A CONSTRUCTION OF MANY UNCOUNTABLE RINGS USING SFP DOMAINS AND ARONSZAJN TREES

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ABSTRACT

The paper is in two parts. In Part I we describe a construction of a certain kind of subdirect product of a family of rings. We endow the index set of the family with the partial order structure of an SFP domain, as introduced by Plotkin, and provide a commuting system of homomorphisms between those rings whose indices are related in the ordering. We then take the subdirect product consisting of those elements of the direct product having finite support in the sense of this domain structure. In the special case where the homomorphisms are isomorphisms of a fixed ring S, our construction reduces to taking the Boolean power of S by a Boolean algebra canonically associated with the SFP domain.

We examine the ideals of a ring obtainable in this way, showing for instance that each ideal is determined by its projections onto the factor rings. We give conditions on the underlying SFP domain that ensure that the ring is atomless. We examine the relationship between the $L_{\infty\omega}$-theory of the ring and that of the SFP domain.

In Part II we prove a 'non-structure theorem' by exhibiting $2^{\aleph_1}$ pairwise non-embeddable $L_{\infty\omega}$-equivalent rings of cardinality $\aleph_1$ with various higher-order properties. The construction needs only ZFC, and uses Aronszajn trees to build many different SFP domains with bases of cardinality $\aleph_1$.

Preface

This paper presents a blend of ideas from ring theory, set-theoretic combinatorics and computer science. It is divided into two parts; Part I will perhaps be of more interest to algebraists, and Part II to logicians.

In Part I we develop a method of constructing a subdirect product of certain families of rings. To do this we impose a partial order structure on the index set of the family. We take this poset structure to be that of an SFP domain, a notion introduced in **[14]** and well-known to domain theorists in computer science. The construction we give is related to the *Boolean power* construction (see **[4]**), and reduces to this in special cases (see Theorem I.4.2). It tends to produce rings with many orthogonal central idempotents, so is most at home when constructing Boolean or von Neumann regular rings.

We analyse the ideals of the resulting subdirect product and show that *inter alia* they carry information about the underlying poset structure of the index set. So if two such rings are isomorphic then their underlying SFP domains must be fairly similar.

We exploit this in Part II. Using a variant of the construction of Aronszajn trees in set theory we will construct, using ZFC only, $2^{\aleph_1}$ pairwise 'dissimilar' SFP domains. If all component rings are assumed to be countable, any subdirect products obtained with them will be pairwise non-embeddable rings. We can impose further conditions on the domains or the component rings themselves to obtain stronger results.

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A 'sample theorem' is:

THEOREM. *Let S be a countable Boolean ring. There are $2^{\aleph_1}$ pairwise non-embeddable Boolean rings R of cardinality $\aleph_1$ extending S. Each such ring R is existentially closed and without non-trivial injective endomorphisms ('rigid'), and each of its maximal ideals has a countable set of generators.*

This is proved in § 6 of Part II. The theorem suggests that there are too many such rings to classify fully. It is thus a *non-structure theorem* in the spirit of, for example, the result of **[15]** that if T is a non-superstable complete first-order theory of cardinality κ then there are 2^λ pairwise non-elementarily embeddable models of T of cardinality λ for all regular λ greater than κ (this has now been extended to all $\lambda > \kappa$). The construction is not limited to Boolean rings. Corollary I.1.4 will show that if K is the class of commutative rings, von Neumann regular rings, or existentially closed commutative rings, then the resulting subdirect product also lies in K. If $S \in K$, we can use this to build $2^{\aleph_1}$ pairwise non-embeddable but $L_{\infty\omega}$-equivalent rings R of cardinality $\aleph_1$ in K extending S (by taking a subdirect product of rings in K). Each ring R can be given a degree of rigidity, and each maximal, prime and irreducible ideal of R will be countably generated.

The work in this paper simplifies the construction in the doctoral thesis **[8]** of the first author, which used the continuum hypothesis. The argument there was more complicated and less general because SFP domains were not used. The motivation for **[8]** came from the paper of Ziegler **[17]**. Recall that a ring R is *atomless* if it has no principal maximal ideals. If R is a countable Boolean ring, then R is atomless if and only if the injective hull of R regarded as a left R-module has no indecomposable direct summand (cf. the example preceding 5.11 of **[17]**). In this case it is easily seen that R has $2^{\aleph_0}$ (that is, $2^{|R|}$) maximal ideals. (A generalisation to arbitrary countable rings was given in **[17]**—see, in particular, § 5 and 7.1(1), 7.2, 8.3.) Our initial objective was to show that this fails when $|R| = \aleph_1$. This is established by the 'sample theorem' above. Each R of the theorem is Boolean and existentially closed, and hence atomless [**7**, 6.3.9, Ex. 6.3.2]. But every maximal ideal is countably generated, so they are at most $2^{\aleph_0}$ in number—this can be less than $2^{\aleph_1} = 2^{|R|}$ (for example, if we assume the continuum hypothesis). The construction in **[3]** gives an atomless Boolean ring of cardinality $\aleph_1$ also illustrating this, but Jensen's $\Diamond$ (diamond) is used. On the other hand, unlike our construction, the Boolean algebra constructed in **[3]** has no uncountable set of pairwise incomparable elements. (The Boolean algebras that we build in Part II have no countable dense subalgebras. By [**3**, Theorem 3], if B is a Boolean algebra of cardinality ω_1 with no countable dense subalgebra, then B has an uncountable set of pairwise incomparable elements.)

It would be interesting to prove an intrinsic characterisation theorem for rings arising by our construction, analogous to that for varieties and reduced products. Possibly the work of Smyth **[16]** and Jung **[11]** would be relevant.

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Grossberg, M. Prest, J. C. Robson, S. J. Vickers and the referee of an earlier draft of part of this paper. The first author further thanks D. Gabbay, who read a draft of the paper and made a series of valuable suggestions, and also the U.K. Science and Engineering Research Council, King's College, Cambridge, and many friends, for financial and moral support without which the Ph.D. would not have been completed.

PART I. SFP SYSTEMS

This part of the paper contains the results of a more algebraic nature. We will define the notion of an SFP system of rings, and study some of the properties of its limit.

Let us describe the approach in rather more detail than above. Let $(P, \leqslant)$ be a poset such that for every $p \in P$ we have a ring R_p. Suppose further that for every $p, q \in P$ with $p \leqslant q$, we have a ring homomorphism v_{pq}: $R_p \to R_q$. We require that the v_{pq} (for $p \leqslant q$ in P) form a commuting system in the usual sense.

Assume that P has a least element, $\perp$, say. Then the presence of the maps v allows us to embed the ring $R_\perp$ diagonally into the direct product $\prod (R_p: p \in P)$, via $r \mapsto (v_{\perp p}(r)$: $p \in P)$ for $r \in R_\perp$. We would like to generalise this as follows. Let $N \subseteq P$ be finite. Can we embed the finite direct product $\prod (R_n$: $n \in N)$ diagonally into the full direct product?

So let $r \in \prod (R_n$: $n \in N)$. We need to define its image r' in $\prod (R_p$: $p \in P)$. By analogy with the case $N = \{\perp\}$, for each $p \in P$ we would like to define $r'(p)$ to be $v_{np}(r(n))$, where n is an appropriate element of N, depending on p. To force a unique choice of n for each p we will assume that N has the following property:

$$\text{for all } p \in P \text{ there is a unique maximal element of } \{n \in N:\ n \leqslant p\}.$$

We call such an N a *support*. This would hold, for example, if N is linearly ordered and $\perp \in N$. We write this maximal element as p/N. We can now define r' to be $(v_{p/N,p}(r(p/N))$: $p \in P)$. Then N is in effect a *finite support* of r' in $\prod (R_p$: $p \in P)$.

So we consider the set R^* of all elements of $\prod (R_p$: $p \in P)$ having a finite support in this sense. We require that R^* be a subring of $\prod (R_p$: $p \in P)$. To obtain closure under $+$ and $-$ we will need any two finite supports to be contained in a third, and to avoid redundancy of any R_p we will formally require that

(*) *any finite subset of P extends to a finite support $N \subseteq P$.*

For example, if P is linearly ordered with a least element, this is trivially true. So we could take P to be $(\mathbb{Q} \cup \{-\infty\}, <)$, each R_p to be the ring $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and all v_{pq} to be the identity map. In this case R^* turns out to be the countable atomless Boolean ring. See Remark 4.3(2).

However, the condition (*) holds in much more general cases and is closely related to the SFP domains of Plotkin **[14]**. Any P satisfying (*) extends canonically to an SFP domain by adding where necessary a least upper bound h for each directed subset D of P. These extra points h turn out to be very useful: $\langle R_d, v_{de}$: $d \leqslant e$ in $D\rangle$ forms a direct system and it is technically convenient to define R_h to be its direct limit, and extend v accordingly. Hence we will work with SFP domains throughout.

It is easy to show that if the 'component rings' R_p $(p \in P)$ have various properties then so does R^*. Examples of properties preserved in this way are commutative, Boolean, von Neumann regular, and existentially closed commutative (see Corollary 1.4). The cardinality of R^* is also related to the cardinalities of P and of the R_p (see Proposition 1.5), and the $L_{\infty\omega}$-theory of R^* is determined by the $L_{\infty\omega}$-theory of P together with the R_p and the maps ν_{pq} (§ I.6).

So far the construction could be undertaken for any model-theoretic structure. We consider rings because we can fruitfully study their ideals. See §§ 2–4 below. (Generalisations to structures such as lattices are probably possible here.) An important class of ideals arises as follows. If I is a (left) ideal of R_s for some $s \in P$, then the set $I@s = \{r \in R^*: r(s) \in I\}$ is a left ideal of R^*. Ideals of this form are called *full ideals*: they are in a sense 'locally determined'. We can recover I and s from $I@s$, so the full ideals are closely related to the poset structure of P. They are a kind of basis for the set of all ideals of R^*. Using the extra elements h of P we can show that any maximal, prime or irreducible ideal of R^* must be full, and every ideal of R^* is the intersection of the full ideals that contain it.

The layout of this part of the paper is as follows. In § 1 we discuss SFP domains and formally lay out the subdirect product construction. In §§ 2–4 we discuss ideals of R^*, and in § 5 we use these results to enforce that R^* has a property related to *atomlessness*. Finally, in § 6 we discuss $L_{\infty\omega}$-equivalence.

1. *Definition of an SFP system*

In this section we give most of the definitions that we will need, plus some examples and useful lemmas for illustration.

Algebraic dcpos

Recall that a partially ordered set, or *poset*, is a (usually non-empty) set equipped with a reflexive transitive binary relation, written here as '$\leq$'. A poset $(D, \leq)$ is *directed* if for all finite subsets X of D there is $d \in D$ with $d \geq x$ for all $x \in X$. Equivalently, D is non-empty (take $X = \emptyset$) and whenever $d_1, d_2 \in D$, then there is $d_3 \in D$ with $d_3 \geq d_1$ and $d_3 \geq d_2$. It will help to bear in mind that directed sets are always non-empty.

A non-empty poset P is said to be *directed complete* (a 'dcpo') if any directed subset D of P has a least upper bound in P. That is, there is $u \in P$ such that for all $v \in P$ we have $v \geq u$ if and only if $v \geq d$ for all $d \in D$. We write this bound u as $\mathrm{lub}(D)$, or more explicitly $\mathrm{lub}_P(D)$. It is necessarily unique.

An element p of a dcpo P is said to be *finite* if whenever D is a directed subset of P and $p \leq \mathrm{lub}(D)$ then $p \leq d$ for some $d \in D$. We write P^0 for the set of finite elements of P. We call P^0 the *base* of P, and P is said to be *algebraic* if for all $p \in P$, the set $p{\downarrow} = \{q \in P: q \leq p\}$ is such that $p{\downarrow} \cap P^0$ is directed and $\mathrm{lub}(p{\downarrow} \cap P^0) = p$. That is, p is the 'lub' of the set of finite elements beneath it. It follows that in this case P is determined by its base (see below). Algebraic dcpos P with countable base and a minimum element are usually called *domains* in the computer science literature.

Examples of algebraic dcpos are all finite (non-empty) posets and all successor ordinals. If X is a non-empty set then its power set $\wp(X)$, ordered by inclusion, is an algebraic dcpo whose finite elements are just the finite subsets of X, whence the name. The half-open real interval $(0, 1]$ has no finite elements and shows that a dcpo need not be algebraic, as does the dcpo illustrated in Fig. 1.1.

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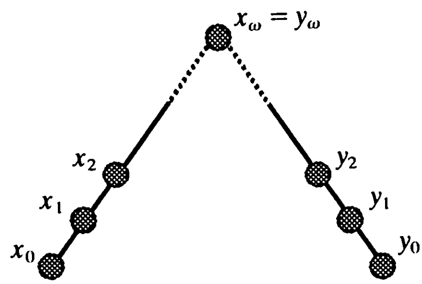

FIG. 1.1

Ideals

Let P be any poset. An *ideal* of P is a subset I of P that is closed downwards (that is, if $x \in P$, $y \in I$ and $x \leq y$ then $x \in I$) and directed. Clearly if $p \in P$ then $p\downarrow$ is an ideal; ideals of this form are said to be *principal*. It is well known that if P is an arbitrary non-empty poset, then the set of ideals of P, ordered by inclusion, forms an algebraic dcpo. Its finite elements are just the principal ideals, and they are in order-isomorphism with P. Hence any P can be 'completed' to an algebraic dcpo by taking this '*ideal completion*'. Moreover, any algebraic dcpo P is isomorphic to the ideal completion of its base P^0. We will often identify a poset with the set of finite elements of its ideal completion. (A similar ideal completion can be undertaken for preorders also.)

Locally directed sets

Let P be a poset. A subset N of P is said to be *locally directed in P* (written $N \trianglelefteq P$) if $p\downarrow \cap N$ is directed for all $p \in P$. Equivalently, $N \trianglelefteq P$ if and only if $N \cap I$ is directed for all ideals I of P.

For example, if P is an algebraic dcpo then $P^0 \trianglelefteq P$. If P contains a least element $\perp$, then any linearly ordered subset N of P with $\perp \in N$ is locally directed in P. A subset N of $\wp X$ is locally directed in $(\wp X, \subseteq)$ if and only if N is closed under finite (including empty) unions. Since $P \trianglelefteq P$ for any P, locally directed does not imply directed. The converse also fails, since if $\perp$ is the least element of P then $N \trianglelefteq P \Rightarrow \perp \in N$. So if $p \in P \setminus \{\perp\}$ then $\{p\}$ is directed but not locally directed in P.

It is easily seen that $\trianglelefteq$ is a reflexive and transitive relation on posets, and that if $N \trianglelefteq P$ and $N \subseteq Q \subseteq P$ then $N \trianglelefteq Q$.

Now assume that P is a dcpo. If $N \trianglelefteq P$ and $p \in P$, we write p/N for $\text{lub}_P(p\downarrow \cap N)$. The lub exists since P is a dcpo. Indeed if N is finite, or, more generally, a dcpo such that $\text{lub}_P(D) = \text{lub}_N(D)$ for all directed $D \subseteq N$, then $P/N \in N$. We can view p/N as the best approximation to p in N. We have $p/N \leq p$ for all $p \in P$. Further, P is algebraic if and only if $P^0 \trianglelefteq P$ and $p/P^0 = p$ for all $p \in P$. If $N \trianglelefteq P$, we can define an equivalence relation $\sim_N$ on P by $x \sim_N y \Leftrightarrow x/N = y/N$. We will see in § 4 that the equivalence classes are related to the well known 'patch' topology on P.

SFP domains

We can now define the strain of poset of interest to us here. A poset P is said to be *nice* if any finite subset X of P can be extended to a finite locally directed subset of P. An *SFP domain* is an algebraic dcpo P such that P^0 is nice. So the ideal completion of a nice poset is an SFP domain, and all SFP domains arise in this way.

An equivalent definition uses the notion of MUB-closure (see Plotkin **[14]**). If $X \subseteq P$, define $\text{MUB}(X) = \{p \in P\colon\ p \text{ is a minimal upper bound of } X\}$. Also define an increasing chain $U^n(X)$ $(n \leq \omega)$ by

$$U^0(X) = X, \quad U^{n+1}(X) = \bigcup \{\text{MUB}(Y)\colon\ Y \subseteq U^n(X)\}, \quad U^\omega(X) = \bigcup_{n<\omega} U^n(X).$$

We call $U^\omega(X)$ the MUB-closure of X. Then it is easily seen that P is SFP if and only if, for all finite $X \subseteq P^0$,

(i) for all $p \in P$ with $X \subseteq p{\downarrow}$ there is $y \in \text{MUB}(X)$ with $y \leq p$,

(ii) $\text{MUB}(X)$ is finite,

(iii) $U^\omega(X)$ is finite.

In fact, in this case $U^\omega(X) \trianglelefteq P^0$. Domains satisfying (i) and (ii) are sometimes called 2/3-SFP. Of course, (iii) implies (ii).

Examples of nice posets are any finite poset, any linear order with a least element, any Boolean algebra, and any tree with finitely many minimal elements. The restriction to finitely many minimal elements is necessary. For if P is a nice poset then take a finite set $N \trianglelefteq P$. Every $p \in P$ lies above some element of N. Hence the minimal elements of P are exactly the minimal elements of N.

Figure 1.2 shows the three main kinds of non-nice poset. See **[16]**.

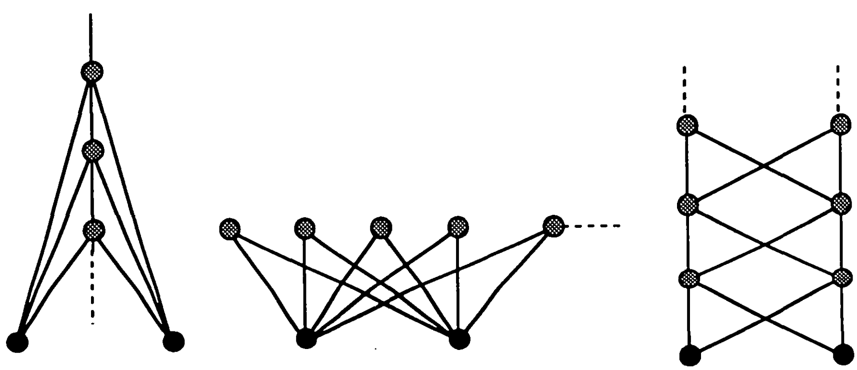

FIG. 1.2

On the left of Fig. 1.2 the two black elements have no minimal upper bound, violating condition (i) above. In the centre poset they have infinitely many minimal upper bounds, violating (ii). The right-hand poset satisfies (i) and (ii) but now the black elements have infinite MUB-closure.

SFP domains were introduced in **[14]** as those arising as inverse limits of projective Sequences of Finite Posets. They are of considerable interest in computer science, where they are used to provide denotational semantics for programming languages. Any domain P can be equipped with a topology (the Scott topology): the open subsets X of P are defined to be those such that

(i) X is closed upwards, and

(ii) if D is a directed subset of P and $\text{lub}(D) \in X$ then $D \cap X \neq \varnothing$.

If D and E are domains, we write $[D \to E]$ for the poset of Scott-continuous functions from D to E, ordered by

$$f \leqslant g \quad \Leftrightarrow \quad f(d) \leqslant g(d) \text{ for all } d \in D.$$

In **[16]** Smyth showed amongst other things that if D is a domain with countable base, then $[D \to D]$ is also a domain with countable base if and only if D is SFP. In this case $[D \to D]$ is also SFP. The SFP domains form the largest Cartesian closed full subcategory of the category of domains (with countable base and a minimum element), the morphisms being the Scott-continuous maps. The restrictions of countable base and minimum element were removed by Jung **[11]**—in this case there are four such maximal subcategories.

SFP systems

We now give our main algebraic definition. An *SFP system* is a triple $\langle P, \rho, \nu \rangle$, where the following hold.

(i) The set P is an SFP domain.

(ii) The map ρ is a map from P into the class of rings (throughout this paper, all rings will have a 1, and $1 \neq 0$). We will write R_p for $\rho(p)$, where ρ is understood.

