Sh:92

Algebra Universalis, **11** (1980) 77–89 Birkhäuser Verlag, Basel

Remarks on Boolean algebras

SAHARON SHELAH

§0. Introduction

We prove some theorems on Boolean algebras. In section 1, we prove that if S is a subset of a (Boolean algebra) B, $|S|$ singular strong limit, then $S^- = \{a - b : a, b \in S\}$ has subset of power $|S|$ which is a pie subset (a subset no two elements of which are comparable). This answers a question of Erdös.

In section 2 we prove the consistency of $ZFC + 2^{\aleph_0} > \aleph_1$ with "there is a Boolean algebra B of cardinality $2^{\aleph_0}$, with no uncountable pie subset, nor an uncountable chain." This answers a question of Rubin.

In section 3, we prove that if $|B| = \lambda^+$, $\lambda = \lambda^{<\kappa}$, B satisfies the κ-chain condition, then $B - \{0\}$ is the union of λ ultrafilters (proper, of course). We shall show somewhere else that this is not necessarily true when $|B| = \lambda^{++}$, and other connected results.

In section 4, we show that if B satisfies the κ-chain condition, $|B| > 2^{<\kappa}$ then B has large independent subsets e.g. of power $> 2^{<\kappa}$. This answers a question of Rubin.

The author is grateful to Kunen for a stimulating discussion of §3 and Rubin for asking him §1, §3, §4.

The referee has added the following information. Concerning §1, Archangel'ski [A] proved that if a space X is of pointwise countable type (in particular if X is compact), then $wx \leq 2^{sX}$, where wX is the weight of X and sX is the spread of X. It follows that if B is a Boolean algebra with $|B|$ singular strong limit, then B has a pie subset of power $|B|$. The result of section 1 improves this result. The same conclusion holds if $\mathrm{cf}\,|B| = \aleph_0$ ($|B|$ not necessarily strong limit). The theorem of section 2 shows however, that the conclusion cannot be extended to arbitrary singular cardinal since $2^{\aleph_0}$ can be made singular.

The author would like to thank the NSF for partially supporting his research by grants 144-H747, MCS-76-08479.

The author is very grateful to the referee D. Monk, who rewrote a large part of the paper, and detected many errors.

Presented by J. D. Monk. Received March 13, 1978. Accepted for publication in final form November 16, 1978.

The author has shown under CH that there is a Boolean algebra of power $\aleph_1$ with no uncountable chains or pie subsets (earlier weaker results were obtained by Berney and Nyikos, Baumgartner, Komgath, and M. Rubin). Baumgartner has shown that it is consistent with ZFC that every uncountable Boolean algebra has an uncountable pie.

The results of section 4 improve results of B. S. Efimov, who has proved, for example, that if B satisfies the κ^+-chain condition and $|B|\geq(\exp\exp\exp\kappa)^+$ then B has an independent set of cardinality $(\exp\kappa)^+$; see e.g., Pierce [P].

1

The following answers positively a question of Erdös.

1.1. CLAIM. If λ is a strong limit singular cardinal, S a family of λ sets, then $S^-=\{A-B : A,\ B\in S\}$ contains a pie subfamily of cardinality λ.

Proof. Let $\lambda=\sum_{i<\kappa}\lambda_i$, $\kappa=\operatorname{cf}\lambda$, $\lambda_i<\lambda$ the λ_i's strictly increasing. By a theorem of Erdös, Hagnal and Rado (see [EHR]), for each n we can find distinct $A_{i,\alpha}\in S$ $(\lambda_i\leq\alpha<\lambda_{i+1},\ i<\kappa)$ such that for any Boolean term $\tau(x_i,\dots,x_n)$:

(i) the truth value of $\tau(A_{i_1,\alpha_1},\dots,A_{i_n,\alpha_n})=0$ depends only on $i_1,\dots,i_n$ and the order and equality relations between the $\alpha_i,\dots,\alpha_n$,
(ii) the truth value of $\tau(A_{i,\alpha_1},\dots,A_{i,\alpha_n})=0$ depends only on the order and equalities between $\alpha_1,\dots,\alpha_n$.

For (ii) note there are only finitely many terms τ.

We use such a family for $n=4$. W.l.o.g. S is a family of subsets of λ.

Let us prove the existence of such pie subfamily. As $A_{0,0}\neq A_{0,1}$, $A_{0,1}-A_{0,0}\neq\phi$ or $A_{0,0}-A_{0,1}\neq 0$, but if we replace each $A_{i,\alpha}$ by $\lambda-A_{i,\alpha}$ nothing changes except that those relations are inverted, so w.l.o.g. $A_{0,1}-A_{0,0}\neq\phi$. So by condition (ii) $\lambda_i\leq\alpha<\beta<\lambda_{i+1}$ implies $A_{i,\beta}-A_{i,\alpha}\neq\phi$. We shall show that $S^*=\{A_{i,2\alpha+1}-A_{i,2\alpha} : \lambda_i\leq\alpha<\lambda_{i+1},\ i<\kappa\}$ is as required. Trivially $S^*\subseteq S^-$ and $|S^*|=\lambda$, so we have to prove S^* is a pie. Let for some $i,j<\kappa$, $\lambda_i\leq\alpha<\lambda_{i+1}$, $\lambda_j\leq\beta<\lambda_{j+1}(i,\alpha)\neq(j,\beta)$ (hence $\alpha\neq\beta$) and

(a) $A_{i,2\alpha+1}-A_{i,2\alpha}\subseteq A_{j,2\beta+1}-A_{j,2\beta}$.

We shall get a contradiction.

By conditions (i) we can assume $2\alpha\neq 2\beta+2$. So by condition (i) also

(b) $A_{i,2\alpha+1}-A_{i,2\alpha}\subseteq A_{j,2\beta+2}-A_{j,2\beta+1}$

intersecting (a) and (b) we get

$$A_{i,2\alpha+1}-A_{i,2\alpha}\subseteq(A_{j,2\beta+1}-A_{j,2\beta})\cap(A_{j,2\beta+2}-A_{j,2\beta+1})$$
$$\subseteq A_{j,2\beta+1}\cap(\lambda-A_{j,2\beta+1})=\phi$$

contradicting an assumption.

