langle Q_\alpha \colon \alpha < \kappa \rangle$.

At stage α we have defined an RCS iteration $\mathbf{P}_\alpha$.

Case 1. $L(\alpha)$ is a $\mathbf{P}_\alpha$-term for a partial ordering R_α such that $\|R_\alpha$ is $\aleph_1$-semi-proper partial ordering $\|^{\mathbf{P}_\alpha} = 1$. Then we let

$$Q_\alpha = R_\alpha * \mathrm{Col}^{\mathbf{P}_\alpha * R_\alpha}\left(\omega_1, 2^{|\mathbf{P}_\alpha * \mathbf{R}_\alpha|}\right)$$

(hence $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * R_\alpha * \mathrm{Col}^{\mathbf{P}_\alpha * R_\alpha}(\omega_1, 2^{|\mathbf{P}_\alpha * \mathbf{R}_\alpha|})$).

Case 2. $L(\alpha)$ is a $\mathbf{P}_\alpha$-term for a partial ordering such that $\|L(\alpha)$ is an $\aleph_1$-semi-proper partial ordering $\|^{\mathbf{P}_\alpha} < 1$. Then, in $V^{\mathbf{P}_\alpha}$, let $\delta = \sup(2^{2^{|L(\alpha)|^+}}, 2^{|\mathbf{P}_\alpha|})$ and let $Q_\alpha = \mathrm{Col}(\omega_1, \delta)$ (hence $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \mathrm{Col}(\omega_1, \delta)$).

Case 3. Otherwise. Let $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * 1$. Let $\mathbf{P} = \mathbf{P}_\kappa$.

Using property c) of RCS iterations we see that $\mathbf{P}$ is κ-c.c. Since we are frequently (i.e. always in cases 1) and 2)) collapsing cardinals, $V^{\mathbf{P}} \models \kappa \leq \aleph_2$. By property d) of RCS iterations, $\mathbf{P}$ is $\aleph_1$-semi-proper. Hence $V^{\mathbf{P}} \models \kappa = \aleph_2$.

We now check that $\mathbf{P}$ satisfies the hypothesis of Lemma 3. From the last paragraph we see that a) is satisfied. To see b), let $\gamma \in \mathrm{OR}$. Let $Q = \mathrm{Col}(\omega, \gamma)$. Choose a γ^+-supercompact embedding j such that $j(L)(\kappa) = Q$. Consider $j(\mathbf{P})$.

By property e) of R.C.S. iterations, $j(\mathbf{P}) = \mathbf{P}_\kappa * Q_\kappa * j(\mathbf{P})_{j(\kappa)}/(\mathbf{P})_{\kappa+1}$, and $j(\mathbf{P})$ is defined in M with respect to $j(L)$ the same way that $\mathbf{P}$ is in V.

Hence at stage κ, when $j(L)(\kappa) = \text{Col}(\omega, \gamma)$ we are in Case 2 of the definition of $j(\mathbf{P})$. Hence $Q_\kappa = \text{Col}(\omega_1, \delta)$ for some $\delta \geq \gamma$. Hence $j(\mathbf{P}) = \mathbf{P}_\kappa * \text{Col}(\omega_1, \gamma) * \text{Col}(\omega_1, \delta - \gamma) * j(\mathbf{P})/j(\mathbf{P})_{\kappa+1}$. By property e) of R.C.S. iterations, in $M^{\mathbf{P}_\kappa * \text{Col}(\omega_1, \gamma)}$, $\text{Col}(\omega_1, \delta - \gamma) * j(\mathbf{P})/j(\mathbf{P})_{\kappa+1}$ is $\aleph_1$-semi-proper.

Hence in $V^{\mathbf{P}}$, if a partial ordering Q preserves stationary subsets of ω_1 then it is $\aleph_1$-semi-proper.

Let the semi-proper forcing axiom (SPFA) be MA for $\Gamma =$ "the class of $\aleph_1$-semi-proper partial orderings".

We will be done if we show $V^{\mathbf{P}} \models$ SPFA, since every partial ordering that preserves stationary subsets of ω is $\aleph_1$-semi-proper.

Claim. $V^{\mathbf{P}} \models$ SPFA.

Let $G \subseteq \mathbf{P}$ be generic and let $Q \in V[G]$ be $\aleph_1$-semi-proper. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be a collection of dense sets in Q in $V[G]$. Let $j: V \to M$ be a $|Q|^+$-supercompact embedding such that $j(L)(\kappa)$ is a $\mathbf{P}$-term for Q such that

$$\| j(L)(\kappa) \text{ is } \aleph_1\text{-semi-proper} \|_{\mathbf{P}} = 1.$$

Let $H \subseteq j(\mathbf{P})$ be a V-generic ultrafilter extending G. Then we can extend j to $\hat{j}: V[G] \to M[H]$. By the definition of $j(\mathbf{P})$ in M, $j(\mathbf{P}) = \mathbf{P}_\kappa * Q * \mathbf{R}$ for some $\mathbf{R}$. Hence, $H = G * G' * H'$ where $G' \subseteq Q$ is generic over $V[G]$. In $M[H]$, consider $j''G' \subseteq j(Q)$.

For each D_α, $G' \cap D_\alpha \neq \varnothing$; hence $j''G' \cap j(D_\alpha) \neq \varnothing$. Since $\text{crit}(j) > \aleph_1$, $j(\langle D_\alpha : \alpha < \omega_1 \rangle) = \langle j(D_\alpha) : \alpha < \omega_1 \rangle$. Hence $M[H] \models$"$jG' \subseteq j(Q)$ is generic for $j(\langle D_\alpha : \alpha < \omega_1 \rangle)$".

Thus

$$M[H] \models \text{there is a filter } F \subseteq j(Q) \text{ such that } F \text{ is generic for } j(\langle D_\alpha : \alpha < \omega_1 \rangle).$$

By elementarity,

$$V[G] \models \text{there is a filter } F \subseteq Q \text{ such that } F \text{ is generic for } \langle D_\alpha : \alpha < \omega_1 \rangle.$$

Hence $V[G] \models$ SPFA. A small variation on this argument shows $V[G] \models$ SPFA$^+$. This completes the proof of Theorem 5. □

We now consider several possible Γ's and show that Martin's Maximum implies MA for these Γ's.

Definition. If Q is a partial ordering, then Q is *bounded* if and only if for all $f: \omega_1 \to \omega_1$, $f \in V^Q$, there is a $g \in V$, $g: \omega_1 \to \omega_1$ such that $f(\alpha) > g(\alpha)$ for all α. (Equivalently, Q preserves ω_1 and for all $f: \omega_1 \to \omega_1$, $f \in V^Q$ there is a $g: \omega_1 \to \omega_1$, $g \in V$ such that g eventually dominates f.)

PROPOSITION 6. *If Q is a bounded partial ordering then Q preserves stationary subsets of ω_1.*

Proof. We show that for every closed unbounded set $C \subseteq \omega_1$ in V^Q there is a closed unbounded set $D \subseteq \omega_1$, $D \in V$ and $D \subseteq C$. Let C be club, $C \in V^Q$. Let $f(\alpha) =$ least element of C above α. Let $g \in V$, $g(\alpha) > f(\alpha)$ for all α.

Let $D = \{\beta\colon \text{for all } \alpha < \beta,\ g(\alpha) < \beta\}$. Then D is closed unbounded and it is easy to check that $D \subseteq C$. □

Proposition 6 proves that Martin's Maximum implies MA for $\Gamma = \{Q\colon Q \text{ is a bounded partial ordering}\}$.

We now turn our attention to $\Gamma =$ "the class of partial orderings Q such that Q doesn't add a real or collapse ω_1". Note that in general there is a partial ordering Q such that Q does not add a real or collapse ω_2 and Q kills a stationary set. However Martin's Maximum implies that there are no such Q.

PROPOSITION 7. *Martin's Maximum implies* MA *for* $\Gamma =$ "*the class of partial orderings Q that do not add reals or collapse $\aleph_2$*".

Proof. Baumgartner has shown that the proper forcing axiom implies that there are no Canadian trees on $\aleph_1$. Todorcevic showed that if there are no Canadian trees and every Aronzähn tree is special then every partial order that adds a subset of ω_1 either collapses ω_2 or adds a real.

Consequently, if Q is a partial ordering that does not add reals or collapse ω_2 then Q adds no new subsets to ω_1. By Proposition 6, Martin's Maximum implies MA for such Q. □

In Section 3 we show the consistency of CH + MA for various Γ's. We use Lemma 3 there also.

2. Applications of Martin's Maximum

We now prove some results using Martin's Maximum. The general outline of these proofs is the same as for applications of Martin's Axiom; e.g., given a partial ordering Q, we verify that it has some property (in this case, Q preserves stationary subsets of ω_1) and then meet ω_1 dense sets by a filter G and argue combinatorially about the filter G.

In the following we abbreviate Martin's Maximum by MM.

LEMMA 8. *Suppose κ is regular, $\kappa \geq \omega_2$ and $A \subseteq \kappa \cap \operatorname{cof}(\omega)$ is stationary. Let $S \subseteq \omega_1$ be stationary and $\lambda > 2^\kappa$ be a regular cardinal. Then for any expansion of $\langle H(\lambda), \varepsilon\rangle$, $\langle H(\lambda), \varepsilon, \Delta, f_i\rangle_{i \in \omega}$, there is an $N \prec$*

$\langle H(\lambda), \varepsilon, \Delta, f_i\rangle_{i\in\omega}$ a countable elementary substructure of $H(\lambda)$ such that $N \cap \omega_1 \in S$ and $\sup N \cap \kappa \in A$.

Proof. Let $M \prec \langle H(\lambda), \varepsilon, \Delta, f_i\rangle_{i\in\omega}$ be an elementary substructure of $H(\lambda)$ such that $\omega_1 \subseteq M$ and $\sup(M \cap \kappa) \in A$. (Such an M exists since there is a club set of uncountable elementary substructures of $\langle H(\lambda), \varepsilon, f_i\rangle_{i\in\omega}$.) Let $\langle \alpha_n \colon n \in \omega\rangle \subseteq M \cap \kappa$ be cofinal in $M \cap \kappa$.

Let $\langle N_\alpha \colon \alpha < \omega_1\rangle$ be a continuous increasing chain of countable elementary substructures of M such that $\langle \alpha_n \colon n \in \omega\rangle \subseteq N_0$. Then for each N_α, $\sup N_\alpha \cap \kappa = \sup M \cap \kappa \in A$. Further, $\{N_\alpha \cap \omega_1 \colon \alpha < \omega_1\}$ is a closed unbounded set in ω_1. Thus for some α, $N_\alpha \cap \omega_1 \in S$. Thus N_α is the required N. □

THEOREM 9. MM *implies*:

If $\kappa \geq \omega_2$ is regular and $A \subseteq \kappa \cap$ cof ω is stationary, then A contains a closed set of order type ω_1.

Proof. Let $\mathbf{P} = \langle\{p | p\colon \alpha + 1 \to A, \alpha < \omega_1$ and p is an increasing continuous function$\}, \subseteq\rangle$. Standard lemmas imply that for any $p \in \mathbf{P}$ and $\beta < \omega_1$, there is a $q \Vdash p$ such that $\beta \in \operatorname{dom} q$. Hence forcing with $\mathbf{P}$ adds a closed set $C \subseteq A$ such that o.t. $C = \omega_1$. Further, for any $p \in \mathbf{P}$ and $\gamma \in \kappa$ there is a $q \Vdash p$ such that $\gamma < \sup \operatorname{range} q$.

We claim that $\mathbf{P}$ preserves stationary subsets of ω_1.

Let $p \in \mathbf{P}$ and $S \in V$, $S \subseteq \omega_1$ be a stationary set. Let C be a term for a closed unbounded subset of ω_1. Let $N \prec \langle H(\lambda), \varepsilon, \Delta, \mathbf{P}, \cdots\rangle$ be a countable elementary substructure of $H(\lambda)$, $\lambda \gg \kappa$, such that $p \in N$, $\delta = N \cap \omega_1 \in S$ and $\sup N \cap \kappa \in A$.

Let $\langle p_n \colon n \in \omega\rangle \subseteq N$ be a generic sequence for N such that $p_0 = p$. Then $\bigcup_{n\in\omega} \operatorname{dom} p_n = \delta$ and $\bigcup_{n\in\omega} \operatorname{range} p_n$ is cofinal in $N \cap \kappa$. Hence the function $q\colon \delta + 1 \to \kappa$ defined by $q = \bigcup_{n\in\omega} p_n \cup \{\langle \delta, \sup N \cap \kappa\rangle\}$ is a continuous function with range a subset of A. Hence $q \in \mathbf{P}$ and for each n, $q \Vdash p_n$. Thus by Lemma $*$, $\mathbf{P}$ preserves stationary subsets of ω_1.

Let $\mathcal{D} = \langle D_\alpha \colon \alpha < \omega_1\rangle$ be defined by $D_\alpha = \{p \in \mathbf{P} \colon \alpha \in \operatorname{dom} p\}$. Let G be a filter generic for $\mathcal{D}$. Then $\bigcup G\colon \omega_1 \to \kappa$ is an increasing continuous function with range included in A. Hence A contains a closed set of order type ω_1. □

We note that Ben-David remarked that the conclusion of Theorem 1 and $\diamondsuit(\operatorname{cof}(\omega))$ implies $\diamondsuit(\operatorname{cof}(\omega_1))$.

The conclusion of Theorem 9 is known as "Friedman's Problem". Shelah [Sh1] has shown it consistent for $\kappa = \omega_2$ from a measurable cardinal and for general regular κ from two supercompacts.

A closely related theorem is:

THEOREM 10. *If* $\kappa \geq \omega_2$ *is regular and* MM *holds then* $\kappa^{\omega_1} = \kappa$. *In particular* $2^{\aleph_0} = \aleph_2$.

Proof. Let $\langle S_\alpha: \alpha < \omega_1 \rangle$ be a disjoint maximal antichain in $\mathscr{P}(\omega_1)/NS_{\omega_1}$ such that $\bigcup_{\alpha < \omega_1} S_\alpha = \omega_1$. Let $\langle A_\alpha: \alpha < \kappa \rangle$ be a partition of $\kappa \cap \mathrm{cof}(\omega)$ into κ disjoint stationary subsets.

We will build a one-to-one function $i: [\kappa]^{\omega_1} \to \kappa$. This clearly suffices.

Let $f \in [\kappa]^{\omega_1}$. Define the partial ordering $\mathbf{P}_f$ by: $p \in \mathbf{P}_f$ if and only if for some $\delta < \omega_1$, $p: \delta + 1 \to \kappa$, p is increasing and continuous and for all $\beta \leq \delta$, if $\beta \in S_\alpha$ then $p(\beta) \in A_{f(\alpha)}$. $\mathbf{P}_f$ is ordered by inclusion.

Claim. If $p \in \mathbf{P}_f$ and $\delta > \sup \mathrm{dom}\, p$, $\delta \in \omega_1$ and $\gamma < \kappa$, then there is a $q \Vdash p$ such that $\delta \in \mathrm{dom}\, q$ and $q(\delta) > \gamma$.

Proof. We prove this by induction on δ. If δ is a successor, $\delta = \beta + 1$, this is immediate.

Assume that it is true for all $\delta' < \delta$; let $\gamma \in \kappa$ and suppose that $\delta \in S_\alpha$. Let $N \prec \langle H(\lambda), \varepsilon, \Delta \cdots \rangle$ be a countable elementary substructure of $H(\lambda)$, such that $\gamma < \sup N \cap \kappa \in A_{f(\alpha)}$ and $\delta, p \in N$.

Let $\langle \alpha_n: n \in \omega \rangle \subseteq N \cap \kappa$ be cofinal in $N \cap \kappa$. Using our induction hypothesis inside N we can build a sequence of conditions $\langle p_n: n \in \omega \rangle \subseteq N$ such that $p_{n+1} \Vdash p_n$, $p_0 = p$ and $\bigcup_{n \in \omega} \mathrm{dom}\, p_n = \delta$ and $\bigcup_{n \in \omega} \mathrm{range}\, p_n$ is cofinal in $N \cap \kappa$. Let $q = \bigcup_{n \in \omega} p_n \cup \{\langle \delta, \sup N \cap \kappa \rangle\}$. Then q is continuous, $q(\delta) \in N_{f(\alpha)}$ and hence $q \in \mathbf{P}_f$ is as desired. □

Claim. $\mathbf{P}_f$ preserves stationary subsets of ω_1.

Proof. Let $S \subseteq \omega_1$ be stationary, $C \in \mathbf{P}_f$ be a term for a club subset of ω_1 and $p \in \mathbf{P}_f$. As usual we will be done if we can show that there is an $N \prec \langle H(\lambda), \varepsilon, \Delta, C, p \rangle$ such that $N \cap \omega_1 \in S$ and there is a strong master condition for N, q, extending p.

Since $\langle S_\alpha: \alpha < \omega_1 \rangle$ is a maximal antichain there is an α such that $S \cap S_\alpha$ is stationary. Let $N \prec \langle H(\lambda), \varepsilon, \Delta, \mathbf{P}_f, f_{i \in \omega} \rangle$ be a countable elementary substructure of $H(\lambda)$ such that $\delta = N \cap \omega_1 \in S \cap S_\alpha$ and $\gamma = \sup N \cap \kappa \in A_{f(\alpha)}$.