(iii) The map ν is a map defined on those pairs $(p, q) \in P^2$ with $p \leqslant q$. Each $\nu(p, q)$ is a ring homomorphism from R_p into R_q. (All ring homomorphisms in this paper preserve 0 and 1.) We write $\nu(p, q)$ as ν_{pq}. We require further that

(a) ν_{pp} is the identity on R_p for all $p \in P$,

(b) $\nu_{qr} \circ \nu_{pq} = \nu_{pr}$ if $p \leqslant q \leqslant r$ in P,

(c) if D is a directed subset of P, with least upper bound $u \in P$, then R_u is the direct limit of the direct system $\langle R_d, \nu_{de}\colon d \leqslant e \text{ in } D \rangle$, and for all $d \in D$, the map ν_{du} is the canonical ring homomorphism from R_d into R_u.

REMARK 1.1. Let P be a nice poset. Suppose we have a triple (P, ρ, ν) satisfying (ii) and (iii) (a), (b). Then we can canonically complete it to an SFP system by

(1) embedding P canonically into its ideal completion Q,

(2) defining R_q for $q \in Q \backslash P$ to be the direct limit

$$\varinjlim \langle R_p, \nu_{pr}\colon p \leqslant r \text{ in } P \cap q{\downarrow} \rangle,$$

and

(3) defining $\nu_{qq'}$ for $q \leqslant q'$ in Q to be the 'limit' of the $\nu_{pp'}$ for $p, p' \in P$ with $p \leqslant q$, $p' \leqslant q'$.

(Notice that if D is a directed subset of Q with $\mathrm{lub}_Q(D) = q$, then as each $p \in P \cap q{\downarrow}$ is finite in Q, there is $d \in D$ with $d \geqslant p$. It follows that $\varinjlim \langle R_d\colon d \in D \rangle = R_q$. Hence Condition (iii(c)) holds.) All SFP systems arise in this way. So an SFP system $\langle P, \rho, \nu \rangle$ is determined by its 'finite' part: by P^0, R_p and $\nu_{pp'}$ for $p \leqslant p'$ in P^0.

Limits of SFP systems

Let $\langle P, \rho, \nu \rangle$ be an SFP system, and let $N \trianglelefteq P$. Recall that if $p \in P$, the element p/N is defined to be $\mathrm{lub}_P(p{\downarrow} \cap N)$. An element $r \in \prod \langle R_p\colon p \in P \rangle$ is said to have *support* N if $r(p) = \nu_{(p/N),p}[r(p/N)]$ for all $p \in P$. We define the *limit* of $\langle P, \rho, \nu\rangle$, or $\lim\langle P, \rho, \nu\rangle$, to be the subdirect product consisting of those elements of $\prod R_p$ that have a finite support N for some $N \trianglelefteq P^0$. Since P is an SFP domain, any two finite locally directed subsets of P^0 are contained in a third, and it follows that the limit of $\langle P, \rho, \nu\rangle$ is a subring of $\prod R_p$. Clearly it is also identifiable with a subring of $\prod\langle R_p\colon p \in P^0\rangle$, since P^0 supports any element of the limit of $\langle P, \rho, \nu\rangle$.

We will generally write R_P for the limit of $\langle P, \rho, \nu\rangle$. Obviously, for any $p_0 \in P$ the projection $(r \mapsto r(p_0))$ of R_P onto R_{p_0} is a surjective ring homomorphism.

As an example, if $P^0 = (\mathbb{Q}, <) \cup \{-\infty\}$ and all R_p are $\mathbb{Z}_2$ then R_P is the unique countable atomless Boolean ring. (A Boolean ring R is atomless if and only if whenever $r \neq 0$ in R then there is $s \in R$ with $r \neq s.r = s \neq 0$.) See Remark 4.3(2) below.

Subsystems

Let P be an SFP domain. If $Q \subseteq P$, we write $Q \preccurlyeq P$, and say that Q is a *subdomain* of P, if

Q is itself an SFP domain under the ordering induced from P,

$Q^0 \subseteq P^0$,

Q is a locally directed subset of P ($Q \trianglelefteq P$),

if D is a directed subset of Q then $\mathrm{lub}_Q(D) = \mathrm{lub}_P(D)$.

Note that these conditions imply that $P^0 \cap Q \subseteq Q^0$, so that we have $P^0 \cap Q = Q^0$ in fact. Clearly $\preccurlyeq$ is reflexive and transitive, and if $N \subseteq P$ is finite then $N \preccurlyeq P$ if and only if $N \trianglelefteq P^0$.

PROPOSITION 1.2. *Suppose that we have an SFP system $\langle P, \rho, \nu\rangle$. Let $Q \preccurlyeq P$. Then $\langle Q, \rho \restriction Q, \nu \restriction Q^2\rangle$ is an SFP system. Moreover, its limit ring R_Q is canonically isomorphic to the subring of R_P consisting of those elements supported by Q.*

Proof. To show that $\langle Q, \rho \restriction Q, \nu \restriction Q^2\rangle$ is an SFP system we only need to check that if $D \subseteq Q$ is directed then

$$R_{\mathrm{lub}_Q(D)} = \underrightarrow{\lim}\langle R_d, \nu_{de}\colon d \leqslant e \text{ in } D\rangle.$$

But this is clear, since $\langle P, \rho, \nu\rangle$ is an SFP system and $\mathrm{lub}_Q(D) = \mathrm{lub}_P(D)$.

Now if $r \in R_Q$, there is a finite $N \preccurlyeq Q$ supporting r. By transitivity of $\preccurlyeq$ we have $N \preccurlyeq P$, so r extends naturally to an element $r' \in R_P$ given by

$$r'(p) = \nu_{p/N,p}[r(p/N)] \quad \text{for } p \in P.$$

The map $r \mapsto r'$ is a ring embedding from R_Q into R_P, and clearly its image is precisely the set of elements of R_P supported by a finite locally directed subset of Q. We must show that this is the set of all elements of R_P supported by Q.

Certainly if $s \in R_P$ is supported by a finite set $N \trianglelefteq Q$ then s is supported by Q. Conversely, let $s \in R_P$ be supported by Q. Let $N \preccurlyeq P$ be a finite support of s. We show that $N \cap Q \trianglelefteq Q$, and $N \cap Q$ is a finite support of s in R_P.

If $p \in P$, define p_i $(i < \omega)$ by

$$p_0 = p, \quad p_{2i+1} = p_{2i}/Q, \quad p_{2i+2} = p_{2i+1}/N.$$

Then it is easy to show by induction on i that, for all $i < \omega$,

(a) $p_i \geqslant x$ for all $x \in N \cap Q \cap p{\downarrow}$, and

(b) $s(p) = \nu_{p_i,p}(s(p_i))$.

But $p_0 \geqslant p_1 \geqslant \ldots$, so as N is finite and $p_{2i} \in N$ for all $i > 0$, the sequence p_i is eventually equal to n for some $n \in N \cap Q$. Clearly $n = \text{lub}(N \cap Q \cap p\downarrow)$ and $s(p) = v_{np}(s(n))$. It follows that $N \cap Q \trianglelefteq Q$, and $N \cap Q$ supports s.

Hence the image $(R_Q)'$ is precisely the set of elements of R_P supported by Q.

In future we identify R_Q with the subring $(R_Q)'$ of R_P, whenever $Q \preccurlyeq P$.

A special case is where Q is finite, that is, $Q = N$, a finite locally directed subset of P^0. Then clearly $R_N \cong \prod \langle R_n \colon n \in N \rangle$, a finite direct product. If N, N' are finite locally directed subsets of P^0, and $N \subseteq N'$, then $N \preccurlyeq N'$, and so (if we make the identification) R_N is a subring of $R_{N'}$. Since P is SFP, the following is clear:

PROPOSITION 1.3. *The system $\langle R_N \colon N \trianglelefteq P^0$ is finite$\rangle$ is a direct system of rings under inclusion, and its direct limit is naturally isomorphic to R_P.*

COROLLARY 1.4. *Let $\langle P, \rho, v \rangle$ be an SFP system.*

(i) *If P has a least element, $\perp$, say, then $R_\perp$ is a subring of R_P.*

(ii) *The following classes K of rings are closed under SFP systems, in the sense that if $R_p \in K$ for all $p \in P^0$ then $R_P \in K$ also:*

(a) *the class of commutative rings;*

(b) *the class of von Neumann regular rings (that is, $R \models \forall x\, \exists y (xyx = x)$);*

(c) *the class of Boolean rings;*

(d) *the class of rings that are existentially closed in the class of commutative rings;*

(e) *the class of existentially closed rings in the class of Boolean rings.*

Proof. (i) Suppose that $\perp \in P$ is such that $\perp \preccurlyeq p$ for all $p \in P$. Clearly $\{\perp\} \preccurlyeq P$. The result now follows from Proposition 1.2.

(ii) By Proposition 1.3 it is enough to show that the classes cited are closed under finite direct products and direct limits—or at least, direct limits of direct systems with injective morphisms. This is clear for (a), (b) and (c), where there is no use of injectivity. We prove (d).

Let L be a first-order signature and Σ a class of L-structures that is closed under isomorphism. Recall (from, for example, [7]) that an L-structure $M \in \Sigma$ is said to be *existentially closed* in Σ if whenever $M \subseteq N \in \Sigma$ and $\varphi(\bar{x})$ is an existential formula of L, then

$$\text{for all } \bar{a} \in M, \text{ if } N \models \varphi(\bar{a}) \text{ then already } M \models \varphi(\bar{a}).$$

Clearly the class of existentially closed structures is closed under isomorphism. By considering disjunctive normal forms we need only consider formulas $\varphi(\bar{x})$ of the form $\exists \bar{y} \psi(\bar{x}, \bar{y})$ where ψ is a conjunction of atomic and negated atomic formulas.

It is easy to see that if Σ is closed under direct limits of the form $\varinjlim \langle M_i, v_{ij} \colon i \leqslant j \text{ in } I \rangle$ where the v_{ij} are injective, then a direct limit of existentially closed structures is existentially closed. The class of commutative rings is closed under such limits, so to prove (d) it suffices to prove that if A, B are existentially closed commutative rings (that is, they are existentially closed in the class of commutative rings) then so is $A \times B$.

Suppose that C is a commutative ring containing $A \times B$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ in $A \times B$. Then since C is commutative, e_1 is a central idempotent of C. It follows that the left ideal Ce_1 of C is a commutative ring in its own right, with identity e_1. It has a subring $(A \times B)e_1$, which is isomorphic to A via $(a, b)e_1 \mapsto a$. Similarly, Ce_2 is a commutative ring with a subring $(A \times B)e_2$ isomorphic to B.

Now since $e_1 e_2 = 0$ and $e_1 + e_2 = 1$ we have $C \cong Ce_1 \times Ce_2$ via $c \mapsto (ce_1, ce_2)$. It follows that

(**) if $\alpha(\bar{x})$ is an atomic formula of L and $\bar{c} \in C$, then

$$C \models \alpha(\bar{c}) \quad \Leftrightarrow \quad Ce_i \models \alpha(\bar{c}e_i) \text{ for } i = 1, 2.$$

Similarly, if $\bar{c} \in A \times B$ then

$$A \times B \models \alpha(\bar{c}) \quad \Leftrightarrow \quad (A \times B)e_i \models \alpha(\bar{c}e_i) \text{ for } i = 1, 2.$$

If α is an atomic formula, define α^1 to be α and α^0 to be $\neg\alpha$. Let $\psi(\bar{x}, \bar{y})$ above be $\bigwedge_{j<m} \alpha_j(\bar{x}, \bar{y})^{n_j}$, where the α_j are atomic formulas of the signature $\{+, -, \times, 0, 1\}$ of rings, and $n_j = 0$ or 1. Suppose that $C \models \psi(\bar{a}, \bar{c})$ for $\bar{a} \in A \times B$, $\bar{c} \in C$. Then by (**), there are $p_j, q_j \in \{0, 1\}$ with $p_j q_j = n_j$ for all $j < m$, such that $Ce_1 \models \bigwedge_j \alpha_j(\bar{a}e_1, \bar{c}e_1)^{p_j}$ and $Ce_2 \models \bigwedge_j \alpha_j(\bar{a}e_2, \bar{c}e_2)^{q_j}$.

As $(A \times B)e_1 \cong A$, we can identify them and regard A as a subring of Ce_1. Because A is existentially closed, there is $\bar{c}_1 \in A$ such that $A \models \bigwedge_j \alpha_j(\bar{a}e_1, \bar{c}_1)^{p_j}$. Similarly, we can find $\bar{c}_2 \in B$ with analogous properties for B. Take $\bar{d} \in A \times B$ with $\bar{d}e_1 = \bar{c}_1$, $\bar{d}e_2 = \bar{c}_2$. Then $A \models \bigwedge_j \alpha_j(\bar{a}e_1, \bar{d}e_1)^{p_j}$ and $B \models \bigwedge_j \alpha_j(\bar{a}e_2, \bar{d}e_2)^{q_j}$. Hence by (**) again, $A \times B \models \bigwedge_j \alpha_j(\bar{a}, \bar{d})^{n_j}$.

Hence $A \times B$ is an existentially closed commutative ring, as required.

(e) The proof is the same as that of (d).

Note that for Boolean rings, *existentially closed* is the same as *atomless*. See, for example, [**7**, 6.3.9, Ex. 6.3.2]. Since many of the SFP domains we use have a least element $\perp$, SFP systems can often be used to produce rings extending a given ring $R = R_\perp$ (Corollary 1.4(i)).

A similar proof gives a slightly more general preservation result, namely that if all R_p satisfy $\varphi = \forall\bar{x}\, \exists\bar{y}(\bigwedge_i \pi_i \to \pi)$ where π_i and π are equations, then R_P also satisfies φ. This includes Corollary 1.4(ii)(a)–(c).

There is an easy cardinality result that also follows from Proposition 1.3.

PROPOSITION 1.5. *Suppose that $\langle P, \rho, v\rangle$ is an SFP system in which each ring R_p is countable, and P is infinite. Then $|R_P| = |P^0|$.*

Proof. If N is finite and $N \preccurlyeq P$ then R_N is countable. Since P is infinite, so is P^0. Consequently, by Proposition 1.3 we have

$$|R_P| \leqslant \sum \{|R_N|:\ N \preccurlyeq P,\ N \text{ finite}\} \leqslant \omega \,.\, |\{N \preccurlyeq P:\ N \text{ finite}\}| \leqslant |P^0|.$$

Conversely, define finite sets $N_i \preccurlyeq P$ $(i < |P^0|)$ by induction on i as follows. Given that N_j have been defined for all $j < i$, choose $p_i \in P^0 \setminus \bigcup_{j<i} N_j$ and take a finite set $N_i \preccurlyeq P$ containing p_i. Define for each i, an element $r_i \in R_P$ by $r_i \in R_{N_i}$, $r_i(p_i) = 1$ and $r_i(q) = 0$ for all $q \in N_i \setminus \{p_i\}$.

Suppose that $r_i = r_j = r$ for some i, j with $i \leq j < |P^0|$. Let $n = p_j/N_i$. Then as N_i supports r, we have $1 = r(p_j) = \nu_{n,p_j}(r(n))$. Hence $r(n) = 1$ and $n = p_i$, so that $p_i \leq p_j$. Similarly, $p_j \leq p_i$. Hence $p_i = p_j$, and by choice of $p_j \notin \bigcup_{k<j} N_k$ we obtain $i = j$.

So if $i < j < |P^0|$ then $r_i \neq r_j$. It follows that $|R_P| \geq |P^0|$, which completes the proof.

2. *Ring ideals*

In this and the following two sections we examine the relationship between (ring) ideals of the limit ring R_P of an SFP system $\langle P, \rho, \nu \rangle$ and the underlying SFP domain P of the system. The relationship is close and will be crucial for the work in later sections and in Part II. Unless otherwise stated, all ring ideals will be left ideals, though most of our results apply to two-sided ideals as a special case.

The study has three aspects. In § 3 we examine the class of *full ideals* of R_P. An ideal is full if it is of the form $\{r \in R_P\colon\ r(p) \in I\}$ for some $p \in P$ and some ideal I of R_p. So the full ideals are linked naturally to the elements of P, and because of this they will be used heavily in Part II of this paper. We show that all maximal, prime and irreducible ideals of R_P are full. Then in § 4 we use the correspondence between full ideals and elements of P to motivate the link between SFP limits and Boolean powers. Stone duality is involved.

We begin in this section by showing that an ideal of the limit ring R_P of an SFP system $\sigma = \langle P, \rho, \nu \rangle$ is determined by its projections onto the component rings R_p $(p \in P)$. Full ideals are important in the proof. We also obtain a characterisation of the ideal of R_P generated by a given ideal of the limit ring of an SFP subsystem of σ.

Ideals of limit rings of SFP systems

NOTATION. Let P be an SFP domain, and let $\langle P, \rho, \nu \rangle$ be an SFP system with limit ring R_P. We will generally use 'J' to denote an ideal of R_P and 'I' for an ideal of a component ring R_p $(p \in P)$. If J is an ideal of R_P and $q \in Q \leq P$, we will write J_Q for $J \cap R_Q$, and $J_Q(q)$ for the projection $\{r(q)\colon\ r \in J_Q\}$ of J_Q onto the qth component ring R_q. We write simply $J(q)$ for $J_P(q)$.

Let J be an ideal of R_P. Then certainly, $J(p)$ is an ideal of R_p for all $p \in P$, and so we obtain a family of projected ideals, $(J(p)\colon\ p \in P)$. The main aim of this section is to prove that this family determines J.

DEFINITION. Let $\langle P, \rho, \nu \rangle$ be an SFP system with limit ring R_P and let $J \subseteq R_P$ be an ideal. Define

$$J\$ = \{r \in R_P\colon\ r(p) \in J(p) \text{ for all } p \in P\}.$$

Evidently, $J\$$ is an ideal of R_P, and $J \subseteq J\$ = J\$\$$. We will prove the following theorem.

THEOREM. *Let $\langle P, \rho, \nu \rangle$ be an SFP system with limit ring R_P and let $J \subseteq R_P$ be an ideal. Then $J\$ = J$.*

That is, if J' is any ideal of R_P with $J'(p) = J(p)$ for all $p \in P$, then $J' = J$. This is a key result and will greatly simplify our work later.

We establish it in three stages. The case where P is finite is easy and is proved in Lemma 2.1; it essentially says that in this case, $J \cong \prod(J(p)\colon p \in P)$. Note that any finite poset is an SFP domain. In Theorem 2.2 we establish some properties of ideals $J\$$ such that $J\$(p) \neq R_p$ for a unique $p \in P$. These are the *full ideals*. In Theorem 2.3 we use them to prove the full version of the theorem.

LEMMA 2.1. *Let P be any finite poset and let $\langle P, \rho, v\rangle$ be an SFP system with limit ring R_P. Let J be an ideal of R_P. Then $J = J\$$.*

Proof. It is clear that $J \subseteq J\$$. We prove that $J \supseteq J\$$. For each $p \in P$ define a central idempotent $e_p \in R_P$ by

$$e_p(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & \text{if } x \in P\setminus\{p\}. \end{cases}$$

If $r \in J\$$ then $r(p) \in J(p)$ for all $p \in P$. So for each p there is $s_p \in J$ with $s_p(p) = r(p)$. Then $r = \sum_{p\in P}(e_p \,.\, s_p) \in J$, as required.

Now fix any SFP system $\langle P, \rho, v\rangle$.

DEFINITION. If $p \in P$ and I is a proper ideal of R_p, we write $I@p$ for the set $\{r \in R_P\colon\ r(p) \in I\}$.

Note that $I@p$ is a proper ideal of the limit ring R_P. Strictly it depends on P also, and we will sometimes write '$I@p$ in R_P'. By definition, $I@p = (I@p)\$$.

If $p' \in P$ and I' is an ideal of $R_{p'}$, then $I@p = I'@p'$ implies that $p = p'$ and $I = I'$. For if $p \neq p'$, then as P is algebraic, $p{\downarrow} \cap P^0 \neq p'{\downarrow} \cap P^0$. Assume without loss of generality that there is $q \in P^0 \cap (p{\downarrow}\setminus p'{\downarrow})$. As P is an SFP domain, there is a finite set $N \preccurlyeq P$ (that is, $N \trianglelefteq P^0$) containing q. Hence $p/N \neq p'/N$. We can find $r \in R_N$ such that $r(p/N) = 0$ and $r(p'/N) = 1$. Then $r \in I@p\setminus I'@p'$, a contradiction. Hence $p = p'$, and it easily follows that $I = I'$.

