On normal ideals and Boolean Algebras

In [Sh 1] 3.1 we prove: If $\mathcal{B}$ is a Boolean algebra of power $\kappa^+, \kappa = \kappa^{<\kappa}$, and $\mathcal{B}$ satisfies the κ-chain condition then $\mathcal{B} - \{0\}$ is the union of κ ultrafilters (why not "$\mathcal{B}$ of power λ^{++}"? see [Sh 3] mainly 2.4, p.245). We here replace "κ-chain condition" by a weaker condition we introduce here (κ-SD, (see Definition 1), which says that for almost all $\mathcal{B} \subseteq \mathcal{B}$ of power κ, $\mathcal{B} <\!\!\circ\, \mathcal{B}$ (for the right interpretation of almost).

The other theorem (6) is that $2^{\aleph_0} < 2^{\aleph_1}$ implies $\mathcal{D}_{\omega_1}$ (the club filter on ω_1), cannot be $\aleph_2$- dense. We then observe we cannot improve this to $[2^{\aleph_0} < 2^{\aleph_0} \Longrightarrow \mathcal{D}_{\omega_1}$ not $\aleph_2$-saturated] as by Forman Magidor Shelah [FMS], a universe V, $V \models$"$\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated understructibly under c.c.c. forcing" was obtained and discuss the large cardinal needed. For proving Theorem 6 we use normal filters connected with variants of the weak diamonds (see Devlin Shelah [DS], Shelah [Sh 2]) and prove a more general such theorem. Compare with a recent result of Woodin: from ADR + "ϑ regular" he gets the consistency of "$\mathcal{D}_{\omega_1} + X$ is $\aleph_1$-dense" for some stationary $X \subseteq \omega_1$. The conception of this work is closely connected with Forman Magidor and Shelah [FMS], and also Shelah and Woodin [SW], and [Sh 5]; it was done subsequently to most of [FMS].

Notation: $\mathcal{P}(\lambda) = \{A: A \subseteq \lambda\}$, it is a Boolean algebra and we sometimes say λ instead of $\mathcal{P}(\lambda)$. $\mathcal{B}$ denotes a Boolean algebra; the filter $E \subseteq \mathcal{B}$ generated is $\langle E \rangle_{\mathcal{B}} = \{x \in \mathcal{B}$: there are $n < \omega$, $x_1 \in E, \ldots, x_n \in E$ such that $\bigcap_{i=1}^{n} x_\ell \leq x\}$, it is proper if $0 \notin \langle E \rangle_{\mathcal{B}}$; an ultrafilter is a maximal proper filter. Let $\mathcal{B}_1 <\!\!\circ\, \mathcal{B}_2$ means $\mathcal{B}_1$ is a subalgebra of $\mathcal{B}_2$, and every maximal antichain of $\mathcal{B}_1$ is a maximal antichain of $\mathcal{B}_2$, or what is equivalent: for every $x \in \mathcal{B}_2, x \neq 0$ there is $y \in \mathcal{B}_1, y \neq 0$ such that $(\forall z \in \mathcal{B}_1)[0 < z \leq y \rightarrow x \cap z \neq 0]$. Let $\mathcal{B}_1 <\!\!\circ^* \mathcal{B}_2$ means that $\mathcal{B}_1$ is subalgebra of $\mathcal{B}_2$ and $\{x \in \mathcal{B}_2 : \{y \in \mathcal{B}_1 : y \cap x = 0\}$ is dense in $\mathcal{B}_1\}$ is dense below no $z \in \mathcal{B}_1, z \neq 0$.

For a regular $\lambda > \aleph_0$ let $\mathcal{D}_\lambda$ be the filter (on $\mathcal{P}(\lambda)$) generated by the closed unbounded subsets of λ. For I an ideal of $\mathcal{B}$ let $\mathcal{B}/I$ be the quotient algebra, similarly we define $\mathcal{B}/\mathcal{D}$, $\mathcal{D}$ a (proper) filter on $\mathcal{B}$.

