Periodica Mathematica Hungarica Vol. 28 (1), (1994), pp. 39–42

A NOTE ON A SET-MAPPING PROBLEM OF HAJNAL AND MÁTÉ

P. KOMJÁTH* (Budapest) and S. SHELAH (Jerusalem)

Abstract

It is consistent that there exists a set mapping F with $\beta < F(\beta,\alpha) < \alpha$ for $\beta + 2 \le \alpha < \omega_2$ with no uncountable free sets.

For our current purposes, a *set mapping* is a function F such that $\mathrm{Dom}(F) = [X]^2$, $\mathrm{Ran}(F) \subseteq [X]^\lambda$ (or $\subseteq [X]^{<\lambda}$) for some set X and cardinal λ. Here, if X is a set, λ a cardinal, $[X]^\lambda = \{Y \subseteq X : |Y| = \lambda\}, [X]^{<\lambda} = \{Y \subseteq X : |Y| < \lambda\}$. For a set mapping as above, a set $Y \subseteq X$ is *free* if $y \notin F(x_1, x_2)$ for any *different* $x_1, x_2, y \in Y$. For theorems and problems about free sets see [2] and [3]. In [3], A. Hajnal and A. Máté asked if it is consistent that there is a set mapping $F : [\omega_2]^2 \to [\omega_2]^{<\omega}$ such that for $\beta < \alpha < \omega_2$, $F(\beta, \alpha)$ is a subset of the ordinal interval $(\beta, \alpha) = \{\gamma : \beta < \gamma < \alpha\}$ and no uncountable free set exists. Here we prove, using models of Abraham–Shelah [1] that such functions can consistently exist. The models of [1] were created to show the consistency of the following statement. There exists a family of closed, unbounded sets $\{E_\alpha : \alpha < \omega_2\}$ in ω_1 such that the intersection of any uncountably many of them is *finite*.

THEOREM 1. *It is consistent that there exists a set mapping $F : [\omega_2]^2 \to [\omega_2]^{\le\aleph_0}$ with $F(\beta,\alpha) \subseteq (\beta,\alpha)$ and with no uncountable free set.*

We give two proofs for the Theorem, both using models of [1].

Assume that for $\alpha < \omega_2$ a function $\varphi_\alpha : \alpha \to \omega_1$ is given such that $\varphi_\alpha^{-1}(i)$ is countable for every $i < \omega_1$. Assume further that $E_\alpha \subseteq \omega_1$ is a closed, unbounded set for $\alpha < \omega_2$. Put for $\beta < \alpha < \omega_2$ $\quad \gamma \in F(\beta, \alpha)$ if $\beta < \gamma < \alpha$ and $\varphi_\alpha(\gamma) < \min(E_\alpha - [\varphi_\alpha(\beta) + 1])$.

FIRST PROOF. Let V be a model of ZFC, fix φ_α as above for $\alpha < \omega_2$. The first model of [1] extends V to a model V'' containing closed, unbounded sets $E_\alpha (\alpha < \omega_2)$,

* Research supported by Hungarian National Research Fund No. 1805 and 1908.

Mathematics subject classification numbers, 1991. Primary 03E05; Secondary 03E35, 04A20.

Key words and phrases. Set mappings, forcing.

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

and also V'' contains Cohen reals $\{c_\alpha : \alpha < \omega_2\}$ such that if $X \in V''$ is a countable subset of ω_2 then $X \in V' = V[c_\alpha : \alpha < \omega_2]$. If $X \in V[c_\alpha : \alpha \in B], X \subseteq \omega_2$ is infinite, then $X \not\subseteq E_\alpha$ for $\alpha \notin B$. Assume that $\{\alpha_\xi : \xi < \omega_1\}$ is the increasing enumeration of a free set. $A = \{\alpha_\xi : \xi < \omega^2\} \in V[c_\alpha : \alpha \in B]$ for some countable $B \subseteq \omega_2$. Put $\gamma_n = \sup\{\alpha_{\omega n+j} : j < \omega\}$, then $\{\gamma_n : n < \omega\} \in V[c_\alpha : \alpha \in B]$. If $n < \omega, j < k < \omega, \alpha > \sup\{\gamma_n : n < \omega\}$, $\alpha \notin B$, and $\{\alpha_{\omega n+j}, \alpha_{\omega n+k}, \alpha\}$ is free, then $\varphi_\alpha(\alpha_{\omega n+j}) < \varphi_\alpha(\alpha_{\omega n+k})$ and are separated by an element of E_α. Therefore, if $t(n) = \sup\{\varphi_\alpha(\alpha_{\omega n+k}) : k < \omega\}$, then $X = \{t(n) : n < \omega\} \subseteq E_\alpha$ (as E_α is closed), and $X \in V[c_\xi : \xi \in B]$, as $A, \varphi_\alpha \in V[c_\xi : \xi \in B]$. But these contradict $\alpha \notin B$. ■