DEFINITION. If J is a proper ideal of R_P, we say that J is *full* (in R_P) if $J = I@p$ for some p, I. Clearly I will be a proper ideal of R_p. Since p and I are unique, we can define $\sigma J = p$ (the *site* of J), and $\Delta J = I$ (the *defect* of J).

Next we show that full ideals are well-behaved with respect to their intersections with limits of subsystems.

THEOREM 2.2. *Let $J \subseteq R_P$ be an ideal. Then the following are equivalent:*

(i) *J is full in R_P;*

(ii) *J_N is full in R_N for each finite $N \preccurlyeq P$;*

(iii) *J_Q is full in R_Q for each $Q \preccurlyeq P$.*

Moreover, if any of (i)–(iii) *hold, and $Q \preccurlyeq P$, $\sigma J = p$ and $\sigma(J_Q) = q$, then we have*

(iv) $q = p/Q$,

(v) $\Delta(J_Q) = (v_{qp})^{-1}(\Delta J)$.

Proof. (i) $\Rightarrow$ (ii). Assume that J is full in R_P. Let $J = I @ p$ (for some $p \in P$ and $I \subseteq R_p$). Let $N \leqslant P$ be finite and let $n = p/N$. If $r \in R_N$, then

$$\begin{aligned} r \in J \quad &\Leftrightarrow \quad r(p) = v_{np}[r(n)] \in I \\ &\Leftrightarrow \quad r(n) \in v_{np}^{-1}(I) \\ &\Leftrightarrow \quad r \in [v_{np}^{-1}(I)] @ n \quad \text{in } R_N. \end{aligned}$$

Hence $J_N = [v_{np}^{-1}(I)] @ n$ in R_N. This proves (ii), and also (iv) and (v) in the case where Q is finite.

(ii) $\Rightarrow$ (iii). Assume (ii) and take $Q \leqslant P$. Let $N \leqslant Q$ be finite. By transitivity of '$\leqslant$', $N \leqslant P$, and so J_N is full in R_N for all finite $N \leqslant Q$.

Now if $N, N' \leqslant Q$ and $N \subseteq N'$, then $N \leqslant N'$. It follows from the proof of (i) $\Rightarrow$ (ii) that

$$(\dagger) \qquad \sigma(J_N) = \sigma(J_{N'})/N \leqslant \sigma(J_{N'}).$$

So as Q is SFP, the set $D = \{\sigma J_N\colon N \text{ finite}, N \leqslant Q\}$ is directed. Let $q = \mathrm{lub}_Q(D)$.

Claim 1. If $N \leqslant Q$ is finite, then $\sigma J_N = q/N$.

Proof of Claim. Clearly $q \geqslant \sigma J_N \in N$. Hence $\sigma J_N \leqslant q/N$. For the converse inequality, note that as $q/N \leqslant q$ and q/N is a finite element of Q, there is a finite $N' \leqslant Q$ such that $\sigma J_{N'} \geqslant q/N$. By (†) we may assume that $N' \supseteq N$, and so $\sigma J_N = \sigma J_{N'}/N \geqslant q/N$. This proves the claim.

Now let $I = \{r(q)\colon r \in J_Q\}$. Clearly I is an ideal of R_q.

Claim 2. $J_Q = I @ q$ in R_Q.

Proof of Claim. It is clear that $J_Q \subseteq I @ q$; we pass to $J_Q \supseteq I @ q$. Let $r \in R_Q$ be such that $r(q) \in I$. So there is $s \in J_Q$ with $s(q) = r(q)$. Since $R_q = \lim_{\rightarrow}\langle R_{q'}\colon q' \in q\!\downarrow \cap\, Q^0\rangle$ and Q is SFP, we can find finite $N \leqslant Q$ supporting r and s, and such that $s(q/N) = r(q/N)$. But $s \in J_N$, and, by Claim 1, J_N is full with site q/N. Hence $r \in J_N$ also. This proves the claim.

So by the claim, J_Q is full in R_Q, which proves (iii).

(iii) $\Rightarrow$ (i). This is trivial.

It remains to prove (iv) and (v) for infinite $Q \leqslant P$. Let $J \subseteq R_P$ be full, and let $\sigma J = p$. Then J_Q is full, of the form $I @ q$.

If $N \leqslant Q$ is finite then we may already apply (iv), to get $q/N = \sigma J_N$. But also $N \leqslant P$, so similarly $\sigma J_N = p/N$. Hence $p/N = q/N$ for all finite $N \leqslant Q$. Since Q is SFP, it follows that $p\!\downarrow \cap\, Q^0 = q\!\downarrow \cap\, Q^0$. Taking least upper bounds on both sides, we obtain $p/Q = q$, proving (iv).

For (v), we must show that $I = v_{qp}^{-1}(\Delta J)$. Take $a \in R_q$, and choose a finite set $N \leqslant Q$ and an element $r \in R_Q$ supported by N, such that $r(q) = a$. By the above, $p/N = q/N$. So $r(p) = v_{qp}(r(q))$, and hence

$$a \in I \quad \Leftrightarrow \quad r \in J \quad \Leftrightarrow \quad r(p) \in \Delta J \quad \Leftrightarrow \quad r(q) = a \in v_{qp}^{-1}(\Delta J).$$

We now move from full ideals to arbitrary ideals. As before we let P be any SFP domain and $\langle P, \rho, v\rangle$ an SFP system with limit ring R_P.

THEOREM 2.3. *Let J be any left ideal of R_P. Then $J = J\$$. In other words,*

$$J = \bigcap \{J(p) @ p\colon p \in P\}.$$

Proof. We already agree that $J \subseteq J\$$. For the converse it suffices to prove:

$$(**) \qquad J = \bigcap \{J' \colon J' \text{ a full ideal of } R_P, J' \supseteq J\}.$$

For assume that $r \in J\$$. Let $I @ q$ be any full ideal containing J, where $q \in P$ is arbitrary. Clearly $J(q) \subseteq I$. So $r \in I @ q$. Hence $J\$ \subseteq J'$ for all full ideals $J' \supseteq J$. Given (**) we obtain $J\$ \subseteq J$ as required.

We only need to prove '$\supseteq$' of (**). Let $r \in R_P \backslash J$. It suffices to find a full ideal $J' \supseteq J$ with $r \notin J'$.

Using Zorn's lemma choose a left ideal J' of R_P which is maximal with respect to

$$J' \supseteq J, \quad r \notin J'.$$

We show that J' is a full ideal of R_P. If it is not, then, by Theorem 2.2, there is a finite set $N \preccurlyeq P$ such that J'_N is not full in R_N. Since it is certainly proper, by Lemma 2.1 there are distinct $n_1, n_2 \in N$ such that $J'_N(n_i)$ is a proper ideal of R_{n_i} $(i = 1, 2)$. Define $e_1 \in R_N$ by $e_1(n_1) = 1$ and $e_1(n) = 0$ for all $n \in N \backslash \{n_1\}$. Set $e_2 = 1 - e_1$. Then e_1 and e_2 are orthogonal central idempotents of R_P, and $e_1 + e_2 = 1$. By our choice of the n_i we have $e_i(n_i) = 1 \notin J'_N(n_i)$ $(i = 1, 2)$. So certainly $e_1, e_2 \notin J'$. By maximality of J' we have

$$r = j_i + r_i e_i \quad \text{for some } j_i \in J' \text{ and } r_i \in R_P \ (i = 1, 2).$$

So

$$r = e_1 r + e_2 r = e_1(j_2 + r_2 e_2) + e_2(j_1 + r_1 e_1) = e_1 j_2 + e_2 j_1 \in J'.$$

This is a contradiction. Hence J' is a full ideal of R_P, which completes the proof.

Theorem 2.3 proves that any ideal J of R_P is determined by its components $J(p)$ (for $p \in P$). Now let $Q \preccurlyeq P$. Since J_Q determines its components $J_Q(q)$ $(q \in Q)$, each $J_Q(q)$ is determined by the $J(p)$ $(p \in P)$. But intuitively, each $J_Q(q)$ should only depend on the $J(p)$ for those $p \in P$ with $p/Q = q$. This is certainly so in the special case where J is full in R_P; for by Theorem 2.2, $J_Q(q)$ is the intersection of all ideals of R_q of the form $\nu_{qp}^{-1}(J(p))$ for $p \in P$ with $p/Q = q$. We now show that the same holds for arbitrary left ideals. (This result is an aside; we will not need it later.)

THEOREM 2.4. *Let $Q \preccurlyeq P$ and let J be an ideal of R_P. Then for each $q \in Q$,*

$$J_Q(q) = \bigcap \{\nu_{qp}^{-1}(J(p)) \colon p \in P, p/Q = q\}.$$

Proof. For $q \in Q$ define $I_q = \bigcap \{\nu_{qp}^{-1}(J(p)) \colon p \in P, p/Q = q\}$. So I_q is a left ideal of R_q. We must show that

$$(*) \qquad J_Q(q) = I_q \quad \text{for all } q \in Q.$$

It is easy to prove that $J_Q(q) \subseteq I_q$. Let $a \in J_Q(q)$ for some $q \in Q$. Then there is $r \in J_Q$ with $r(q) = a$. Clearly, $\nu_{qp}(a) = r(p) \in J(p)$ for all $p \in P$ with $p/Q = q$. So $a \in I_q$.

To prove that $I_q \subseteq J_Q(q)$, suppose for a contradiction that there exist $q \in Q$ and $a \in I_q \backslash J_Q(q)$. Using Zorn's Lemma as in Theorem 2.3 take a left ideal J' of R_P that is maximal subject to $J' \supseteq J$ and $a \notin J'_Q(q)$.

Claim. The ideal J'_Q is full in R_Q and $\sigma(J'_Q) = q$.

Proof of Claim. If this is not true, there is a finite $N \preccurlyeq Q$ such that J'_N is not full in R_N. As before, take orthogonal idempotents $e_1, e_2 \in R_N \backslash J'$, central in R_P and such that $e_1 + e_2 = 1$. By maximality of J' there are $j_i \in J'$ and $r_i \in R_P$ such that

$$j_i + r_i.e_i \in R_Q \quad \text{and} \quad (j_i + r_i.e_i)(q) = a \quad (i = 1, 2).$$

Consider the element $s = e_1(j_2 + r_2e_2) + e_2(j_1 + r_1e_1)$. Since $e_1, e_2 \in R_Q$, we have $s \in R_Q$. Also, $s(q) = e_1(q).a + e_2(q).a = [(e_1 + e_2)(q)].a = a$. But also $s = e_1j_2 + e_2j_1 \in J'$. So $s \in J'_Q$ and $s(q) = a$, a contradiction to the choice of J'. Hence J'_Q is full in R_Q, and clearly $\sigma J'_Q = q$. This proves the claim.

Take $r \in R_Q$ with $r(q) = a$. We will show that $r \in J'$. Hence we will have $a \in J'_Q(q)$, contradicting the choice of J' and completing the proof of (*). By Theorem 2.3, it suffices to show that $r(p) \in J'(p)$ for all $p \in P$. So pick $p \in P$. Suppose first that $p/Q = q$. Then $r(p) = \nu_{qp}(a)$. Since $a \in I_q$, it follows that $r(p) \in J(p) \subseteq J'(p)$. Suppose next that $p/Q = q' \neq q$. By the claim, $J'_Q(q') = R_{q'}$. But by the analogy of the proof of '$\subseteq$' of (*) for J', we have $J'_Q(q') \subseteq \nu^{-1}_{q'p}(J'(p))$. Hence $J'(p) = R_p$, so certainly $r(p) \in J'(p)$. So $r \in J'$, as required.

We can add some straightforward corollaries of Theorem 2.3 that will be needed later. First we extend our previous notation. If $X \subseteq R_P$ is any *set*, and $p \in P$, we write $X(p)$ for the projection $\{r(p)\colon r \in X\}$. Note that if $Q \preccurlyeq P$ and $X \subseteq R_Q$, then by Proposition 1.2 each element of X is supported by Q, so we have $X(p) = \nu_{p/Q,p}(X(p/Q))$ for all $p \in P$.

We can determine the left ideal of R_P generated by a left ideal of R_Q, a result needed in § 3 of Part II. Corollary 2.6 is a special case; it will be used in Proposition 5.3.

COROLLARY 2.5. *Let $Q \preccurlyeq P$ and I be a left ideal of R_Q.*

(i) *The left ideal J of R_P generated by I is given by*

(*) *$J(p)$ is the left ideal of R_p generated by $I(p)$ (for all $p \in P$).*

(ii) *Suppose that $I = I' @ q$ in R_Q (for some $q \in Q$ and left ideal I' of R_q), and for all $p \in P$ with $p \neq q$ and $p/Q = q$, the left ideal of R_p generated by $\nu_{qp}(I')$ is improper. Then I generates the left ideal $I' @ q$ in R_P.*

Proof. (i) For each $p \in P$ write $\langle I(p) \rangle$ for the left ideal of R_p generated by $I(p)$. Then let $J = \{r \in R_P\colon r(p) \in \langle I(p) \rangle$ for all $p \in P\}$. Certainly J is a left ideal of R_P. It suffices to show that

(a) $J(p) = \langle I(p) \rangle$ for all $p \in P$, and

(b) I generates J in R_P.

(a) Let $p \in P$. Clearly $J(p) \subseteq \langle I(p) \rangle$. Conversely, we clearly have $I \subseteq J$, so that $I(p) \subseteq J(p)$. But $J(p)$ is an ideal of R_p, so $\langle I(p) \rangle \subseteq J(p)$.

(b) We know already that $I \subseteq J$. Let J' be a left ideal of R_P containing I; we show that $J \subseteq J'$. Let $p \in P$. Clearly $I(p) \subseteq J'(p)$, so as before, $\langle I(p) \rangle \subseteq J'(p)$. Using (a), we obtain $J(p) \subseteq J'(p)$. This holds for all $p \in P$, so by Theorem 2.3 we obtain $J \subseteq J'$, and the proof is complete.

(ii) This is a special case of (i). We will use it in Theorem II.3.2.

COROLLARY 2.6. *Assume that $Q \preccurlyeq P$ and let the left ideal I of R_Q generate the left ideal J of R_P. Then*

(i) $J(q) = I(q)$ *for all* $q \in Q$,

(ii) $J_Q = I$.

Proof. Part (i) is a special case of Corollary 2.5(i). Hence for each $q \in Q$ we have $I(q) \subseteq J_Q(q) \subseteq J(q) = I(q)$, so $J_Q(q) = I(q)$. Part (ii) now follows by Theorem 2.3.

3. *Full ideals and central idempotents*

Fix an SFP system $\langle P, \rho, v \rangle$ and consider its limit ring, R_P. Whilst R_P can have many full ideals with the same site, we now show that this is not so if we restrict to the elements of R_P that take values 0, 1 only. (These elements form a Boolean subring of central idempotents of R_P.) Thus we can extract the site of an ideal from these elements; this will be needed in Theorem II.1.7.

We will also show that maximal, prime and irreducible ideals of R_P are full.

DEFINITION. We write $(R_P)^*$ for the set $\{r \in R_P\colon r(p) \in \{0, 1\}$ for all $p \in P\}$. If $X \subseteq R_P$, we write X^* for $X \cap R_P^*$.

PROPOSITION 3.1. *Let I, J be full ideals of R_P. Then $\sigma I = \sigma J$ if and only if $I^* = J^*$.*

Proof. Assume that $\sigma I = \sigma J$. Then if $r \in (R_P)^*$, we have $r \in I$ if and only if $r(\sigma I) \in \Delta I$. But ΔI is a proper ideal of $R_{\sigma I}$, so this holds if and only if $r(\sigma I) = 0$. Since the same holds for J, we have $r \in I$ if and only if $r \in J$, so $I^* = J^*$.

Conversely, suppose that $\sigma I \neq \sigma J$. Since P is SFP, we can find a finite set $N \preccurlyeq P$ such that $\sigma I/N \neq \sigma J/N$. Let $r \in (R_P)^*$ be supported by N, and given by, for all $n \in N$, $r(n) = 0$ if $n = \sigma I/N$, and $r(n) = 1$ otherwise. Then $r \in I^* \setminus J^*$ so that $I^* \neq J^*$.

Full ideals include the maximal, prime and irreducible ideals of R_P. Let us say that an ideal I of a ring S is *whole* if $S \setminus I$ contains no pair of orthogonal central idempotent elements (that is, there do not exist $x, y \in S \setminus I$, commuting multiplicatively with every element of S, and such that $x^2 = x$, $y^2 = y$, $xy = 0$).

PROPOSITION 3.2. (i) *If I is a maximal, prime, or irreducible left (or right) ideal of a ring S, then I is whole. If I is a maximal two-sided ideal of S, then I is whole.*

(ii) *If I is a proper whole ideal of R_P then I is full.*

Proof. (i) This is straightforward. As an example we prove that an irreducible left ideal I of S is whole. If I is not whole, take orthogonal central idempotents $e_1, e_2 \notin I$. Let $r \in (I + Se_1) \cap (I + Se_2)$. So for some $i_j \in I$ and $s_j \in S$, we have $r = i_j + s_j e_j$ for $j = 1, 2$. Then $re_1 = (i_2 + s_2 e_2).e_1 = i_2 e_1 = e_1 i_2 \in I$. Hence $s_1 e_1 = i_1 e_1 + s_1 e_1^2 - i_1 e_1 = re_1 - e_1 i_1 \in I$. It follows that $r = i_1 + s_1 e_1 \in I$. Consequently, I is not irreducible.

(ii) This follows from the proof of Theorem 2.3, or equally from the Claim of Theorem 2.4.

REMARK 3.3. Let $p \in P$ and let I be a left ideal of R_p.

(1) Clearly, I is prime in R_p if and only if $I @ p$ is prime in R_P.

(2) If J is an ideal of R_P, then evidently $J \supseteq I @ p$ if and only if J is full, $\sigma J = p$ and $\Delta J \supseteq I$. There is thus a one-to-one inclusion-preserving correspondence between the ideals of R_p containing I, and the ideals of R_P containing $I @ p$. Hence $I @ p$ is maximal, maximal two-sided or irreducible in R_P if and only if I has the respective property in R_p.

4. *Stone duality and Boolean powers*

Here we show that in the special case of an SFP system in which all component rings are equal to a fixed ring R, and all connecting homomorphisms are the identity, the limit ring is the Boolean power of R by a Boolean algebra naturally associated with P. This remains true in the more general construction when the component structures $\rho(p)$ need not be rings. We prove the result using Stone duality in a canonical Boolean ring built as an SFP limit.

Boolean powers have been extensively studied. For information see **[4]**.

FACT 4.1. There is a well-known natural correspondence between Boolean rings and Boolean algebras. Let R be a Boolean ring. We can turn R into a Boolean algebra by defining the Boolean complement r^* to be $1-r$, $r \wedge s$ to be rs, and $r \vee s$ to be $r+s-rs$. Conversely, we can turn a Boolean algebra B into a Boolean ring by defining $a+b=(a \vee b) \wedge (a \wedge b)^*$ (symmetric difference) and $ab = a \wedge b$. The ideals of a Boolean ring are exactly the ideals of the corresponding Boolean algebra.

Recall that if B is a Boolean algebra, the Stone space $S(B)$ is the set of maximal ideals of B. It has a natural topology; as a basis of closed and open sets, we can take the sets of maximal ideals of the form $\{I\colon b \in I\}$ for some $b \in B$. These closed and open sets form a set Boolean algebra which by Stone duality is naturally isomorphic to B. The isomorphism is $b \mapsto \{I\colon b^* \in I\}$.

DEFINITION. (1) Let S be a ring and P an SFP domain. Write $S^{\langle P \rangle}$ for the limit of the SFP system $\langle P, \rho, \nu \rangle$ where $\rho(p) = S$ for all $p \in P$, and $\nu_{pq} = \mathrm{id}_S$ whenever $p \leqslant q$ in P.

(2) If $Q \trianglelefteq P$, define an equivalence relation $\sim_Q$ on P by

$$p \sim_Q p' \quad \Leftrightarrow \quad p/Q = p'/Q.$$

(3) Define $\mathrm{BA}_0(P)$ to be the set of equivalence classes of the $\sim_N$, for finite $N \leqslant P$. That is,

$$\mathrm{BA}_0(P) = \bigcup \{P/\sim_N\colon\ N \text{ finite}, N \leqslant P\} \subseteq \wp(P).$$

(4) Define $\mathrm{BA}(P)$ to be the Boolean algebra of subsets of P generated by $\mathrm{BA}_0(P)$.