Let $\langle p_n: n \in \omega \rangle \subseteq N$ be a generic sequence for N such that $p_0 = p$. Then it is easy to verify that $q = \bigcup_{n \in \omega} p_n \cup \{\langle \delta, \gamma \rangle\}$ is a condition forcing p_n for each n. Hence by Lemma $*$, $\mathbf{P}_f$ preserves stationary subsets of ω_1.

Thus we are in a position to apply Martin's Maximum to $\mathbf{P}_f$. Let $D_\delta \subseteq \mathbf{P}_f$ be defined by $D_\delta = \{p \in \mathbf{P}_f: \delta \in \mathrm{dom}\, p\}$. Let $G \subseteq \mathbf{P}_f$ be generic for $\mathscr{D} = \langle D_\delta: \delta < \omega_1 \rangle$. Then $F = \bigcup G: \omega_1 \to \kappa$ is a continuous function such that if $\delta \in S_\alpha$ then $F(\delta) \in A_{f(\alpha)}$. Let $\gamma_f = \sup \mathrm{range}\, F$. Hence $A_{f(\alpha)} \cap \gamma_f$ contains the continuous image of a stationary subset of ω_1 and hence is stationary. Further, $\bigcup_{\alpha<\omega_1} A_{f(\alpha)} \cap \gamma_f$ contains a closed unbounded set in γ_f. Thus for $\beta < \kappa$, $A_\beta \cap \gamma_f$ is stationary in γ_f if and only if $\beta \in$ range f. Since f is increasing we can recover f from its range. Hence γ_f uniquely determines f.

Define i: $[\kappa]^{\omega_1} \to \kappa$ by $i(f) = \gamma_f$. We have just argued that i is always defined and is one-to-one. □

COROLLARY 11. *If* MM *holds, then for singular cardinals* κ, $\kappa^{\operatorname{cof}(\kappa)} = \max(\kappa^+, 2^{\operatorname{cof}(\kappa)})$.

Proof. By standard arguments $\kappa^{\operatorname{cof}(\kappa)} \le (\kappa \times 2^{\operatorname{cof}\kappa})^{\operatorname{cof}\kappa}$, so if $2^{\operatorname{cof}(\kappa)} \ge \kappa$ then $\kappa^{\operatorname{cof}(\kappa)} = 2^{\operatorname{cof}(\kappa)}$.

We prove by induction on κ that if $\kappa > 2^{\operatorname{cof}(\kappa)}$ then $\kappa^{\operatorname{cof}(\kappa)} = \kappa^+$.

If $\operatorname{cof}(\kappa) = \omega$ or ω_1, then $\kappa^{\operatorname{cof}(\kappa)} = \kappa^+$ since $(\kappa^+)^{\omega_1} = \kappa^+$.

If $\operatorname{cof}(\kappa) > \omega_1$ then there is a closed unbounded set $C \subseteq \kappa$ such that if $\mu \in C$ then $\operatorname{cof}(\mu) < \operatorname{cof}(\kappa)$ and $\mu > 2^{\operatorname{cof}(\kappa)}$. By induction, $\mu^{\operatorname{cof}\mu} = \mu^+$. By Silver's theorem [Si1] "on the G.C.H. at singular cardinals of uncountable confinality" $\kappa^{\operatorname{cof}(\kappa)} = \kappa^+$. □

This corollary can be regarded as heuristic evidence for the necessity of a supercompact cardinal in the proof of the consistency of Martin's Maximum.

Using the techniques of [M1] one can show that if "ZFC + there is a supercompact cardinal" is consistent then so is "ZFC + there is a supercompact cardinal κ such that there is a cofinal set $A \subseteq \kappa$ of strong singular limit cardinals with the property that $\alpha \in A$ implies $2^\alpha > \alpha^+$". In the latter model, if $\mathbf{P}_\kappa$ is the partial ordering defined in Theorem 5 for adding MM, then by Corollary 11 for all $\beta < \kappa$, $V^{\mathbf{P}_\beta} \models \neg$MM. Further, $\langle V_\kappa, \varepsilon\rangle \models$ ZFC and no set forcing can force MM to hold in $\langle V_\kappa, \varepsilon\rangle$.

Saturation properties of ideals have a wide literature ([K1], [F1], [F2], [F-L], [M] etc). A natural ideal to study is the non-stationary ideal on a regular cardinal κ.

Steel and Van Wesep in [S-VW] showed that relative to the theory "$\mathrm{AD}_{\mathbf{R}}$ + θ-regular + ZF + DC" it is consistent for the non-stationary ideal on ω_1 to be $\aleph_2$-saturated.

We show:

THEOREM 12. *If* MM *holds then* NS_{ω_1} *is* $\aleph_2$*-saturated.*

Later we shall show that for various Γ's such that MA for Γ is consistent with CH, MA for Γ implies there is a stationary set S such that $\mathrm{NS}_{\omega_1} \restriction S$ is $\aleph_2$-saturated.

Before proving Theorem 12 we remark that if $\mathcal{I}$ is a normal, κ-complete ideal on κ and $B = \mathcal{P}(\kappa)/\mathcal{I}$ and $\langle A_\alpha\colon \alpha < \kappa\rangle$ are $\mathcal{I}$-positive sets then we can represent the Boolean sum $\Sigma_{\alpha<\kappa}[A_\alpha]_{\mathcal{I}}$ by $\nabla_{\alpha<\kappa}A_\alpha = \{\beta\colon$ there is an $\alpha < \beta$ such that $\beta \in A_\alpha\}$. In other words $\Sigma_{\alpha<\kappa}[A_\alpha]_{\mathcal{I}} = [\nabla_{\alpha<\kappa}A_\alpha]_{\mathcal{I}}$.

Proof of Theorem 12. Let $\langle A_\alpha\colon \alpha < \omega_2\rangle$ be a putative antichain in $\mathcal{P}(\omega_1)/_{\mathcal{I}}$. Without loss of generality we may assume that it is a maximal antichain.

Let $\mathbf{P} = \mathrm{Col}(\omega_1, \omega_2) * Q$ where Q is defined in $V^{\mathrm{Col}(\omega_1, \omega_2)}$ as follows.

Let $G\colon \omega_1 \to \omega_2^V$ be the canonical generic object. Then define $\nabla_G A_\alpha = \{\beta\colon$ there is an $\alpha < \beta$, $\beta \in A_{G(\alpha)}\}$. Since $\nabla_G A_\alpha \supseteq A_{G(0)}$, $\nabla_G A_\alpha$ is stationary in $V^{\mathbf{P}}$. Let Q be the partial ordering for shooting a closed set through $\nabla_G A_\alpha$ with countable conditions (See [B-H-K]). So $q \in Q$ if and only if $q\colon \alpha + 1 \to \nabla_G A_\alpha$ for some countable α and q is continuous and increasing. Note that there is a dense set $D \subseteq \mathbf{P}$ of conditions of the form (p, q) where $q \in V$.

Claim. $\mathbf{P}$ preserves stationary subsets of ω_1.

Proof. Let $S \subseteq \omega_1$ be a stationary set, $\dot{C} \in V^{\mathbf{P}}$ be a term for a closed unbounded set and $p \in \mathbf{P}$. As usual we will be done when we show that there is a $q \Vdash p$ such that $q \Vdash \dot{C} \cap S \neq \varnothing$.

Since $\langle A_\alpha\colon \alpha < \omega_2\rangle$ is a maximal antichain, there is an $\alpha < \omega_2$ such that $S \cap A_\alpha$ is stationary.

Let $\lambda \geq 2^{2^{|\mathbf{P}|^+}}$ and $N \prec \langle H(\lambda), \varepsilon, \Delta, \mathbf{P}, \langle A_\alpha\colon \alpha < \omega_2\rangle, S \ldots\rangle$ be a countable elementary substructure of $H(\lambda)$ such that $\{p, \alpha\} \subseteq N$ and $\delta = N \cap \omega_1 \in A_\alpha \cap S$.

Let $\langle\langle p_n, q_n\rangle\colon n \in \omega\rangle \subseteq N$ be a generic sequence for N such that $p = \langle p_0, q_0\rangle$ and $p_1(\sup \mathrm{dom}(p_0)) = \alpha$. Then $p^* = \bigcup_{n\in\omega} p_n$ is a condition in $\mathrm{Col}(\omega_1, \omega_2)$. Further $\bigcup_{n\in\omega}\mathrm{dom}\, q_n = \delta$ and $\sup \bigcup_{n\in\omega}\mathrm{range}\, q_n = \delta$. Since $\delta \in A_\alpha$ and $\delta > \sup(\mathrm{dom}\, p_0)$, $p^* \Vdash \delta \in \nabla_G A_\alpha$. Hence $p^* \Vdash q^* = \bigcup_{n\in\omega} q_n \cup \{\langle\delta, \delta\rangle\}$ is a continuous increasing function with range in $\nabla_G A_\alpha$". So $p^* \Vdash q^* \in Q$. Then for each n, $(p^*, q^*) \Vdash (p_n, q_n)$; so by Lemma $*$, the claim holds.

Let $\mathcal{D} = \langle D_\alpha\colon \alpha < \omega_1\rangle$ be the following collection of dense sets:

$D_\alpha = \{(p, q)\colon \alpha \in \mathrm{dom}\, p$ and $\alpha \in \mathrm{dom}\, q\}$. Let $H \subseteq \mathbf{P}$ be generic for $\mathcal{D}$. Let $G = \bigcup\{p\colon$ there is a q such that $(p, q) \in H\}$ and $C = \bigcup\{q\colon$ there is a p, $(p, q) \in H\}$. Then $G \in V$, $G\colon \omega_1 \to \omega_2$ and C is a closed subset of ω_1. Further, $\nabla_G A_\alpha = \{\beta\colon$ for some $\alpha < \beta$, $\beta \in A_{G(\alpha)}\} \supseteq C$. Hence $\Sigma_{\alpha<\kappa}[A_{G(\alpha)}] = [\nabla_G A_\alpha] = 1$. But the range of G has cardinality $\aleph_1$; so some A is incompatible with $\Sigma_{\alpha<\kappa}[A_{G(\alpha)}]$, a contradiction! □

We will later show that under MM the non-stationary ideal is "c.c.c. indestructible".

A combinatorial key to the preceding results is the equivalence of $\aleph_1$-semi-properness and the preserving of stationary subsets of ω_1. We now examine this property more carefully.

If $S \subseteq [H(\lambda)]^\omega$ then we say that S *reflects to a set of size* $\aleph_1$ if and only if there is an $X \subseteq H(\lambda)$, $\omega_1 \subseteq x$, $|X| = \aleph_1$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$. (Equivalently, if $S \subseteq [Z]^\omega$ is stationary then S reflects to a set of size ω_1 if and only if for all $Y \subseteq Z$, $|Y| = \omega_1$, there is an X such that $Y \subseteq X$, $|X| = \omega_1$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$.)

We remark that MA^+ for Γ: "the class of ω-closed partial orderings" implies that for every regular λ, every stationary subset of $[H(\lambda)]^\omega$ reflects to a set of size $\aleph_1$.

To see this we apply MA^+ to $\mathbf{P} = \mathrm{col}(\omega_1, H(\lambda))$.

By Proposition 4, in $V^{\mathbf{P}}$ we get a function $f\colon \omega_1 \xrightarrow[\text{onto}]{1-1} H(\lambda)^V$ such that $\{\alpha\colon f''\alpha \in S\}$ is stationary in ω_1. Hence by MA^+ we get a function $F \in V$, $f\colon \omega_1 \xrightarrow{1-1} H(\lambda)$ such that $\{\alpha\colon f''\alpha \in S\}$ is stationary in ω_1. Hence, taking $Xf''\alpha$ we get the desired result. Surprisingly MM is enough to get this result. (In fact this proof shows that MA^+ for $\Gamma =$ "ω-closed partial orderings" implies that for any stationary subset $S \subseteq P_{\omega_1}(H(\lambda))$ there is a stationary set $T \subseteq P_{\omega_2}(H(\lambda))$ such that for all $x \in T$, $S \cap P_{\omega_1}(x)$ is stationary. For each $g\colon H(\lambda)^{<\omega} \to H(\lambda)$ we use the term $\dot{S}^* \in V^{\mathbf{P}}$, $S^* = \{\alpha\colon f''\alpha \in S$ and $f''\alpha$ is closed under $g\}$.)

THEOREM 13. *Assume* MM. *Then for every regular* λ *and every stationary set* $S \subseteq [H(\lambda)]^\omega$, S *reflects to a set of size* ω_1.

Proof. Since the non-stationary ideal on ω_1 is $\aleph_2$-saturated, there are $\aleph_1$ stationary subsets of ω_1, $\langle A_\alpha\colon \alpha < \omega_1\rangle$, such that:

a) For each α there is a closed unbounded set C_α in $[H(\lambda)]^\omega$ such that $A_\alpha \cap \{x \cap \omega_1\colon x \in C_\alpha \cap S\} = \varnothing$.

b) For every A if there is a closed unbounded $C \subseteq [H(\lambda)]^\omega$ with $A \cap \{x \cap \omega_1\colon x \in C \cap S\} = \varnothing$ then $A - \nabla_{\alpha<\omega_1} A_\alpha$ is non-stationary.

Let $\mathbf{P} = Q * R$ where $Q = \mathrm{Col}(\omega_1, |H(\lambda)|)$ and R is defined in V^Q as follows. Let $f \in V^Q$ be the generic function $f\colon \omega_1 \xrightarrow[\text{onto}]{1-1} H(\lambda)$. By Proposition 4, $\{\alpha\colon f''\alpha \in S\}$ is stationary. Let R be the partial ordering for shooting a closed set through $\{\alpha\colon f''\alpha \in S\} \cup \nabla_{\alpha<\omega_1} A_\alpha$, with countable conditions. So $r \in R$ if and only if for some countable δ, r is a continuous, increasing function $r\colon \delta + 1 \to \{\alpha\colon f''\alpha \in S\} \cup \nabla_{\alpha<\omega_1} A_\alpha$.

We claim that $\mathbf{P}$ preserves stationary subsets of ω_1. Note that there is a dense set in $\mathbf{P}$ of conditions of the form $(q, r) \in Q * R$ where $r \in V$. Let

$B \subseteq \omega_1$ be stationary, $\dot{C} \in V^Q$ be a term for a stationary set and $p \in \mathbf{P}$. As usual we will be done if we find a $p^* \Vdash p$ such that $p^* \Vdash C \cap S \neq \varnothing$. We show this in the usual way by a master-condition argument.

Case 1. $B \cap \nabla_{\alpha < \omega_1} A_\alpha$ is stationary. Then by the usual arguments if $N \prec H((2^{2^\lambda})^+)$ is a countable elementary substructure and $\delta = N \cap \omega_1 \in B \cap \nabla_{\alpha < \omega_1} A_\alpha$ then N has a master condition $p^* \Vdash p$. So $p^* \Vdash \delta \in \dot{C} \cap B$.

Case 2. Otherwise. Then for every closed unbounded set $D \subseteq [H(\lambda)]^\omega$ there is an $N \in D \cap S$ such that $N \cap \omega_1 \in B$. Let $N \prec \langle H((2^{2^\lambda})^+)$, $\varepsilon, \Delta, S, B, \ldots \rangle$ such that $N \cap H(\lambda) \in S$ and $\delta = N \cap \omega_1 \in B$. There is such an N by Lemma 0. Let $\langle p_n \colon n \in \omega \rangle$ be a generic sequence for N such that $p_0 = p$. Let $p_n = \langle q_n, r_n \rangle$. Then $\bigcup \operatorname{dom} q_n = \delta$ and $\bigcup \operatorname{range} q_n = N \cap H(\lambda)$. Hence if $q^* = \bigcup_{n \in \omega} q_n$ then $q^* \Vdash \delta \in \{\alpha \colon f''\alpha \in S\}$. Let $r^* = \bigcup_{n \in \omega} r_n \cup \{\langle \delta, \delta \rangle\}$. Since $\bigcup_{n \in \omega} \operatorname{dom} r_n = \delta$ and $\sup \bigcup_{n \in \omega} \operatorname{range} r_n = \delta$, r^* is a continuous function. Further, $q^* \Vdash \operatorname{range} r^* \subseteq \{\alpha \colon f''\alpha \in S\} \cup \nabla_{\alpha < \omega_1} A_\alpha$. Hence $p^* = (q^*, r^*) \in \mathbf{P}$.

Since $p^* \Vdash p_n$ for each n, p^* is a master condition for n and $p^* \Vdash \delta \in \dot{C} \cap B$.