2

The following answers a question of Rubin.

CLAIM. It is consistent with ZFC, that $2^{\aleph_0}>\aleph_1$, and there is a Boolean algebra of power continuum, which contains neither uncountable chains, nor uncountable pie. (In fact $2^{\aleph_0}$ may have any possible value.)

Remark. We can say more on the Boolean algebra we have, but we do not try.

Proof. Let $\aleph_\alpha^{\aleph_0}=\aleph_\alpha$, and let $\langle x_i : i<\aleph_\alpha\rangle$ be a system of distinct objects. We define a partial order P: its elements are pairs (w, B) where w is a finite subset of $\aleph_\alpha$ and

(1) B is a Boolean algebra generated by $\{x_i : i\in w\}$,
(2) $B-\{x_i : i\in w\}$ is disjoint from $\{x_i : i<\aleph_\alpha\}$,
(3) for each $i\in w$, x_i does not belong to the subalgebra of B generated by $\{x_j : j\neq i,\ j\in w\}$.

The order is defined naturally $(w_1, B_1)\leq(w_2, B_2)$ if $w_1\subseteq w_2$ and B_1 is a subalgebra of B_2 (or, in fact, embeddable into B_2 over w_1). Notice $(w, B)\in P$ implies B is finite. Let $(w, B)\restriction w_1$ be $(w\cap w_1, B')$ where B' is the Boolean subalgebra of B generated by $\{x_i : i\in w\cap w_1\}$, so $w(w, B)\restriction w_1\leq(w, B)$.

FACT 1. P satisfies the countable chain condition.

For let $(w_i, B_i)\in P$ $(i<\omega_1)$, so by a well known theorem w.l.o.g. for some finite w, B $i\neq j\Rightarrow w_i\cap w_j=w$, and $(w_i, B_i)\restriction w=(w, B)$. Now let B' be the free product of B_1, B_2 over B, and then $(w_1\cup w_2, B)\in P$, $(w_l, B_l)\leq(w_1\cup w_2, B')$ $(l=1,2)$ so (w_1, B_1), (w_2, B_2) are compatible.

FACT 2. $\{(w, B) : i\in w\}$ is dense in P for each $i<\aleph_\alpha$.

So we force with P and get a generic set G, a model $V[G]=V^P$, and a Boolean algebra $B^*=\bigcup\{B : (w, B)\in G\}\in V^P$ generated by $\{x_i : i<\aleph_\alpha\}$. As is well-known, cardinals are not collapsed in V^P, and the continuum is at most $\aleph_\alpha$. It is seen to be exactly $\aleph_\alpha$ because the sets

$$s_\lambda = \{n \in w : x_{\lambda+n} \wedge x_{\lambda+n+1} = 0 \text{ in } B^*\}$$

are distinct for distinct limit ordinals $\lambda < \aleph_\alpha$. The main properties of B^* will be established using the following reduction. Let S be a list of $\aleph_1$ distinct elements of B^* (in V^P). So in V it has a name **S**, and let $p \in P$, then for each i, there is $p_i \geq p$, $p_i \Vdash$ "$\tau_i(x_{\alpha(i,1)}, \ldots, x_{\alpha(i,n_i)}) = \tau^i$ is the i^{th} element of **S**" (τ_i-Boolean terms) w.l.o.g. $\tau_i = \tau$, $n_i = n$, and let $p_i = (w_i, B_i)$, so w.l.o.g. $\alpha(i, l) \in w_i$, and by Fact 1, w.l.o.g. for some w, $w_i \cap w_j = w$ for $i < j < \omega_1$, and $(w_i, B_i) \restriction w = B$, so w.l.o.g. $p = (w, B)$.

Let $w_i - w = \{\beta(i, l) : l < m_i\}$, so w.l.o.g. $m_i = m$ and the mapping f^i_j, $f^i_j \restriction \{x_i : i \in w\} = id$, $f^i_j(x_{\beta(j,l)}) = x_{\beta(i,l)}$ induce an isomorphism from B_j onto B_i taking τ^j to τ^i. Clearly $\tau^i \in B_i - B$.

For proving B^* has no uncountable chain, suppose S is a list of $\aleph_1$ comparable elements of B^*, and p forces this fact. Applying the above reduction, it suffices now to prove: there is q, $p_0, p_1 \leq q \in P$, $q \Vdash^P$ "τ^0, τ^1 are incomparable." For this let B' be the free product of B_0, B_1 over B, and $q = (w_0 \cup w_i, B')$ is as required.

For proving B^* has no uncountable pie, suppose S is a list of $\aleph_1$ incomparable elements of B^*, and p forces it. Applying the above reduction it now suffices to prove there is a q, $p_0, p_1 \leq q \in P$, $q \Vdash^P$ "$\tau^0 \leq \tau^1$." For this let B' be the Boolean algebra generated by $\{w_i : i \in w_0 \cup w_1\}$ with the relations of B_0, of B_1 and $\tau^0 \leq \tau^1$. We have to check that in B' no x_i is generated by the other x_j's. (We can reduce this to the case when B is trivial, and then check directly.)

§3. Decomposing a Boolean algebra to ultrafilters

3.1 THEOREM. *If $\lambda = \lambda^{<\kappa}$, $|B| = \lambda^+$, B a Boolean algebra satisfying the κ-chain condition, κ regular $>\aleph_0$ then $B - \{0\}$ is the union of $\leq\lambda$ (proper) filters. Moreover any (proper) filter generated by $<\kappa$ elements is included in at least one of them.*

Proof. The completion of B has power $\leq \|B\|^{<\kappa} = \lambda^+$ so w.l.o.g. B is complete. Let B_i $(i < \lambda^+)$ be increasing continuous, $B = \bigcup_{i<\lambda^+} B_i$ $|B_i| \leq \lambda$ and B_{i+1} is complete. Hence cf $\delta \geq \kappa$ implies B_δ is complete. We can assume w.l.o.g. $|B_{i+1} - B_i| = \lambda$, B_0 trivial.