For example, if P is linearly ordered then $\mathrm{BA}(P)$ is the Boolean algebra generated by the half-open intervals of P of the form $[x, y)$ (for $x \leqslant y$ in P^0). The case where P is a tree is similar.

We remark that $\mathrm{BA}(P)$ is isomorphic to the Boolean algebra of subsets of P^0 generated by $\bigcup \{P^0/\sim_N\colon\ N \text{ finite}, N \trianglelefteq P^0\}$.

THEOREM 4.2. *Let P be any SFP domain.*

(i) *The domain P is in canonical bijection with the set of maximal ideals of* $\mathbb{Z}_2^{\langle P\rangle}$, *via* $p \mapsto 0@p$. (*As usual,* $\mathbb{Z}_2$ *is the two element ring* $\mathbb{Z}/2\mathbb{Z}$.)

(ii) *We have* $\mathbb{Z}_2^{\langle P\rangle} \cong \mathrm{BA}(P)$, *regarding the latter as a Boolean algebra* (*cf. Fact* 4.1). *The isomorphism is* $r \mapsto \{p \in P\colon r(p) = 1\}$.

(iii) *Let S be any ring. Then* $S^{\langle P\rangle}$ *is the Boolean power* $S[B(P)]^*$.

Proof. (i) By Remark 3.3(2), $0@p$ is a maximal ideal of $\mathbb{Z}_2^{\langle P\rangle}$ for each $p \in P$. The map $(p \mapsto 0@p)$ is clearly injective. If J is a maximal ideal of $\mathbb{Z}_2^{\langle P\rangle}$, then by Proposition 3.2, J is full, so $J = I@p$ for some $p \in P$ and proper ideal I of $\mathbb{Z}_2$. Clearly $I = 0$, and thus the map $(p \mapsto 0@p)$ is surjective.

(ii) By Stone duality, $\mathbb{Z}_2^{\langle P\rangle}$ (viewed as a Boolean algebra using Fact 4.1) is isomorphic to the Boolean algebra of those sets of maximal ideals of $\mathbb{Z}_2^{\langle P\rangle}$ of the form

$$\langle r\rangle := \{I\colon I \text{ a maximal ideal of } \mathbb{Z}_2^{\langle P\rangle},\ r \in I\}$$

for $r \in \mathbb{Z}_2^{\langle P\rangle}$. The isomorphism is $r \mapsto \langle 1 - r\rangle$.

By (i), the maximal ideals are in bijection with P, via their sites. Under this bijection, $\langle 1 - r\rangle$ goes to the set $[r] := \{p \in P\colon r(p) = 1\} \subseteq P$. It suffices to show that the map $\sigma\colon r \mapsto [r]$ is a bijection from $\mathbb{Z}_2^{\langle P\rangle}$ to $\mathrm{BA}(P)$.

Let $r \in \mathbb{Z}_2^{\langle P\rangle}$, and let $N \preccurlyeq P$ be a finite support of r. Let $N^1 = \{n \in N\colon r(n) = 1\}$. Clearly, $[r] = \{p \in P\colon p/N \in N^1\}$. Hence $[r]$ is a (finite) union of $\sim_N$-classes, so $[r] \in \mathrm{BA}(P)$.

Certainly σ is injective. To prove surjectivity, let $X \in \mathrm{BA}(P)$. Let $N_i \preccurlyeq P$ $(i < k)$ be finite sets such that X lies in the subalgebra of $\mathrm{BA}(P)$ generated by the $\sim_{N_i}$-classes $(i < k)$. As P is SFP, we can take a finite $N \preccurlyeq P$ containing each N_i. It is easily seen that X is a union of $\sim_N$-classes. Let $r \in R_N$ be defined by $r(n) = 1$ if $n \in N \cap X$, and $r(n) = 0$ for $n \in N \setminus X$. Then for all $p \in P$ we have

$$r(p) = 1 \quad\Leftrightarrow\quad p/N \in X \quad\Leftrightarrow\quad p \in X.$$

Hence $X = [r]$, as required. It is easily checked that σ is an isomorphism of Boolean algebras.

(iii) We can take $S[B(P)]^*$ to be the subring of S^P consisting of those $r \in S^P$ such that there exist $k < \omega$ and pairwise disjoint X_i $(i < k)$ in $\mathrm{BA}(P)$ with $\bigcup_{i<k} X_i = P$ and $r(p) = r(q)$ for all $p, q \in X_i$ (all $i < k$).

Let $r \in S[B(P)]^*$. Take k and X_i $(i < k)$ as above. By the argument in (ii) above, there is a finite $N \preccurlyeq P$ such that each X_i is a union of $\sim_N$-classes. Clearly N is a support for r. Hence $r \in S^{\langle P\rangle}$. Conversely, if $r \in S^{\langle P\rangle}$, let $N \preccurlyeq P$ be a finite support for r. It is clear that the $\sim_N$-classes all lie in $\mathrm{BA}(P)$ and partition P, and that r is constant on each class. So $r \in S[B(P)]^*$, which completes the proof.

REMARK 4.3. (1) If P and Q are SFP domains, $\mathrm{BA}(P) \cong \mathrm{BA}(Q)$ does not imply that $P \cong Q$. For example, let P be the ideal completion of the tree ${}^{<\omega}\omega$ (the set of all finite sequences of natural numbers, ordered by 'initial segment'), and let Q be the ideal completion of the set $\mathbb{Q}_{\geqslant 0}$ of non-negative rational numbers with the usual ordering. Certainly the Boolean algebras $\mathrm{BA}(P)$ and $\mathrm{BA}(Q)$ are both countable. We now indicate that each is *atomless* (that is, for each non-empty S there is a non-empty S_1 properly contained in S).

We saw that BA(Q) is the Boolean algebra generated by the half-open intervals of $\mathbb{Q}_{\geq 0}$; this is clearly atomless. As for P, let $N \preccurlyeq P$ be finite, let $n \in N$ and let $S = \{p \in P\colon p/N = n\}$ be the corresponding $\sim_N$-class. As n has ω immediate successors in ${}^{<\omega}\omega$, we can take distinct immediate successors n_1, n_2 of n with $n_1, n_2 \notin N$, and find finite $M \preccurlyeq P$ containing N, n_1 and n_2. Then $S_1 = \{p \in P\colon p/M = n_1\} \in \text{BA}(P)$ and $\varnothing \subset S_1 \subset S$. It follows that BA($P$) is also atomless.

Hence $\text{BA}(P) \cong \text{BA}(Q) \cong B$, the countable atomless Boolean algebra. There is up to isomorphism a unique such algebra—see **[5]**. So by Theorem 4.2(iii), for all rings S we have $S^{\langle P \rangle} \cong S^{\langle Q \rangle}$, although P and Q are not isomorphic posets. This shows that in general we cannot recover the poset structure of an SFP domain P from a limit ring R_P. We will pursue this in Part II.

(2) Let P be the ideal completion of ${}^{<\omega}\omega$ or of $\mathbb{Q}_{\geq 0}$, as above. Then by Theorem 4.2(ii), $\mathbb{Z}_2^{\langle P \rangle} \cong \text{BA}(P)$, the countable atomless Boolean ring (by Fact 4.1). So we have determined the limit ring in this case.

(3) *Topology on P.* By Fact 4.1 and Theorem 4.2(i), (ii), P is in natural bijection with the Stone space $S(\text{BA}(P))$. Hence the topology on the latter induces a homeomorphic topology on P. It is in fact the 'patch' topology referred to in, for example, **[6]**, whose construction bears some similarity to ours. The proof of Theorem 4.2(ii) shows that BA(P) is a basis of closed and open sets. In fact, $\text{BA}_0(P)$ is also a basis of closed and open sets, for as in Theorem 4.2(ii), any finite intersection of elements of $\text{BA}_0(P)$ is a finite union of elements of $\text{BA}_0(P)$. For any $Q \preccurlyeq P$, any $\sim_Q$-class is closed in the topology. Hence (taking $Q = P$) we see that every singleton subset of P is closed: the topology is *regular.*

(4) By a theorem of Baldwin and Lachlan **[2]**, if S is a finite or countable ω-categorical ring and P is an SFP domain such that BA(P) is the countable atomless Boolean algebra, then $S^{\langle P \rangle} = S[B(P)]^*$ is also ω-categorical. (Note that ${}^{<\omega}\omega$ is not an ω-categorical poset.)

5. *Densely decomposable ideals*

Here we develop a way to obtain an atomless Boolean ring as the limit of an SFP system in the case where all component rings are Boolean. As in **[17]** we use densely decomposable ideals to generalise the notion of *atomless* to arbitrary rings. Again, unless otherwise stated, all ring ideals will be left ideals.

DEFINITION. Let R be any ring, and I a proper left ideal of R. Then I is said to be *densely decomposable* if whenever J is a left ideal of R properly extending I, there are left ideals $X, Y \subseteq J$ properly extending I, with $X \cap Y = I$.

EXAMPLE 5.1. Let R be a Boolean ring. Then the ideal 0 is densely decomposable if and only if R is atomless. So for an ideal of a Boolean ring, being densely decomposable is the same as having atomless quotient, and is in a sense opposite to being irreducible.

We wish to find conditions for ideals of the limit ring of an SFP system to be densely decomposable.

DEFINITION. Let R be a ring and $I \subseteq J$ left ideals of R. We say that J *splits over* I if there are left ideals $X, Y \subseteq J$ with $X \supset I$, $Y \supset I$, $X \cap Y = I$. If $S \supseteq I$ is any subset of R, we say that S *strongly splits over* I if J splits over I for all left ideals J with $I \subset J \subseteq S$.

Clearly I is a densely decomposable ideal of R if and only if any set S containing I strongly splits over I.

Let $\langle P, \rho, \nu \rangle$ be an SFP system. An ideal of R_P can be densely decomposable for two reasons. First, its projections onto the component rings R_p ($p \in P$) might already make it densely decomposable. For example, if $p \in P$ and I is an ideal of R_p then $I @ p$ is densely decomposable if and only if I is densely decomposable in R_p. Second, it can be densely decomposable because of the SFP system structure of R_P. We now separate the two causes. As in § 2, if I is an ideal of R_P and $n \in N \preccurlyeq P$, we write $I_N(n)$ for the projection $\{r(n)\colon r \in I_N\}$ of I_N $(= I \cap R_N)$ onto R_n.

DEFINITION. Let I be a left ideal of R_P. We define $I^\wedge$ to be the set

$$\{r \in R_P\colon \text{for any finite support } N \preccurlyeq P \text{ of } r, \text{ there is at most one } n \in N \text{ with } r(n) \notin I_N(n)\}.$$

So $I \subseteq I^\wedge$. If I is a proper ideal of R_P, then by Theorem 2.2, I is full if and only if $I^\wedge = R_P$. If $r \in R_P$ and $i \in I^\wedge$ then clearly $ri \in I^\wedge$. Hence $I^\wedge$ is the union of the left ideals contained in it.

LEMMA 5.2. *Let I be a left ideal of R_P. The following are equivalent:*
(i) *I is a densely decomposable ideal of R_P;*
(ii) *$I^\wedge$ strongly splits over I in R_P.*

Proof. We only need prove that (ii) implies (i). Let $J \supset I$ be a left ideal of R_P. We must prove that J splits over I. If $J \subseteq I^\wedge$, this is clear by assumption. Assume that $J \nsubseteq I^\wedge$. There exist $r \in J$ and a finite support $N \preccurlyeq P$ of r such that for some distinct $y, z \in N$ we have $r(y) \notin I_N(y)$ and $r(z) \notin I_N(z)$. Define $e_y \in R_N$ by $e_y(x) = 1$ if $x = y$, and $e_y(x) = 0$ otherwise. Let Y be the left ideal of R_P generated by I and $e_y.r$. Define e_z and Z similarly. Then e_y and e_z are orthogonal central idempotents of R_P. We clearly have $I \subset Y$, $I \subset Z$ and $Y \cup Z \subseteq J$. Hence the following claim proves the lemma.

Claim. $Y \cap Z = I$.

Proof of Claim. Let $s \in Y \cap Z$. So $s = i_y + r_y(e_y.r) = i_z + r_z(e_z.r)$ for some i_y, $i_z \in I$ and r_y, $r_z \in R_P$. Multiplying by e_z, we obtain $e_z i_y = e_z i_z + r_z(e_z.r)$. Hence $r_z(e_z.r) = e_z(i_y - i_z) \in I$. Hence $s = i_z + r_z(e_z.r) \in I$, which proves the claim.

Hence whether I is densely decomposable depends only on $I^\wedge$. Clearly, the smaller $I^\wedge$ is, the more likely I is to be densely decomposable.

Recall from § 4 the definition of the Boolean algebra BA(P) of subsets of P. By Theorem 4.2, $\mathrm{BA}(P) \cong \mathbb{Z}_2^{\langle P \rangle}$. One way to force $I^\wedge$ to be small is to require that BA(P) be atomless. Examples of P such that BA(P) is atomless are the ideal completion of any dense linear ordering I with a least element, and the ideal completion of any tree T (with a root) such that every $t \in T$ has infinitely many immediate successors in T (cf. Remark 4.3). We will exploit this in Part II.

DEFINITION. Let $I \subseteq R_P$ be a left ideal. We say that I is *locally generated* if there is a finite set $N \preccurlyeq P$ such that I_N generates I in R_P.

For example, any finitely generated ideal of R_P is locally generated.

PROPOSITION 5.3. *Let S be a ring and P an SFP domain such that* BA(P) *is atomless. Let I be any proper locally generated left ideal of $S^{\langle P\rangle}$. Then I is densely decomposable.*

Proof. By Lemma 5.2 it is enough to show that $I^\wedge = I$. Let $r \in R_P \setminus I$. We show that $r \notin I^\wedge$. Take finite $N \preccurlyeq P$ supporting r and such that I_N generates I. As $r \notin I_N$, by Lemma 2.1 there is $n \in N$ with $r(n) \notin I_N(n)$.

Now by assumption the $\sim_N$-class of n is not an atom in BA(P). So we can choose a finite set $M \preccurlyeq P$ containing N and such that the $\sim_N$-class of n is the union of more than one $\sim_M$-class. It follows that the set

$$n^M = \{m \in M\colon\ m \neq n \text{ and } m/N = n\} \neq \varnothing.$$

By Corollary 2.6(ii), the left ideal of R_M generated by I_N is in fact I_M. So by Corollary 2.5, $I_M(m) = I_N(m/N) \subseteq S$ for all $m \in M$. Now M also supports r. Take $m \in n^M$. We have $r(n) \notin I_M(n)$, and $r(m) = r(n) \notin I_N(n) = I_M(m)$. Hence $r \notin I^\wedge$, as required.

REMARK 5.4. We can evidently generalise Proposition 5.3 to the case where $\langle P, \rho, \nu\rangle$ is an SFP system such that if $N \preccurlyeq P$ is finite then for each $n \in N$ there is $m \in P^0 \setminus N$ such that $m/N = n$ and ν_{nm} is a (surjective) isomorphism.

6. $L_{\infty\omega}$-*equivalence of SFP systems and their limits*

Here we define a canonical model-theoretic structure M_σ from an SFP system $\sigma = \langle P, \rho, \nu\rangle$. We prove that if $\sigma_i = \langle P_i, \rho_i, \nu_i\rangle$ $(i = 1, 2)$ are SFP systems and M_{σ_1} and M_{σ_2} are $L_{\infty\omega}$-equivalent then so are the limit rings of σ_1 and σ_2. We will also provide a simple sufficient condition for M_{σ_1} and M_{σ_2} to be $L_{\infty\omega}$-equivalent, namely that $(P_1)^0$ and $(P_2)^0$ are $L_{\infty\omega}$-equivalent and the σ_i are sufficiently similar SFP systems. In Part II we will construct $2^{\aleph_1}$ SFP domains P_i $(i < 2^{\omega_1})$ such that $(P_i)^0$ and $(P_j)^0$ are $L_{\infty\omega}$-equivalent for all $i < j < 2^{\omega_1}$, and yet the limit rings of any SFP systems built on the P_i are pairwise non-embeddable. These limits will none-the-less be $L_{\infty\omega}$-equivalent if the SFP systems are sufficiently similar. This means crudely that although the limit rings are different, the differences are hard to detect.

Recall from, for example, [5] the definition of $L_{\infty\omega}$-equivalence. Let L be any signature. The infinitary language $L_{\infty\omega}$ is built from L by allowing formulas with finite strings of quantifiers but conjunctions and disjunctions of arbitrary length. Two L-structures M, N are said to be $L_{\infty\omega}$-*equivalent* (written $M \equiv_{\infty\omega} N$) if they satisfy the same sentences of $L_{\infty\omega}$.

We can usefully characterise $L_{\infty\omega}$-equivalence in terms of a game between two players, '$\forall$' and '$\exists$', played on two L-structures M and N. The game $G(M, N)$ has ω moves. At each move in a play, player $\forall$ chooses an element from one structure, M or N. Then $\exists$ completes the move by choosing an element from the other structure. After the play is over, the result is two tuples $\bar{m} \in M$, $\bar{n} \in N$ of

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length ω, possibly with repetitions: the ith elements m_i, n_i of $\bar{m}$, $\bar{n}$ respectively consist of the elements chosen in the ith move of the game from M, N respectively. (No record is kept of which player chose which element.) So $\bar{m}$ and $\bar{n}$ define a relation $\theta = \{(m_i, n_i): i < \omega\} \subseteq M \times N$. Player $\exists$ wins the play of the game if and only if θ is a partial isomorphism, that is, θ is a partial function from M to N, and for all quantifier-free first-order formulas $\varphi(\bar{x})$ of L and all $\bar{a} \in \operatorname{dom}(\theta)$, we have $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\theta(\bar{a}))$.

FACT 6.1. The structures M and N are $L_{\infty\omega}$-equivalent if and only if $\exists$ has a winning strategy in the game $G(M, N)$. See **[12]** or **[13]** for details.

DEFINITION. Let $\sigma = \langle P, \rho, \nu \rangle$ be an SFP system. Define a structure $M_\sigma = (P^0, (R_p: p \in P^0))$ in the signature $\{\leq, \rho^*, \nu^*, +^*, \times^*, 0^*, 1^*\}$. The domain of M_σ is the disjoint union of P^0 and the R_p ($p \in P^0$). The binary relation symbol $\leq$ is interpreted as the partial ordering on P^0; $M_\sigma \models p \leq q$ if and only if $p, q \in P^0$ and $p \leq q$. Also, ρ^* is a binary relation symbol, and $M_\sigma \models \rho^*(p, r)$ if and only if $p \in P^0$ and $r \in R_p$. Similarly, ν^* is a binary relation symbol corresponding to ν; we define $M_\sigma \models \nu^*(r, s)$ if and only if $r \in R_p$ and $s \in R_q$ for (necessarily unique) $p, q \in P^0$ with $p \leq q$ and $\nu_{pq}(r) = s$. The ternary relation symbols $+^*$, $\times^*$ for sum and product are defined on each R_p in the obvious way: $M_\sigma \models +^*(r, s, t)$ if and only if $r, s, t \in R_p$ for some $p \in P^0$ and $r + s = t$, and similarly for $\times^*$. Also, 0^* and 1^* are unary relation symbols and $M_\sigma \models 0^*(r)$ if and only if $r = 0 \in R_p$ for some $p \in P^0$ (and similarly for 1^*).

We say that SFP systems σ_1, σ_2 are *$L_{\infty\omega}$-equivalent* if $M_{\sigma_1} \equiv_{\infty\omega} M_{\sigma_2}$.

THEOREM 6.2. *Let $\sigma_i = \langle P_i, \rho_i, \nu_i \rangle$ be $L_{\infty\omega}$-equivalent SFP systems with limit rings R_i ($i = 1, 2$). Then $R_1 \equiv_{\infty\omega} R_2$ in the signature $\{+, \times, 0, 1\}$ of rings.*

Proof. By hypothesis and Fact 6.1 we may take a winning strategy for $\exists$ in the game $G(M_{\sigma_1}, M_{\sigma_2})$. We will describe a winning strategy for $\exists$ in the game $G(R_1, R_2)$. We use a play of $G(R_1, R_2)$ to generate a play of $G(M_{\sigma_1}, M_{\sigma_2})$. Player $\exists$'s strategy in this game will then suggest moves for her in the main game $G(R_1, R_2)$. The method is well known.