Second proof. We show that if $\square_{\omega_1}$ holds and there are ω_2 closed, unbounded sets such that the intersection of any uncountably many of them is finite, then the statement of Theorem 1 holds. This suffices for the proof as both models in [1] are gotten by cardinal presering forcing extension, and we can start from a model of $V = L$.

As $\square_{\omega_1}$ holds, there are closed sets $C_\alpha \subseteq \alpha$ for $\alpha < \omega_2$ such that $0 \in C_\alpha$ $(\alpha > 0)$, $\sup(C_\alpha) = \alpha$ for α limit, if $\beta \in C_\alpha$ then $C_\beta = \beta \cap C_\alpha$, and also β is successor iff its index is a successor in C_α's increasing enumeration.

For $\alpha < \omega_2$ we construct by induction $\varphi_\alpha : \alpha \to \omega_1$ as follows. $\varphi_0 = \emptyset$. If α is limit, $\varphi_\alpha = \bigcup\{\varphi_\beta : \beta \in C_\alpha\}$. If $\alpha = \beta + 1$, $\gamma = \max(C_\alpha)$, $\tau = \mathrm{tp}\,(C_\alpha)$, we let

$$\varphi_\alpha(\varepsilon) = \begin{cases} \varphi_\gamma(\varepsilon) & \text{if } \varepsilon < \gamma \\ \max(\tau, \varphi_\beta(\varepsilon)) & \text{if } \gamma \le \varepsilon < \beta \\ \tau & \text{if } \varepsilon = \beta. \end{cases}$$

It is easy to prove by induction on α that

(1) $$\text{if } \beta \in C_\alpha \text{ then } \varphi_\beta \subseteq \varphi_\alpha;$$

(2) $$\text{if } \varepsilon \text{ is in the } \tau'\text{th interval of } C_\alpha, \text{ then } \varphi_\alpha(\varepsilon) \ge \tau;$$

(3) $$|\varphi_\alpha^{-1}(i)| \le \omega \text{ for } i < \omega_1.$$

(1), (2) are obvious, (3) for $\mathrm{cf}\,(\alpha) \le \omega$ is easy, for $\mathrm{cf}\,(\alpha) = \omega_1$ follows from 2.

Claim. *If $\beta \le \alpha, \beta$ is limit, then there exists a $y = y(\beta, \alpha) < \omega_1$ and $\beta' < \beta$ such that for $\beta' < \varepsilon < \beta, \varphi_\alpha(\varepsilon) = \max(y, \varphi_\beta(\varepsilon))$.*

Proof of Claim. By induction on α, for a fixed β. The statement is obvious for $\alpha = \beta$.

Assume that $\alpha = \bar{\alpha} + 1$. If $\beta \le \max(C_\alpha)$, we can apply the inductive hypothesis. If $\gamma = \max(C_\alpha) \le \beta, \tau = \mathrm{tp}\,(C_\alpha)$, for an and-segment $\varphi_\alpha(\varepsilon) = \max(\tau, \varphi_{\bar{\alpha}}(\varepsilon))$, on az end-segment $\varphi_{\bar{\alpha}}(\varepsilon) = \max(y, \varphi_\beta(\varepsilon))$ for some y, i.e. $\varphi_\alpha(\varepsilon) = \max(\max(y, \tau), \varphi_\beta(\varepsilon))$.