Let $B_{i+1} - B_i = \{a^i_\alpha : \alpha < \lambda\}$ (without repetitions).

Let $[\lambda]^{<\kappa} = \{A \subseteq \lambda : |A| < \kappa\}$. Then there are functions $f_\alpha : \lambda^+ \to [\lambda]^{<\kappa} (\alpha < \lambda)$ such that for every $A \subseteq \lambda^+$ with $|A| < \kappa$ and every $f : A \to [\lambda]^{<\kappa}$ there is an $\alpha < \lambda$ such that f_α extends f. To see this, we may apply a theorem of Engelkind and

Kalowicz [EK] (see e.g. [CN] 3.16) to get a sequence of functions $f'_\beta:\lambda\to[\lambda]^{<\kappa}(\beta<\lambda^+)$ such that for any $A\subseteq\lambda^+$ with $|A|<\kappa$ and any $f:A\to[\lambda]^{<\kappa}$ there is an $\alpha<\lambda$ such that $f'_\beta(\alpha)=f(\beta)$ for all $\beta\in A$. Letting $f_\alpha(\beta)=f'_\beta(\alpha)$ for all $\alpha<\lambda$, $\beta<\lambda^+$ gives the above desired functions.

Now we define by induction on $i<\lambda^+$ an ultrafilter D^i_ζ of B_i (for $\zeta<\lambda$) D^i_ζ increasing and continuous (in i). For $i=0$, i limit there is no problem. Supposing D^i_ζ has been defined, if $D^i_\zeta\cup\{a^i_\alpha:\alpha\in f_\zeta(i)\}$ has the finite intersection property, extend it to an ultrafilter D^{i+1}_ζ of B_{i+1}, otherwise extend D_i to an ultrafilter D_{i+1} of B_{i+1}. Let $D_\zeta=D^{\lambda^+}_\zeta$.

For $i<\lambda^+$ choose a set $S_i\subseteq i+1$ as follows: if i is a successor $S_i=\{i\}$, if i limit, $\operatorname{cf} i\geq\kappa$, $S_i=\{i\}$, and if i limit, $\operatorname{cf} i<\kappa$, let $\alpha(i,j)(j<\operatorname{cf} i)$ be increasing with j, each $\alpha(i,j)$ is a successor $<i$, and $i=\bigcup_{j<\operatorname{cf} i}\alpha(i,j)$, and let $S_i=\{\alpha(i,j):j<\operatorname{cf} i\}$. For each $a\in B$ let $i=i(a)^{j<\operatorname{cf} i}$ be the unique $i<\lambda^+$ such that $a\in B_{i+1}-B_i$.

Now let $C=\{c_\gamma:\gamma<\gamma(0)\}\subseteq B(\gamma(0)<\kappa)$ have the finite intersection property and we shall prove $C\subseteq D_\zeta$ for some $\zeta<\lambda$. Define by induction $C_n: C_n=C$, C_{n+1} is the Boolean subalgebra of B generated by $C_n\cup\{f(a,B_i),\ g(a,B_i):a\in C_n, i\in S_{i(a)}\}$, and let $C^*=\bigcup_{n<\omega}C_n$ (on $f(a,B_i)$, $g(a,B_i)$ see §4 Notation). As $|S_i|<\kappa$ for every i, and κ is regular, we can prove by induction on n that $|C_n|<\kappa$, and as $\kappa>\aleph_0$, clearly $|C^*|<\kappa$. Clearly C^* is a Boolean subalgebra, so let D^* be an ultrafilter of C^*, $C\subseteq D^*$. Define $A=\{i(c):c\in C^*\}$, and $f:A\to[\lambda]^{<\kappa}$ be: $f(i)=\{\alpha<\lambda:a^i_\alpha\in D^*\}$ (clearly $f(i)\subseteq\lambda$ and $|f(i)|<\kappa$ as $D^*\subseteq C^*$, $|C^*|<\kappa$).

So for some $\zeta<\lambda$, $f\subseteq f_\zeta$.

We now prove by induction on i that $D^*\cap B_i\subseteq D^i_\zeta$. For $i=0$ this is trivial (as B_0 is trivial) for i limit it is immediate, so let us prove for $i+1$, assume we have proved for i. If $B_{i+1}\cap C^*\subseteq B_i$, again the conclusion follows from the induction hypothesis, so we can assume $C^*\cap(B_{i+1}-B_i)\neq\phi$, so $i\in A$. By the definition of D^{i+1}_ζ it suffices to prove $D^i_\zeta\cup\{a^i_\alpha:\alpha\in f_\zeta(i)\}$ has the finite intersection property. Let w be any finite subset of $f_\zeta(i)$, and $a=\bigcap_{\alpha\in w}a^i_\alpha$, so it suffices to prove a is not disjoint to any member of D^i_ζ. If $a\in B_i$ this is immediate ($a\in D^*$ as D^* is an ultrafilter in C^* and $a^i_\alpha\in D^*$ for $\alpha\in w$), so let $a=a^i_{\alpha(0)}$, so clearly $a^i_{\alpha(0)}\in D^*$ hence $\alpha(0)\in f_\zeta(i)$. Suppose $c\in D^i_\zeta$, $c\cap a^i_{\alpha(0)}=0$.