Since $\mathbf{P}$ preserves stationary subsets of ω_1, using MM we can find a generic object G for $\mathscr{D} = \langle D_\alpha \colon \alpha < \omega_1 \rangle$ where $D_\alpha = \{p \in \mathbf{P} \colon p = \langle q, r \rangle$ and $\alpha \in \operatorname{dom} q \cap \operatorname{dom} r$ and $\alpha \in \operatorname{range} q\}$. Let f be the canonical function $f \colon \omega_1 \to H(\lambda)$ coming from G and $C = \bigcup \{r \colon$ there is a $q \in Q$, $(q, r) \in G\}$. Then C is a closed unbounded set and $C \subseteq \{\alpha \colon f''\alpha \in S\} \cup \nabla_{\alpha < \omega_1} A_\alpha$. Thus we will be done if we can show that $\omega_1 - (\nabla_{\alpha < \omega_1} A_\alpha)$ is stationary, since this will show that $S \cap \mathscr{P}_{\omega_1}(\operatorname{range} f)$ is stationary.

For each α we have $C_\alpha \subseteq [H(\lambda)]^\omega$ such that $A_\alpha \cap \{x \cap \omega_1 \colon x \in C_\alpha \cap S\} = \varnothing$. Then $\nabla_{\alpha < \omega_1} A_\alpha \cap (\Delta_{\alpha < \omega_1} \{x \cap \omega_1 \colon x \in C_\alpha \cap S\}) = \varnothing$ and

$$\Delta_{\alpha < \omega_1} \{x \cap \omega_1 \colon x \in C_\alpha \cap S\} \supseteq \left\{x \cap \omega_1 \colon x \in \left(\Delta_{\alpha < \omega_1} C_\alpha\right) \cap S\right\}.$$

But $\Delta_{\alpha < \omega_1} C_\alpha$ is closed and unbounded in $[H(\lambda)]^\omega$. Hence $\Delta_{\alpha < \omega_1} C_\alpha \cap S$ is stationary, so that $\{x \cap \omega_1 \colon x \in (\Delta_{\alpha < \omega_1} C_\alpha) \cap S\}$ is stationary and disjoint from $\nabla_{\alpha < \omega_1} A_\alpha$. □

Shelah has shown in [Sh3] that if every stationary subset of $[\aleph_2]^{<\omega_1}$ reflects $2^{\aleph_0} \leq \aleph_2$. This gives an alternate proof that MM and MA^+ for $\Gamma =$ "ω-closed partial orderings" imply $2^{\aleph_0} = \aleph_2$.

Reflecting stationary subsets of $[H(\lambda)]^\omega$ is the crux of the equivalence between $\aleph_1$-semi-properness and the preserving of stationary subsets of ω_1, as the following proposition shows:

Proposition 14. *Suppose for a cofinal set of regular cardinals λ every stationary subset of $[H(\lambda)]^\omega$ reflects to a set of size ω_1; then for all partial orderings* **P**, **P** *is $\aleph_1$-semi-proper if and only if* **P** *preserves stationary subsets of ω_1.*

Proof. Suppose **P** preserves stationary subsets of ω_1 and **P** is not $\aleph_1$-semi-proper. Then for some regular $\lambda > |\mathbf{P}|$, for which every stationary subset of $[H(\lambda)]^\omega$ reflects, there is a stationary subset S of $[H(\lambda)]^\omega$ and $p \in \mathbf{P}$ such that $N \in S$ implies there is no semi-master condition $q \Vdash p$ for N.

Unravelling the definition we see that $p \Vdash$ "If $N \in S$ then $N[G] = \{\tau^{V[G]}: \tau \subset N, \tau \text{ a } \mathbf{P}\text{-term}\} \cap \omega_1 \neq N \cap \omega_1$." Since S reflects to a set of size $\aleph_1$ there is a function $f: \omega_1 \xrightarrow{1-1} H(\lambda)$ such that $T = \{\alpha: f''\alpha \in S\}$ is stationary.

In $V[G]$, let $C = \{N \prec H(\lambda)^{V[G]}: N$ is closed under the function sending $\tau \in V^{\mathbf{P}}$ to its realization $\tau^{V[G]}$ and N is closed under f and $f^{-1}\}$. Then C is a closed unbounded set. Since **P** preserves stationary subsets of ω_1, T is stationary in $V^{\mathbf{P}}$ and hence there is an $N \in C$ such that $\delta = N \cap \omega_1 \in T$. Let $N' = f''\delta$. Then $N' \in S$ and $N' \cap \omega_1 = \delta$. Further $N'[G] \cap \omega_1 = \delta$ since N is closed under the function-realizing terms in N. But $N' \in S$ implies $N'[G] \cap \omega_1 \neq N' \cap \omega_1$, a contradiction. □

We let (†) abbreviate the proposition "for all partial orderings **P**, **P** preserves stationary subsets of ω_1 if and only if **P** is $\aleph_1$-semi-proper." Then (†) is itself a combinatorial principle of some strength as we shall show.

Corollary 15. MM *implies* (†) *and* SPFA$^+$ *implies* MM. (*So* MM$^+$ *if and only if* SPFA$^+$.)

We remark that we could have given an alternate proof of the consistency of MM as follows:

We show that Lemma 3 implies MA$^+$ for Γ = "countably closed partial orderings." By Proposition 14, MA$^+$ for Γ = "countably closed partial orderings" implies (†). Hence it is enough to show the consistency of SPFA$^+$. The argument given was our original argument. We present it as it generalizes to get precipitous ideals on larger cardinals.

Let the *Strong Chang Conjecture* be the following property:

For every structure $\mathscr{A} = \langle A; \omega_1, f_i\rangle_{i \in \omega}$ of type $(\aleph_2, \aleph_1)$ there is a closed unbounded set $C \subseteq \omega_1$ such that $\alpha \in C$ implies that there is an $\mathscr{L} \prec \mathscr{A}$ of type $(\aleph_1, \aleph_0)$ such that $\mathscr{L} \cap \omega_1 = \alpha$.

The following result appears in [Sh1]:

THEOREM (Shelah).

a) *Namba forcing preserves stationary subsets of* ω_1.

b) *If Namba forcing is* $\aleph_1$*-semi-proper then the Strong Chang Conjecture holds.* (*In fact Shelah obtains much stronger results than* b).)

If $\mathscr{I}$ is an $\aleph_2$-saturated ideal on ω_1, then $\mathscr{I}$ is c.c.c. indestructible if and only if whenever $\mathbf{P}$ is a c.c.c. partial ordering then $\bar{\mathscr{I}} = \{x \subseteq \omega_1: x \in V^{\mathbf{P}}$ and there is a $y \in \mathscr{I}$ such that $x \subseteq y\}$ is $\aleph_2$-saturated. ($\bar{\mathscr{I}}$ is the ideal in $V^{\mathbf{P}}$ induced by $\mathscr{I}$.)

A question in [B-T2] is whether there can be c.c.c. indestructible ideals on ω_1. In [F-M1], Foreman and Magidor show that there can be a $\aleph_2$-saturated ideal on ω_1 that is not c.c.c. indestructible.

In [B-T2] there is a Chang's Conjecture-type criterion for the c.c.c. indestructibility of an $\aleph_2$-saturated ideal on ω_1. We present another one which holds under MM.

THEOREM 16 (ZFC). *Suppose the Strong Chang Conjecture holds, S is a stationary subset of* ω_1 *and* $\mathrm{NS}_{\omega_1} \upharpoonright S$ *is* $\aleph_2$*-saturated. Then* $\mathrm{NS}_{\omega_1} \upharpoonright S$ *is* c.c.c. *indestructible.*

If $\mathscr{I}$ is an $\aleph_2$-saturated ideal on ω_1 and $G \subseteq \mathscr{P}(\omega_1)/\mathscr{I}$ is generic, let j: $V \to M = V^{\aleph_1}/G$ be the generic ultrapower. Then Laver [L2] and Baumgartner-Taylor [B-T2] showed the following criterion of c.c.c. indestructibility:

THEOREM. $\|\bar{\mathscr{I}}$ *is* $\aleph_2$*-saturated*$\|_{\mathbf{P}} = 1$ *if and only if* $\|j(\mathbf{P})$ *is* c.c.c. *in* $V[G]\|_{\mathscr{P}(\omega_1)/\mathscr{I}} = 1$.

Proof of Theorem 16. Let $\mathbf{P}$ be a c.c.c. partial ordering. By the theorem of Laver, Baumgartner-Taylor, we must see that for any generic $G \subseteq \mathscr{P}(\omega_1)/\mathscr{I}$, $V[G] \models j(\mathbf{P})$ is c.c.c.

Since $\mathrm{NS}_{\omega_1} \upharpoonright S$ is $\aleph_2$-saturated $\omega_1^{V[G]} = \omega_2^V$. Hence $j(\mathbf{P})$ is not c.c.c. if and only if in $V[G]$ there are functions $\langle f_\alpha: \alpha < \omega_2^V \rangle$ such that $f_\alpha \in V$, $f_\alpha: \omega_1 \to \mathbf{P}$ and for all $\alpha, \beta \in \omega_2$, $I_{\alpha,\beta} = \{\delta: f_\alpha(\delta)$ is incompatible with $f_\beta(\delta)\} \in G$. Let $\langle \dot{f}_\alpha: \alpha < \omega_2^V \rangle$ be a term in V for such a sequence. Let $T \subseteq S$ be a stationary set such that $[T]_{\mathrm{NS}_{\omega_1} \upharpoonright S} \Vdash I_{\alpha,\beta} \in G$ for all $\alpha < \beta < \omega_2^V$. By the standard theory of saturated ideals (see [J1]) there is a sequence of functions $\langle g_\alpha: \alpha < \omega_2 \rangle \in V$, g_α: $\omega_1 \to \mathbf{P}$ such that $[T] \Vdash \{\delta: \dot{f}_\alpha(\delta) = g_\alpha(\delta)\} \in G$. Hence for $\alpha, \beta \in \omega_2$, $[T] \Vdash \{\delta: g_\alpha(\delta)$ and $g_\beta(\delta)$ are incompatible$\} \in G$.

Since our ideal is $\mathrm{NS}_{\omega_1} \upharpoonright S$ there are closed unbounded sets $C_{\alpha,\beta}$ such that for all $\delta \in C_{\alpha,\beta} \cap T$, $g_\alpha(\delta)$ and $g_\beta(\delta)$ are incompatible.

Let $\lambda > 2^{\omega_2}$ and let $\mathscr{A} \prec \langle H(\lambda), \omega_1, \varepsilon, \Delta, \langle g_\alpha: \alpha < \omega_2\rangle, \langle C_{\alpha,\beta}: \alpha, \beta < \omega_2\rangle\rangle$ be an elementary substructure of $H(\lambda)$ such that $|\mathscr{A}| = \omega_2$ and $\omega_2 \subseteq \mathscr{A}$.

Since the Strong Chang Conjecture holds there is an $\mathscr{L} \prec \mathscr{A}$ such that $\delta = \mathscr{L} \cap \omega_1 \in T$ and $|\mathscr{L} \cap \omega_2| = \omega_1$. Then for all $\alpha, \beta \in \mathscr{L} \cap \omega_2$, $C_{\alpha,\beta}$ is unbounded in δ and hence $\delta \in C_{\alpha,\beta}$. Thus $f_\alpha(\delta)$ and $f_\beta(\delta)$ are incompatible. But then $\{f_\alpha(\delta): \alpha \in \mathscr{L} \cap \omega_2\}$ is an antichain in $\mathbf{P}$ of size ω_1, a contradiction. □

COROLLARY 17. *If* MM *holds then* NS_{ω_1} *is* $\aleph_2$*-saturated and* c.c.c. *indestructible.*

Proof. Assume MM. By Shelah's theorem and Corollary 15, the Strong Chang Conjecture holds. Hence the hypothesis of Theorem 16 hold for NS_{ω_1}. □

It is not known how to describe the quotient algebra $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ exactly under MM, but the following theorem yields some information.

THEOREM 18. *Suppose* MA *holds for* c.c.c. *partial orderings. Let* $\mathscr{I}$ *be an* $\aleph_2$*-saturated ideal on* ω_1 *and* $\mathbf{P} = \mathscr{P}(\omega_1)/\mathscr{I}$. *Let* $G \subseteq \mathbf{P}$ *be generic and* r *a real,* $r \in V[G]$, $r \notin V$. *Then* $V[r] = V[G]$.

Remark. This says that in a strong sense every new real in $V[G]$ is a minimal V-degree.

Proof. Let $j: V \to M \simeq V^{\omega_1}/G \subseteq V[G]$ be the generic ultraproduct. Then, by standard arguments $\mathbf{R}^M = \mathbf{R}^{V[G]}$ (see [J1]). Let r be a real, $r \in V[G] \sim V$. Let $f: \omega_1 \to \mathbf{R}^V$ be a function such that $[f]_M = r$ and $f \in V$.

By [B-T-W], $\mathscr{I}$ is selective and hence f is one-to-one on a set of measure one for $\mathscr{I}$.

For $s \in \mathbf{R}$ let $\mathrm{Seq}(s)$ be the set of sequence numbers of s by any standard Gödel numbering.

A standard application of MA shows that for any $X \subseteq \omega_1$ there is an $a_x \subseteq \omega$ such that $\alpha \in X$ if and only if $a_x \cap \mathrm{seq}(f(\alpha))$ is finite.

As usual $j(f)(\omega_1) = r$. (See [F2].) Thus for $X \subseteq \omega_1$, $X \in G$ if and only if $\omega_1 \in j(X)$ if and only if $\mathrm{Seq}(j(f)(\omega_1)) \cap j(a_x)$ is finite if and only if $\mathrm{Seq}(r) \cap a_x$ is finite. Hence from r we can recover G. □

COROLLARY 19. MA *implies that if* $\mathscr{I}$ *is an* $\aleph_2$*-saturated ideal on* ω_1 *then*
a) $\mathscr{P}(\omega_1)/\mathscr{I}$ *is not* $\aleph_1$*-dense,*
b) $\mathscr{P}(\omega_1)/\mathscr{I} \not\equiv \mathscr{B}(\mathrm{Col}(\omega, <\omega_2))$.

Proof. Both an $\aleph_1$ dense ideal and $\mathrm{Col}(\omega, <\omega_2)$ add Cohen reals. □

Note. a) was known and appeared in [T1]. b) contradicts published results of Woodin in [W1]. □

3. Versions of Martin's Maximum with CH

Our techniques combined with work of Shelah in [Sh1] give versions of Martin's Maximum consistent with the continuum hypothesis. We will briefly explicate this here; a more complete version will appear in [Sh-W] now in preparation.

In [Sh1], Shelah defines E-complete forcing. We now give a simpler definition that is a special case of E-completeness.

Let $\mathbf{P}$ be a partial ordering and $S \subseteq \omega_1$ be a stationary set. $\mathbf{P}$ is S-closed if and only if there is a closed unbounded set of $[H(2^{2^{|\mathbf{P}|^+}})]^\omega$ such that whenever $N \cap \omega_1 \in S$ and $\langle p_n: n \in \omega\rangle \subseteq N$ is a generic sequence for N then there is p such that for all n, $p \Vdash p_n$.

The canonical example of an S-closed forcing is the partial ordering for shooting a closed unbounded set through S with countable conditions.

Proposition 20. *Suppose* $\mathbf{P}$ *is an* S*-closed forcing; then* $\mathbf{P}$ *is* (ω, ∞)*-distributive.*

Proof. Let $\tau = \langle \tau_n: n \in \omega\rangle$ be a term for a new ω-sequence of ordinals. Let $N \prec H(2^{2^{|\mathbf{P}|^+}})$ be such that $N \cap \omega_1 \in S$ and N is countable and $\tau \in N$. Let $\langle p_n: n \in \omega\rangle$ be a generic sequence for N such that for some p, $p \Vdash p_n$ for all n. Then for each n, p decides the value of τ_n. Hence there is a sequence of ordinals $\langle \alpha_n: n \in \omega\rangle \in V$ such that $p \Vdash \tau_n = \alpha_n$. □

The following theorem is due to Shelah.

Theorem (Shelah). *If* $\mathbf{P}_\kappa$ *is an iteration of length* κ *with countable supports such that each factor is* S*-closed then* $\mathbf{P}$ *is* S*-closed.*

In [Sh1] we see that for an (ω, ∞)-distributive iteration, revised countable supports are the same as countable supports.

Theorem 21. *If there is a supercompact cardinal* κ *and* S *is a stationary subset of* ω_1 *then there is an* $\aleph_1$*-semi-proper,* (ω, ∞)*-distributive partial ordering* $\mathbf{P}$ *such that in* $V^{\mathbf{P}}$, MA *for* $\Gamma =$ "*all partial orderings* Q *such that* Q *is* S*-closed and preserves stationary subsets of* ω_1" *holds.*

Proof. We iterate along a Laver function L as we did in the proof of Theorem 5. Our partial ordering $\mathbf{P}$ will be an iteration of length κ with countable supports.

At stage α: If $L(\alpha)$ is a term in $V^{\mathbf{P}_\alpha}$ for an S-closed, semi-proper partial ordering Q_α, then $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * Q_\alpha * \mathrm{Col}(\omega_1, 2^{|\mathbf{P}_\alpha * Q_\alpha|^+})$. Otherwise $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \mathrm{Col}(\omega_1, 2^{|L(\alpha)| \times |\mathbf{P}_\alpha|^+})$.