If α is limit, $\gamma \in C_\alpha - \beta, \varphi_\gamma \subseteq \varphi_\alpha$, and we can use the inductive hypothesis. ■

Let $\{E_\alpha : \alpha < \omega_2\}$ be a sequence of closed, unbounded sets in ω_1. Define F as is defined before the first proof of the Theorem.

LEMMA 2. *If $\{\alpha_\xi : \xi < \omega_1\}$ is the increasing enumeration of some free set, then*

(a) *for some infinite $A \subseteq \omega_1$, $A \subseteq E_{\alpha_\xi}$ for uncountably many ξ;*

(b) *there is a closed, unbounded $D \subseteq \omega_1$, such that if $\delta \in D$, $\delta \leq \xi$, then E_{α_ξ} contains an end-segment of $D \cap \delta$;*

(c) *there is a closed, unbounded $D \subseteq \omega_1$ such that for every $\delta \in D$, for uncountably many $\xi, D \cap \delta \subseteq E_{\alpha_\xi}$.*

Clearly, Lemma 2 concludes the second proof of Theorem 1.

PROOF. (b) implies (c), by Fodor's theorem. (c) clearly implies (a), so we prove (b). Put $\alpha = \sup\{\alpha_\xi : \xi < \omega_1\}$, $C_\alpha = \{\beta_\xi : \xi < \omega_1\}$, an increasing enumeration. We define the following set E. Put $\delta \in E$, if δ is limit, $\beta_\delta = \sup\{\alpha_\xi : \xi < \delta\}$, $\xi < \delta$ iff $\varphi_\alpha(\alpha_\xi) < \delta$. Define D as follows. Put $\delta \in D$ if δ is a limit point of E.

Assume that $\delta \in D, \delta \leq \xi, \gamma = \sup\{\alpha_\xi : \xi < \delta\}$. Then $\alpha_\xi \geq \gamma, \gamma \in C_\alpha$, therefore $\varphi_\gamma \subseteq \varphi_\alpha$. If $\xi_1 < \xi_2 < \delta$, as $\alpha_{\xi_2} \notin F(\alpha_{\xi_1}, \alpha_\xi)$, we get that $\varphi_{\alpha_\xi}(\alpha_{\xi_1}) < \varphi_{\alpha_\xi}(\alpha_{\xi_2})$ and they are separated by E_{α_ξ}. Therefore, if $\xi_1 < \xi_2 < \delta$ are large enough, $\varphi_\gamma(\alpha_{\xi_1}) < \varphi_\gamma(\alpha_{\xi_2})$ and they are separated by E_{α_ξ}. If $\eta \in D \cap \delta$ is large enough, then

$$\eta = \sup\{\varphi_\alpha(\alpha_\xi) : \xi < \eta\} = \sup\{\varphi_\gamma(\alpha_\xi) : \xi < \eta\}$$

so $\eta \in E_{\alpha_\xi}$ and we are done. ■

COROLLARY 3. *There consistently exists a function f such that $\beta < f(\beta, \alpha) < \alpha$ for $\beta + 2 \leq \alpha < \omega_2$ with no uncountable free set.*

PROOF. If F is as in Theorem 1, we let $p = (s, g) \in P$ if s is a finite subset of ω_2, g is a partial function on $s, g(\beta, \alpha) \in F(\beta, \alpha)$. Put $(s', g') \leq q(s, g)$ iff $s' \supseteq s$, $g' \supseteq g$. It is easy to see that $(P, \leq)$ is ccc, and $\cup G$ covers ω_2 for a generic subset G. Put $f = \cup\{g : (s, g) \in G\}$ for a generic G. Assume that 1 forces that Y is an uncountable free set. We can find $p_\xi | \vdash \alpha_\xi \in Y$ for some different $\alpha_\xi (\xi < \omega_1)$, and we can also assume that p_ξ is of the form $(s \cup s_\xi, g_\xi)$ with $\{s, s_\xi : \xi < \omega_1\}$ pairwise disjoint and with the g_ξ's agreeing on s. There are $\xi_1 < \xi < \xi_2, \alpha_\xi \in f(\alpha_{\xi_1}, \alpha_{\xi_2})$, extend $p_{\xi_1}, p_\xi, p_{\xi_2}$ to a $q = (s', g)$ such that $\alpha_\xi = g(\alpha_{\xi_1}, \alpha_{\xi_2})$. ■

The Claim in Theorem 1 implies the following property of the functions $\{\varphi_\alpha : \alpha < \omega_2\}$.