If i is a successor or $\operatorname{cf} i\geq\kappa$, then $i\in S_{i(a)}$ hence $f(a^i_{\alpha(0)},B_i)$, $g(a^i_{\alpha(0)},B_i)\in C^*$ and B_i is complete. So $c\cap a^i_{\alpha(0)}=0$ implies $a^i_{\alpha(0)}\leq 1-c$ hence $g(a^i_{\alpha(0)},B_i)\leq 1-c$ hence $g(a^i_{\alpha(0)},B_i)\cap c=0$. But as $a^i_{\alpha(0)}\leq g(a^i_{\alpha(0)},B_i)$ and $a^i_{\alpha(0)}\in D^*$, and $a^i_{\alpha(0)}$, $g(a^i_{\alpha(0)},B_i)\in C^*$ (by the definition of C_n, C^*) clearly $g(a^i_{\alpha(0)},B_i)\in D^*$, but as $g(a^i_{\alpha(0)},B_i)\in D^i$, by the induction hypothesis on i $g(a^i_{\alpha(0)},B_i)\in D^i_\zeta$. But also $c\in D^i_\zeta$, contradicting $g(a^i_{\alpha(0)},B_i)\cap c=0$.

So we are left with the case i limit, $\operatorname{cf} i<\kappa$. For some $j<\operatorname{cf} i$, $c\in B_{\alpha(i,j)}$. and $B_{\alpha(i,j)}$ is complete (as $\alpha(i,j)$ is a successor) and $f(a^i_{\alpha(0)},B_{\alpha(i,j)})$, $g(a^i_{\alpha(0)},B_{\alpha(i,j)})\in C^*$, and the contradiction is similar.

So in both cases $D^i_\zeta \cup \{a^i_\alpha : \alpha \in f_\zeta(i)\}$ has the finite intersection property, so by the definition of D^{i+1}_ξ, $\{a^i_\alpha : \alpha \in f_\xi(i)\} \subseteq D^{i+1}_\zeta$, hence by the choice of ζ, $D^* \cap B_{i+1} \subseteq D^{i+1}_\zeta$.

So the induction on i works. For $i = \lambda^+$ we get $D^* \subseteq D_\zeta = D^{\lambda^+}_\zeta$ as we want, thus finishing the proof.

§4

This section answers a question of Rubin.

Context of the section. Let B_* be a Boolean algebra satisfying the κ-chain condition, κ regular.

AIM. We want to find big free subsets of B_*.

NOTATION. Let $a^0 = a$, $a^1 = 1 - a$.

Let B^*_c be the completion of B_*. For a set $A \subseteq B^c_*$ and $a \in B^c_*$ let $B \subseteq B^c_*$ be the completion in B^c_* of the Boolean subalgebra A generates, and

$$f(a, A) = \sup\{b \in B : b \leq a\}$$

$$g(a, A) = \inf\{b \in B : b \geq a\}$$

$$h(a, A) = g(a, B) - f(a, B), \qquad \bar{f}(a, A) = \langle f(a, A), g(a, A)\rangle$$

They exist as B is complete. Clearly:

(*) (1) $f(a, A) \leq g(a, A)$ and if $a \notin B$, strict inequality holds,
(2) for every $x \in B$ for which $0 < x < h(a, A)$ we have $a \cap x \neq 0$, and $(1-a) \cap x \neq 0$,
(3) if $f(a, A), g(a, A) \in A_1 \subseteq A$ then $f(a, A) = f(a, A_1)$, $g(a, A) = g(a, A_1)$.

4.1. OBSERVATION. If for some $B_1 \subseteq B^c_*$, b_1, b_2, $a_i \in B^c_*$ $(i < \alpha)$, and for each i

$$(b_1, b_2) = \bar{f}(a_i, B_1 \cup \{a_j : j < i\}),$$

then for any finite $w \subseteq \alpha$, any function $t : w \to \{0, 1\}$, and any $x \in B_1$ for which $0 < x \leq b_2 - b_1$ we have $x \cap \bigcap_{i \in w} a_i^{t(i)} \neq 0$.

Proof of 4.1. We do this by induction on the number of elements of w. The induction step is by (*) (2).

4.2. CLAIM. If $\lambda \leq |B_*|$, λ regular and $(\forall \mu < \lambda)[\mu^{<\kappa} < \lambda]$ then B_* has a free subset of λ elements.

Proof. Choose distinct $a_i \in B_*$ $(i < \lambda)$, let B_i be the completion of the subalgebra $\{a_j : j < i\}$ generates. Because of the κ-chain condition $|B_i| \leq |i|^{<\kappa} < \lambda$, so by deleting some $i < \lambda$ we may assume that B_i is strictly increasing. Now it is not continuous, but again by the κ-chain condition $\delta \in S_0 = \{\delta : \mathrm{cf}\, \delta \geq \kappa,\ \delta \in \lambda\}$ implies $B_\delta = \bigcup_{i<\delta} B_i$. Clearly b_i^1, $b_i^2 = \bar{f}(a_i, \{a_j : j < i\}) \in B_i$.

It is well known that S_0 is a stationary subset of λ (notice $\lambda > \kappa$ as $(\forall \mu < \lambda)$ $(\mu^{<\kappa} < \lambda)$ and $2^{<\kappa} \geq \kappa$) so by Fodor Theorem on some stationary $S' \subseteq S_0$, b_i^1, b_i^2 are constants. (More formally, let F be a one-to-one mapping from B_λ onto λ, then $S_1 = \{\delta : F \text{ maps } \bigcup_{i<\delta} B_i \text{ onto } \delta\}$ is closed unbounded. So $S_0 \cap S_1$ is stationary and on it the functions $\delta \mapsto F(b_\delta^l)$ are regressive, so we can apply Fodor Theorem as stated.)

Now let $i(0)$ be the first element of S', so $B_{i(0)}$, $\{a_i : i \in S'\}$ satisfy the hypothesis of 4.1, and by its conclusion $\{a_i : i \in S'\}$ is free.

4.3. CLAIM. If $a_i \in B_*^c$, $(i < \lambda)$, $(\forall \mu < \lambda)(\mu^{<\kappa} < \lambda)$, λ regular, then there are b_1, $b_2 \in B_*^c$ such that: $B \subseteq B_*^c$, $\|B\| < \lambda$, b_1, $b_2 \in B$ implies

$$\{i < \lambda : \bar{f}(a_i, B \cup \{a_j : j < i\}) = \langle b_1, b_2 \rangle\}$$

is of power λ.