By Shelah's theorems on revised countable support iterations, $\mathbf{P}_\kappa$ is an $\aleph_1$-semi-proper partial ordering that is S-closed.

As in the proof of Theorem 5, $\mathbf{P}_\kappa$ satisfies the hypothesis of Lemma 3. Further, in $V^{\mathbf{P}_\kappa}$ we have MA for the classes of Q which are $\aleph_1$-semi-proper and S-closed. Hence, if $\mathbf{P} = \mathbf{P}_\kappa$, $\mathbf{P}$ is (ω, ∞)-distributive, $\aleph_1$-semi-proper and $V^{\mathbf{P}}$ satisfies MA^+ for $\Gamma =$ "the classes of Q which are S-closed and preserve stationary subsets of ω_1". □

Since the $\mathbf{P}$ in Theorem 21 is $\aleph_1$-semi-proper, if S was costationary in V then $\tilde{S}$ is stationary in $V^{\mathbf{P}}$. Thus it is consistent to have MA for this Γ and $\tilde{S}$ stationary. The following proposition is the S-closed version of Theorem 12.

Proposition 22. *Suppose S is stationary and costationary and* MA *for* $\Gamma =$ "*the class of partial orders that are S-closed and preserve stationary subsets of* ω_1". *Then* $NS_{\omega_1} \restriction \tilde{S}$ *is* $\aleph_2$*-saturated.*

Proof. Let $\langle A_\alpha : \alpha < \gamma \rangle$ be a maximal antichain in $\mathrm{NS}_{\omega_1} \restriction \tilde{S}$ with $\gamma \geq \omega_2$. We apply MA to $\mathbf{P} = \mathrm{Col}(\omega_1, \gamma) * Q$ where Q is the forcing in $V^{\mathrm{Col}(\omega_1, \gamma)}$ for shooting a closed set through $\nabla_G A_\alpha \cup \tilde{S}$ with countable conditions ($\nabla_G A_\alpha$ is defined as before, G being the canonical generic object).

The $\mathbf{P}$ is S-closed and preserves stationary subsets of ω_1. As in Theorem 12 we get a contradiction. □

Further, the ideal $\mathrm{NS}_{\omega_1} \restriction \tilde{S}$ is c.c.c. indestructible as in Corollary 17.

Corollary 23. *If κ is supercompact then in* $V^{\mathrm{Col}(\omega_1, <\kappa)}$ *there is an* $\aleph_2$*-ideal on* ω_1.

Proof. If $\mathbf{P}$ is the partial ordering defined in Theorem 21 then $\mathrm{Col}(\omega_1, <\kappa)$ can be embedded in $\mathbf{P}$ as a complete subalgebra. This is true since $\mathbf{P}$ is (ω, ∞)-distributive and cofinally often in $\mathbf{P}$ we force with arbitrarily large portions of $\mathrm{Col}(\omega_1, <\kappa)$.

Hence, in $V^{\mathrm{Col}(\omega_1, <\kappa)}$ we can do an $\aleph_2$-c.c. forcing $Q = \mathbf{P}/\mathrm{Col}(\omega_1, <\kappa)$ to add an $\aleph_2$-saturated ideal, $\mathscr{I}^*$. But then $\mathscr{I} = \{x : \|x \in \mathscr{I}^*\| = 1\}$ is $\aleph_2$-c.c. in $V^{\mathrm{Col}(\omega_1, <\kappa)}$. (See [K1].) □

A note on history is appropriate here. Ideals were known to have consequences for Lebesgue measurability of sets of reals in $L(\mathbf{R})$. Magidor, in [M2] showed that if there is a measurable cardinal and a precipitous ideal on ω_1 then every Σ^1_3 set of reals is Lebesgue measurable. Foreman, in [F2], showed that if there is a $2^{\aleph_0}$-dense, normal and fine ideal on $[(2^{\aleph_0})^+]^{\aleph_1}$ then every set of reals in $L(\mathbf{R})$ is Lebesgue measurable, has the property of Baire and $L(\mathbf{R}) \models \omega \to (\omega)^\omega$. Woodin had shown in unpublished work that under CH an ω_1-dense ideal on ω_1 suffices for these consequences.

Woodin, aware of this work and of Theorem 12, proved the following proposition. It was proved simultaneously with the third author's realization that his technique of S-complete forcing could be used together with the results of Sections 1 and 2 to prove Theorem 21. In a phone call to the first author, Woodin, unaware of Theorem 21 and its consequences, announced his proposition. We state Woodin's proposition in somewhat greater generality then he first proved it (his original statement involved $\mathrm{Col}(\omega_1, < \kappa)$ and the non-stationary ideal).

PROPOSITION (Woodin). *Suppose κ is weakly compact and there is a κ-c.c. partial ordering $\mathbf{P}$ such that in $V^{\mathbf{P}}$ there is a generic elementary embedding $j: V \to M$ with $j(\omega_1) = \kappa$ and $(\mathbf{R})^{V^{\mathbf{P}}} \subseteq M$. Then*

$L(\mathbf{R})^V \models$. *Every set is Lebesgue measurable, has the property of Baire, and $\omega \to (\omega)^\omega$.*

The proof of this proposition uses the following theorem.

THEOREM (Folk). *Suppose κ is weakly compact and $\mathbf{P}$ is a κ-c.c. partial ordering such that $V^{\mathbf{P}} \models \kappa = \omega_1$. Then for every generic $G \subseteq \mathbf{P}$ there is a generic $H \subseteq \mathrm{Col}(\omega_1 < \kappa)$ such that $\mathbf{R}^{V[G]} = \mathbf{R}^{V[H]}$.*

Proof. Since $\mathbf{P}$ is κ-c.c. and κ is weakly compact, every real in $V[G]$ is generic for an intermediate extension $V^{\mathscr{B}}$ where $\mathscr{B}$ is a complete subalgebra of $\mathscr{B}(\mathbf{P})$ and $|\mathscr{B}| < \kappa$. (This is standard; see [J1] or [Mi] for a proof.) Hence for all reals $r \in V[G]$ there are an inaccessible $\gamma < \kappa$ and a V-generic object $H_\gamma \subseteq \mathrm{Col}(\omega, < \gamma)$, $H_\gamma \in V[G]$, such that $r \in V[H_\gamma]$.

Let G^* be $V[G]$ generic for $\mathrm{Col}(w, \kappa)$. In $V[G^*]$, $|\mathbf{R}^{V[G]}| = \omega$ and there is a cofinal sequence of inaccessibles $\langle \gamma_n \colon n \in \omega \rangle \subseteq \kappa$. In $V[G][G^*]$ choose a sequence $H_n \subseteq \mathrm{Col}(\omega, < \gamma_n)$ such that

1) H_n is V-generic, $H_n \in V[G]$,

2) for each real $r \in V[G]$ there is an n such that $r \in V[H_n]$. (We use the homogeneity of the Levy algebra to do this.) By the chain condition, for any antichain $A \subseteq \mathrm{Col}(\omega, < \kappa)$ there is an n such that $A \subseteq \mathrm{Col}(\omega, < \gamma_n)$. Hence $\bigcup H_n \subseteq \mathrm{Col}(\omega, < \kappa)$ is generic. Thus $L(\mathbf{R})^{V[G]} \subseteq L(\mathbf{R})^{V[H]}$. Since every $H_{\gamma_n} \in V$, $L(\mathbf{R})^{V[H]} \subseteq L(\mathbf{R})^{V[G]}$. □

We now prove Woodin's proposition.

Proof. Since κ is weakly compact and $j(\omega_1) = \kappa$ and **P** is κ-c.c., **P** satisfies the hypothesis of the Folk theorem. Hence for each $G \subseteq \mathbf{P}$ generic, for some generic $H \subseteq \text{Col}(\omega, <\kappa)$, $L(\mathbf{R})^{V[G]} = L(\mathbf{R})^{V[H]}$. Thus $j: L(\mathbf{R})^V \to L(\mathbf{R})^M = L(\mathbf{R})^{V[H]}$ is an elementary embedding of $L(\mathbf{R})^V$ into $L(\mathbf{R})^{V[H]}$ where H is generic for the Levy collapse. By Solovay's results in [Sol], $L(\mathbf{R})^{V[H]} \models$. Every set of reals in $L(\mathbf{R})$ is Lebesgue measurable, has the property of Baire and $\omega \to (\omega)^\omega$. Since $L(\mathbf{R})^V \equiv L(\mathbf{R})^{V[H]}$ we are done. □

If κ is supercompact, Corollary 23 implies that in $V^{\text{Col}(\omega_1, <\kappa)}$ there is an $\aleph_2$-saturated ideal $\mathscr{I}$ on ω_1. Let $Q = \mathscr{P}(\omega_1)/\mathscr{I}$ in $V^{\text{Col}(\omega_1, \kappa)}$. Then $\mathbf{P} = \text{Col}(\omega_1 < \kappa) * Q$ satisfies the hypothesis of Woodin's proposition:

First, $\mathbf{P} * Q$ is κ-c.c. Let $G * H \subseteq \mathbf{P} * Q$ be generic. By standard theory of saturated ideals there is an elementary embedding $j: V[G] \to M^* \subseteq V[G * H]$ sending ω_1 to κ and $V[G * H] \models \mathbf{R} \subseteq M^*$.

Let $M = \bigcup_{\alpha \in OR} j(R^V_\alpha)$. Then $j \restriction V: V \to M$. Since $V[G] \models \mathbf{R} \subseteq V$, $M^* \models \mathbf{R} \subseteq M$. Hence

$$\mathbf{R}^{V[G * H]} \subseteq M.$$

Thus, as a corollary of Woodin's Proposition and Corollary 23 we get:

Corollary. *If there is a supercompact cardinal κ then every set of reals in $L(\mathbf{R})$ is Lebesgue measurable, has the property of Baire and $L(\mathbf{R}) \models \omega \to (\omega)^\omega$.*

Shelah and Woodin have since weakened the hypothesis on κ a great deal [Sh-W].

It is also easy to see, when these techniques are used, that if κ is a supercompact cardinal and Q is any partial ordering and $G \subseteq Q$ is generic, then for some γ, $H \subseteq \text{Col}(\omega, <\gamma)$ generic there are an elementary embedding

$$j: L(\mathbf{R})^{V[G]} \to L(\mathbf{R})^{V[H]}$$

and an elementary embedding $k: L(\mathbf{R})^V \to L(\mathbf{R})^{V[H]}$. Hence the theory of $L(\mathbf{R})$ is invariant under set forcing.

Magidor has shown that MM implies that all $\underset{\sim}{\Sigma}^1_3$s sets of reals are Lebesgue measurable.

Finally we note another version of MA for a class of partial orderings that is consistent with CH.

If κ is a supercompact cardinal, MA^+ for ω-closed partial orderings holds in $V^{\text{Col}(\omega_1, <\kappa)}$. As noted earlier, MA^+ for ω-closed partial orderings implies (†). We shall show that it implies that the non-stationary ideal on ω_1 is almost $\aleph_2$-saturated.

Definition ([B-T]). A κ-complete ideal $\mathscr{I}$ on κ is κ^+-preserving if and only if forcing with $\mathbf{P} = \mathscr{P}(\kappa)/\mathscr{I}$ preserves κ^+. $\mathscr{I}$ is presaturated if and only if $\mathscr{I}$ preserves κ^+ and is precipitous.

THEOREM [B-T]. *If* $2^\kappa = \kappa^+$ *and* $\mathscr{I}$ *preserves* κ^+ *then*
a) $\mathscr{I}$ *is precipitous.*
b) *If* $j\colon V \to M \subseteq V[G]$ *is the generic elementary embedding then*

$$M^\kappa \cap V[G] \subseteq M$$

Thus presaturated ideals have many of the same desirable properties that saturated ideals have.

LEMMA 24. *Let* $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i\rangle_{i\in\omega}$ *be a fully Skolemized expansion of* $H(\lambda)$. (f_i: $[H(\lambda)]^n \longrightarrow H(\lambda)$ *for some* n.) *Let* $N \prec \mathscr{A}$ *be an elementary substructure of* $\mathscr{A}$, $x \in N$ *and* $\alpha < \sup N \cap \mathrm{OR}$. *Let* $\mathscr{L}$ *be an expansion of* $\mathscr{A}$, $\mathscr{L} = \langle H(\lambda), \varepsilon, f_i, g_j\rangle_{i,j\in\omega}$ *such that the functions* g_j *are closed under composition with the* f_i*'s and include Skolem functions for* $\mathscr{L}$. *Suppose that* $\langle N, g_j \restriction N\rangle \prec \mathscr{L}$. *Then*

$$\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap x = \mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap x.$$

Proof. For the conclusion of the lemma we may assume that

$$g_j\colon H(\lambda) \times \mathrm{OR} \to x.$$

Let $\gamma \in N \cap \mathrm{OR}$, $\gamma > \alpha$. If $y \in N$ then the function $g_j(y, -) \restriction \gamma \in N$ for each j, since $(N, g_j \restriction N) \prec \mathscr{L}$. Now $\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap x = \{g_j(y, \alpha)\colon y \in N\}$. But $g_j(y, -) \restriction \gamma \in \mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\})$ and $\alpha \in \mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\})$. Hence $g_j(y, \alpha) \in \mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\})$. Thus $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap x = \mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap x$. □

This lemma is useful in that it lets us change the quantifier "almost all" to "all" for subsets of $[H(\lambda)]^\kappa$ (κ a regular cardinal).

Suppose $\mathscr{A} = \langle H(\kappa), \varepsilon, \Delta \ldots\rangle$ is a structure such that for almost all $N \in [H(\lambda)]^\kappa$ there is an α such that $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap x = N \cap x$. Then by adding a predicate C for the closed unbounded set witnessing this we get that for all $N \prec \langle H(\lambda), \varepsilon, \Delta, C, \{x\}\rangle = \mathscr{L}$ there is an α such that $\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap x = N \cap x$.

THEOREM 25. MA^+ *for* ω*-closed partial orderings implies that the non-stationary ideal on* ω_1 *is presaturated.*

Proof. We will show:

Claim. Let T be a stationary set. If $\langle A_n\colon n \in \omega\rangle$ is an ω-sequence of maximal antichains below T in $\mathbf{P} = \mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ then there is a stationary set

$S \subseteq T$ such that for all n,

$$A_n \restriction S = \{[x \cap S]: x \in A_n\}$$

has cardinality $\aleph_1$.

[B-T] has a proof that the claim suffices. For the readers convenience we give it here.

Assume the claim. Then, if $\langle \tau_n: n \in \omega \rangle$ is a term for a sequence of functions in $V[G]$ with $\|\tau_n \in V, \tau_n: \omega_1 \to \mathrm{OR}\|_{\mathbf{P}} = 1$, we can find stationary set S and functions $\langle f_n: n \in \omega \rangle \in V$, $f_n: \omega_1 \to \mathrm{OR}$ so that

$$[S] \Vdash \{\alpha: f_n(\alpha) = \tau_n(\alpha)\} \in G.$$

Hence, if $[S] \Vdash \{\alpha: \tau_{n+1}(\alpha) < \tau_n(\alpha)\} \in G$ then there is a closed unbounded set $C_n \subseteq \omega_1$ such that for all $\alpha \in C_n \cap S$, $f_{n+1}(\alpha) < f_n(\alpha)$. Let $\beta \in \bigcap_{n \in \omega} C_n \cap S$. Then for all n, $f_{n+1}(\beta) < f_n(\beta)$. This contradicts regularity. Hence NS_{ω_1} is precipitous.

To see that $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ preserves ω_2, we suppose not. Let $\tau \in V^{\mathbf{P}}$ be a term for a function from ω onto ω_2. Let A_n be a maximal antichain deciding the values of $\tau(n)$. Then there is a stationary set $S \subseteq \omega_1$ and a set $P \subseteq \omega_2$ of cardinality ω_1 such that $[S] \nVdash$ range $\tau \subseteq P$. This contradicts surjectivity.

We prove the claim. Let $T \subseteq \omega_1$ be a stationary set and $\langle a^n_\alpha: \alpha < \gamma_n \rangle = A_n$ be a sequence of maximal antichains in T. Let κ be a regular cardinal $\kappa > \sup_{n \in \omega} \gamma_n$. Let $G \subseteq \mathrm{Col}(\omega_1, \kappa)$ be generic. Then we can form $\nabla_G A_n = \{\alpha$: there is a $\beta < \alpha$, $\alpha \in a^n_{G(\beta)}\}$. Suppose that in $V[G]$, $\bigcap_{n \in \omega} \nabla_G A_n$ is stationary. Then by MA^+ for countably closed partial orderings, in V we could get a function $G: \omega_1 \to \kappa$ such that $S = \bigcap_{n \in \omega} \nabla_G A_n$ is stationary. But then $|A_n \restriction S| \leq |\mathrm{range}\ G| = \omega_1$. Hence we would be done.