1 Definition : Let $\mathcal{B}$ be a Boolean algebra of cardinality κ^+, $\mathcal{B} = \bigcup_{\alpha<\kappa^+} \mathcal{B}_\alpha$, $\mathcal{B}_\alpha$ increasing continuous, each $\mathcal{B}_\alpha$ of cardinality $\leq \kappa$. We say $\mathcal{B}$ is κ-SD if $\{\alpha :$ if $cf\ \alpha = cf\ \kappa$ then $\mathcal{B}_\alpha <\!\!\circ \mathcal{B}\}$ belong to $\mathcal{D}_{\kappa^+}$. We say $\mathcal{B}$ is almost κ-SD if $\{\alpha : cf\ \alpha = cf\ \kappa$ and $\mathcal{B}_\alpha <\!\!\circ \mathcal{B}\} \neq \phi \mod \mathcal{D}_{\kappa^+}$. We say $\mathcal{B}$ is almost κ-WSD if for some stationary $S \subset \{\alpha\ cf\ \alpha = cf\ \kappa\}$, for every $i < j, [i \in S, j \in S \Longrightarrow \mathcal{B}_i <\!\!\circ^* \mathcal{B}_j]$. We say $\mathcal{B}$ is κ-WSD if we can choose an S above such that $S \cup \{\alpha : cf\ \alpha \neq cf\ \kappa\} \in \mathcal{D}_\lambda$.

1A Remark: 1) We can define naturally κ-SD, κ-WSD for $\mathcal{B}$ of cardinality $> \kappa^+$, see the proof of Theorem 2 and Claim 3.

2) if $\kappa = \kappa^{<\kappa}$, $\mathcal{B}$ satisfies the κ-chain condition, $\mathcal{B}$ has cardiality κ^+ then $\mathcal{B}$ is κ-SD.

2. Theorem : If $\mathcal{B}$ is κ-SD, $\kappa = \kappa^{<\kappa}$ then $\mathcal{B} - \{0\}$ is the union of κ, ultrafilters.

Proof : Let $\mathcal{B} = \bigcup_{\alpha<\kappa^+} \mathcal{B}_\alpha$, $\mathcal{B}_i$ increasing continuous, $\mathcal{B}_i$ of cardinality $\leq \kappa$.

As $\kappa = \kappa^{<\kappa}$, and as we can replace $\mathcal{B}$ by any extension satisfying the same conditions, w.l.o.g. $\mathcal{B}$ is closed under unions of $< \kappa$ elements.

Let $S = \{i < \kappa^+ : i = 0, i$ is a successor ordinal or i is a limit ordinal with cofinality $\kappa\}$.

By renaming the $\mathcal{B}_i$ we can assume;

(α) if $i \in S$ then $\mathcal{B}_i \lessdot \mathcal{B}$ and $\mathcal{B}_i$ is $(<\kappa)$-complete, i.e. if $\alpha < \kappa$, $a_\gamma \in \mathcal{B}_i$ for $\gamma < \alpha$, then $\bigcup_{\gamma,\alpha} a_\gamma \in \mathcal{B}_i$ (where $\bigcup_{\gamma<\alpha} a_\gamma$ is taken in $\mathcal{B}$).

Let $\chi = (2^{\kappa^+})^+$ and w.l.o.g. $\mathcal{B}_i \in H(\chi)$. Now for each $y \in \mathcal{B}, y \neq 0$ we define by induction on $n < \omega$, an elementary submodel N_n^y of $(H(\chi), \in)$ such that :

(i) $y \in N_n^y$, $\langle \mathcal{B}_i : i < \kappa^+ \rangle \in N_n^y$.

(ii) N_n^y has cardinality $< \kappa$ but $N_n^y \cap \kappa$ is an ordinal.

(iii) $N_n^y \prec N_{n+1}^y$ and $N_n^y \in N_{n+1}^y$ (remember $N_n^y \in H(\chi)$).