Remark. We can consistently replace "of power λ" by "stationary."

Proof. Suppose not; then for every pair $\bar{b} = \langle b_1, b_2 \rangle$ in B_*^c there is $B_{\bar{b}} \subseteq B_*^c$, $|B_{\bar{b}}| < \lambda$, b_1, $b_2 \in B_{\bar{b}}$, with $\{i < \lambda : \bar{f}(a_i, B_{\bar{b}} \cup \{a_j : j < i\}) = \bar{b}\}$ of power $< \lambda$. Using (*) (3) we may assume that if $\bar{b} = \bar{f}(a_i, B_{\bar{b}} \cup \{a_j : j < i\})$ then $a_i \in B_{\bar{b}}$. Then we can find $B' \subseteq B_*^c$, $a_i \in B'$ $(i < \lambda)$, $|B'| = \lambda$, B' complete, and $\bar{b} \in B' \times B'$ implies $B_{\bar{b}} \subseteq B'$. Let $B' = \bigcup_{i<\lambda} B_i'$, B_i' increasing and continuous $|B_i'| < \lambda$, and for $\delta \in S_0 = \{i < \lambda : \mathrm{cf}\, i \geq \kappa\}$, B_δ' is complete.

$$S_1 = \{\delta : \delta \text{ limit};\ i < \delta \leftrightarrow a_i \in B_\delta' \quad \text{and} \quad \bar{b} \in B_\delta' \times B_\delta' \Rightarrow B_{\bar{b}} \subseteq B_\delta'\}$$

is closed and unbounded, so $S_1 \cap S_0 \neq 0$; choose $\delta \in S_1 \cap S_0$. Thus B_δ' is complete and $a_\delta \notin B_\delta'$. Let $\bar{b} = \bar{f}(a_\delta, B_\delta') \in B_\delta' \times B_\delta'$. By (*) (3), $\bar{b} = \bar{f}(a_\delta, B_{\bar{b}} \cup \{a_j : j < \delta\})$, so $a_\delta \in B_{\bar{b}} \subseteq B_\delta'$, a contradiction.

4.4. CONCLUSION. Suppose $\lambda \leq \|B_*\|$, $(\forall \mu < \lambda)[\mu^{<\kappa} < \lambda]$ and let $\chi = \mathrm{cf}\, \lambda$, $\lambda = \sum_{i<\chi} \lambda_i$, λ_i increasing, $\lambda_i = \lambda_i^{<\kappa}$, Then there are $b_i \in B_*^c$, $(i < \chi)$ and (distinct)

$a_{i,\alpha}(\alpha<\lambda_i^+)$ such that:

(i) $a_{i,\alpha}\in B_*$,
(ii) letting B_j (for $j<\chi$) be the Boolean algebra generated by $\{b_i : i<\chi\}\cup\{a_{i,\alpha} : i<j,\ \alpha<\lambda_i^+\}$, then for any $x\in B_i$, $0<x\le b_i$ and any non-trivial Boolean combination τ of $a_{i,\alpha}(\alpha<\lambda_i^+)$ $x\cap\tau\ne\phi$, hence also $x\cap(1-\tau)\ne 0$,
(iii) $a_{i,\alpha}\le b_i$.

Proof. Choose distinct $c_{i,\alpha}(\alpha<\lambda_i^+,\ i<\chi)$ in B_*, and for each i use 4.3 for $c_{i,\alpha}(\alpha<\lambda_i^+)$ so there are suitable b_1^i, b_2^i. We let $b_i=b_2^i-b_1^i$.

Now we define by induction on i, $a_{i,\alpha}(\alpha<\lambda_i^+)$. If we have defined for every $j<i$, B_i is defined, so apply the choice of b_1^i, b_2^i and 4.3 to it. So

$$S_i=\{\alpha<\lambda_i^+ : \langle b_1^i, b_2^i\rangle=\bar f(c_{i,\alpha}, B_i\cup\{c_{i,\beta}:\beta<\alpha\})\}$$

has power λ_i^+. Let $S_i=\{\zeta(i,\alpha):\alpha<\lambda_i^+\}$, $\zeta(i,\alpha)$ increasing with α, and $a_{i,\alpha}=c_{i,\zeta(i,2\alpha+1)}-c_{i,\zeta(2\alpha)}\in B_*$. It is easy to check b_i, a_i, are as required.

4.5. CLAIM. If κ is weakly compact, (hence strongly inaccessible), B_* a Boolean algebra satisfying the κ-chain condition, *then* in B there are κ free elements.

Proof. Let $a_i\in B_*$ $(i<\kappa)$ be distinct, and w.l.o.g. assume $|B_*|=\kappa$. Let B_*^c be the completion of B_*: thus $|B_*^c|=\kappa$, so we may assume that $B_*^c=\kappa$. Let $B_*^c=\bigcup_{i<\kappa}B_i$, where B_i is increasing and continuous $|B_i|<\kappa$, and if $A\subseteq B_i$ then $\sup A\in B_{i+1}$. Thus if $\delta<\kappa$ is a limit cardinal, $A\subseteq B_\delta$, and $|A|<\operatorname{cf}\delta$, then $\sup A\in B_\delta$. Let

$$S_1=\{\delta<\kappa:\delta \text{ is a strong limit cardinal, } B_\delta=\delta, \text{ and } a_i\in B_\delta \text{ iff } i<\delta\}.$$

Thus S_i is closed unbounded. Let

$$S_2=\{\delta\in S_i:\sigma \text{ is a strongly inaccessible cardinal and } B_\delta \text{ satisfies the } \delta\text{-chain condition}\},$$

From κ weakly compact it follows that S_2 is stationary. In fact, to show this we can use the $\prod_1^1$-indescribability of κ: let C be any closed unbounded subset of κ, and let σ be a $\prod_1^1$-sentence such that

$$\langle V_\kappa, \in, C\cap S_1, \cup, \cap, -\rangle\vDash\sigma$$

says that $\langle \kappa, \cup, \cap, - \rangle$ is a BA, κ is a limit ordinal, κ is regular, $C \cap S_1$ is unbounded in κ, and for all $X \subseteq \kappa$ consisting of pairwise disjoint elements of the BA there is a $\delta \in X \cap S_1$ such that $X \subseteq \delta$. Then there is an $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, C \cup S_1 \cap \alpha, \cup \restriction \alpha, \cap \restriction \alpha, - \restriction \alpha \rangle \models \sigma.$$

Clearly $\alpha \in C \cap S_1 \cap S_2$, as desired. So S_2 is stationary. For $\delta \in S_2$, B_δ is a complete subalgebra of B^c_*, and we can continue as in 4.2.