Thus we must show that $\bigcap_{n \in \omega} \nabla_G A_n$ is stationary in $V[G]$. We do this by an application of Lemma $*$. Suppose that for each $x \in H(\lambda)$ there is a countable $N \prec H(\lambda)$ such that for all n, $\delta = N \cap \omega_1 \in a^n_\alpha$ for some $\alpha \in N$ and $x \in N$. Since $\mathrm{Col}(\omega_1, \kappa)$ is countably closed we have a strong master condition q for N. Then for each $\alpha \in N$, $\alpha < \kappa$ implies that α is in the range of $q \restriction \delta$. Hence $q \Vdash \delta \in \bigcap_{n \in \omega} \nabla_G A_n$. On the other hand, for any term $\dot{C} \in V^{\mathrm{Col}(\omega_1, \kappa)}$ for a closed unbounded set, if $C \in N$ then $q \Vdash \delta \in \dot{C}$. Hence $q \Vdash (\bigcap_{n \in \omega} \nabla_G A_n) \cap \dot{C} \neq \varnothing$.

Fix $x \in H(\lambda)$. We must see that there is a countable $N \prec \langle H(\lambda), \varepsilon, \Delta, x \rangle$ such that for all n, $\delta = N \cap \omega_1 \in a^n_\alpha$ for some α. We prove this using (†).

By the remarks preceding Theorem 13, (†) holds. From (†) we will deduce that for any n, and any expansion of $H(\lambda)$, $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ there is a closed unbounded set of $N \prec \mathscr{A}$, C_n, such that for all $N \in C_n$, if $N \cap \omega_1 \in T$ there is an α, $N \cap \omega_1 \in a^n_\alpha$ and $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \omega_1 = N \cap \omega_1$.

If there are such sets, by Lemma 24, we can assume that there is an expansion $\mathscr{L} = \langle H(\lambda), \varepsilon, g_i \rangle_{i \in \omega}$ of $\mathscr{A}$ such that for all $N \prec \mathscr{L}$ and for all n, if $N \cap \omega_1 \in T$ there is an α such that $\delta = N \cap \omega_1 \in a^n_\alpha$ and $\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap \omega_1 = N \cap \omega_1$. Thus, by adding such α's, one at a time, to an $N \prec \mathscr{L}$ and closing under Skolem functions we get an $N \prec H(\lambda)$ as desired.

Thus, we must see that there are such sets C_n. For each antichain A_n, let $\mathbf{P}$ be the partial ordering $\mathrm{Col}(\omega_1, \kappa) * Q$ where $Q \in V^{\mathbf{P}}$ shoots a club set through $\nabla_G A_n$ with countable conditions. By (†), $\mathbf{P}$ is semi-proper. Let C_n be the closed unbounded set of $N \prec \mathscr{A}$ that has partial master conditions. Let $N \in C_n$, $N \cap \omega_1 \in T$ and $p = (r, q)$ be a partial master condition. Then if $\delta = N \cap \omega_1 p$, $\Vdash \delta \in \nabla_G A_n$; hence for some $\alpha \in \kappa$ and $\beta < \delta$, $r(\beta) = \alpha$ and $\delta \in a^n_\alpha$. Let $G \subseteq \mathbf{P}$ be generic, with $p \in G$. Then $N[G] \cap \omega_1 = N \cap \omega_1$. But $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \subseteq N[G]$, so $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \omega_1 = N \cap \omega_1$. Thus $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \omega_1 = N \cap \omega_1$, for this α. □

This argument is the prototype of many arguments to show that various ideals on a cardinal μ or $[\mu]^{<\lambda}$ are precipitous or presaturated. The strategy is always to expand a structure N to include elements of an antichain in N without increasing $N \cap \mu$.

4. Precipitous ideals

As we saw in Theorem 25, and in Shelah's theorems about Namba forcing, (†) is a strong combinatorial principle in its own right. We now elaborate on this to produce models where the non-stationary ideal on a regular cardinal μ (and $[\mu]^\omega$ etc.) is precipitous. We start by stating a standard lemma:

LEMMA. *If $\mathscr{I} \subseteq \mathscr{P}(Z)$ is an ideal on Z then $\mathscr{I}$ is precipitous if and only if there is no set $S \in \mathscr{I}^+$ and no tree $T \subseteq (2^Z)^{<\omega}$ labelled with $\mathscr{I}$-positive sets $\langle A_\eta : \eta \in T \rangle$, $A_\eta \subseteq Z$, such that*

a) $A_\varnothing = S$.

b) *For each $\eta \in T$, $\{A_{\eta \frown \alpha} : \eta \frown \alpha \in T\}$ is a maximal antichain below A_η and,*

c) *for all $f: \omega \to 2^Z$, if for all n, $f \restriction n \in T$, then $\bigcap_{n \in \omega} A_{f \restriction n} = \varnothing$ (see* [J1], *p.* 439).

Thus to prove that an ideal is precipitous, we must show that there is no such tree. If T is such a tree we let $\mathcal{A}_n = \{A_\eta : \eta \in T \text{ and } l(\eta) = n\}$. Then by b, $\mathcal{A}_n$ is an $\mathscr{I}$-maximal antichain below S and $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$.

THEOREM 26. (†) *implies that* NS_{ω_1} *is precipitous.*

Proof. Suppose not. Let $\lambda \gg \omega_1$. Let $\langle A_n : n \in \omega \rangle$ be the sequence of antichains coming from a tree T that witnesses NS_{ω_1} is not precipitous. Let $S = A_\varnothing$. For each maximal antichain $A \subseteq \mathscr{P}(\omega_1)$, consider the forcing

$$\mathbf{P} = \mathrm{col}(\omega_1, 2^{\omega_1}) * Q$$

where Q is the partial ordering for shooting a closed set through $\nabla_G A$. Then $\mathbf{P}$ is $\aleph_1$-semi-proper by (†). Hence as we argued in the proof of Theorem 25, for any expansion $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ there is a club set $C \subseteq [H(\lambda)]^\omega$ such that if $N \in C$ then there is an $a \in A$ such that

a) $N \cap \omega_1 \in a$,

b) $\mathrm{Sk}^{\mathscr{A}}(N \cup \{a\}) \cap \omega_1 = N \cap \omega_1$.

Claim. For any expansion $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ of $H(\lambda)$ there is a club set $C \subseteq [H(\lambda)]^\omega$ such that for all $N \in C$ and all maximal antichains $A \subseteq \mathscr{P}(\omega_1)$, $A \in N$ implies that there is an $a \in A$ and

a) $N \cap \omega_1 \in a$.

b) $\mathrm{Sk}^{\mathscr{A}}(N \cup \{a\}) \cap \omega_1 = N \cap \omega_1$.

Proof. Otherwise there would be a particular maximal antichain A and a stationary set $T \subseteq [H(\lambda)]^\omega$ such that for all $N \in T$, $A \in N$ and for all $a \in A$, if $N \cap \omega_1 \in a$ then $\mathrm{Sk}^{\mathscr{A}}(N \cup \{a\}) \cap \omega_1 \neq N \cap \omega_1$. This contradicts the last paragraph. The claim follows. □

Let C be a club set in $[H(\lambda)]^\omega$ witnessing the claim for $\mathscr{A} = \langle H(\lambda), \varepsilon, T \rangle$. Let $\mathscr{L} = \langle H(\lambda), \varepsilon, T, f_i \rangle_{i \in \omega}$ be such that all countable elementary substructures $N \prec \mathscr{L}$ are in C.

Then by Lemma 24, if $N \prec \mathscr{L}$ and $\alpha < 2^{\omega_1}$ then

$$\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap \omega_1 = \mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \omega_1.$$

Let $N \prec \mathscr{L}$ be a countable set such that $\delta = N \cap \omega_1 \in S$. We will build a function $f : \omega \to 2^{\omega_1}$ such that for all n, $f \restriction n \in T$ and $\delta \in A_{f \restriction n}$. This will be a contradiction.

Suppose we have defined $f \restriction n$, such that $\mathrm{Sk}^{\mathscr{L}}(N \cup f \restriction n) \cap \omega_1 = N \cap \omega_1$. Then $\{\tilde{A}_{f \restriction n}\} \cup \{A_{f | n^\frown \alpha} : f \restriction n^\frown \alpha \in T\}$ is a maximal antichain that lies in $\mathrm{Sk}^{\mathscr{L}}(N \cup f \restriction n)$. Hence there is an α such that $\delta \in A_{f \restriction n^\frown \alpha}$ and $\mathrm{Sk}^{\mathscr{L}}(N \cup f \restriction n \cup \{\alpha\}) \cap \omega_1 = \delta$. Let $f(n) = \alpha$. □

One might ask about cardinals above ω_1. Gittik and Shelah have done considerable work on this problem (see [G1], [Sh1]).

It turns out however, that with a sufficiently large cardinal a Levy collapse is sufficient to make NS precipitous:

THEOREM 27. *Suppose κ is a supercompact cardinal and $\mu < \kappa$ is regular. Then in $V^{\mathrm{Col}(\mu, <\kappa)}$ the non-stationary ideal on μ is precipitous.*

The original proof of this used a version of (†) that holds in $V^{\mathrm{Col}(\mu, <\kappa)}$ and an argument similar to Theorem 25. The direct proof is simpler so we give it. We need to use a particular stationary set in $[H(\lambda)]^{<\lambda}$ that is very resilient.

Let $N \prec \langle H(\lambda), \varepsilon, \Delta, f_i \rangle_{i \in \omega}$. Then N is *internally approachable* (IA) if and only if there is a sequence $\langle N_\alpha: \alpha < \delta \rangle$ such that $N = \bigcup_{\alpha < \delta} N_\alpha$ and if $\beta < \delta$, then $\langle N_\alpha: \alpha < \beta \rangle \in N$.

By cardinality considerations, $|\delta| \le |N|$. We note that all countable N are internally approachable. Let IA $= \{N \prec H(\lambda): N$ is internally approachable$\}$. Note that the definition of IA is independent of whether we are working in $[H(\lambda)]^{<\lambda}$ or $[H(\lambda)]^{<\mu}$ for some $\mu < \lambda$.

The following lemma yields the salient facts about IA.

LEMMA 28. *Let $\gamma < \lambda$ be uncountable regular cardinals.*

a) IA *is stationary in* $[H(\lambda)]^{<\gamma}$.

b) *If λ' is regular, $\gamma < \lambda' < \lambda$ and $N \prec H(\lambda)$, $N \in$ IA and $\lambda' \in N$ then $N \cap H(\lambda') \in$ IA.*

(*Note there are two* IA's *here—one for λ and one for λ'.*)

c) $\{N \cap \gamma: N \in \mathrm{IA}\}$ *includes a club set.*

d) *If $S \subseteq$ IA is stationary in $[H(\gamma)]^{<\gamma}$ and σ is any ordinal then S is stationary in $V^{\mathrm{Col}(\gamma, <\sigma)}$.*

Proof. a) Let C be a club set in $[H(\lambda)]^{<\gamma}$. Let $\langle N_i: i \in \omega \rangle \subseteq C$ be such that $N_{i+1} \supseteq N_i \cup \{N_i\}$. Then $\bigcup_{i \in \omega} N_i \in C \cap$ IA.

b) Let $N \prec H(\lambda)$, $N \in$ IA and $\lambda' \in N$. Then $N = \bigcup_{\alpha < \delta} N_\alpha$ and for each $\beta < \delta$, $\langle N_\alpha: \alpha < \beta \rangle \in N$. Let $N' = N \cap H(\lambda')$. Then $N' \prec H(\lambda')$. We claim that for each $\beta < \delta$, $\langle N_\alpha \cap H(\lambda'): \alpha < \beta \rangle \in N'$.

Since $\langle N_\alpha: \alpha < \beta \rangle \in N$, $N \models \langle N_\alpha \cap H(\lambda'): \alpha < \beta \rangle$ is a $<\gamma$-sequence of elements of $H(\lambda')$. Hence, $N \models \langle N_\alpha \cap H(\lambda'): \alpha < \beta \rangle \in H(\lambda')$. Thus, $\langle N_\alpha \cap H(\lambda'): \alpha < \beta \rangle \in N' = N \cap H(\lambda')$.

c) Let $\langle N_\alpha: \alpha < \gamma \rangle \subseteq [H(\lambda)]^{<\gamma}$ be a continuous tower of elementary substructures of $H(\lambda)$ such that

$$\langle N_\beta: \beta < \alpha \rangle \in N_{\alpha+1}.$$

Then $\{N_\alpha: \alpha$ is a limit ordinal $< \gamma\} \subseteq$ IA and is continuous. Hence

$$\{N_\alpha \cap \gamma: \alpha \text{ is a limit point } < \gamma\}$$

is a club set in γ.

d) Let $S \subseteq \mathrm{IA} \cap [H(\lambda)]^{<\gamma}$ be a stationary set. Let $\dot{C} \in V^{\mathrm{Col}(\gamma, <\sigma)}$ be a term for a club set in $[H(\lambda)^V]^{<\gamma}$. Let $\lambda^* \gg \sigma$ be regular. Let

$N \prec \langle H(\lambda^*), \varepsilon, \Delta, \mathrm{Col}(\omega_1, < \sigma), \dot{C}, S\rangle$ be an elementary substructure of cardinality $< \gamma$ such that $N \cap H(\lambda) \in S$. (We use Lemma 0 to get such an N.)

Then $N \cap H(\lambda) = \bigcup_{\alpha<\delta} N_\alpha$ for some sequence $\langle N_\alpha: \alpha < \delta\rangle$ and for all $\beta < \delta$, $\langle N_\alpha: \alpha < \beta\rangle \in N$. Choose a sequence of conditions $\langle p_\alpha: \alpha < \beta\rangle \subseteq \mathrm{Col}(\gamma, < \sigma)$ such that:

a) If $\alpha > \alpha'$ then $p\alpha \Vdash p\alpha'$.

b) There is an $M_\alpha \in N \cap [H(\lambda)]^{<\gamma}$ such that $p_\alpha \Vdash$ "$N_\alpha \subseteq M_\alpha$ and $M_\alpha \supseteq \bigcup_{\beta<\alpha} M_\beta$ and $M_\alpha \in C$".

c) For all $\beta < \delta$, $\langle p_\alpha: \alpha < \beta\rangle \in N$.

Such a sequence is easy to build if at stage α we choose $p_{\alpha+1}$ the Δ-least condition of $\mathrm{Col}(\gamma, < \sigma)$ such that for some M_α, b) holds. Then we choose the Δ-least such M_α.

Since $M_\alpha \in N$, $N \vDash |M_\alpha| < \gamma$; hence $|M_\alpha| \in N$. But $N \cap \gamma \in \mathrm{OR}$. Hence $M_\alpha \subseteq N$. Since $M_\alpha \subseteq H(\lambda)$, $M_\alpha \subseteq N \cap H(\lambda)$.

Let $p \in \mathrm{Col}(\gamma, < \sigma)$ be such that for all $\alpha < \delta$, $p \Vdash p_\alpha$. (Recall $\delta < \gamma$ by cardinality considerations.)

Then $p \Vdash C \cap (N \cap H(\lambda))$ is unbounded in $N \cap H(\lambda)$. Hence $p \Vdash N \cap H(\lambda) \in \dot{C}$. Hence $p \Vdash C \cap S \neq \varnothing$. □

(The theorem above is also true for the strongly closed unbounded filter on $[\mu]^{<\gamma}$. Instead of working with a term for a closed unbounded set $\dot{C}$ we work with a term for a countable sequence of functions $\langle f_i: i \in w\rangle$. We build a sequence $\langle p_\alpha: \alpha < \delta\rangle$ such that: a) For all $\beta < \alpha$, $\langle p_\alpha: \alpha < \beta\rangle \in N$.

b) For all $\vec{n} \in N_\alpha$ and all i, there is an m such that $p_\alpha \Vdash f(\vec{n}) = m$. Then $\bigcup_{\alpha<\delta} p_\alpha \Vdash N$ is closed under $\langle f_i: i \in \omega\rangle$.)

Proof of Theorem 27. We will work as in Theorem 26 to build a path through any tree of antichains. Let $\mathbf{P} = \mathrm{Col}(\mu, < \kappa)$.

Main Claim. Let $\lambda \gg \mu$. In $V^{\mathbf{P}}$ let $\mathscr{A} = \langle H(\lambda), \varepsilon, \Delta, f_i\rangle_{i\in\omega}$ be any expansion of $H(\lambda)$. Then for almost all $N \prec \mathscr{A}$, $N \in [H(\lambda)]^{<\mu} \cap \mathrm{IA}$, if $\langle A_\alpha: \alpha < \mu^+\rangle \in N$ is a maximal antichain in $\mathscr{P}(\mu)/\mathrm{NS}_\mu$ then there is an $\alpha < \mu^+$ such that

a) $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \mu = N \cap \mu$.

b) $N \cap \mu \in A_\alpha$.