More fully, let $\forall$ begin by choosing (without loss of generality) $r_1 \in R_1$. Player $\forall$'s choice gives rise to the following finite sequence of elements of M_{σ_1}: those in an arbitrary finite support $N_1 \trianglelefteq P_1^0$ for r_1, listed in some arbitrary order, together with the sequence $r_1(n_1)$ of elements of the R_{n_1} ($n_1 \in N_1$). Player $\exists$ treats them as successive moves of $\forall$ in a play of $G(M_{\sigma_1}, M_{\sigma_2})$ and uses her winning strategy in this game to choose corresponding elements of M_{σ_2}. This correspondence gives a partial isomorphism from M_{σ_1} to M_{σ_2}. Moreover, as the strategy is winning, the elements chosen corresponding to the elements n_1 form a locally directed subset N_2 of P_2^0. Hence the elements corresponding to the elements $r_1(n_1)$ give rise to an element r_2 of R_2; r_2 is supported by N_2 and $r_1(n_1)$ corresponds to $r_2(n_2)$ for each corresponding pair (n_1, n_2). Player $\exists$'s reply in the main game $G(R_1, R_2)$ is this element r_2.

In each subsequent move $\forall$'s choice generates a further finite sequence of elements of a structure M_{σ_i} ($i = 1$ or 2). We can assume that the set of all elements so far chosen in each $P_i^0 \subseteq M_{\sigma_i}$ ($i = 1, 2$) is a locally directed subset. On each occasion $\exists$ continues with her strategy to obtain corresponding elements of the other structure. Note that at each stage, *all* elements so far chosen in M_{σ_1} are in partial isomorphism with the corresponding ones in M_{σ_2}.

After ω moves, tuples of ω elements $\bar{a}_1 \in M_{\sigma_1}$, $\bar{a}_2 \in M_{\sigma_2}$ will have been generated. The map $\bar{a}_1 \mapsto \bar{a}_2$ is a partial isomorphism from M_{σ_1} to M_{σ_2}. It is now easy to see that the corresponding elements of the R_i $(i = 1, 2)$ are also in partial isomorphism. Hence the strategy described is winning for $\exists$. The result follows by Fact 6.1.

COROLLARY 6.3. *Let P_1 and P_2 be SFP domains with $P_1^0 \equiv_{\infty\omega} P_2^0$, and let R_1, R_2 be $L_{\infty\omega}$-equivalent rings. Then the limit rings $R_1^{\langle P_1 \rangle}$ and $R_2^{\langle P_2 \rangle}$ are $L_{\infty\omega}$-equivalent.*

Proof. Define SFP systems $\sigma_i = \langle P_i, \rho_i, \nu_i \rangle$ $(i = 1, 2)$ by

$$\rho_i(p) = R_i \quad \text{for all } p \in P_i,$$

$$\nu_i(p, q) = \mathrm{id}_{R_i} \quad \text{for all } p \leq q \text{ in } P_i.$$

It is evident that σ_1 and σ_2 are $L_{\infty\omega}$-equivalent, and by definition, $\lim(\sigma_i) = R_i^{\langle P_i \rangle}$ $(i = 1, 2)$. The result follows by Theorem 6.2.

This shows that under suitable restrictions on the rings and morphisms of the SFP systems σ_1 and σ_2, to get the limit rings to be $L_{\infty\omega}$-equivalent it suffices to begin with SFP domains having $L_{\infty\omega}$-equivalent bases. We will apply this in § 4 of Part II.

The same proof as in Theorem 6.2 shows that if R_i are $L_{\infty\omega}$-equivalent rings and B_i are $L_{\infty\omega}$-equivalent Boolean algebras $(i = 1, 2)$ then $R_1[B_1]^* \equiv_{\infty\omega} R_2[B_2]^*$.

PART II. NON-STRUCTURE THEOREMS

Here we return to investigating the effect of the SFP domain of an SFP system on its limit ring. We want, in particular, to find a way of changing the underlying domain that necessarily changes (the isomorphism type of) the ring. Our approach is to ask how much of the domain structure gets to be encoded in the limit ring in such a way that we can recover it purely ring-theoretically. For if we use two different domains, and they are recoverable intrinsically from the two limit rings in sufficient detail to reveal their differences, then the rings must be different as rings.

We saw in Remark I.4.3 that however we may alter the domain in a system, there is no guarantee of getting different limit rings in the countable situation—when the base of the SFP domain and also each component ring is countable. Suppose, for instance, that we build atomless Boolean rings (as in Remark I.4.3). Up to isomorphism there is a unique countable such ring. Hence the domain structure here cannot exert any effect.

However, we will see that things are different if we allow the base of the domain to be uncountable. We briefly sketch the 'non-structure theorem' that will occupy this part of the paper. (Our description here is not accurate in detail.) The technique is well known. To simplify matters, assume that we have two SFP domains P, P^1 that are in fact a certain kind of tree of height ω_1, that all R_p

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($p \in P \cup P^1$) are the trivial ring $\mathbb{Z}_2$, and that all v_{pq} are isomorphisms. We can express P as a union $\bigcup_{i<\omega_1} P_i$, where P_i is the subtree of P consisting of the elements of height at most i. The P_i are an increasing chain of SFP subdomains of P. Write R for $\mathbb{Z}_2^{\langle P\rangle}$, and (if $i<\omega_1$) R_i for $\mathbb{Z}_2^{\langle P_i\rangle}$, where the notation $S^{\langle P\rangle}$ for the SFP power of S by P is as defined in § 4 of Part I. Thus we have $R=\bigcup_{i<\omega_1} R_i$.

Take a full ideal I of R. So $I=\{r \in R\colon r(s)=0\}$ for some $s \in P$. It turns out that $I \cap R_i$ is full for all $i<\omega_1$. So for each $i<\omega_1$ there is $s_i \in P_i$ such that $I \cap R_i=\{r \in R_i\colon r(s_i)=0\}$, and $s_i \leqslant s_j \leqslant s$ if $i<j<\omega_1$. The same holds for R^1, defined similarly using P^1.

Assume now that $\theta\colon R \to R^1$ is a ring isomorphism. Then the set $C=\{i<\omega_1\colon \theta(R_i)=R_i^1\}$ is a club (a large set) in ω_1. Moreover, for each $i \in C$ the image $\theta(I \cap R_i)$ is a full ideal of R_i^1 (because here, as in Theorem I.4.2(i), 'full' is the same as 'maximal'). So there are $s_i^1 \in P_i^1$ $(i \in C)$ such that $\theta(I \cap R_i)=\{r \in R_i^1\colon r(s_i^1)=0\}$. Define $P \restriction C=\{p \in P\colon \text{height}(p) \in C\}$ (and similarly for P^1). Then θ induces a partial map Θ from $P \restriction C$ to $P^1 \restriction C$ by $s_i \mapsto s_i^1$. By considering all full ideals I, we find that Θ extends to a bijection from $P \restriction C$ to $P^1 \restriction C$, and it is order-preserving. Thus the existence of an isomorphism from R to R^1 forces the underlying SFP domains to be closely related; there is a club $C \subseteq \omega_1$ such that $P \restriction C \cong P^1 \restriction C$.

So in order to produce many non-isomorphic rings R, it suffices to find many trees P such that no two are isomorphic on any club. In **[1]** this is done for Aronszajn trees, using the hypothesis of $2^{\aleph_0}<2^{\aleph_1}$ (weak diamond). Our construction here is in some ways similar, but a weaker result suffices and we do not need any set-theoretic hypotheses beyond ZFC. (The trees we construct are not strictly Aronszajn trees; in fact it is consistent with $MA+2^{\aleph_0}>\aleph_1$ that any two Aronszajn trees are isomorphic on some club **[1]**.) However, our construction is made more complicated because we consider the more general case of ring embeddings θ and arbitrary rings R_p. In this setting Θ becomes a relation between the restricted trees.

The layout of Part II is as follows. In § 1 the appropriate form of tree is defined and the relation Θ discussed. In § 2 we construct many different trees using an Aronszajn-style argument, and use them to produce many different rings. Finally, we establish some higher-order properties of the rings. We show that each of their full ideals can be made countably generated (§ 3), and that the rings themselves can be made pairwise $L_{\infty\omega}$-equivalent (§ 4) and to some degree rigid (§ 5).

1. *Spruce trees and conformal relations*

Trees

Most of the following definitions will be familiar but we include them for convenience. A *tree* is a non-empty poset $(T, \leqslant)$ such that the set $\hat{t}=\{u \in T\colon u<t\}$ of predecessors of any $t \in T$ is well-ordered (and hence linearly ordered). We will refer to the elements of a tree as *nodes*. The *height* of a node $t \in T$, written $\text{ht}_T(t)$ or $\text{ht}(t)$, is the order type of $\hat{t}$. If i is an ordinal, we write $T(i)$ for the set of nodes of T of height i, the ith *level* of T.

More generally, if $S \subseteq T$ is closed downwards (that is, if $t<u$ and $u \in S$ then $t \in S$), we write $S(i)$ for $S \cap T(i)$, and $\text{ht}_T(S)$ for the least ordinal i such that

$S(i)=\varnothing$. If X is a set of ordinals, we define $S \restriction X$ to be $\{s \in S\colon \mathrm{ht}_T(s) \in X\}$. So for example, if i is an ordinal then $S \restriction i$ is the set of elements of S whose height is less than i. (Since S, if non-empty, is a tree in its own right, the notation $S(i)$ etc. would be ambiguous if S were not closed downwards in T.)

If $t, t' \in T$, we say that t' is an *immediate successor* of t if $t' > t$ and $\mathrm{ht}(t') = \mathrm{ht}(t) + 1$. Then also t is an *immediate predecessor* of t'. A *terminal node* is one without any successors in T. A *branching node* is a node with at least two immediate successors. A node $t \in T$ is said to be *green* in T if T contains a branching node b with $b \geqslant t$.

A tree T is called *normal* if whenever $t, u \in T$ have limit height and $\hat{t} = \hat{u}$, then $t = u$. Our convention is that every ordinal is exactly one of: 0, successor, limit.

A *branch* of a tree T is a maximal linearly ordered subset of T. A branch β is said (unusually) to be *cofinal in* T if every node of β is green in T. If T is normal, this means that the branching nodes are 'cofinal' in β: if $i < \mathrm{ht}(\beta)$ then there is a branching node $b \in \beta$ of height at least i in T.

REMARK. Let T be a tree with a least element $\perp$. Then any $S \subseteq T$ with $\perp \in S$ is locally directed in T. If T is a dcpo then T is an SFP domain, the finite elements being those not of limit height.

Spruce trees

We can now define the type of tree that interests us here. A *spruce tree* is a normal tree T satisfying:

(i) every branch of T has height ω_1;

(ii) each node of T has exactly one non-branching immediate successor;

(iii) T has no cofinal branches;

(iv) for all $i < j < \omega_1$ and every branching node b of height i in T, there are exactly $\aleph_0$ branching nodes of T of height j above b;

(v) $T(0)$ has just one node '$\perp$', which is a branching node, and hence by (iv), each higher level of T has exactly $\aleph_0$ branching nodes.

An example of a spruce tree is an Aronszajn tree (cf. [9] and below) but with each branch and node extended individually by new non-branching nodes up to height ω_1. In general, however, not all predecessors of a branching node will be branching. In § 2 the existence of many spruce trees is established.

Fix a spruce tree T.

DEFINITION. A node of T is said to be *basic* if

(i) it is a branching node and

(ii) it is finite in the sense of § I.1, that is, its height is not a limit ordinal.

We write $B(T)$ for the set of basic nodes of T; by (v) above, $B(T)$ is non-empty.

Note that $B(T)$ is a poset, by restriction of the ordering of T. Recall from § I.1 that an *ideal* of $B(T)$ is a downwards-closed directed subset of $B(T)$. The *ideal completion* of $B(T)$, written $\mathrm{Idl}(B(T))$, is just the set of ideals of $B(T)$, ordered as a poset by inclusion. We identify $b \in B(T)$ with the principal ideal $\{b' \in B(T)\colon b' \leqslant b\}$ of $\mathrm{Idl}(B(T))$. This identification preserves the ordering on $B(T)$.

PROPOSITION 1.1. *There is a canonical embedding λ of the ideal completion of $B(T)$ into T. The restriction $\lambda \upharpoonright B(T)$ is the inclusion map from $B(T)$ into T.*

Proof. Let I be an ideal of $B(T)$. Then I is non-empty and linearly ordered. Let β be any branch of T containing I. Since T has no cofinal branches, I is countable. But β has height ω_1 in T. Hence $\{t \in \beta: t \geqslant i \text{ for all } i \in I\}$ is non-empty. Let t be its least element. Since T is normal, t does not depend on β, but only on I. We write t as $\mathrm{lub}_T(I)$.

We define λ: $\mathrm{Idl}(B(T)) \to T$ by $\lambda(I) = \mathrm{lub}_T(I)$. Clearly if $I \subseteq J$ then $\lambda(I) \leqslant \lambda(J)$ in T. Let I and J be distinct ideals, and take $i \in I \setminus J$ (without loss of generality). Clearly there is no $j \in J$ with $j \geqslant i$. Consequently, $\mathrm{lub}(J) \not\geqslant i$, and since i has successor height in T, we have $\mathrm{lub}(J) \neq i$. But $\mathrm{lub}(I) \geqslant i$. Hence λ is injective, and so an order-preserving embedding. Clearly if $b \in B(T)$ then the ideal $I = \{b' \in B(T): b' \leqslant b\}$ corresponding to b satisfies $\lambda(I) = b$. Hence $\lambda \upharpoonright B(T)$ is just inclusion.

DEFINITION. We say that a node t of T is a *limit node* if $t = \mathrm{lub}_T(I)$ for some ideal I of $B(T)$. We write $L(T)$ for the set of limit nodes.

REMARK. We have $B(T) \subseteq L(T)$. Clearly $L(T) \cong \mathrm{Idl}(B(T))$ is a dcpo (cf. § I.1). Since $L(T)$ is a tree with a unique least element, it is in fact an SFP domain. Notice that $B(T) \trianglelefteq L(T) \trianglelefteq T$, and $B(T) = [L(T)]^0$. In general, $L(T)$ is not closed downwards in T.

We will use the SFP domain $L(T)$ to build SFP systems. The remainder of T is used to keep track of what is going on. To do this we need to deal with the subtrees of T of countable height.

Recall that if δ is a limit ordinal and X_i $(i < \delta)$ are arbitrary sets, the X_i are said to form a *continuous chain* if $X_i \subseteq X_j$ for each $i < j < \delta$, and $X_j = \bigcup \{X_i\colon\ i < j\}$ for each limit ordinal $j < \delta$. The *union* of the chain is defined to be $\bigcup \{X_i\colon\ i < \delta\}$.

If $i < \omega_1$, we define $L(T)_i$ to be the set of elements of $L(T)$ with height at most i in T. Then $L(T)_i \leqslant L(T)$ (see § I.1 for this notation). Similarly define $B(T)_i = B(T) \cap (T \upharpoonright i + 1)$. As $B(T)$ has no nodes of limit height, the $B(T)_i$ $(i < \omega_1)$ form a continuous chain with union $B(T)$. The chain $(L(T)_i\colon\ i < \omega_1)$ is not continuous, but its union is $L(T)$. We have $(L(T)_i)^0 = B(T)_i$ for each $i < \omega_1$.

Spruce trees and SFP systems

Now take an SFP system $\langle L(T), \rho, \nu\rangle$ such that each R_t (for $t \in B(T)$) is countable. Let its limit ring be R. Writing R_i for $R_{L(T)_i}$, the limit of the subsystem based on $L(T)_i$, we see that the R_i form a continuous chain of countable subrings of R, with union R.

We define for each $i < \omega_1$ a projection π_i: $R \to R_i$, given as follows. If $r \in R$, then by definition r is a function from $L(T)$ into $\bigcup \{R_t\colon\ t \in L(T)\}$. Then $\pi_i(r)$ is just the restriction $r \upharpoonright L(T)_i$ of r to the set $L(T)_i$.

We must show that $\pi_i(r) \in R_i$. Let $N \leqslant L(T)$ be a finite support of r in R, and define $N' = N \cap L(T)_i$. Then $\perp \in N'$ and so $N' \leqslant L(T)_i$. Clearly if $x \in L(T)_i$ then $x/N = x/N'$. It follows that N' supports $\pi_i(r)$ in R_i. So $\pi_i(r) \in R_i$, as required.

Each π_i is a surjective ring homomorphism and is the identity on R_i.

Full ideals

The notion of full ideals becomes a little more complicated in this setting, since now we have $\aleph_1$ different rings and we can no longer tell from its site which ring a full ideal lies in. So we refine the notion of site, using the part of the tree T that lies outside $L(T)$.

Recall that each $t \in T$ has a unique non-branching immediate successor, t^+ say. Hence if $\mathrm{ht}(t) = i$, we can define a node $t^{[j]}$ for each $i \leq j < \omega_1$, by induction on j:

$t^{[i]} = t$,

$t^{[j+1]} = (t^{[j]})^+$,

if j is a limit ordinal, $t^{[j]}$ is the unique node of height j with $t^{[j]} > t^{[k]}$ for all k with $i \leq k < j$; this is well defined as T is spruce.

Note that although certainly $t^{[j]}$ is not a branching node if j is a successor ordinal, it may be a branching node if j is limit. If $j > i$, we have $t^{[j]} \notin L(T)$.

In the light of this we can define a map ζ: $T \to T$ as follows: $\zeta(t)$ is the lowest node $t' \leq t$ such that $t = t'^{[\mathrm{ht}(t)]}$. We clearly have the following proposition.

Proposition 1.2. *We have $\zeta(T) = L(T)$ and $\zeta^2(t) = \zeta(t)$ for all $t \in T$. For all $i < \omega_1$, the restriction $\zeta \restriction (T(i))$: $T(i) \to L(T)_i$ is a bijection, whose inverse is given by $t \mapsto t^{[i]}$.*

Now if $i < \omega_1$ then the set of possible sites for full ideals of R_i is $L(T)_i$, and this is in bijection with $T(i)$ via ζ. So if I is a full ideal of R_i with site $s \in L(T)_i$, we define the *tree site* of I, written τI, to be $s^{[i]}$.

Tree sites behave well with respect to subrings. We have:

Proposition 1.3. (i) *If $i < j < \omega_1$ and J is a full ideal in R_j, then $J \cap R_i$ is full in R_i and $\tau(J \cap R_i) \leq \tau J$. (Note that $\tau(J \cap R_i)$ is determined by this inequality, since it has height i.)*

(ii) *If $i < j < \omega_1$ and I is a full ideal of R_i, then the ideal $J = \pi_i^{-1}(I) \cap R_j$ is full in R_j, and $\tau J = (\tau I)^{[j]}$. We write $I^{[j]}$ for this ideal.*

Proof. (i) Let $\sigma J = p \in L(T)_j$. Since $L(T)_i \preccurlyeq L(T)_j$, by Theorem I.2.2 we see that $J \cap R_i$ is full in R_i with site $q = p/L(T)_i$. We must show that $p^{[j]} \geq q^{[i]}$.

If $\mathrm{ht}(p) \leq i$ then $p \in L(T)_i$ and $q = p$, so the result is clear. So suppose that $\mathrm{ht}(p) > i$. We show that $p \geq q^{[k]}$ for all k satisfying $\mathrm{ht}(q) \leq k \leq i$, by induction on k.

If $k = \mathrm{ht}(q)$ or k is a limit ordinal then this is trivial. Assume that $k+1 \leq i$ and $p \geq q^{[k]}$. If $p \ngeq q^{[k+1]}$ then there is $b \leq p$ with $\mathrm{ht}(b) = k+1$ and $b \neq q^{[k+1]}$. The immediate predecessor of b is $q^{[k]}$, so b must be a branching node in T. Hence $b \in L(T)_i$ and so $q = p/L(T)_i \geq b$. As $b > q$, this is a contradiction. So $p \geq q^{[k+1]}$, which completes the induction.

(ii) For all $r \in R_j$ we have

$$r \in J \quad \Leftrightarrow \quad r \restriction L(T)_i \in I \quad \Leftrightarrow \quad r(\sigma I) \in \Delta I.$$

So $J = \Delta I \,@\, \sigma I$ in R_j. Hence I and J have the same site and defect, though they lie in different rings. We have $\tau J = (\sigma I)^{[j]} = (\sigma I)^{[i][j]} = (\tau I)^{[j]}$.