4.6. DEFINITION. $P(\kappa, \lambda)$ means: if B is a Boolean algebra satisfying the κ-chain condition, $a_i \in B$, $a_i \neq 0$ $(i < \lambda)$ *then* there is $S \subseteq \lambda$, S of power λ such that $\{a_i : i \in S\}$ has the finite intersection property.

4.7. CONCLUSION. Suppose $(\forall \mu < \lambda)[\mu^{<\kappa} < \lambda]$, $\lambda \leq |B_*|$,

(1) In B_* there is a free set of power λ if λ is regular or $P(\kappa, \operatorname{cf} \lambda)$, λ singular.

(2) If the condition of (1) fails then the conclusion does not necessarily hold.

Proof. (1) If λ is regular use 4.2; otherwise apply 4.4, and get $b_i \in B^c_*$, $a_{i,\alpha} \in B_*$ $(i < \chi = \operatorname{cf} \lambda,\ \alpha < \lambda_i^+)$. By 4.6 w.l.o.g. $\{b_i : i < \chi = \operatorname{cf} \lambda\}$ has the finite intersection property, and then $\{a_{i,\alpha} : \alpha < \lambda_i^+,\ i < \chi\}$ is $\subseteq B_*$ (by 4.4(i)) and is free (by 4.4(ii) and the assumption on the b_i's).

(2) We suppose $\operatorname{cf} \lambda = \chi < \lambda$, but not $P(\kappa, \chi)$. Clearly there is a Boolean algebra B of power χ, satisfying the κ-chain condition, and $0 \neq b_i \in B (i < \chi)$ such that for no $S \subseteq \chi$, $|S| = \chi$ does $\{b_i : i \in S\}$ have the finite intersection property. Now let, B_* be generated by B and x_α $(\alpha < \lambda)$ freely except the relations that hold in B and $x_\alpha \leq b_i (\bigcup_{j<i} \lambda_j^+ \leq \alpha < \lambda_i^+)$, where $\lambda = \sum_{i<\chi} \lambda_i$, λ_i increasing, $\lambda_i^{<\kappa} = \lambda_i$.

We want to prove B is a counterexample. For this we have to prove B satisfies the κ-chain condition and it has no independent set of λ elements.

First we prove it satisfies the κ-chain condition. Suppose $D = \{d_i : i < \kappa\} \subseteq B$ is a set of κ pairwise disjoint non-zero elements. By the definition of B we can find $c_i \in B$, and Boolean term $\tau_i = \tau_i(x_i, \ldots, x_{n(i)})$, and $\alpha(i, 1), \ldots, \alpha(i, n(i))$ which are distinct and $< \lambda$, such that $d_i \geq c_i \cap \tau_i(x_{\alpha(i,1)}, \ldots, x_{\alpha(i,n(i))}) \neq 0$, and for every $l = 1, n(i)$, $c_i \leq b_{\xi(i,l)}$ or $c_i \cap b_{\xi(i,l)} = 0$ where $\xi(i, l) = \min \{\xi : \alpha(i, l) < \lambda_\xi^+\}$. As we can assume $n(i)$ is minimal, $c_i \leq b_{\xi(i,l)}$. As we can replace D by any subset of the same cardinality w.l.o.g. $\tau_i = \tau_i$, $n(i) = n$; $\alpha(i_1, l_1) = \alpha(i_2, l_2)$ implies $l_1 = l_2$.

As B satisfies the κ-chain condition there are $i(1) \neq i(2) < \kappa$ such that $c_{i(1)} \cap c_{i(2)} \neq 0$. It is easy to check (by B's definition) that

$$\left\{ x_\alpha : \bigcup_{\xi < j(i(1))} \lambda_\xi^+ \leq \alpha < \lambda_{j(i(1))}^+ \text{ or } \bigcup_{\xi < j(i(2))} \lambda_\xi^+ \leq \alpha < \lambda_{j(i(2))}^+ \right\}$$

Sh:92

is independent. From this it is easy to check that

$$d_{i(1)} \cap d_{i(2)} \geq c_{i(1)} \cap c_{i(2)} \cap \tau(x_{\alpha(i(1),1)}, \ldots, x_{\alpha(i(2),n)}) \cap \tau(x_{\alpha(i(2),1)}, \ldots, x_{\alpha(i(2),n)}) \neq 0$$

(remember $c_{i(1)} \cap c_{i(2)}$ is $\neq 0$, and $\leq b_{\xi(i(1),l)}$, $b_{\xi(i(2),l)}$, and $\alpha(i(1), l_1) = \alpha(i(2), l_2)$ implies $l_1 = l_2$).