Proof. Otherwise by normality we get a stationary set $S \subseteq \mathrm{IA}$ and a particular maximal antichain $\langle A_\alpha: \alpha < \mu^+\rangle$ such that for all $N \in S$, if $N \cap \mu \in A_\alpha$ then

$$\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \mu \neq N \cap \mu.$$

Let $j: V \to \dot{M}$ be a λ-supercompact embedding. Then, since $\mathbf{P} = \mathrm{Col}(\mu, < \kappa)$ is κ-c.c., if $G \subseteq \mathbf{P}$ is generic then there is an $H \subseteq j(\mathbf{P}) =$

$\mathrm{Col}(\mu, < j(\kappa))$ generic such that $G \subseteq H$ and j can be extended to $\hat{j}$: $V[G] \to M[H]$. Let $V' = V[G]$. Since $S \subseteq \mathrm{IA}$, S is stationary in $[H(\lambda)^{V'}]^{<\mu}$ in $M^{j(\mathbf{P})}$. Let f: $\mu \xrightarrow[\text{onto}]{1-1} H(\lambda)^{V'}$ and

$$T = \{\delta < \mu: f''\delta \in S \text{ and } \delta = f''\delta \cap \mu\};$$

then T is stationary.

Let $\langle A^j_\alpha: \alpha < j(\kappa)\rangle = j(\langle A_\alpha: \alpha < \kappa\rangle)$. By elementarity, $M[H] \models \langle A^j_\alpha: \alpha < j(\kappa)\rangle$ is a maximal antichain in $\mathscr{P}(\mu)/\mathrm{NS}_\mu$. Thus there is an α such that $A^j_\alpha \cap T$ is stationary.

In $M[H]$, let $C = \{N \prec j(\mathscr{A})$: $|N| < \mu$, $\alpha \in N$ and N is closed under f, f^{-1} and $j \restriction H(\lambda)^{V[G]}\}$. Then C is a club set in $[H(j(\lambda))]^{<\mu}$. Choose $N \in C$ such that $\delta = N \cap \mu \in T \cap A_\alpha$. Let $N' = f''\delta$; then $N' \in S$. Further, $j(N') = j''N' \subseteq N$ and $N \cap \mu = N' \cap \mu = (j''N') \cap \mu$, since $\mathrm{crit}(j) = \kappa > \mu$.

Now $\mathrm{Sk}^{j(\mathscr{A})}(j(N') \cup \{\alpha\}) \subseteq N$; hence $\mathrm{Sk}^{j(\mathscr{A})}(j(N') \cup \{\alpha\}) \cap \mu = j(N') \cap \mu$. But then

$$M[H] \models \text{there is an } \alpha < \mu^+, \text{ such that } j(N') \cap \mu \in A^j_\alpha \text{ and}$$
$$\mathrm{Sk}^{j(\mathscr{A})}(j(N') \cup \{\alpha\}) \cap \mu = j(N') \cap \mu.$$

So

$$V[G] \models \text{“there is } \alpha < \mu^+,$$
$$N' \cap \mu \in A_\alpha \text{ and } \mathrm{Sk}^{\mathscr{A}}(N' \cup \{\alpha\}) \cap \mu = N' \cap \mu.\text{”}$$

But $N' \in S$, a contradiction. □

By the main claim and Lemma 24, we can expand $H(\lambda)$ to $\mathscr{L} = \langle H(\lambda), \varepsilon, \Delta, f_i\rangle_{i \in \omega}$ such that for all elementary substructures $N \prec \mathscr{L}$ with $|N| < \mu$ and all maximal antichains $\langle A_\alpha: \alpha < \mu^+\rangle \in N$ there is an α such that

a) $N \cap \mu \in A_\alpha$,

b) $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \mu = N \cap \mu$.

One can also pick $\mathscr{L}$ such that if N is a substructure of $\mathscr{L}$, $N \in \mathrm{IA}$ and $\alpha < \mu^+$, then the closure of $N \cup \{\alpha\}$ under the operation of $\mathscr{L}$ is in IA. See below in the proof of Theorem 29.

We now work exactly as in Theorem 26. Let $T \subseteq (\mu)^{<\omega}$ be a tree labelled with stationary sets $\langle A_\eta: \eta \in T\rangle$ such that $\{A_{\eta^\frown\alpha}: \eta^\frown\alpha \in T\}$ is a maximal antichain below A_η. We show that there is a function f: $\omega \to \mu^+$ such that for all n, $f \restriction n \in T$ and $\bigcap_{n \in \omega} A_{f \restriction n} \neq \varnothing$.

Let $N \prec \mathscr{L}$, $T \in N$, $|N| < \mu$ and $N \cap \mu \in T_\varnothing$. Then as before we can build a sequence $\langle \alpha_n: n \in \omega\rangle \subseteq \mu^+$ such that $\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha_n: n < m\}) \cap \mu = N \cap \mu$ for all finite m and $N \cap \mu \in A_{\langle \alpha_0, \alpha_1 \cdots \alpha_n\rangle}$ for all n. □

COROLLARY. *If* ZFC + *there is a supercompact cardinal is consistent then so is* ZFC + *for every regular cardinal* κ, NS_κ *is precipitous*. (*Compare* [F1].)

Proof. We only use $(2^{2^{2^\kappa}})^+$-supercompactness (at most) in the proof of Theorem 27. If there is a supercompact cardinal κ, $V_\kappa \models$ there is a class of α that is $(2^{2^{2^\alpha}})^+$-supercompact.

By iterating Levy collapses with Easton-supports we can make a generic object G such that $V_\kappa[G] \models$ If α is the successor of a regular cardinal then α is $(2^{2^{2^\alpha}})^+$-supercompact in V.

Since, if $2^\mu = \mu^+$, μ^+-closed forcing preserves precipitousness, $V_\kappa[G] \models$ ZFC for every regular cardinal κ, NS_κ is precipitous. □

Higher type ideals have very nice consequences for the set-theoretic universe. (See [F2]).

THEOREM 29. *Let κ be a supercompact cardinal and $\omega < \gamma < \mu < \kappa$ be regular cardinals. Then in $V^{\mathrm{Col}(\mu, <\kappa)}$ there is a stationary set $S \subseteq [\mu] < \gamma$ such that $\mathrm{NS}_{[\mu]^{<\gamma}} \upharpoonright S$ is precipitous. Further $\mathrm{NS}_{[\mu]^\omega}$ is precipitous.*

We note that this is one theorem where we get a stronger result by considering the filter of "strongly" closed unbounded sets. (See the introduction for comments about "strongly" closed unbounded sets.) Woodin has remarked that this theorem gives generalizations of Namba forcing for cardinals above ω_2 by considering $N \in \mathrm{IA}$ with $N \cap \alpha$ having various cofinalities, where α is some cardinal less than μ.

Proof. We will show that in $V^{\mathrm{Col}(\mu, <\kappa)}$, $\mathrm{NS}_{[H(\mu)]^{<\gamma}} \upharpoonright \mathrm{IA}$ is precipitous. Since $|H(\mu)| = \mu$ in $V^{\mathrm{Col}(\mu, <\kappa)}$, this proves the theorem.

Our method will be as in Theorems 26 and 27. We will build a branch through any tree of antichains and an $N \prec H(\lambda)$ with $N \cap H(\mu)$ in the intersection of this branch. We first show that if $\lambda \gg \mu$ then the projection of $\mathrm{NS}([H(\lambda)]^{<\gamma}) \cup \{\widetilde{\mathrm{IA}}\}$ onto $\mathrm{NS}([H(\mu)]^{<\gamma})$ is $\mathrm{NS}([H(\mu)]^{<\gamma}) \cup \{\widetilde{\mathrm{IA}}\}$.

Claim a) If $C \subseteq [H(\lambda)]^{<\gamma}$ is a closed unbounded set then $\{x \cap H(\mu): x \in C \cap \mathrm{IA}\} \supseteq D \cap \mathrm{IA}$ for some $D \subseteq [H(\mu)]^{<\gamma}$ that is closed and unbounded.

b) If $D \subseteq [H(\mu)]^{<\gamma}$ is closed and unbounded then there is a closed unbounded set $C \subseteq [H(\mu)]^{<\gamma}$ such that $\{x \cap H(\mu): x \in C$ and $x \in \mathrm{IA}\} \subseteq D \cap \mathrm{IA}$.

Proof. By Lemma 0 there is a function $f: H(\lambda)^{<\omega} \to H(\lambda)$ such that if $N \in [H(\lambda)]^{<\gamma} \cap \mathrm{IA}$, $N \cap \gamma \in \gamma$ and N is closed under f then $N \in C$. Also we can make sure that if $M \in \mathrm{IA} \cap H(\mu)^{<\gamma}$ then the closure of M under f is in IA. Such an f is defined by induction (for $\delta < \gamma$) f_δ: $H(\mu \xrightarrow{<\gamma} H(\lambda)^{<\gamma}$ where f_0 is the original function guaranteeing that $N \in C$ and $f_\delta(M) =$ the closure of M under $\langle f_\rho | \rho < \delta\rangle$. Let f code $\langle f_\delta | \delta < \gamma\rangle$ such that if N is closed under f, then N is closed under f_δ for $\delta \in N \cap \gamma$. This f is easily seen to satisfy the

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requirements. Let $D = \{M \in [H(\mu)]^{<\gamma}\colon M \prec \mathscr{L},\ M \cap \gamma \in \gamma\}$. We claim that for all $M \in D \cap \mathrm{IA}$ there is an $N \in C \cap \mathrm{IA}$ such that $N \cap H(\mu) = M$.

Let $\mathscr{A} = \langle H(\lambda), \varepsilon, \Delta, f, g_i \rangle_{i \in \omega}$ be a fully skolemized structure. Let $\mathscr{L} = \langle H(\mu), \varepsilon, \Delta, h_j \rangle_{j \in \omega}$ be such that if $M \prec \mathscr{L}$ then $\mathrm{Sk}^{\mathscr{A}}(M) \cap H(\mu) = M$. Let $D = \{M \in [H(\mu)]^{<\gamma}\colon M \prec \mathscr{L}, M \cap \gamma \in \gamma\}$. We claim that for all $M \in D \cap \mathrm{IA}$, there is an $N \in C \cap \mathrm{IA}$ such that $N \cap H(\mu) = M$.

If $M \in D \cap \mathrm{IA}$, then the closure of M under f is as required. This proves the claim.

We now prove the main claim analogous to the one in Theorem 27.

Main Claim Let $G \subseteq \mathrm{Col}(\mu, <\kappa)$ be generic. In $V[G]$, let $\lambda \gg (2^{2^\kappa})$ and $\mathscr{A} = \langle H(\lambda), \varepsilon, \Delta, f_i \rangle_{i \in \omega}$. Then for almost all $N \prec [H(\lambda)]^{<\gamma}$, if $N \in \mathrm{IA}$ then for all $\langle A_\alpha\colon \alpha < \mu \rangle \in N$, $\langle A_\alpha\colon \alpha < \mu \rangle$ a maximal antichain in

$$\mathscr{P}([H(\mu)]^{<\gamma})/\mathrm{NS} \cup \{\widetilde{\mathrm{IA}}\}$$

then there is an $\alpha < \mu$ such that

a) $\mathrm{Sk}(N \cup \{\alpha\}) \cap H(\mu) = N \cap H(\mu)$,

b) $N \cap H(\mu) \in A_\alpha$.

Proof. Otherwise, there is a stationary set $S \subseteq \mathrm{IA} \cap [H(\mu)]^{<\gamma}$ and a fixed maximal antichain $\langle A_\alpha\colon \alpha < \mu^+ \rangle$ such that if $N \in S$ then $\langle A_\alpha\colon \alpha < \mu^+ \rangle \in N$ and if $N \cap H(\mu) \in A_\alpha$ then

$$\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap H(\mu) \neq N \cap H(\mu).$$

Let $j\colon V \to M$ be a λ^+-supercompact embedding. Let $V' = V[G]$. Then in $M^{\mathrm{Col}(\mu, <j(\kappa))}$, $|H(\lambda)^{V'}| = \mu$ and S is a stationary subset of $[H(\lambda)^{V'}]^{<\gamma}$. Let $f\colon H(\mu) \to H(\lambda)^V$ be a bijection.

Let $T = \{N \in [H(\mu)^{V'}]^{<\gamma}\colon f''N \in S \text{ and } N = f''N \cap H(\mu)^{V'}\}$; then T is stationary and $T \subseteq \mathrm{IA}$ by Lemmas 0 and 28.

Let $\langle A^j_\alpha\colon \alpha < j(\kappa)\rangle = j(\langle A_\alpha\colon \underset{\sim}{\alpha^{<\kappa}} \rangle)$. Then $\langle A^j_\alpha\colon \alpha < j(\kappa)\rangle$ is a maximal antichain in $\mathscr{P}([H(\mu)]^{<\gamma})/\mathrm{NS} \cup (\widetilde{\mathrm{IA}})$; hence for some α, $A^j_\alpha \cap T$ is stationary. Let $C = \{N \prec H(j(\lambda))^{M^{\mathrm{Col}(\mu, <j(\kappa))}}, |N| < \gamma$ and $\alpha \in N$ and N is closed under f, f^{-1} and $j \restriction H(\lambda)^{V'}\}$. By Lemma 0, there is an $N \in C$, $N \cap H(\mu) \in T \cap A^j_\alpha$. Let $N' = N \cap H(\mu)$ and $N^* = f''N'$. Then $N^* \in S$.

Since $|N^*| < \mu$, $j(N^*) = j''N^*$ and $j(N^*) \subseteq N$. Further, $j(N^*) \cap H(\mu) = N \cap H(\mu)$. Hence $\mathrm{Sk}^{j(\mathscr{A})}(j(N^*) \cup \{\alpha\}) \cap H(\mu) = N \cap H(\mu)$. So by elementarity, $V' \vDash$ "there is an α, $N^* \cap H(\mu) \in A_\alpha$ and $\mathrm{Sk}^{\mathscr{A}}(N^* \cup \{\alpha\}) \cap H(\mu) = N^* \cap H(\mu)$". But this is a contradiction since $N^* \in S$. This proves the main claim.

By Lemma 24, we can expand $\mathscr{A}$ to an $\mathscr{L} = \langle H(\lambda), \varepsilon, g_i \rangle_{i \in \omega}$ such that for all $N \prec \mathscr{L}$, if $N \in \mathrm{IA}$ and $\langle A_\alpha \colon \alpha < \mu^+ \rangle \in N$ is a maximal antichain in $\mathscr{P}([H(\mu)]^{<\gamma})/\mathrm{NS} \cup \{\widetilde{\mathrm{IA}}\}$ then for some $\alpha < \mu^+$,

a) $\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap H(\mu) = N \cap H(\mu)$,

b) $N \cap H(\mu) \in A_\alpha$.

Hence, as in Theorem 27, this allows us to build a branch with non-empty intersection through any tree of antichains. □

Huge, cardinal-type ideals have been studied extensively. (See [F2] and [M2].) Magidor in [M3] showed:

THEOREM (Magidor). $(\kappa, \lambda) \to (\kappa', \lambda')$ *if and only if there is a normal, fine, countably complete ideal on* $[\kappa]^{\kappa'}$ *concentrating on* $[\lambda]^{\lambda'}$.

(Recall $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$ is the statement that every structure of type (κ, λ) has an elementary substructure of type (κ', λ').)

The ideal in Magidor's theorem always exists. Chang's conjecture is needed to show that it is a proper ideal.

The ideal is easy to describe, namely: $X \in \check{\mathscr{I}}$ if and only if $X \subseteq [\kappa]^{\kappa'}$, $|x \cap \lambda| = \lambda'$ and there is a structure $\mathscr{A} = \langle \kappa, \lambda, f_i \rangle_{i \in \omega}$ such that $X = \{y \prec \mathscr{A} \colon \text{o.t. } x = \kappa' \text{ and } |x| = \lambda'\}$. This ideal is seen to be analogous to the non-stationary ideal on $[\gamma]^{<\delta}$ for cardinals λ, δ.

When "huge" ideals are precipitous they can imply the G.C.H., $\aleph_\omega$ is Jonsson etc. [F2]. This makes it desirable to show that they can be precipitous.

Frequently proofs of the consistency of Chang conjecture type transfer properties yield the stronger result that there is a precipitous normal ideal as in Magidor's theorem. The next theorem shows that modulo a supercompact cardinal this is equivalent to the transfer property.

THEOREM 30. *Suppose* κ *is a supercompact cardinal and suppose that* $\mu < \kappa$ *and* $(\mu, \gamma) \twoheadrightarrow (\mu', \gamma')$ *for regular cardinals* $\mu' < \mu$ *and* $\gamma' < \lambda$. *Then in* $V^{\mathrm{Col}(\mu, <\kappa)}$ *there is a precipitous ideal on* $[\mu]^{\mu'}$ *concentrating on* $[\gamma]^{\gamma'}$.

Thus, modulo a supercompact cardinal, Chang's conjecture is equivalent to a precipitous huge ideal.

Note. We will show that the minimal normal and fine ideal are precipitous.

COROLLARY 31. *If* "ZFC + *there is a supercompact*" *is consistent then so is* "ZFC + *there is a normal and fine precipitous ideal on* $[\aleph_2]^{\aleph_1}$ *concentrating on* $[\aleph_1]^{\aleph_0}$."

Proof of corollary from theorem. By Silver's theorem on Chang's conjecture, [Si2], from an Erdös cardinal one can force $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_2)$. Since the first

Erdös cardinal is less than the first supercompact cardinal and the forcing in Silver's theorem is of small cardinality, Silver's forcing yields $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ and preserves the supercompactness of the cardinal. Hence by Theorem 30, a further forcing yields a precipitous ideal. □

Huge-type precipitous ideals were shown to be consistent from huge cardinals in [F2].