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Clubs

Let $C \subseteq \omega_1$. We say that C is a *club* (in ω_1) if it is closed and unbounded in ω_1. That is:

(cl) if C_0 is a countable subset of C, then $\bigcup C_0 \in C$ (of course, $\bigcup C_0$ is $\text{lub}_{\omega_1}(C_0)$);

(ub) for each $i < \omega_1$ there is $c > i$ with $c \in C$.

Examples of clubs are ω_1 itself, and the set of countable limit ordinals. We can go further. If C is any subset of ω_1, we write ∂C for the set of limit points of C. Thus, ∂C is the set of all ordinals of the form $\bigcup \{c_i \colon i < \omega\}$ for some strictly increasing sequence c_i ($i < \omega$) in C. So (cl) above just says that $\partial C \subseteq C$. We then have:

FACT. If C is a club then so is ∂C.

Note that (ub) implies that C is uncountable. We can think of clubs as 'large' subsets of ω_1. We have:

FACT [**10**, § 7]. Any countable intersection of clubs is a club.

We remark that if T is a spruce tree and C a club in ω_1, then $T \restriction C$ is normal and satisfies all conditions except possibly (ii) and (v) of the definition of 'spruce'. A node of $T \restriction C$ is green in $T \restriction C$ if and only if it is green in T.

We will also use the following lemma on clubs.

FACT [**7**, Lemma 5.2.2]. Let $f \colon \omega_1 \to \omega_1$ be a map. Then

$$\{i < \omega_1 \colon \forall j < i(f(j) < i)\}$$

is a club in ω_1.

The proofs of these facts are not hard.

Ring embeddings and conformal relations

Now suppose that U is another spruce tree. Take an SFP system $\langle L(U), \rho', \nu' \rangle$, and write S_u for $\rho'(u)$ ($u \in L(U)$) and S for its limit. Suppose that each S_u is countable. We have a continuous chain $S = \bigcup \{S_i \colon i < \omega_1\}$, as for R. We abuse notation by using the symbol ζ to refer to the maps on T and on U, distinguishing them by context. But π always refers to R.

Recall that if $X \subseteq S$ then $X^* = \{s \in X \colon s(u) \in \{0, 1\}$ for each $u \in L(U)\}$. That is, $X^* = \mathbb{Z}_2^{\langle L(U) \rangle} \cap X$. Clearly the S_i^* ($i < \omega_1$) also form a continuous chain, with union S^*.

PROPOSITION 1.4. *Suppose that $\theta \colon S \to R$ is a ring embedding. Then there is a club C of limit ordinals in ω_1 such that for each $i \in C$,*

(i) $\theta(S_i) = R_i \cap \theta(S)$, *and*

(ii) *if $j < i$ then $\pi_j \theta(S^*) = \pi_j \theta(S_i^*)$.*

Proof. If $j < \omega_1$, let $f(j)$ be the least $k < \omega_1$ such that

$$\theta(S_j) \subseteq R_k,$$
$$R_j \cap \theta(S) \subseteq \theta(S_k),$$
$$\pi_j\theta(S^*) = \pi_j\theta(S_k^*).$$

We can find such a k because the left-hand side of each of these is countable. Then by the fact above, $C' = \{i < \omega_1\colon \forall j < i(f(j) < i)\}$ is a club in ω_1. We can take $C = \partial(C')$.

Now let θ and C satisfy the conditions of the proposition. Define a binary relation $\Theta \subseteq T \times U$ as follows. If t, u have equal height i in T, U respectively, and $i \in C$, then $t \Theta u$ if and only if there is a full (left) ideal I of R_i such that

$$\tau I = t,$$
$$J = \theta^{-1}(I \cap \theta(S_i)) \text{ is a full ideal of } S_i, \text{ and } \tau J = u.$$

We say that the ideal I *represents* the pair (t, u). We have $J = \theta^{-1}(I)$ by definition of C.

DEFINITION. Let T, U be arbitrary trees. A relation $\Phi \subseteq T \times U$ is said to be *height-preserving* if whenever $t \Phi u$ then $\mathrm{ht}_T(t) = \mathrm{ht}_U(u)$. A height-preserving relation Φ is said to be *homomorphic* if whenever $t \Phi u$, $t' \leqslant t$, $u' \leqslant u$ and t' and u' have equal heights, then $t' \Phi u'$.

Clearly, Θ is height-preserving.

PROPOSITION 1.5. *The binary relation Θ is a homomorphic relation contained in $T \restriction C \times U \restriction C$. Moreover, in the notation above, if $i, j \in C$, $i < j$ and J is a left ideal of R_j representing (t, u), then $J \cap R_i$ represents (t', u').*

Proof. Suppose that $i < j$ in C and that the elements $t \in T(j)$, $u \in U(j)$ are related by Θ. Take an ideal J representing (t, u). Then J is full in R_j, and has (tree) site t. By Proposition 1.3, $J \cap R_i$ is full in R_i and has site t'. Similarly, $\theta^{-1}(J) \cap S_i$ is full in S_i with site u'. But as $i \in C$, we have $\theta^{-1}(J) \cap S_i = \theta^{-1}(J \cap R_i)$. Hence $t' \Theta u'$ holds.

DEFINITION. Let T and U be trees of height ω_1. A height-preserving relation $\Phi \subseteq T \times U$ is said to be *surjective* if for all $u \in U$ there is $t \in T$ such that $t \Phi u$. We then write that $\Phi\colon T \to U$ is a surjective relation.

PROPOSITION 1.6. *The relation $\Theta\colon T \restriction C \to U \restriction C$ is surjective.*

Proof. Let $u \in U(i)$ for $i \in C$, and let $\zeta(u) = z$. Let I be full in S_i with site z and defect 0 (that is, $I = 0 @ z$ in S_i).

Claim. The set $\theta(I)$ generates a proper left ideal of R_i.

Proof of claim. If the claim is not true, there are $n_0 < \omega$ and $a_n \in I$, $r_n \in R_i$ $(n < n_0)$ such that

$$\sum_{n<n_0} r_n . \theta(a_n) = 1.$$

Now for each n we have $a_n(z)=0$. We can take a finite set $N \preccurlyeq L(U)_i$ such that each a_n is supported by N, and $a_n(z')=0$, where $z'=z/N$. Define $d \in S_i$ as follows: d is supported by N, $d(x)=1$ if $x \in N$, $x \neq z'$, and $d(z')=0$. Then $d \neq 1$, but $a_n.d=a_n$ for each $n<n_0$.

Now let $e=\theta(d) \in R_i$. Since θ is an embedding, $e \neq 1$. But we have

$$e=\left[\sum r_n.\theta(a_n)\right].e=\sum r_n.(\theta(a_n.d))=\sum r_n.\theta(a_n)=1,$$

a contradiction. This proves the claim.

By Zorn's lemma there is a maximal left ideal J of R_i extending $\theta(I)$. By Proposition I.3.2, J is full in R_i. Then $\theta^{-1}(J)$ is a proper left ideal of S_i and extends I, so it is full with site z. So if $t=\tau J$, we have $t\,\Theta\, u$, the pair (t, u) being represented by J.

Continuity

So far we have shown that a ring embedding θ: $S \to R$ induces a height-preserving homomorphic surjective relation Θ: $T \restriction C \to U \restriction C$. Such a Θ need not preserve much structure; for example, the trivial relation $\bigcup_{i \in C} T(i) \times U(i)$ has these properties. We need a stronger preservation result.

The key to obtaining one is the following observation. For simplicity identify S with $\theta(S)$, so that $S \subseteq R$. Take two full ideals I_1, I_2 of R_i for some $i \in C$, and suppose that $I_1 \cap S_i = I_2 \cap S_i$ and that this ideal is full in S_i. If $j>i$ in C, we know from Proposition 1.3(ii) that there is a 'canonical' full ideal $I_1^{[j]}$ of R_j associated with I_1. Similarly, $I_2^{[j]}$ is associated with I_2. Would we still have $I_1^{[j]} \cap S_j = I_2^{[j]} \cap S_j$?

In general we would not. However, if I_1 is itself of the form $K_1^{[i]}$ for some ideal K_1 of R_k for some $k<i$, then we might expect I_1 to be 'determined' by K_1. So if $I_2 = K_2^{[i]}$ in the same way, and I_1 and I_2 'agree' in some way on S_i, we might hope that $I_1^{[j]}$ and $I_2^{[j]}$ also agree on S_j. In fact we will show in Theorem 1.7 that if $K_1^{[i]} \cap S_i$ and $K_2^{[i]} \cap S_i$ *have the same site* as full ideals of S_i, then the same holds for $K_1^{[j]} \cap S_j$, $K_2^{[j]} \cap S_j$. The proof relies on Proposition 1.3(ii) to show that K determines the site of I, and on Proposition I.3.1. Theorem 1.7 is needed in the preservation lemma (2.2 below), which shows that Θ does preserve enough features to prove anti-structure theorems.

First we need the following definition.

Definition. Let T and U be spruce trees and C a club in ω_1. A homomorphic relation Φ: $T \restriction C \to U \restriction C$ is said to be *continuous* if for all $i, j \in C$ with $i<j$ and all $u \in U(i)$, there is a node $u^{[\Phi,j]} \geqslant u$ of height j in U and such that for all $t \in T(i)$ with $t\,\Phi\, u$ and $\zeta(t)<t$, if there exists $u' \in U(j)$ such that $u'>u$ and $t^{[j]}\,\Phi\, u'$, then $u'=u^{[\Phi,j]}$. We do not require that $u^{[\Phi,j]}=u^{[j]}$.

Roughly, this says that a small change in nodes in $T \restriction C$ (viz. going from $t^{[j]}$ to $t'^{[j]}$, where t and t' are related via Φ to the same node $u \in U \restriction C$) results in only a small change (no change) in their Φ-relatives above u in U; if $t^{[j]}$ and $t'^{[j]}$ are related to any node above u, then they are related to only one, and the same one. Hence the name 'continuous'.

Theorem 1.7. *The relation* Θ: $T \restriction C \to U \restriction C$ *is continuous.*

Proof. Suppose that $i<j$ in C and $u \in U(i)$, and let $t_1, t_2 \in T(i)$ be such that $t_l \Theta u$ and $\zeta(t_l)<t_l$ $(l=1,2)$. Suppose that $u_l \in U(j)$ with $u_l > u$ are such that $t_l^{[j]} \Theta u_l$ for $l=1,2$. We must show that $u_1=u_2$.

For $l=1,2$ choose a full left ideal J_l of R_j representing the pair $(t_l^{[j]}, u_l)$. Then $\sigma J_l = \zeta t_l < t_l$. As i is a limit ordinal, there is $k<i$ such that $\sigma J_l \in T \restriction k$ for each l. Set $K_l = J_l \cap R_k$. By Proposition 1.3, $J_l = \pi_k^{-1}(K_l) \cap R_j$, that is, we have $J_l = K_l^{[j]}$ in the notation of Proposition 1.3. Also set $I_l = K_l^{[i]}$ (which by the proposition is $J_l \cap R_i$). These definitions are illustrated in Fig. 1.1.

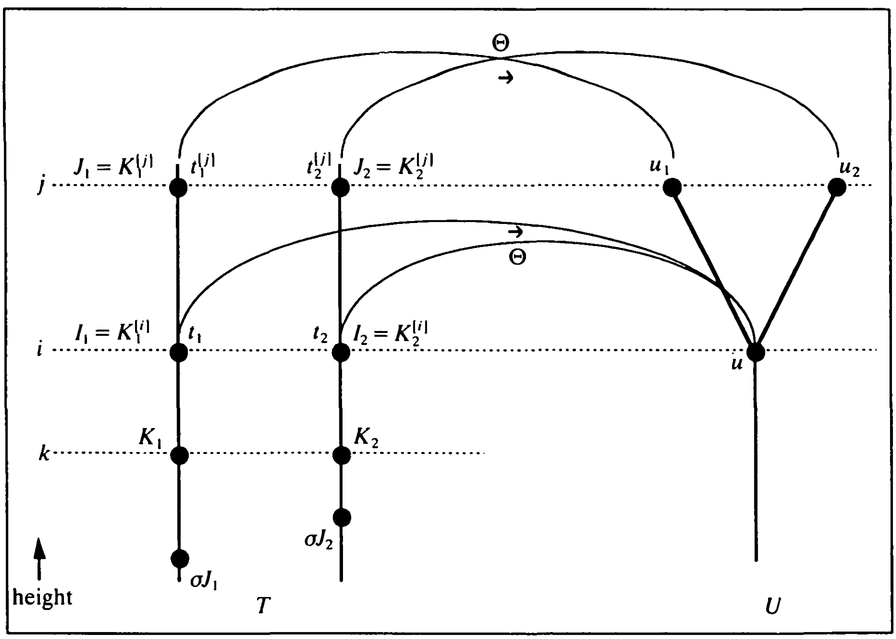

FIG. 1.1

Assume for a contradiction that $u_1 \neq u_2$. By Proposition I.3.1, $[\theta^{-1}(J_1)]^* \neq [\theta^{-1}(J_2)]^*$, so without loss of generality we may assume that there is $s \in S_j^*$ with $\theta(s) \in J_1 \backslash J_2$. Hence $\pi_k\theta(s) \in K_1 \backslash K_2$. By definition of the club C (cf. Proposition 1.4(ii)), there is $s' \in S_i^*$ with $\pi_k\theta(s') = \pi_k\theta(s)$. Hence $\theta(s') \in I_1 \backslash I_2$. Consequently, $[\theta^{-1}(I_1)]^* \neq [\theta^{-1}(I_2)]^*$.

But $u_l \geqslant u \in U(i)$, and by Proposition 1.5, Θ is homomorphic. So $\theta^{-1}(I_l)$ is full in S_i with tree site u, for each l. By Proposition I.3.1 again, $[\theta^{-1}(I_1)]^* = [\theta^{-1}(I_2)]^*$. This is a contradiction. So $u_1 = u_2$, as required.

We can now adapt surjectivity to green nodes. A relation Φ: $T \to U$ is said to be *surjective on green nodes* if whenever $u \in U$ is green then there is a green node $t \in T$ with $t \Phi u$.

PROPOSITION 1.8. *Suppose that* Φ: $T \to U$ *is a homomorphic, surjective and continuous relation, where* T *and* U *are spruce trees. Then* Φ *is surjective on green nodes.*

Proof. Let $u \in U$ be a green node of height i. Using property (iv) of spruceness we may first choose a branching node $u' > u$ of height j in U, and then an ordinal $k > j$ and $u'' \in U(k)$ with $u'' \neq u'^{[\Phi,k]}$.

Now as Φ is surjective, there is $t'' \in T(k)$ related to u'' via Φ. Let t', t be the predecessors of t'' of heights j, i in T respectively. As Φ is homomorphic, $t' \Phi u'$ and $t \Phi u$.

If t is not green in T, then $\zeta(t') \leq t < t'$ and also $t'' = t^{[k]}$. So by continuity the only node related to t'' by Φ is $u^{[\Phi,k]}$. This is a contradiction, and proves the proposition.

DEFINITION. *A relation* Φ: $T \rightarrow U$ on spruce trees is said to be *conformal* if it is height-preserving, homomorphic, surjective on green nodes, and continuous.

EXAMPLES. Any tree isomorphism is conformal. The results above show that Θ: $T \restriction C \rightarrow U \restriction C$ is conformal.

Conformal relations preserve sufficient tree structure for us to prove our non-structure results. We will see this in the remaining sections; see, in particular, Theorem 2.7 below.

2. *Aronszajn trees*

An *Aronszajn tree* is a tree A of height ω_1, such that each level $A(i)$ (for all $i < \omega_1$) is countable, but without uncountable branches. See **[9]** or **[10]** for the classical construction of an Aronszajn tree. We modify it slightly to obtain a large family of 'pseudo-Aronszajn' trees such that there is no conformal relation defined on any club between any pair of the family. Hence by the results of § II.1, the limits of any SFP systems built on the trees will be pairwise non-embeddable. We also show how to make the trees fairly rigid with respect to conformal relations. In § 5 we will use this to produce rings that are also fairly rigid. The trees we build are spruce and so not strictly Aronszajn, but they retain enough 'Aronszajn-ness' to ensure that rings built on them have the Aronszajn-like property that every maximal ideal (more generally, every full ideal) is countably generated.

To make the trees different we will use the devices of 'grids' and 'nests'. We will define nests later; first we look at grids.

DEFINITION. A *grid* is a pair $\Gamma = \langle G, \gamma \rangle$, where $G \subseteq \partial\partial\omega_1$ is a set of limit limit ordinals, and γ: $G \times \omega \rightarrow \partial\omega_1$ is a map that provides for each $j \in G$ a strictly increasing sequence of countable limit ordinals $\gamma(j, n) = j_n$ $(n < \omega)$ with $\bigcup \{j_n: n < \omega\} = j$.

It will also be useful to define a node t of limit height i in a spruce tree T to be *cofinal* if $\hat{t}$ is a cofinal branch of $T \restriction i$; that is, there are branching nodes of unbounded height in $\hat{t}$.

We now present our main construction. The statement of the theorem contains some terms that will be defined below.

THEOREM 2.1. *Let $\Gamma = (G, \gamma)$ be a grid. Then there is a spruce tree $A = A(\Gamma)$ with the following properties:*

(α) *if $i < j < \omega_1$ and $\xi \in A(i)$ is a sequence node with* $\sup(\xi) < q \in \mathbb{Q}$, *then there is a sequence node $\eta \in A(j)$ with $\xi < \eta$ and* $\sup(\eta) < q$;

(κ) *if $i < \omega_1$, $A(i)$ contains at most $\aleph_0$ sequence nodes;*

(ν) *for all $i \in \partial\omega_1$ the number of distinct cofinal green nodes of height i in A is*

$$|\gamma^{-1}(i)|.\aleph_0 \text{ if } i \in G,$$

$$|\gamma^{-1}(i)| + \aleph_0 \text{ if } i \notin G.$$

So, for example, if $i \in G \backslash \mathrm{im}(\gamma)$ then there are no cofinal green nodes in $A(i)$.

Proof. Unlike in the classical Aronszajn construction the nodes of A will be of two kinds: sequence nodes or blank nodes.

Sequence nodes are certain elements of ${}^{<\omega_1}\mathbb{Q} = \{\eta\colon \exists i < \omega_1(\eta\colon i \to \mathbb{Q})\}$. So ${}^{<\omega_1}\mathbb{Q}$ is the set of countable sequences of rationals. If $\eta \in {}^{<\omega_1}\mathbb{Q}$, we write $\mathrm{len}(\eta)$ for $\mathrm{dom}(\eta)$, and $\sup(\eta)$ for $\sup\{\eta(i)\colon i < \mathrm{len}(\eta)\} \in \mathbb{R} \cup \{\infty, -\infty\}$. Each sequence node η will be a *bounded increasing sequence*; that is, $\sup(\eta) < \infty$ and $\eta(i) > \sup(\eta \restriction i)$ for all $i < \mathrm{len}(\eta)$. The letters η, ξ will denote sequence nodes.

Blank nodes are 'filler' nodes. We can increase the height of a sequence node in the tree by inserting blank nodes beneath it.

Each node of A will be either sequence or blank, but not both. The sequence nodes will be precisely the branching nodes. It will be clear that if $\Gamma = (\emptyset, \emptyset)$ then deleting the blank nodes from $A(\Gamma)$ gives a classical Aronszajn tree.

We will construct A by induction on levels. We must specify which elements of A are related in the tree ordering. As in the standard Aronszajn tree, if $\xi, \eta \in A$ are sequence nodes then $\xi < \eta$ will hold in the tree if and only if ξ is a proper initial segment of η. However, blank nodes are not sequences and we will specify explicitly how the tree ordering relates them. Since blank nodes may occur beneath sequence nodes, we will have $\mathrm{ht}(\eta) \geq \mathrm{len}(\eta)$ for every sequence node $\eta \in A$, whereas in the classical case we have equality.

We now begin the construction of A. We define $A(0)$, the 0th level of A, to be $\{\langle\rangle\}$, where $\langle\rangle$ is the empty sequence, a sequence node with supremum $-\infty$. If $A(i)$ has been defined, we construct $A(i+1)$ as follows. First, for every node $a \in A(i)$ we put a single blank node a^+ into $A(i+1)$ above a. This gives property (ii) of the definition of 'spruce'. Then for each sequence node $\eta \in A(i)$ and every $q \in \mathbb{Q}$ with $q > \sup(\eta)$, we put the sequence node $\eta\hat{\ }q$ (the sequence η followed by q) into $A(i+1)$. This adds countably many sequence nodes above η. Clearly (α) and (κ) are preserved.