(*) If $\delta < \omega_1$ is limit, $\{x(i) : i < \delta\}$ is increasing, $x(i) < \omega_2$, then there exist $j(i) < \omega_1$ such that if $\alpha \geq \sup\{x(i) : i < \delta\}$, then there is a $y < \omega_1$ such that for $i < \delta$ large enough $\varphi_\alpha(x(i)) = \max(y, j(i))$.

We show that (*) is independent.

THEOREM 4. *If the existence of a Mahlo cardinal is consistent then it is consistent that there are no functions $\varphi_\alpha : \alpha \to \omega_1$ with $|\varphi_\alpha^{-1}(i)| \leq \omega$ for $i < \omega_1$, satisfying (*).*

PROOF. Assume GCH. Let κ be a Mahlo cardinal, let P be the Lévy collapse of κ onto ω_2. Fix for $\beta < \kappa$, $\mathrm{cf}(\beta) = \omega$, a converging sequence $x(\beta, i) \nearrow \beta (i < \omega)$. Assume that for $\{x(\beta, i) : i < \omega\}$ some $\{j(\beta, i) : i < \omega\}$ exists, as in (*). One can find, by standard arguments, (see [4]), an inaccessible cardinal $\alpha < \kappa$, an intermediate model V' in which $\alpha = \omega_2$ and $\{j(\beta, i) : i < \omega\}$ are determined for $\beta < \alpha$. For a stationary $S \subseteq \{\beta < \alpha : \mathrm{cf}(\beta) = \omega\}$, $j(\beta, i) = j(i)$ $(i < \omega)$. The final model is forced over V' by a countably closed poset, Q. Let $q \in Q$ be a condition forcing that $\varphi_\alpha^{-1}(j(i)) \subseteq \gamma$ for some $\gamma < \alpha$ $(i < \omega)$. In the final model, there is a $j < \omega$ for every $\beta \in S$, $\beta > \gamma$, such that $x(\beta, j) > \gamma$, for $j < i < \omega$, $\varphi_\alpha(x(\beta, i)) = \max\{y(\beta), j(i)\}$, but q forces that it is different from $j(i)$, so it is $y(\beta)$.

By a well-known lemma, S stays stationary after forcing by Q. Then, there is an $i < \omega, j \geq i$, an unbounded $S' \subseteq S$ such that for $\beta \in S'$, $x(\beta, i) = x$, $\{x(\beta, i+1) : \beta \in S'\}$ are different. Therefore, if $\beta \in S'$, $\varphi_\alpha(x(\beta, i+1)) = y(\beta) = \varphi_\alpha(x)$, so $\varphi_\alpha^{-1}(\varphi_\alpha(x))$ contains uncountably many elements, a contradiction. ■

REFERENCES

[1] U. ABRAHAM, S. SHELAH, On the intersection of closed, unbounded sets, *Journal of Symb. Logic*, **51** (1986), 180–189.

[2] P. ERDŐS, A. HAJNAL, On the structure of set mappings, *Acta Math. Acad. Sci. Hung.*, **9** (1958), 111–131.

[3] A. HAJNAL, A. MÁTÉ, Set mappings, partitions, and chromatic numbers, *Logic Colloquium '73*, Bristol, North-Holland, 1975, 347–379.

[4] T. JECH, *Set Theory* Acad. Press, 1978.

(Received October 1, 1991)

DEPARTMENT OF COMP. SCI
R. EÖTVÖS UNIVERSITY
BUDAPEST, MÚZEUM KRT 6–8
1088, HUNGARY

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, GIVAT RAM
ISRAEL