If $(\mu, \gamma) \twoheadrightarrow (\mu', \gamma')$ then for all regular $\lambda \geq \mu$, we get an ideal on $[H(\lambda)]^{\mu'}$ concentrating on $[\gamma]^{\gamma'}$ analogously to Magidor's theorem. Namely, a set $X \subseteq \{x \in [H(\lambda)]^{\mu'}: |x \cap \gamma| = \gamma'\}$ is in the dual of $\mathscr{I}$ if and only if there is a structure $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ such that

$$X \supseteq \left\{x \in \left[H(\lambda)\right]^{\mu'}: |x \cap \gamma| = \gamma' \text{ and } x \prec \mathscr{A}\right\}.$$

Since $(\mu, \gamma) \twoheadrightarrow (\mu', \gamma')$, this is a proper ideal. We will call this the *non-stationary ideal* on $[H(\lambda)]^{\mu'}$. A set $S \subseteq [H(\lambda)]^{\mu'}$ is *stationary* if and only if for all expansions $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ of $H(\lambda)$ there is an $x \in S$, $|x \cap \gamma| = \gamma'$ and $x \prec \mathscr{A}$. Similarly we define a set $S \subseteq [\mu]^{\mu'} \cap \{x: |x \cap \gamma| = \gamma'\}$ to be *stationary* if and only if for all expansions $\mathscr{A} = \langle \mu, f_i \rangle_{i \in \omega}$ there is an $X \prec \mathscr{A}$, $x \in S$. A set will be called closed and unbounded if its complement is not stationary.

We will prove Theorem 30 with the same method as we proved Theorems 27 and 29. We must define a notion of *internally approachable* appropriate in this context.

A set $N \in [H(\lambda)]^{\mu'}$ is *internally approachable* if and only if there is a continuous increasing sequence $\langle N_\alpha: \alpha < \sup N \cap \mu \rangle \subseteq [H(\lambda)]^{\mu'}$ such that for each $\beta \in N \cap \mu$, $\langle N_\alpha: \alpha < \beta \rangle \in N$, $|N_\beta| < \mu$ and $\bigcup_{\alpha < \sup N \cap \mu} N_\alpha \supseteq N$. We will let IA stand for the collection of $N \in [H(\lambda)]^{\mu'}$ that are internally approachable. We claim that IA is stationary in $[H(\lambda)]^{\mu'}$ and projects to a closed unbounded set in $[\mu]^{\mu'}$. To see that IA is stationary we let $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i, \Delta \rangle_{i \in \omega}$ be an expansion of $H(\lambda)$.

Let $\langle N_\alpha: \alpha < \mu \rangle$ be a continuous increasing sequence of elementary substructures of $\mathscr{A}$ such that each N_α has cardinality $< \mu$ and $\langle N_\alpha: \alpha \leq \beta \rangle \in N_{\beta+1}$. Let $M = \bigcup N_\alpha$ and let $\mathscr{L}$ be the result of expanding $\langle M, \varepsilon, f_i, \Delta \rangle_{i \in \omega}$ by the function $g(\beta) = \langle N_\alpha: \alpha < \beta \rangle$.

Since $|\mathscr{L}| = \mu$ we can choose a Chang elementary substructure N of $\mathscr{L}$ of type (μ', γ'). Since $N \prec \mathscr{L}$, $N \models$"for all x there is a $\beta < \mu$ such that for some N_α in the sequence $g(\beta)$, $x \in N_\alpha$". Further such an α must exist in N. Hence $N \subseteq \bigcup_{\alpha \in N \cap \mu} N_\alpha$ and, since $g(\beta) \in N$ for $\beta \in N \cap \mu$, $\langle N_\alpha: \alpha < \beta \rangle \in N$. Thus N is an elementary substructure of $\mathscr{A}$ and N is internally approachable.

To see that IA projects to a closed unbounded set in $[\mu]^{\mu'}$ (i.e. $\{N \cap \mu: N \in \text{IA} \cap [H(\lambda)]^{\mu'}\}$ is closed and unbounded in $[\mu]^{\mu'}$) it is enough to see that for any stationary set $S \subseteq [\mu]^{\mu'}$ and any expansion of $H(\lambda)$, $\mathscr{A} = \langle H(\lambda), \varepsilon, \Delta, f_i \rangle_{i \in \omega}$, there is an $N \prec \mathscr{A}$, $N \in \mathrm{IA} \cap [H(\lambda)]^{\mu'}$ such that $N \cap \mu \subset S$.

Again we build $\langle N_\alpha \colon \alpha < \mu \rangle$, a continuous increasing sequence of elementary substructures of $\langle H(\lambda), \varepsilon, \Delta, f_i \rangle_{i \in \omega}$ such that $|N_\alpha| < \mu$ and for each $\beta < \mu$, $\langle N_\alpha \colon \alpha \leq \beta \rangle \in N_{\beta+1}$. Let $\mathscr{L}$ expand $\bigcup_{\alpha < \mu} N_\alpha$ by the function $g(\beta) = \langle N_\alpha \colon \alpha < \beta \rangle$. As usual we find functions $\langle g_i \colon i \in \omega \rangle$ with domain $[\mu]^{<\omega}$ so that if $x \subseteq \mu$ is closed under $\langle g_i \colon i \in \omega \rangle$ then $\mathrm{Sk}^{\mathscr{L}}(x) \cap \mu = x$.

Since $S \subseteq [\mu]^{\mu'}$ is stationary there is an $x \in S$ closed under $\langle g_i \colon i \in \omega \rangle$. But then $\mathrm{Sk}^{\mathscr{L}}(x) \cap \mu = x$ and $\mathrm{Sk}^{\mathscr{L}}(x) \prec \mathscr{L}$. Thus $N = \mathrm{Sk}^{\mathscr{L}}(x)$ is internally approachable and $N \cap \mu \in S$ as desired.

We now need a lemma like Lemma 28 d.

LEMMA 32. *Assume* $(\mu, \gamma) \twoheadrightarrow (\mu', \gamma')$ *and* $\delta \in \mathrm{OR}$. *Suppose* $\lambda \gg \mu$ *and* $S \subseteq [H(\lambda)]^{\mu'} \cap \mathrm{IA}$ *is stationary. Then in* $V^{\mathrm{Col}(\mu, <\delta)}$,

a) $(\mu, \delta) \twoheadrightarrow (\mu', \gamma')$,

b) S *is stationary in* $[H(\lambda)^V]^{\mu'}$ (i.e. *any expansion of* $H(\lambda)^V$ *has an elementary substructure of type* (μ', γ') *in* S).

Proof. A proof of a) appears in [F1].

b) It suffices to see b) for $\delta > \lambda$. Let $\mathscr{A} = \langle H(\lambda)^V, \varepsilon, f_i \rangle_{i \in \omega}$ be an expansion of $H(\lambda)^V$ in $V^{\mathrm{Col}(\mu, <\delta)}$. In V, if $\lambda' > \lambda$, the non-stationary ideal on $[H(\lambda')]^{\mu'}$ projects onto the non-stationary ideal on $[H(\lambda)]^{\mu'}$. Thus, if $p \in \mathrm{Col}(\mu, <\delta)$ and $\dot{\mathscr{L}} = \langle H(\lambda), \dot{f}_i \rangle$ is a term for the structure $\mathscr{A}$ and $\lambda' \gg \delta$, there is an elementary substructure $N \prec \langle H(\lambda'), \varepsilon, \Delta, \dot{\mathscr{L}}, \{p\} \rangle$ such that $N \cap H(\lambda) \in S$.

Since $S \subseteq \mathrm{IA}$ there is a sequence $\langle N_\alpha \colon \alpha < \sup N \cap \mu \rangle$ of sets of size $< \mu$ so that $\bigcup_{\alpha < \sup N \cap \mu} N_\alpha \supseteq N \cap H(\lambda)$ and for each $\beta \in N \cap \mu$, $\langle N_\alpha \colon \alpha < \beta \rangle \in N$. Working inside N, we can build a tower of conditions $\langle p_\alpha \colon \alpha < \sup N \cap \mu \rangle$ extending p so that p decides all of each $\dot{f}_i \restriction N_\alpha$. (Note that the tower is not in N but for each $\beta \in N \cap \mu$, $\langle p_\alpha \colon \alpha < \beta \rangle \in N$.) To do this we use the fact that $N \vDash |N_\alpha| < \mu$ so that we can extend any condition to a condition that decides all of $f_i \restriction N_\alpha$. Then for $\beta \in N \cap \mu$, $\langle p_\alpha \colon \alpha < \beta \rangle$ is the lexicographically least sequence such that p_α decides all of $\dot{f}_i \restriction N_\alpha$. Since $\langle N_\alpha \colon \alpha < \beta \rangle \in N$, $\langle p_\alpha \colon \alpha < \beta \rangle \in N$.

Since $\mathrm{Col}(\mu, <\delta)$ is $<\mu$-closed there is a condition q such that for all $\alpha < \sup N \cap \mu$, $q \Vdash p_\alpha$.

Then $q \Vdash$ "$N \cap H(\lambda)$ is closed under each f_i" because if $x \in N \cap H(\lambda)$ then for some $\alpha \in N \cap \mu$, $x \in N_\alpha$. Hence $p_\alpha \Vdash f_i(x) = y$ for some $y \in N \cap \dot{H}(\lambda)$. So $q \Vdash f_i(x) \in N \cap H(\lambda)$. Thus $q \Vdash$ "$N \cap H(\lambda) \prec \mathscr{A}$" and $N \cap H(\lambda) \in S$. Hence S is stationary in $V^{\mathrm{Col}(\mu, <\delta)}$. □

Proof of Theorem 30. As usual, we will be done if we can show some version of the *main claim*.

Main Claim. Assume the hypothesis of the theorem. In $V' = V^{\mathrm{Col}(\mu, <\kappa)}$, let $\lambda \gg \kappa$ and $\mathscr{A} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ be an expansion of $H(\lambda)$. Then for a closed unbounded set $C \subseteq [H(\lambda)]^{\mu'}$, whenever $x \in C$ is internally approachable and $\langle A_\alpha \colon \alpha < \mu^+ \rangle \in x$ is a maximal antichain in $\mathscr{P}([\mu]^{\mu'})/\mathrm{NS}$ there is an α such that

a) $\mathrm{Sk}^{\mathscr{A}}(x \cup \{\alpha\}) \cap \mu = x \cap \mu$,

b) $x \cap \mu \in A_\alpha$.

Proof. Let $j \colon V \to M$ be a λ^+-supercompact embedding. If the lemma is false, let $S \subseteq [H(\lambda)]^{\mu'} \cap \mathrm{IA}$ be a stationary set and $\langle A_\alpha \colon \alpha < \kappa \rangle$ be a maximal antichain in $\mathscr{P}([\mu]^{\mu'})/\mathrm{NS}$ such that for all $x \in S$, $\langle A_\alpha \colon \alpha < \kappa \rangle \in S$ and for all α, if $x \cap \mu \in A_\alpha$ then $\mathrm{Sk}^{\mathscr{A}}(x \cup \{\alpha\}) \cap \mu \neq x \cap \mu$.

We can extend j to $j' \colon V' \to M' = M^{\mathrm{Col}(\mu, <j(\kappa))}$. By Lemma 32, in M', S is stationary in $[H(\mu)^{V'}]^{\mu'}$. Let $f \colon \mu \to H(\lambda)^{V'}$ be a bijection and $\langle A^j_\alpha \colon \alpha < j(\kappa) \rangle = j(\langle A_\alpha \colon \alpha < \kappa \rangle)$. Then $T = \{x \in [\mu]^{\mu'} \colon f''x \in S \text{ and } x = f''x \cap \mu\}$ is stationary in $[\mu]^{\mu'}$. Hence for some α, $T \cap A^j_\alpha$ is stationary.

Let $C = \{N \prec j(\mathscr{A}) \colon \alpha \in C$ and N is closed under f, f^{-1} and $j \restriction H(\lambda)^{V'}\}$. Then C is closed and unbounded and hence for some $x \in C$, $x \cap \mu \in T \cap A^j_\alpha$.

Let $N = f''x \cap \mu$; then, $\mathrm{Sk}^{j(\mathscr{A})}(j''N \cup \{\alpha\}) \cap \mu \subseteq x \cap \mu = j''N \cap \mu$ and $j''N = j(N)$. Hence $M' \vDash$"there is an α, $j(N) \cap \mu \in A^j_\alpha$ and $\mathrm{Sk}^{j(\mathscr{A})}(j(N) \cup \{\alpha\}) \cap \mu = j(N) \cap \mu$." But then $N \in S$ and $V' \vDash$"there is an α, $N \cap \mu \in A_\alpha$ and $\mathrm{Sk}^{\mathscr{A}}(N \cup \{\alpha\}) \cap \mu = N \cap \mu$", a contradiction. This proves the main claim.

By Lemma 24, we can expand $\langle H(\lambda), \varepsilon \rangle$ to an $\mathscr{L} = \langle H(\lambda), \varepsilon, f_i \rangle_{i \in \omega}$ such that for every $N \prec \mathscr{L}$ with $|N| = \mu'$ and $|N \cap \gamma| = \gamma'$ and every maximal antichain $\langle A_\alpha \colon \alpha < \mu^+ \rangle \in N$, there is an α such that $N \cap \mu \in A_\alpha$ and $\mathrm{Sk}^{\mathscr{L}}(N \cup \{\alpha\}) \cap \mu = N \cap \mu$.

This allows us to build a branch with non-empty intersection through any tree of antichains, thus proving Theorem 30. □

Previous to this work Jech asked two questions that in light of Theorems 26–30 look very attractive. He asked whether, assuming that there is a supercompact cardinal κ, one can prove either

a) NS_κ is precipitous,

b) NS_{ω_1} is precipitous.

Unfortunately both are false:

THEOREM 33. *If* $\kappa^\kappa = \kappa$, $2^\kappa = \kappa^+$ *then there is a* $<\kappa$*-closed*, κ^+-c.c. *forcing* $\mathbf{P}$ *such that for all normal ideals* $\mathscr{I}$ *in V the normal closure of* $\mathscr{I}$ *in* $V^{\mathbf{P}}$ *is not precipitous*.

We first prove two lemmas:

LEMMA 34. *Suppose that $\mathscr{I}$ is a normal, κ-complete ideal on κ and* **P** *is a $<\kappa$-closed forcing; then the normal closure of $\mathscr{I}$ in $V^{\mathbf{P}}$ is a proper ideal.*

Proof. The normal closure of $\mathscr{I}$ is the collection of sets included in some set of the form $\nabla\langle X_\alpha: \alpha < \kappa\rangle$ for a sequence $\langle X_\alpha: \alpha < \kappa\rangle \subseteq \mathscr{I}$ in $V^{\mathbf{P}}$. If the normal closure is not a proper ideal then there is a sequence $\langle X_\alpha: \alpha < \kappa\rangle \subseteq \mathscr{I}$, $\langle X_\alpha: \alpha < \kappa\rangle \in V^{\mathbf{P}}$ such that $\kappa \subseteq \nabla\langle X_\alpha: \alpha < \kappa\rangle$. Let $\langle \tau_\alpha: \alpha < \kappa\rangle \in V^{\mathbf{P}}$ be a term for such a sequence.

In **P**, build a sequence of conditions $\langle p_\alpha: \alpha < \kappa\rangle$ such that $p_\alpha \Vdash p'_\alpha$ for $\alpha > \alpha'$ and for each α, $p_\alpha \| \tau_\alpha$, $p \Vdash \dot{\tau}_\alpha = X_\alpha$ for some $x_\alpha \in \mathscr{I}$. Then $\nabla_{\alpha<\kappa} x_\alpha \supsetneq \kappa$ since $\mathscr{I}$ is proper in V. Let $\delta \in \kappa \sim \nabla_{\alpha<\kappa} x_\alpha$. Then, if $\beta > \delta$, $p_\beta \Vdash \delta \notin \nabla_{\alpha<\kappa} x_\alpha$, a contradiction. □

The following lemma is standard and we omit the proof.

LEMMA 35. *Let κ be a regular cardinal. There is a sequence of functions $\langle O_\alpha: \alpha < \kappa^+\rangle$, $O_\alpha: \kappa \to \kappa$, such that whenever $\mathscr{I}$ is a normal κ-complete precipitous ideal on κ then O_α represents α in the generic ultrapower.*

Recall, an ideal is not precipitous if and only if there is a tree of maximal antichains where the intersection of the sets that lie on any branch of the tree is empty. Thus to show that an ideal $\mathscr{I}$ is not-precipitous it is enough to show that there are sets $\langle A_\eta: \eta \in (\kappa^+)^{<\omega}\rangle$ and functions $\langle f_\eta: \eta \in (\kappa^+)^{<\omega}\rangle$ such that:

a) If η extends ν then $A_\eta \subseteq A_\nu$.

b) $\{A_{\eta^\frown\alpha}: \alpha \in \kappa^+\}$ is an almost disjoint maximal antichain below A_η.

c) $f_\eta: A_\eta \to \kappa^+$ and for all $\gamma \in A_{\eta^\frown\alpha}$, $f_{\eta^\frown\alpha}(\gamma) < f_\eta(\gamma)$.