Now assume that $j < \omega_1$ is a limit ordinal and we have built $A(i)$ for all $i < j$. There are two cases.

Case I: $j \notin G$. In this case we follow the classical construction. So for each sequence node $\eta \in A \restriction j$ and each rational $q > \sup(\eta)$, we choose a rational q' with $q > q' > \sup(\eta)$ and a strictly increasing sequence of ordinals i_n ($n < \omega$) with $i_0 = \mathrm{ht}(\eta)$ and $\bigcup\{i_n\colon n < \omega\} = j$. We then define sequence nodes $\eta_n \in A(i_n)$ ($n < \omega$) by induction on n. We set $\eta_0 = \eta$. If η_n has been defined, we use (α) to find a sequence node $\eta_{n+1} \in A(i_{n+1})$ with $\eta_{n+1} > \eta_n$ and $\sup(\eta_{n+1}) < q'$. Then the union η_ω of the sequences η_n is an increasing sequence of rationals, and

$\sup(\eta_\omega) \leq q' < q$. We put η_ω into $A(j)$, so η_ω will lie above the branch of $A \upharpoonright j$ defined by the η_n.

We then add a single blank node above each remaining branch of $A \upharpoonright j$. This gives amongst other things property (i) of the definition of 'spruce'. Clearly (α) and (κ) are preserved by the construction.

REMARK. In fact, (α) clearly ensures that there is more than one choice for η_{n+1} at each stage. Hence there are $2^{\aleph_0}$ possible choices of η_ω.

Case II: $j \in G$. Our aim is to make level j of A 'special' by using the fact that $j \in G$, whilst all the time preserving (α). Write j_m for $\gamma(j, m)$ $(m < \omega)$. Let $\eta \in A \upharpoonright j$ be a sequence node, let $q \in \mathbb{Q}$ with $\sup(\eta) < q$, and let $m < \omega$ be the least m such that $j_m > \mathrm{ht}(\eta)$. Since j_m is a limit ordinal, we can use the argument of Case I to choose an increasing series of sequence nodes η_n $(n < \omega)$ in $A \upharpoonright j_m$ with $\eta_0 = \eta$, $\bigcup\{\mathrm{ht}(\eta_n)\colon n < \omega\} = j_m$ and $\sup(\eta_\omega) < q$, where η_ω is the union of the sequences η_n.

Now by the Remark above, there are $2^{\aleph_0}$ possible choices of η_ω, so by property (κ) we can choose one such that $\eta_\omega \notin A \upharpoonright j$. It follows that the branch of $A \upharpoonright j_m$ determined by the η_n has only blank nodes above it in $A \upharpoonright j$, so it determines a branch β of $A \upharpoonright j$. We then put the sequence node η_ω into $A(j)$ above β. We do this for all $\eta \in A \upharpoonright j$ and all $q > \sup(\eta)$. This preserves (α) and (κ).

We complete the construction by adding a single blank node above each remaining branch of $A \upharpoonright j$, as in Case I. The conditions (α), (κ) remain undisturbed.

Let A be the resulting tree of height ω_1. We must check that it is spruce. All clauses of the definition except perhaps (iii) are obviously true. Clause (iii) follows as in the classical Aronszajn construction, because a cofinal branch of A would give rise to an uncountable strictly increasing sequence of rationals, which is impossible as $\mathbb{Q}$ is countable.

We finally check that A satisfies (v). For $i \leq j < \omega_1$ let $Y(i, j) = \{b \in A(i)\colon b$ is cofinal and green, and $b^{[j]}$ is the lowest sequence node above b in $A\}$. Evidently, the $Y(i, j)$ are pairwise disjoint, and for each $i < \omega_1$ the set of cofinal green nodes in $A(i)$ is exactly $\bigcup\{Y(i, j)\colon i \leq j < \omega_1\}$.

Let $i \in \partial\omega_1$; we evaluate $|Y(i, j)|$ for each j by referring to the construction. Firstly, $Y(i, i)$ is the set of cofinal sequence nodes in $A(i)$, so $|Y(i, i)|$ is 0 if $i \in G$, and $\aleph_0$ if $i \notin G$. Secondly, let $j > i$. If $Y(i, j)$ is non-empty then let $b \in Y(i, j)$. Since the sequence node $b^{[j]}$ is not cofinal, we must have $j \in G$, and as b is cofinal, we have $i = \gamma(j, n)$ for some $n < \omega$. But if $i = \gamma(j, n)$ for some j, n, then $|Y(i, j)| \geq \aleph_0$, as $Y(i, j)$ gets $\aleph_0$ elements for each sequence node $\xi \in A \upharpoonright i \setminus \bigcup_{m<n} A \upharpoonright \gamma(j, m)$. Since the map $b \mapsto b^{[j]}$ is injective, it follows from property (κ) that $Y(i, j)$ is at most countable. So $|Y(i, j)| = \aleph_0$ if $i = \gamma(j, n)$ for some $j \in G$ and $n < \omega$, and $|Y(i, j)| = 0$ otherwise.

Totting up, we see that the number of cofinal green nodes in $A(i)$ is $|\gamma^{-1}(i)|.\aleph_0$ for $\bigcup_{j>i} Y(i, j)$, plus an extra $\aleph_0$ for $Y(i, i)$ if $i \notin G$. This proves (v) and completes the proof of Theorem 2.1.

We will use (v) to show that if Γ and Γ' are sufficiently different grids then there is no conformal relation defined on any club between $A(\Gamma)$ and $A(\Gamma')$.

Suppose that C is a club and Φ: $A(\Gamma) \restriction C \to A(\Gamma') \restriction C$ is a conformal relation. We would hope that if $i \in C$ then the ith levels of $A(\Gamma)$ and $A(\Gamma')$ are 'similar'. For comparison we want to use the cofinal green nodes, because we can control them using (v) of Theorem 2.1. Suppose that $b \in A(\Gamma')$ is a cofinal green node of height i. As Φ is surjective on green nodes, $A(\Gamma)$ will contain a green node a of height i with $a \Phi b$, but as Φ is a relation, it does not follow that a is cofinal. However, we can show that if $i \in \partial C$ and $A(\Gamma')$ contains *uncountably many* cofinal green nodes b of height i, then $A(\Gamma)$ contains *at least one* cofinal green node a of height i with $a \Phi b$ for some such b. To do this we use our second device, the nest.

DEFINITION. Let T be a tree. A *nest* in T is a set N of green nodes of T such that

$N \subseteq T(i)$ for some $i < \text{ht}(T)$,

N is uncountable,

$\{t \in T\colon \exists n \in N(n > t)\}$ is countable.

Note that if T is a spruce tree, C is a club in ω_1 and $i \in \partial C$, then a set $N \subseteq T(i)$ is a nest in T if and only if N is a nest in $T \restriction C$.

LEMMA 2.2 (preservation lemma). *Let T, U be spruce trees, let C be a club in ω_1 and suppose that Φ: $T \restriction C \to U \restriction C$ is a conformal relation. Let $i \in \partial C$ be such that there is a nest $N \subseteq U(i)$. Then $T(i)$ contains a cofinal green node m with $m \Phi n$ for some $n \in N$.*

Proof. The argument is similar to that of Theorem 1.7. Let $N \subseteq U(i)$ be a nest. Since there are only countably many nodes in U lying below the elements of N, the set $N^* = \{n \in N\colon \forall n' < n \ (n \neq n'^{[\Phi, i]})\}$ is also a nest. Take $n \in N^*$. By surjectivity for green nodes, there is a green node $m \in T(i)$ with $m \Phi n$. Suppose for a contradiction that m is not cofinal in T. Thus $\zeta(m) < m$. As $i \in \partial C$, we may choose $m' \in T \restriction C$ such that $\zeta(m) < m' < m$. Then also $\zeta(m') = \zeta(m) < m'$, and $m'^{[i]} = m$.

As Φ is homomorphic, we have $m' \Phi n'$ for some $n' \in U(\text{ht}_T(m'))$ with $n' < n$. By continuity of Φ we must have $n = n'^{[\Phi, i]} \notin N^*$. This is a contradiction, which proves the lemma.

The relationship of cofinal green nodes to nests is given by the following proposition.

PROPOSITION 2.3. *Let T be a spruce tree. Let $i < \omega_1$ and suppose that $N \subseteq T(i)$. Then N contains a nest in T if and only if*

(a) *i is a limit ordinal, and*

(b) *there are uncountably many cofinal green nodes in N.*

Proof. To prove (a) and (b) we can assume that N is already a nest. Since T is spruce, every node of T has countably many immediate successors. It follows that (a) holds. If (b) fails, then there are uncountably many nodes $n \in N$ such that $\zeta(n) < n$ (cf. Proposition 1.2). By Proposition 1.2, all the $\zeta(n)$ are distinct, so there must be uncountably many nodes lying below nodes in N, which contradicts the assumption that N is a nest. Hence (b) holds too.

Conversely, if (a) and (b) hold then take an uncountable set N' of cofinal green nodes in N. As $T \restriction i$ contains only countably many branching nodes, it is easily seen that N' is a nest.

We now relate this to our construction.

Definition. A grid $\Gamma = (G, \gamma)$ is said to be *fine* if

(i) $\gamma^{-1}(i)$ is uncountable for all $i \in \mathrm{im}(\gamma)$,

(ii) $\mathrm{im}(\gamma) = (\partial\omega_1)\backslash G$.

It is easy to see that if (G, γ) is fine then G must be uncountable, and for any uncountable $G \subseteq \partial\partial\omega_1$ we can find a γ such that (G, γ) is a fine grid.

We will usually work with fine grids from now on.

Corollary 2.4. *Let $A = A(\Gamma)$, $A' = A(\Gamma')$ be spruce trees, where $\Gamma = (G, \gamma)$, $\Gamma' = (G', \gamma')$ are fine grids. Suppose that C is a club in ω_1 and Φ: $A \restriction C \to A' \restriction C$ is a conformal relation. Then $G \cap \partial C \subseteq G' \cap \partial C$.*

Proof. Pick $i \in G \cap \partial C$; we show that $i \in G'$. As Γ is fine, we have $i \notin \mathrm{im}(\gamma)$. So by (v) of Theorem 2.1, there are no cofinal green nodes in $A(i)$. Hence by the preservation lemma, there is no nest in $A'(i)$. By Proposition 2.3, it follows that there are at most countably many cofinal green nodes in $A'(i)$. If $i \notin G'$ then as Γ' is fine, $i \in \mathrm{im}(\gamma')$ and so $\gamma'^{-1}(i)$ is uncountable. Hence by (v) of Theorem 2.1, there are uncountably many cofinal green nodes in $A'(i)$, a contradiction. Hence $i \in G'$ (and in fact there are no cofinal green nodes in $A'(i)$).

Recall, for example, from [**10**], that a *stationary* subset of ω_1 is a set that has non-empty intersection with every club in ω_1. We quote:

Fact 2.5 [**10**, Theorem 85]. There exist $\aleph_1$ pairwise disjoint stationary subsets of ω_1.

This is usually attributed 'essentially' to Ulam, since the easiest proof uses an Ulam matrix. The theorem was later strengthened by Solovay. Clearly the intersection of a club with a stationary set is stationary. Hence we can find pairwise disjoint stationary subsets S^k ($k < \omega_1$) of $\partial\partial\omega_1$.

Now it is easy to find subsets $X^i \subseteq \omega_1$ ($i < 2^{\omega_1}$) such that if $i \neq j$ then $X^i \backslash X^j$ is non-empty. Define $G^i = \bigcup\{S^k\colon k \in X^i\}$ for each $i < 2^{\omega_1}$. We see that each $G^i \subseteq \partial\partial\omega_1$, and if $i \neq j$ then $G^i \backslash G^j$ is stationary. For each G^i choose γ^i such that $\Gamma^i = (G^i, \gamma^i)$ is a fine grid, and set A^i to be $A(\Gamma^i)$. Write LA^i for $L(A^i)$.

Proposition 2.6. *If $i, j < 2^{\omega_1}$ and $i \neq j$ then there is no conformal relation Φ: $A^i \restriction C \to A^j \restriction C$ for any club C in ω_1.*

Proof. Suppose that C is a club in ω_1 and Φ: $A^i \restriction C \to A^j \restriction C$ is a conformal relation. By Corollary 2.4, $G^i \cap \partial C \subseteq G^j$. But $G^i \backslash G^j$ is stationary and ∂C is a club. This is a contradiction.

THEOREM 2.7. *Suppose that for each* $i<2^{\omega_1}$ *we have an SFP system* $\langle LA^i, \rho^i, v^i\rangle$ *with each* $\rho^i(a)$ *a countable ring, and with limit ring* R^i. *Suppose that* $i, j<2^{\omega_1}$ *are distinct. Then there is no ring embedding* $\theta\colon R^i\to R^j$. *Hence the rings* R^i $(i<2^{\omega_1})$ *are pairwise non-embeddable.*

Proof. By the results of § II.1, such a θ would give rise to a conformal relation $\Theta\colon A^j\restriction C\to A^i\restriction C$ for some club $C\subseteq\omega_1$. This contradicts Proposition 2.6.

REMARK 2.8. If $i<2^{\omega_1}$ let $G''=(\bigcup_{j<2^{\omega_1}}G^j)\backslash G^i$. Evidently we can weaken part (ii) of the definition of *fine* to $\mathrm{im}(\gamma^i)=G''$. See Remark 2.11 below.

Rigidity

It is clear from the proof of Theorem 2.1 that it is not essential to use the same grid throughout the construction, or indeed even within a single level. We will now modify the construction accordingly, to produce a spruce tree A such that if $C\subseteq\partial\omega_1$ is a club and $\Phi\colon A\restriction C\to A\restriction C$ is a conformal relation then $a\,\Phi\,a$ for all green $a\in A\restriction C$. (That is, A is 'conformally rigid'—but note that there may also be $b\neq a$ with $a\,\Phi\,b$ or $b\,\Phi\,a$.) This is enough to produce rigid rings, as we will see in § 5.

First take pairwise disjoint stationary sets $S_{in}\subseteq\partial\partial\omega_1$ (for $i<\omega_1$, $n<\omega$). By deleting elements, we can assume that $\min(S_{in})>i$ for all i, n. Define $G=\bigcup\{S_{in}\colon i<\omega_1, n<\omega\}$. Now A is built by induction on levels. We assign the set $S_{\langle\rangle}=\bigcup_{n<\omega}S_{0n}$ to the empty sequence, $\langle\rangle$. As each higher sequence node ξ is introduced, we assign a new set S_{in} to ξ, where $i=\mathrm{ht}(\xi)$. This is possible as $A(i)$ contains only countably many sequence nodes. We can then write this S_{in} as S_ξ. We can arrange to use all the sets S_{in} in this way.

When S_ξ has been defined, we also choose two grids $V_\xi=(S_\xi, v_\xi)$ and $W_\xi=(S_\xi, w_\xi)$. We require that

$$\mathrm{im}(w_\xi)\cap\partial\partial\omega_1=\varnothing,$$

$$\mathrm{im}(v_\xi)\cap\partial\partial\omega_1=S_\xi,$$

$$v_\xi^{-1}(i)\text{ is uncountable for all } i\in S_\xi.$$

These conditions are easy to arrange.

The construction of A at non-limit levels is as in Theorem 2.1. We build the limit level j of A as follows. If $j\notin G$, we apply 'Case I' of Theorem 2.1; this is the classical Aronszajn case. Suppose instead that $j\in G$. Then $j\in S_{in}$ for some $i<j$ and $n<\omega$. Hence we have already defined S_{in} to be S_ξ for some unique sequence node $\xi\in A(i)$. For each sequence node $\eta\in A\restriction j$ and rational $q>\sup(\eta)$, we want to include a sequence node η' in $A(j)$ with $\eta'>\eta$ and $\sup(\eta')<q$. We apply Case II of Theorem 2.1, but using the grid V_ξ if $\eta\geqslant\xi$ and W_ξ otherwise.

Let A be the result of the construction. We have:

LEMMA 2.9. *Let* $i\in S_\xi$ *for some sequence node* $\xi\in A$. *Then:*

(i) *there is a nest in* $A(i)$ *above* ξ,

(ii) *if* $a\in A(i)$ *is a cofinal green node then* $a>\xi$.

Proof. (i) As $i\in S_\xi$, we have $i\in\mathrm{im}(v_\xi)$. Hence there are an uncountable set $I\subseteq S_\xi$ and $n<\omega$ such that $v_\xi(j, n)=i$ for all $j\in I$. Let $j\in I$. Choose a sequence node $\eta \in A \restriction i$ with $\eta \geqslant \xi$ and $\mathrm{ht}(\eta) \geqslant v_\xi(j, m)$ for all $m < n$. By construction, there is a cofinal branch β of $A \restriction i$ with $\eta \in \beta$, such that the sequence node

$$\bigcup \beta := \bigcup \{\eta' \colon \ \eta' \text{ a sequence node, } \eta' \in \beta\}$$

is in $A(j)$ (and above β). Let a_j be the unique node in $A(i)$ lying below $\bigcup \beta$. Then a_j is cofinal and green and $a_j > \xi$. Since, moreover, every node $a < \bigcup \beta$ of height at least i is a blank node, the a_j $(j \in I)$ are all distinct. Hence by Proposition 2.3, $\{a_j \colon j \in I\}$ is a nest above ξ in $A(i)$.

(ii) Let $a \in A(i)$ be a cofinal green node. Since $i \in G$, Case II was used to construct $A(i)$, and in doing this, no cofinal green nodes were introduced. So a must have been made green at some later stage. That is, there are a sequence node $\eta \in A$, $\gamma \in \{v_\eta, w_\eta\}$, $j \in S_\eta$ and $n < \omega$, such that $\gamma(j, n) = i$ and the sequence node $\bigcup \hat{a}$ was put into $A(j)$ (above a).

Now $i \in S_\xi \subseteq \partial\partial\omega_1$. As $\mathrm{im}(w_\eta) \cap \partial\partial\omega_1 = \varnothing$, we have $\gamma = v_\eta$. Hence by construction, $a > \eta$. But now $i \in \partial\partial\omega_1 \cap \mathrm{im}(v_\eta) = S_\eta$. As the $S_{\eta'}$ are pairwise disjoint, we have $\eta = \xi$. So $\xi < a$, as required.

Corollary 2.10. *Let $\Phi \colon A \restriction C \to A \restriction C$ be a conformal relation, for some club $C \subseteq \partial\omega_1$. Let $t \in A \restriction C$ be a green node. Then $t \Phi t$.*

Proof. Choose a sequence node $\xi \geqslant t$ in A and $i \in S_\xi \cap \partial C$. By Lemma 2.9(i), there is a nest N above ξ in $A(i)$. By Lemma 2.2, there is a cofinal green node $m \in A(i)$ with $m \Phi n$ for some $n \in N$. Hence $m > \xi \geqslant t$ by Lemma 2.9(ii). Because Φ is homomorphic, we obtain $t \Phi t$, as required.

Remark 2.11. Using Remark 2.8, we can combine Corollary 2.10 with Proposition 2.6 to produce $2^{\aleph_1}$ 'conformally different' rigid trees. The method is standard and we will not describe it further.

3. *Countable generation of full ideals*

In the last three sections we study in more detail the limit rings of SFP systems built on the SFP domains LA, for A as in § 2. Already by Proposition I.5.3, if each map v of the system is an isomorphism then each of the locally generated ideals of the limit is densely decomposable. In § 4 below we will show that the limit rings can all be made $L_{\infty\omega}$-equivalent, and in § 5 we build in some rigidity (the rings will have few endomorphisms). In the present section we show that every full ideal can be made countably generated.

Let A be a spruce tree as built in Theorem 2.1, and let $\langle LA, \rho, v\rangle$ be an SFP system such that $\rho(a)$ is countable for all $a \in LA$ (or equivalently for all $a \in B(A)$). Let the SFP system have limit ring R, and let R_i be the limit of the system restricted to LA_i (all $i < \omega_1$). By Proposition I.1.5, R is uncountable, of cardinality $\aleph_1$. None-the-less we will now use Corollary I.2.5(ii) to show that every full ideal (either left or two-sided) of R has a countable set of generators.

First we need a technical lemma on the tree structure of $L(A)$.