So we prove B satisfy the κ-chain condition and now we prove that it has no independent subset of λ elements. Suppose $\{d_{i,\alpha} : \alpha < \lambda_i^+, i < \chi\} \subseteq B_*$ is independent. Now we prove:

ASSERTION. For every $i < \chi$ for some α, β, w, we have: $\alpha < \beta < \lambda_i^+$, $w \subseteq \chi$ $d_{i,\alpha} - d_{i,\beta} \leq \bigcup_{j \in w} b_j$ and $w \cap i = \phi$. For each $\alpha < \lambda_i^+$ there is a Boolean term τ_α, $k_\alpha < \omega$, $a_1^\alpha, \ldots, a_{k_\alpha}^\alpha \in B^*$, $n_\alpha < \omega$, $j(1, \alpha), \ldots, j(n_\alpha, \alpha) < \chi$, $\gamma(1, \alpha) < \lambda_{j(1,\alpha)}^+, \ldots$, $\gamma(n_\alpha, \alpha) < \lambda_{j(n_\alpha,\alpha)}^+$ such that $d_{i,\alpha} = \tau_\alpha(a_1^\alpha, a_2^\alpha, \ldots, a_{k_\alpha}^\alpha, d_{j(1,\alpha),\gamma(1,\alpha)}, \ldots, d_{j(n_\alpha,\alpha),\gamma(n_\alpha,\alpha)})$. As $|B| = \chi$ and χ, $\aleph_0 \leq \lambda_i$ clearly there is $S \subseteq \lambda_i^+$, $|S| = \lambda_i^+$ that for every α, $\beta \in S$

(a) $k_\alpha = k_\beta$, $a_1^\alpha = a_1^\beta, \ldots, a_{k_\alpha}^\alpha = a_{k_\beta}^\beta$,
(b) $n_\alpha = n_\beta$, $j(1, \alpha) = j(1, \beta), \ldots$,
(c) $1 \leq l \leq n$, $j(l, \alpha) < i$ implies $\gamma(l, \alpha) = \gamma(l, \beta)$.

Choose $\alpha < \beta$ in S, $w = \{j(l, \alpha), j(l, \beta) : 1 \leq l \leq n_\alpha, j(l, \alpha) \geq i\}$, so trivially $w \subseteq \chi$, $w \cap i = \phi$, let $c = \bigcup_{j \in w} b_j$ and now we shall prove $d_{i,\alpha} - d_{i,\beta} \leq c$ or equivalently $d_{i,\alpha} - d_{i,\beta} - c = 0$ and thus finish.

$$\begin{aligned} d_{i,\alpha} - d_{i,\beta} - c &= (d_{i,\alpha} - c) - (d_{i,\beta} - c) \\ &\leq \tau_\alpha(a_1^\alpha - c, \ldots, a_{k_\alpha}^\alpha - c, d_{j(1,\alpha),\gamma(1,\alpha)} - c, \ldots) \\ &\quad - \tau_\alpha(a_1^\beta - c, \ldots, a_{k_\alpha}^\beta - c, d_{j(1,\beta),\gamma(1,\beta)} - c, \ldots) = 0. \end{aligned}$$

The last equality holds as:

(i) for $1 \leq l \leq n_\alpha$, $k_\alpha = k_\beta$, $a_l^\alpha = a_l^\beta$ hence $a_l^\alpha - c = a_l^\beta - c$,
(ii) for $1 \leq l \leq n_\alpha = n_\beta$ *if* $j(l, \alpha) < i$, then $j(l, \alpha) = j(l, \beta)$, $\gamma(l, \alpha) = \gamma(l, \beta)$, hence trivially $d_{j(l,\alpha),\gamma(l,\alpha)} - c = d_{j(l,\beta),\gamma(l,\beta)} - c$,
(iii) for $1 \leq l \leq n_\alpha = n_\beta$ *if* $j(l, \alpha) \geq i$ then $j(l, \alpha) = j(l, \beta)$, and $d_{j(l,\alpha),\gamma(l,\alpha)}$, $d_{j(l,\beta)\gamma(l,\beta)} \leq b_{j(l,\alpha)}$, but $b_{j(l,\alpha)} \leq c$, so $d_{j(l,\alpha),\gamma(l,\alpha)} - c = 0 = d_{j(l,\beta),\gamma(l,\beta)} - c$.

So we have proved the assertion, and so we can define for each $i < \chi$, $\alpha_i < \beta_i < \lambda_i^+$, $w_i \subseteq \chi$, $w_i \cap i = \phi$ such that

$$d_{i,\alpha_i} - d_{i,\beta_i} \leq \bigcup_{j \in w_i} b_j.$$

As $\{d_{i,\gamma}: i<\chi, \gamma<\lambda_i^+\}$ is free, $A=\{\bigcup_{j\in w_i} b_j : i<\chi\}$ has the finite intersection property, so some ultrafilter D on B_* includes A. So for each i there is $\zeta(i)\in w_i$, such that $x_{\zeta(i)}\in D$, and $\zeta(i)\geq i$ as $w_i\cap i=\phi$. So $|\{x_{\zeta(i)}: i<\chi\}|=\chi$, and $\{x_{\zeta(i)}: i<\chi\}$ has the finite intersection property, contradiction to their choice.

4.8. CLAIM. (0) If χ is regular $(\forall\mu<\chi)[\mu^{<\kappa}<\chi]$ then $P(\kappa,\chi)$

(1) If $\kappa=\mu^+$, $\mu=\mu^{<\mu}$, $2^\mu=\kappa$ then $P(\kappa,\kappa)$ fails.
(2) If $\mathrm{MA}+2^{\aleph_0}>\lambda$, $\mathrm{cf}\,\lambda>\aleph_0$ then $P(\aleph_1,\lambda)$ holds.
(3) If κ is weakly compact then $P(\kappa,\kappa)$. If $V=L$, κ is strongly inaccessible but not weakly compact (or even if there is a κ-Souslin tree) then $P(\kappa,\kappa)$ fails.
(4) If MA then $P(\aleph_1, 2^{\aleph_0})$ fails.

Proof
(0) Follows easily from the proof of 4.2.
(1) Will appear.
(2) The argument appeared in Juhasz [J] pp. 60–61.