Clause c) guarantees that if $g: \omega \to \kappa^+$ then $\bigcap_{n\in\omega} A_{g\restriction n} = \varnothing$, since if $\gamma \in \bigcap_{n\in\omega} A_{g\restriction n}$ then $\langle f_{g\restriction n}(\gamma): n \in \omega\rangle$ forms a descending ω-sequence of ordinals. The forcing in Theorem 33 consists of approximations to such a tree.

Proof of Theorem 33. **P** will be an iteration of length κ^+. Let $T = (\kappa^+)^{<\omega}$ and $\langle \eta_\alpha: \alpha < \kappa^+\rangle$ be a well-ordering of T such that if η is an initial segment of ν then η comes before ν. The iteration will add a sequence of sets $\langle A_\eta: \eta \in T\rangle$ and functions $\langle f_\eta: \eta \in T\rangle$ such that:

a) $A_\eta \subseteq \kappa$ and $|A_{\eta^\frown\alpha} \cap A_{\eta^\frown\beta}| < \kappa$, if $\alpha \neq \beta$.

b) $f_\eta: A_\eta \to \kappa$ and f_η eventually dominates O_α for each $\alpha \in \kappa^+$.

c) For all α and $\gamma \in A_{\eta^\frown\alpha}$, $f_{\eta^\frown\alpha}(\gamma) < f_\eta(\gamma)$.

The iterations will be with $<\kappa$-supports.

Suppose we have defined $\mathbf{P}_\alpha$ and $\langle A_{\eta_\beta}: \beta < \alpha\rangle$ and $\langle f_{\eta_\beta}: \beta < \alpha\rangle$. To specify $P_{\alpha+1}$ we must define the factor algebra Q_α in $V^{\mathbf{P}_\alpha}$. Suppose $\eta_\alpha = \eta^\frown\gamma$.

Then $\eta = \eta_{\beta_0}$ for some $\beta_0 < \alpha$. We put $q \in Q_\alpha$ if and only if

$$q = \langle a_{\eta_\alpha}, f'_{\eta_\alpha}, \langle b_{\alpha,\beta}: \beta \in y\rangle, \langle S_{\alpha,\beta}: \beta \in x\rangle\rangle$$

where

a) $a_{\eta_\alpha} \subseteq \kappa$, $|a_{\eta_\alpha}| < \kappa$ and $a_{\eta_\alpha} \subseteq A_\eta$,

b) $y \in [\alpha]^{<\kappa}$, $b_{\alpha,\beta} \in \kappa$,

and if $\eta_\beta = \eta^\frown\gamma'$ for some γ' then $a_{\eta_\alpha} \cap A_{\eta_\beta} \subseteq b_{\alpha,\beta}$,

c) $f'_{\eta_\alpha}: a_{\eta_\alpha} \to \kappa$, $x \in [\kappa^+]^{<\kappa}$, $S_{\alpha,\beta} \in \kappa$ and for all $\xi > S_{\alpha,\beta}$, if $\xi \in a_{\eta_\alpha}$ then $f'_{\eta_\alpha}(\xi) > O_\beta(\xi)$,

d) for all $\xi \in a_{\eta_\alpha}$, $f'_{\eta_\alpha}(\xi) < f_\eta(\xi)$.

The ordering is given by $q^* \Vdash q$ where

$$q^* = \langle a^*_{\eta_\alpha}, f^*_{\eta_\alpha}, \langle b^*_{\alpha,\beta}: \beta \in y^*\rangle, \langle S^*_{\alpha,\beta}: \beta \in x^*\rangle\rangle$$

if and only if,

a) $a^*_{\eta_\alpha}$ is an end extension of a_{η_α},

b) $f^*_{\eta_\alpha} \restriction a_{\eta_\alpha} = f'_{\eta_\alpha}$,

c) $y^* \supseteq y$ and for all $\beta \in y$, $b^*_{\alpha,\beta} = b_{\alpha,\beta}$,

d) $x^* \supseteq x$ and for all $\beta \in x$, $S_{\alpha,\beta} = S^*_{\alpha,\beta}$.

Note that a_{η_α} approximates A_{η_α} and f'_{η_α} approximates f_{η_α}. Clause b) in the definition of the partial ordering guarantees that the $A_{\eta^\frown\alpha}$ and $a_{\eta^\frown\beta}$ are almost disjoint.

Clause d) guarantees that $f_{\eta^\frown\alpha} < f_\eta$ on $A_{\eta^\frown\alpha}$. Clause c) guarantees that f_η does not stray into the well-founded part of the generic ultrapower. Let $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * Q_\alpha$.

Claim. $\mathbf{P}$ is $<\kappa$-closed and κ^+-c.c.

Proof. $<\kappa$-closure is true since we are iterating with $<\kappa$-supports and each Q_α is $<\kappa$-closed. Since $\mathbf{P}$ is κ-closed we could have defined $\mathbf{P}$ in the ground model as a product forcing. In fact $\mathbf{P}$ has a dense set, D, of conditions of the form $p = \langle p(\alpha): \alpha \in \operatorname{supp} p\rangle$ where for $\alpha \in \operatorname{supp} p$

$$p(\alpha) = \langle a_{\eta_\alpha}, f'_{\eta_\alpha}, \langle b_{\alpha,\beta}: \beta \in y(\alpha)\rangle, \langle S_{\alpha,\beta}: \beta \in x(\alpha)\rangle\rangle$$

and a_{η_α}, f'_{η_α}, $y(\alpha)$, $\langle b_{\alpha,\beta}: \beta \in y(\alpha)\rangle$ and $\langle S_{\alpha,\beta}: \beta \in x(\alpha)\rangle$ are all elements of V. Further $y(\alpha) = \operatorname{supp} p \cap \alpha$ and $x(\alpha) = \operatorname{supp} p$.

Let $\langle p_\alpha: \alpha < \kappa^+\rangle \subseteq \mathbf{P}$. We want to show that for some α, β, p_α and p_β are compatible. We may assume that each $p_\alpha \in D$ and by a standard Δ-system argument we may further assume that there is a set $F \subseteq \kappa^+$, $|F| < \kappa$ and for all $\alpha < \beta$, $\operatorname{supp} p_\alpha \cap \operatorname{supp} p_\beta = F$ and if $\alpha < \beta$ then $\sup\operatorname{supp} p_\alpha \leq \inf(\operatorname{supp} p_\beta \sim F)$. By the cardinality of κ^κ we may assume that for all α, β and all $\sigma \in F$,

$$(a_{\eta_\alpha})^{p_\alpha} = (a_{\eta_\alpha})^{p_\beta} \text{ and } (f'_{\eta_\alpha})^{p_\alpha} = (f_{\eta_\alpha})^{p_\beta}.$$

Further, by cardinality arguments we may assume that for $\sigma, \delta \in F$,

$$(b_{\sigma,\delta})^{p_\alpha} = (b_{\sigma,\delta})^{p_\beta} \text{ and } (S_{\sigma,\delta})^{p_\alpha} = (S_{\sigma,\delta})^{p_\beta}.$$

We claim that any p and q satisfying these properties are compatible.

Define a condition γ with support $\operatorname{supp} p \cup \operatorname{supp} q$. For $\alpha \in \operatorname{supp} p \sim \operatorname{supp} q$ let $\gamma(\alpha) = p(\alpha)$ and for $\alpha \in \operatorname{supp} q \sim \operatorname{supp} p$ let $\gamma(\alpha) = q(\alpha)$. For $\alpha \in F$, let

$$\gamma(\alpha) = \langle a_{\eta_\alpha}, f'_{\eta_\alpha}, \langle b_{\alpha,\beta}: \beta \in y(\alpha)\rangle, \langle S_{\alpha,\beta}: \beta \in x(\alpha)^q \cup x(\alpha)^p\rangle\rangle.$$

Note that $a^p_{\eta_\alpha} = a^q_{\eta_\alpha}$ and $(f'_{\eta_\alpha})^p = (f'_{\eta_\alpha})^q$ and $\{y(\alpha), \langle b_{\alpha,\beta}: \beta \in y(\alpha)\rangle\}^p = \{y(\alpha), \langle b_{\alpha,\beta}: \beta \in y(\alpha)\rangle\}^q$ and for $\beta \in x(\alpha)^q \cap x(\alpha)^p$, $(S_{\alpha,\beta})^p = (S_{\alpha,\beta})^q$. Then γ is a condition since the restrictions on a coordinate $\gamma(\alpha)$ refer only to a_{η_β} and f'_{η_β} for $\beta < \alpha$, $\beta \in y(\alpha)$, and q and p agree on the a_{η_β}'s for $\beta \in F$. This proves the claim.

As we argued, we will be done if we can show that for any ideal on κ, $\mathscr{I} \in V$ and any $\eta \in T$, $\{A_{\eta^\frown\alpha}: \alpha \in \kappa^+\}$ is an $\mathscr{I}$-maximal antichain below A_η in $V^{\mathbf{P}}$.

Let $S \in V^{\mathbf{P}}$ be a term for an $\mathscr{I}$-positive set $S \subseteq A_\eta$. Then, by the chain condition there is a ψ such that A_η, $S \in V^{\mathbf{P}_\psi}$. Choose the least $\theta > \psi$ such that $\eta_\theta = \eta^\frown\xi$ for some ξ. We will show that $A_{\eta_\theta} \cap S \notin \bar{\mathscr{I}}$.

Let $\langle X_\gamma: \gamma < \kappa\rangle \in V^{\mathbf{P}}$ be a term for a sequence of elements of $\mathscr{I}$ and $p \in \mathbf{P}$ be a condition, $p \Vdash A_{\eta_\theta} \cap S \subseteq \nabla\langle X_\gamma: \gamma < \kappa\rangle$.

Let $G_\psi \subseteq \mathbf{P}_\psi$ be generic with $p \restriction \psi \in G_\psi$. Let $V' = V[G_\psi]$, $\lambda \gg \kappa$ and let $M \prec \langle H(\lambda)^{V'}, \varepsilon, \Delta, \mathbf{P}_\psi, G_\psi, \mathbf{P}, S, \eta, \theta\rangle$ be such that:

a) $M \cap \kappa^+ \in \mathrm{OR}$, $|M| = \kappa$ and $M^{<\kappa} \subseteq M$;

b) $p, \langle X_\gamma: \gamma < \kappa\rangle \in M$.

Let $\langle N_\alpha: \alpha < \kappa\rangle$ be a continuous chain of elementary substructures of M, each of cardinality $< \kappa$ such that

a) $M = \bigcup_{\alpha<\kappa} N_\alpha$.

b) $\{p\} \cup \{\langle X_\gamma: \gamma < \kappa\rangle\} \subseteq N_0$.

c) $\langle N_\beta: \beta < \alpha\rangle \in N_{\alpha+1}$.

Then clause c) implies that whenever α is a limit ordinal then N_α is internally approachable.

Let $S' = S \sim \{\alpha$: for some $\gamma \in N_\alpha$, $\eta^\frown\gamma = \eta_\beta$ for $\beta < \psi$ and $\alpha \in A_{\eta^\frown\gamma}\}$. Then, if $S \cap A_{\eta^\frown\gamma} \in \bar{\mathscr{I}}$ for each γ such that $\eta^\frown\gamma = \eta_\beta$ for some $\beta < \psi$, $S \approx S' \bmod \mathscr{I}$. (Here we use the remarks just before the proof of Theorem 12.)

Let $\delta_\alpha = \sup N_\alpha \cap \kappa$ and $\delta^* = \sup M \cap \kappa^+$. Then

$$C = \left\{\alpha: \text{for all } \gamma \in N_\alpha \cap \kappa^+, \dot{O}_{\delta^*}(\delta_\alpha) > \dot{O}_\gamma(\delta_\alpha)\right\}$$

is a closed and unbounded subset of κ. Further there is a final segment I of A_η

such that $f_\eta > \dot{O}_{\delta*}$ on I. There is a limit ordinal α such that $\delta_\alpha \in C \cap S' \cap I$ and $\delta_\alpha \notin \bigcup(\mathscr{I} \cap N_\alpha)$, since otherwise $C \cap S^* \subseteq \{\alpha$: for some $X \in N_\alpha \cap \mathscr{I}$, $\alpha \in X\} \in \mathscr{I}$. Choose such an α_0. Then $\delta_{\alpha_0} \notin A_{\eta^\frown\gamma}$ for any $\gamma \in N_{\alpha_0}$ for which there is a $\beta < \psi$ and $\eta^\frown\gamma = \eta_\beta$.

Build a sequence of conditions that lie in N_{α_0}, $\langle p_\beta: \beta < \beta^* \rangle \in \mathbf{P}/\mathbf{P}_\psi$ such that $p_0 = p$ and $p_{\beta'} \Vdash p_\beta$ if $\beta' > \beta$ and for all dense open sets $D \subseteq \mathbf{P}/\mathbf{P}_\psi$ that lie in N_{α_0}, there is a β such that $p_\beta \in D$. This is possible since $N_\alpha \in$ IA. (Repeat the argument in Lemma 28d.) Then $\beta^* < \kappa$ and hence there is a master condition $q \Vdash p_\beta$ for each $\beta < \beta^*$. We may assume that q is the coordinatewise union of the $\langle p_\beta: \beta < \beta^* \rangle$. For each $\gamma < \delta_{\alpha_0}$, q decides the value of X_γ and $q \Vdash X_\gamma \in N_{\alpha_0}$. Hence $q \Vdash \delta_{\alpha_0} \notin \nabla_{\gamma<\kappa} X_\gamma$.

Now $\delta_{\alpha_0} \in S$ and $f_\eta(\delta_{\alpha_0}) > \dot{O}_{\delta*}(\delta_{\alpha_0})$. Let

$$q(\theta) = \langle a_{\eta_\theta}, f'_{\eta_\theta}, \langle b_{\theta,\beta}: \beta \in y \rangle, \langle S_{\theta,\beta}: \beta \in x \rangle \rangle$$

be the θth coordinate of q.

Since y, $x \subseteq N_\alpha$, $O_{\delta*}(\delta_{\alpha_0}) > O_\beta(\alpha)$ for each $\beta \in x$ and for each $\beta \in y$, $\delta_\alpha \notin A_\beta$, we can define a condition q^* in Q_θ by

$$\langle a_{\eta_\theta} \cup \{\delta_{\alpha_0}\}, f'_{\eta_\theta} \cup \{\delta_{\alpha_0}, C_{\delta*}(\delta_{\alpha_0})\}\rangle, \langle b_{\theta,\beta}: \beta \in y \rangle, \langle S_{\theta,\beta}: \beta \in x \rangle.$$

Then $q^* \Vdash q(\theta)$ is in Q_θ.

Let $q'(\alpha) = q(\alpha)$ for $\alpha \neq \theta$ and $q'(\theta) = q^*$. We must see that q' is a condition in $\mathbf{P}$. The only problem that could arise is a $\theta' > \theta$, $\theta' \in \operatorname{supp} q$, such that $q' \restriction \theta' \Vdash q(\theta') \notin Q_{\theta'}$. Let

$$q(\theta') = \langle a_{\eta_\theta}, f'_{\eta_{\theta'}}, \langle b_{\theta'}, \beta: \beta \in y(\theta') \rangle, \langle S'_{\theta,\beta}: \beta \in x(\theta') \rangle \rangle.$$

Then $a_{\eta_{\theta'}} \subseteq \delta_{\alpha_0}$ and hence $a_{\eta_{\theta'}}$ cannot conflict with $a_{\eta_\theta} \cup \{\delta_0\}$ and $f'_{\eta_{\theta'}}$ cannot conflict with $f'_{\eta_\theta} \cup \{\langle \delta_{\alpha_0}, 0_{\delta*}(\delta_\alpha) \rangle\}$. Hence $q' \in \mathbf{P}$.

But $q' \Vdash \delta_{\alpha_0} \in (S \cap A_{\eta_\theta}) \sim \nabla_{\gamma<\kappa}\langle X_\gamma: \gamma < \kappa \rangle$ and $q' \Vdash p$, a contradiction. Hence $\{A_{\eta^\frown\alpha}: \alpha < \kappa^+\}$ is a maximal antichain below A_η. □

Now in the proof of Theorem 33, $|\mathbf{P}| = \kappa^+$ and hence if $\lambda > \kappa$ is a supercompact cardinal then λ is supercompact in $V^{\mathbf{P}}$. Further if Q is a $< \kappa$-closed forcing then the normal closure in V^Q of $(\mathrm{NS}_\kappa)^V$ is $(\mathrm{NS}_\kappa)^{V^Q}$. Thus a supercompact cardinal λ does not prove that NS_κ is precipitous for any $\kappa < \lambda$.

If κ is a κ-closed indestructible supercompact cardinal (see [L1]), then forcing with $\mathbf{P}$ leaves κ supercompact. Hence we get a model with a supercompact cardinal κ such that NS_κ is not precipitous. (By Theorem 27, it is consistent to have a supercompact cardinal κ such that NS_κ is precipitous.)

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