Lemma 3.1. *Let T be a spruce tree. Then every node of $L(T)$ has either* 0 *or $\aleph_0$ immediate successors in $L(T)$.*

Proof. Assume that $t \in L(T)$ is not terminal in $L(T)$. If $t \in B(T)$ then clearly t has at least $\aleph_0$ immediate successors in $L(T)$. Assume that $t \in L(T) \backslash B(T)$. There is $b \in L(T)$ with $b > t$. We can assume that $i = \mathrm{ht}_T(b)$ is least possible, so i is a successor ordinal, $b \in B(T)$ and b has an immediate predecessor b^- in T. It is possible that $b^- > t$; clearly, if $b^- > t$ then $b^- \in (T \upharpoonright \partial\omega_1) \backslash L(T)$.

Now b^- is the immediate predecessor of a branching node in T. So by (ii) and (iv) of the definition of 'spruce', b^- is itself branching and has $\aleph_0$ branching immediate successors c in T. By choice of i, each such c is an immediate successor of t in $L(T)$. Hence t has at least $\aleph_0$ immediate successors in $L(T)$.

It remains to prove that no $t \in L(T)$ has uncountably many immediate successors in $L(T)$. Assume for a contradiction that $t \in L(T)$ is a counterexample. Let $\mathrm{ht}_T(t) = i$. As T is spruce, there are arbitrarily large $j < \omega_1$ with $j > i$ such that there is an $L(T)$-immediate successor b of t of height j in T. Clearly j is a successor ordinal; let $j = j^- + 1$ and let $b^- < b$ have height j^- in T. There is no $x \in L(T)$ with $t < x \leqslant b^-$. Hence $b^-/L(T) = t$. It follows that $b^- = t^{[j^-]}$ (cf. the proof of Proposition 1.3). As above, b^- is a branching node of T. As this holds for arbitrarily large j^-, it follows that the branch of T determined by $\{t^{[j]}\colon\ i < j < \omega_1\}$ is cofinal in T. This contradicts the spruceness of T.

We now obtain:

THEOREM 3.2. *Let A, R, R_i be as above. Let J be a full left ideal of R. Then J is countably generated.*

REMARK. Recall that by Proposition I.3.2 the full ideals include the maximal, prime and irreducible left ideals and also the maximal two-sided ideals of R.

Proof. Suppose that $J = I @ a$ in R, for some $a \in LA$ and some ideal $I \subseteq R_a$. There is $i < \omega_1$ such that $a \in LA_i$. By Lemma 3.1, we can choose i so that all of the immediate successors of a in LA (if any) are already in LA_i. It follows that:

$$LA_i \preccurlyeq LA, \quad \{a' \in LA\colon\ a'/LA_i = a\} = \{a\}.$$

By Theorem I.2.2, $J \cap R_i = I @ a$ in R_i. By Corollary I.2.5(ii), $J \cap R_i$ generates J in R. The result follows since by Proposition I.1.5, R_i is countable.

4. $L_{\infty\omega}$-*equivalence*

Here we prove that if two grids Γ^1, Γ^2 are 'sparse' enough then $\mathrm{BA}(\Gamma^1)$ and $\mathrm{BA}(\Gamma^2)$ are $L_{\infty\omega}$-equivalent trees. Since the rings of Theorem 2.7 are the limits of SFP systems of the form $\langle LA(\Gamma), \rho, v\rangle$, this will prove that in the case of Boolean powers (cf. Remark I.4.3) say, the rings R_i ($i < 2^{\omega_1}$) of Theorem 2.7 are all $L_{\infty\omega}$-equivalent.

Formally, Corollary I.6.3 combined with Theorem 4.5 below will give the following:

THEOREM 4.1. *Let Γ^1, Γ^2 be fine, sparse grids, and let R_1, R_2 be $L_{\infty\omega}$-equivalent rings. Then the SFP powers $R_1\langle LA(\Gamma^1)\rangle$ and $R_2\langle LA(\Gamma^2)\rangle$ are $L_{\infty\omega}$-equivalent.*

We mentioned $L_{\infty\omega}$-equivalence in § I.6. There is another characterisation of $L_{\infty\omega}$-equivalence in terms of back-and-forth systems. A *back-and-forth system* between structures M and N is a set Θ of partial isomorphisms from M to N such that:

$\Theta \neq \varnothing$,
if $\theta \in \Theta$ and $a \in M$ then there is $b \in N$ such that $\theta \cup \{(a, b)\} \in \Theta$,
if $\theta \in \Theta$ and $b \in N$ then there is $a \in M$ such that $\theta \cup \{(a, b)\} \in \Theta$.

FACT 4.2 (Karp's theorem [**12**]). The structures M and N are $L_{\infty\omega}$-equivalent if and only if there is a back-and-forth system Θ between M and N.

We will show that $LA(\Gamma^1)$ and $LA(\Gamma^2)$ are $L_{\infty\omega}$-equivalent for sparse Γ^i, by finding a back-and-forth system between them. It will then follow that $BA(\Gamma^1) \equiv_{\infty\omega} BA(\Gamma^2)$. Though LA is definable in A by a first-order formula, $A(\Gamma^1)$ and $A(\Gamma^2)$ will not in general be $L_{\infty\omega}$-equivalent. (If they were, then for all $i < \omega_1$, if $A(\Gamma^1)(i)$ contained a cofinal green node then so would $A(\Gamma^2)(i)$. Hence the $A(\Gamma^i)$ of Theorem 2.7 are not $L_{\infty\omega}$-equivalent.) So we must work directly with the LA, remembering that if $t \in LA$ then maybe $\mathrm{ht}_{LA}(t) < \mathrm{ht}_A(t)$, and t may be a branching node of LA without being branching in A (though it will be green in A).

DEFINITION. An uncountable set $C \subseteq \omega_1$ is said to be *sparse* if

$$\min\{j \in C\colon\ j > i\} > i + i\ (=i.2)$$

for each $i \in C$. A grid (G, γ) is said to be *sparse* if G is sparse.

Clearly any uncountable subset of a sparse set is also sparse. If we define an ω_1-sequence z_j $(j < \omega_1)$ inductively by $z_0 = \omega$, $z_{j+1} = z_j.2 + 1$, and $z_\delta = \bigcup\{z_j\colon\ j < \delta\}$ for limit $\delta < \omega_1$, then $Z = \{z_j\colon\ j < \omega_1\}$ is a sparse club. Hence if $S \subseteq \omega_1$ is stationary then $S \cap Z$ is sparse and stationary. It follows that in Fact 2.5 we can assume that the S^i are subsets of Z, so that the G^i defined prior to Theorem 2.7 are sparse.

Sparseness is used in the following lemma.

LEMMA 4.3. *Let G be a sparse subset of $\partial\partial\omega_1$, and let $\Gamma = (G, \gamma)$ be a fine grid. Write A for $A(\Gamma)$. Let b be a branching node of LA with* $\mathrm{ht}_{LA}(b) = i$. *Then for all ordinals j with $i < j < \omega_1$ and all $q \in \mathbb{Q}$, there is a branching node c of LA with $c > b$,* $\mathrm{ht}_{LA}(c) = j$ *and* $\sup\{\sup(\eta)\colon$ *η a sequence node, $\eta \leqslant c\} > q$.*

Proof. Since b is a branching node of LA, it is green in A. Let η be the lowest sequence node in A with $\eta \geqslant b$. By construction, we may take a sequence node $\eta' \in A$ such that η' is an immediate successor of η in A and $\sup(\eta') > q$. Then $\eta' \in LA$ and $\mathrm{ht}_{LA}(\eta') = i + 1$. This proves the lemma in the case where $j = i + 1$.

Assume that $j > i + 1$. With η' as above, any sequence node greater than η' already has supremum greater than q. So if we replace b by η', it is enough to find a sequence node c above b and of height j in LA.

Let g be the least element of G such that $g > j$ and $g > \mathrm{ht}_A(b)$. By (α) of Theorem 2.1, we can find a sequence node $\xi \in A(g)$ with $\xi > b$. Of course, $g \in G$, so $\hat{\xi}$ is not cofinal in $A \restriction g$. Consequently, $\xi \notin LA$.

Let $\xi_0 = \xi/LA$ and suppose that $\mathrm{ht}_{LA}(\xi_0) = k_0$. There are two cases. If $j \leqslant k_0$ then choose $\xi' \in LA$ so that $\xi' < \xi$ and $\mathrm{ht}_{LA}(\xi') = j$. Any branching immediate successor of ξ is in LA, so ξ' is not a terminal node of LA. Hence by Lemma 3.1, it is a branching node in LA, and we have finished.

If, on the other hand, $j > k_0$, then let k be such that $k_0 + k = j$, and set $j' = g + k$. Using (α) of Theorem 2.1, choose a sequence node $\xi' \in A(j')$ with $\xi' > \xi$. Let $\Xi = \{\xi_0\} \cup \{t \in A\colon \xi < t < \xi'\}$. Now $k \leqslant j < g$. As G is sparse, there is no $g' \in G$ with $g < g' \leqslant j'$. By construction, it follows that every node $t \in \Xi$ is a sequence node, so that $\Xi \subseteq LA$. Clearly Ξ has order type k. Since $\hat{\xi}' \cap LA = (\hat{\xi}_0 \cap LA) \cup \Xi$, we have $\mathrm{ht}_{LA}(\xi') = k + k' = j$, and we can take c to be ξ'.

Corollary 4.4. *Under the assumptions of Lemma* 4.3, *for all limits ordinals* j *with* $i < j < \omega_1$, *there is a terminal node* t *of* LA *with* $t > b$ *and* $\mathrm{ht}_{LA}(t) = j$.

Proof. Take a strictly increasing sequence of successor ordinals j_n $(n < \omega)$ with $j_0 > i$ and $\bigcup_{n<\omega} j_n = j$. As each j_n is a successor ordinal, we may use Lemma 4.3 to define sequence nodes $\xi_n \in LA$ by induction, with $\mathrm{ht}_{LA}(\xi_n) = j_n$, $\xi_0 > b$, $\xi_{n+1} > \xi_n$ and $\sup(\xi_n) > n$ (for all $n < \omega$). Let $t = \mathrm{lub}_{LA}\{\xi_n\colon n < \omega\}$. Then $\mathrm{ht}_{LA}(t) = j$. Further, $\sup\{\sup(\eta)\colon \eta$ a sequence node, $\eta < t\} = \infty$. Hence there can be no sequence node $\eta \geqslant t$ in A, so t must be a terminal node of LA.

For the rest of this section let G^1, G^2 be sparse (stationary) subsets of ω_1, and let $\Gamma^l = (G^l, \gamma^l)$ be fine grids $(l = 1, 2)$. Consider the trees $A^1 = A(\Gamma^1)$, $A^2 = A(\Gamma^2)$ constructed in Theorem 2.1. We will prove:

Theorem 4.5. (i) *The SFP domains* LA^1 *and* LA^2 *are* $L_{\infty\omega}$*-equivalent posets in the signature* $L = \{=, <\}$.

(ii) *The posets* BA^1 *and* BA^2 *are also* $L_{\infty\omega}$*-equivalent.*

Part (i) of Theorem 4.5 will follow immediately from Lemma 4.6 below. Part (ii) follows from part (i) here, since there is a first-order L-formula $\varphi(x)$ such that for any A as in Theorem 2.1, $\{a \in LA\colon LA \models \varphi(a)\} = BA$. The formula φ simply says that x does not have limit height in LA. Part (ii) is what is required for $L_{\infty\omega}$-equivalence of the limit rings.

We begin the proof of the theorem with a definition.

Definition. Let T be any tree. If $U \subseteq T$, we say that U is a *full subtree* of T if U is non-empty and closed downwards in T. If $S \subseteq T$ is non-empty, we write $\langle S \rangle$ for the full subtree $\{t \in T\colon t \leqslant s$ for some $s \in S\}$ of T generated by S. Now T is said to be *finitely generated* if $T = \langle S \rangle$ for some finite $S \subseteq T$. Note that no branch of a finitely generated tree can have limit height.

If U^l is a full subtree of T^l $(l = 1, 2)$, a map $\theta\colon U^1 \to U^2$ is said to be a *strong isomorphism* if θ is bijective and preserves '$<$', and each $u \in U^1$ is a branching node of T^1 if and only if $\theta(u)$ is a branching node of T^2.

For example, if we write $\perp^1, \perp^2$ for the unique least element of T^1 and T^2, then $\{\perp^l\}$ is a finitely generated full subtree of each T^l, and the map $(\perp^1 \mapsto \perp^2)$ is a strong isomorphism. Hence the set Θ of strong isomorphisms between finitely generated full subtrees of LA^1, LA^2 is non-empty. The next lemma shows that Θ is a back-and-forth system between LA^1 and LA^2, and so proves Theorem 4.5.

LEMMA 4.6. *Let T^l be finitely generated full subtrees of LA^l ($l=1,2$), and suppose that θ: $T^1 \to T^2$ is a strong isomorphism from T^1 to T^2.*

(i) *Let $t^1 \in LA^1$. Then there is $t^2 \in LA^2$ such that $\theta \cup \{(t^1, t^2)\}$ extends to a strong isomorphism from $\langle T^1 \cup \{t^1\}\rangle$ to $\langle T^2 \cup \{t^2\}\rangle$.*

(ii) *Exchanging the indices* '1' *and* '2' *in* (i), *we have a similar result.*

Proof. We will only prove (i), as the proof of (ii) is similar. So let T^1, T^2, θ be given, and let $t^1 \in LA^1$. We can assume that $t^1 \notin T^1$; the result is trivial otherwise. Now T^1 has no branches of limit height. So there is a unique largest node $v \in T^1$ with $v < t^1$. We have $T^1 \trianglelefteq LA^1$ and $v = t^1/T^1$ in the notation of § I.1. Let $\theta(v) = w \in T^2$ and let $\mathrm{ht}_{LA^1}(t^1) = h < \omega_1$. It suffices to prove the following:

Claim. There is $t^2 \in LA^2$ with $t^2/T^2 = w$, $\mathrm{ht}_{LA^2}(t^2) = h$, and such that t^1 is a terminal node of LA^1 if and only if t^2 is a terminal node of LA^2.

Given the claim, we can finish as follows. Let U^l be the full subtree of LA^l generated by $T^l \cup \{t^l\}$ ($l=1,2$). Since $\mathrm{ht}(t^2) = h$, we can extend θ canonically to an order-preserving bijection θ': $U^1 \to U^2$. Since by Lemma 3.1 every node of each LA^l is either branching or terminal, θ' will be a strong isomorphism.

Proof of Claim. Since $t^1 \notin T^1$, we see that v is not terminal, so v is a branching node of LA^1. As θ is strong, w is also branching in LA^2, with infinitely many immediate successors. As T^2 is finitely generated, we can take an immediate successor w' of w in LA^2 with $w' \notin T^2$. It suffices to find $t^2 \geq w'$ in LA^2 with the required properties.

If t^1 is a branching node of LA^1 then by Lemma 4.3 there is a branching node t^2 of LA^2 above w' and of height h in LA^2. If t^1 is terminal in LA^1 then h must be a limit ordinal, so we can use Corollary 4.4 to choose a terminal node $t^2 \in LA^2$ above w' of height h. This proves the claim and with it the lemma.

Theorem 4.5 is proved.

5. *Rigidity*

By imposing restrictions on the homomorphisms v_{pq} in the SFP systems and using Corollary 2.10, we can make the $2^{\aleph_1}$ limit rings of Theorem 2.7 somewhat rigid. To conclude our survey we will prove a sample result for Boolean rings. We will define an SFP system with Boolean limit ring R having no non-trivial injective endomorphisms. The example will also illustrate the use of SFP systems in which not all v_{pq} are isomorphisms.

Further cases are discussed in [**8**]. For example, we may set up an SFP system with limit ring R so that any injective endomorphism θ: $R \to R$ satisfies $\theta^{-1}(I) = I$ for every maximal two-sided ideal I of R. (In the case where R is Boolean, this implies that $\theta = \mathrm{id}_R$.)

Take any countable Boolean ring S, and fix a countable set Ξ of maximal ideals of S such that any proper finitely generated ideal I of S is contained in some $K \in \Xi$.

Suppose that A is a 'conformally rigid' tree as considered in Lemma 2.9. We build an SFP system $\sigma = \langle LA, \rho, v\rangle$ on LA as follows. First we partition $\mathbb{Q}$ into sets $\mathbb{Q}_K$ ($K \in \Xi$) such that each $\mathbb{Q}_K$ is dense in $\mathbb{Q}$. For each sequence node $\eta \in BA$ we define R_η to be S. For each $q \in \mathbb{Q}$ with $q > \sup(\eta)$ we define $v_{\eta, \eta^\wedge q}$ so that $\ker(v_{\eta,\eta^\wedge q}) = K$ where $q \in \mathbb{Q}_K$. Since $S/K \cong \mathbb{Z}_2$, this defines $v_{\eta,\eta^\wedge q}$ completely.

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Hence if $a \in LA \backslash BA$ we will have

$$R_a = \lim_{\rightarrow} (R_\eta, \nu_{\eta,\eta'}: \eta < \eta' \text{ in } BA \cap \hat{a}) = \mathbb{Z}_2,$$

and if $b > a$ then ν_{ab} must be the unique embedding of $\mathbb{Z}_2$ into R_b. We have now defined σ completely.

Let $R = \lim(\sigma)$. Then R is an uncountable Boolean ring. If $i < \omega_1$, write R_i for $\lim\langle LA_i, \rho', \nu'\rangle$ as usual, where ρ' and ν' are the appropriate restrictions. Let θ: $R \to R$ be a ring embedding. As in Proposition 1.4 we can find a club $C \subseteq \partial\omega_1$ so that θ induces a conformal relation Θ: $A \restriction C \to A \restriction C$.

We claim that $\theta = \mathrm{id}_R$. Suppose for a contradiction that there is $r \in R$ such that $\theta(r) = r' \neq r$. Take $i \in C$ such that $r, r' \in R_i$. As R_i is Boolean, it is easily seen that at least one of the sets $\{r, 1-r'\}$, $\{1-r, r'\}$ generates a proper ideal of R_i containing just one of r, r'. Assume without loss of generality that $\{r, 1-r'\}$ has this property. Take a finite support $N \preccurlyeq BA_i$ for $\{r, 1-r'\}$. The ideal of R_N generated by $\{r, 1-r'\}$ is proper, so there are $\eta \in N$ and $K \in \Xi$ with $r(\eta)$, $1-r'(\eta) \in K$. (Note that η is a finite element of LA and hence a sequence node in A.) Choose $q \in \mathbb{Q}_K$ with $q > \sup(\eta)$ and $\eta^\wedge q \notin N$, and then choose a green node (say a sequence node) $a \in A(i)$ such that $a > \eta^\wedge q$. So $a/N = \eta$. Write z for $\zeta a \in LA$. Then

$$r(z) = \nu_{\eta z}(r(\eta)) = \nu_{\eta^\wedge q, z} \cdot \nu_{\eta, \eta^\wedge q}(r(\eta)) = 0.$$

Similarly, $(1-r')(z) = 0$.

Now by Corollary 2.10 we have $a \Theta a$. Hence by definition of Θ there are proper ideals I, J of R_z such that in R_i we have $\theta^{-1}(I @ z) = J @ z$. So $\theta(J @ z) \subseteq I @ z$.

Now $r \in J @ z$ in R_i, so $r' \in I @ z$. But $r'(z) = 1$, which contradicts the assumption that I is proper. Hence $\theta = \mathrm{id}_R$ as claimed.

6. *Proof of the 'sample theorem' of the preface*

Let R, S, Ξ be as above. We have shown that any injective endomorphism of R is the identity map. Furthermore, by Theorem 3.2 every maximal ideal of R is countably generated. By Proposition I.1.5 and Corollary I.1.4, $|R| = \aleph_1$ and S embeds into R. We can make R atomless by taking S to be the countable atomless Boolean ring (there is no loss of generality as this ring embeds the original S). Recall that R is atomless if and only if it is existentially closed [**7**, 6.3.9, Ex. 6.3.2]. Alternatively, if we include the zero ideal 0 in Ξ and require that $\nu_{\eta,\eta^\wedge q} = \mathrm{id}_S$ whenever $q \in \mathbb{Q}_0$, then by Remark I.5.4, R will be atomless. We can also combine the construction of A with the techniques of Theorem 2.1 and § 4 to produce $2^{\aleph_1}$ pairwise non-embeddable R. See also [**8**].

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