It is well known that if $\mathrm{MA}+2^{\aleph_0}>\aleph_1$ B a Boolean algebra satisfying the countable chain condition, then among any $\aleph_1$ elements there are $\aleph_1$ pairwise not disjoint ones. Let $P=\{A: A$ a finite subset of $(B-\{0\})\times w$, and for every n,

$A_n=\{a:(a,n)\in A$ for some $n\}$ has non empty intersection$\}$.

Clearly P, ordered by inclusion, satisfies the $\aleph_1$-chain condition, and for every $x\in B-\{0\}$.

$\{A\in P: x\in A_n$ for some $n\}$ are dense. So by MA, for any $X\subseteq B$, $|X|<2^{\aleph_0}$, there is a directed subset G of P, such that $\bigcup\{A_n: n<\omega, A\in G\}\supseteq X$. So letting $D^n=\bigcup\{A_n\cap X: A\in G\}$, clearly $X=\bigcup_n D^n$, D^n has the finite intersection property. If $\mathrm{cf}\,|X|>\aleph_0$ clearly for some n $|D^n|=|X|$, and this is what is required.

(3) If κ is weakly compact, this follows by 4.5. For the second part, Jensen proved that there is a κ-Souslin tree T. For each $\alpha<\kappa$ let x_α be an element of T of level α, $A_\alpha=\{y\in T: y\geq x_\alpha\}$, and B the Boolean algebra that the A_α's generate. We can assume that every x_α has infinitely many immediate successors. Then B, $\{A_\alpha:\alpha<\kappa\}$ shows $P(\kappa,\kappa)$ fail.

(4) This was proved by Erdös and Kunen.

Remark. Our results have the form:

"among any λ elements there are λ free."

(In 4.4. omit (iii), and let in the end $a_{i,\alpha}=c_{i,\zeta(i,\alpha)}$, and use this in 4.7 to get the above mentioned result.)

LEMMA 4.9. *Suppose B_* is a Boolean algebra satisfying the κ-chain, κ regular λ a singular cardinal, $(\forall \mu<\lambda)\mu^{<\kappa}<\lambda$ and $A \subseteq B_*$, $|A|>\lambda$. Then there is a free $A' \subseteq A$, $|A'|=\lambda$.*

Proof. W.l.o.g. let $A=\{a_i : i<\lambda^+\}$, $i<j \Rightarrow a_i \neq a_j$, let $\chi=\operatorname{cf}\lambda$, $\lambda=\sum_{i<\chi}\lambda_i$, $\lambda_0 \geq \chi$, λ_i increasing, $\lambda_i^{<\kappa}=\lambda_i$. Let B_*^c be the completion of B_*. Now we define by induction on $i<\chi$, B_i such that

(i) $B_i \subseteq B_*^c$, $\|B_i\|=\lambda_i$, B_i is a complete subalgebra of B_*^c,
(ii) B_i increases with i,
(iii) if $b^1, b^2 \in B_i$, $\bar b=\langle b^1, b^2\rangle$, and there is $C \subseteq B_*^c$, $b^1, b^2 \in C$, $|C| \leq \lambda_i$ such that $|\{i<\lambda^+ : \bar f(a_i, C)=\bar b\}| \leq \lambda$ then there is such $C=C_{\bar b}^i \subseteq B_i$.

Now for every $i<\chi$, $\bar b \in B_i$ let

$$S_i^{\bar b}=\{\alpha : \bar b=\bar f(a_\alpha, B_i)\}.$$

Clearly $S_i=\bigcup\{S_i^{\bar b} : |S_i^{\bar b}| \leq \lambda\}$ has cardinality $\leq\lambda$, hence there is $\alpha \notin S_i$ for each $i<\chi$. Let $\langle b_i^1, b_i^2\rangle=f(a_\alpha, B_i)$, and it is clear that

(a) $b_i^1, b_i^2 \in B_i$,
(b) $i<j<\chi$ implies $b_i^1 \leq b_j^1 < b_j^2 \leq b_i^2$,
(c) if $b_i^1, b_i^2 \in C \subseteq B_*^c$, $|C| \leq \lambda_i$ then for λ^+ β's $a_\beta \notin C$, $\bar f(a_\beta, C)=\langle b_i^1, b_i^2\rangle$.

Now we define by induction on $\xi<\lambda$, C_ξ and $\alpha(\xi)$ such that,

(α) $C_0=\{b_i^l : i<\chi,\ l=1,2\}$,
(β) $C_\xi=C_0 \cup \{a_{\alpha(\zeta)} : \zeta<\xi\}$,
(γ) $\alpha(\xi)$ increasing,
(δ) if i is minimal such that $\xi<\lambda_i^+$ then

$$\bar f(a_{\alpha(\xi)}, c_\xi)=\langle b_i^1. b_i^2\rangle.$$

This is easy to do and, because of (b) and (α), like 4.1 $\{a_{\alpha(\xi)} : \xi<\lambda\}$ is as required.

CONJECTURE. If λ, κ are as above, $\operatorname{cf}\lambda<\kappa$, then there is a Boolean algebra B, $|B|=\lambda^{<\kappa}=\lambda^{\operatorname{cf}\lambda}$, satisfying the κ-chain condition, with no free subset of cardinality λ^+.

REFERENCES

[A] ARCHANGEL'SKI, Dokl. Akad. SSSR. Tom 199 (1971).
[CN] C. C. COMFORT and S. NEGREPONLIS, *Ultrafilters*.

[EHR] P. ERDÖS, A. HAGNAL and R. RADO, *Partition relations for cardinals*, Acta Math. Acad. Sci. Hungar. *16* (1965), 193–196.

[J] I. JUHASZ, *Cardinal functions in topology*, Math Centre Tracts 34.

[P] PIERCE, Advances in Mathematics, *13* (1974) 323–381.

[EK] E. ENGELKING and M. KARLOWICZ, *Some theorems of set theory and their topological consequences*, Fund. Math *57* (1965) 275–285.

University of California
Berkeley, California
U.S.A.

The Hebrew University
Jerusalem, Israel

University of Wisconsin
Madison, Wisconsin
U.S.A.