Now for every $z, y \in \mathcal{B}, y \neq 0$, natural number n and ordinal $\alpha \in S \cap N_n^y$ we define

$$G_\alpha^n(z;y) = \bigcup\{a \in \mathcal{B}_\alpha : a \in N_n^y \text{ and } (\forall b \in \mathcal{B}_\alpha)[0 < b \leq a \rightarrow b \cap z \neq 0].$$

Let $y \in \mathcal{B}, m < \omega$ we define by induction on $n, m \leq n < \omega$ a set $\mathcal{P}_y^{n,m}$ of terms $\tau = \tau(t)$:

$$\mathcal{P}_y^{m,m} = \{t\}$$

$$\mathcal{P}_y^{n+1,m} = \{G_\alpha^n(\bigcap_{\ell=1}^{k} \tau_\ell, y) : \alpha \in S \cap N_n^y,\ k < \omega \text{ and for } \ell = 1, \ldots, k,\ \tau_\ell \in \mathcal{P}_y^{n,m}\}$$

2A Fact: For $\tau(t) \in \mathcal{P}_y^{n,m}$ and $z \in N_m^y$, $\tau(z)$ is define naturally and it belongs to N_n^y, and if $\tau(t) = G_\alpha^{n-1}(\cdots)$ then $\tau(z) \in \mathcal{B}_\alpha$.

2B Fact; 1) For any $y \in \mathcal{B}$, $m \leq n < \omega$, $z \in N_n^y \cap \mathcal{B}$, $z \neq 0$ and $\tau \in \mathcal{P}_y^{n,m}$ the element $\tau(z)$ is not zero.

2) *if* $m \leq n$, $k < \omega$, $\tau_\ell(t) \in \mathcal{P}_y^{n,m}$ and for $\ell < k$, $z_\ell \in N_m^y \cap \mathcal{B}$, $z_\ell \neq 0$, and $\bigcap_{\ell<k} z_\ell \neq 0$ *then* $\bigcap_{\ell<k} \tau_\ell(z_\ell) \neq 0$.

Proof ; Clearly 1) follows from 2). We prove 2) by induction on n.

When $n = m$, necessarily $\tau_\ell(t) = t$ and there is no problem.

When $n > m$, let $\tau_\ell(t) = G^{n-1}_{\alpha_\ell}(\bigcap_{i<i(\ell)} \tau_{\ell,i}(t),y)$ (where $\alpha_\ell \in N^y_{n-1} \cap S$) so $\tau_{\ell,i}(t) \in \mathcal{P}^{n-1,m}_y$. Let $z_{\ell,i} = \tau_{\ell,i}(z_\ell)$, so $z_{\ell,i} \in N^{n-1}_y$, (by Fact 2A) and by the induction hypothesis on n, $z \overset{def}{=} \bigcap_{\substack{i<i(\ell)\\ \ell<k}} z_{\ell,i} \neq 0$ and clearly $z \in N^y_{n-1} \cap \mathcal{B}$. Clearly $G^{n-1}_{\alpha_\ell}(z,y) \leq G^{n-1}_{\alpha_\ell}(\bigcap_{i<i(\ell)} \tau_{\ell,i}(z_\ell),y)$ for each ℓ. So it suffices to prove that $\bigcap_{\ell<k} G^{n-1}_{\alpha_\ell}(z,y)$. W.l.o.g. $\alpha_0 > \alpha_1 > \cdots > \alpha_{k-1}$, and we define by induction on $\ell \leq k$, an element s_ℓ of $\mathcal{B} \cap N^{n-1}_y$ as follows:

(a) $s_0 = z$,

(b) $s_{\ell+1} \in \mathcal{B}_{\alpha_\ell} \cap N^y_{n-1}$ is such that;

$$(\forall b \in \mathcal{B}_{\alpha_\ell})[0 < b \leq s_{\ell+1} \rightarrow b \cap s_\ell \neq 0]$$

We can find such $s_{\ell+1} \in \mathcal{B}_{\alpha_\ell}$ as $\mathcal{B}_{\alpha_\ell} \lessdot \mathcal{B}$, and we can choose it in N^{n-1}_y as s_ℓ, α_ℓ and $\langle \mathcal{B}_\alpha : \alpha < \kappa^+ \rangle$ belong to N^{n-1}_y, and N^{n-1}_y is an elementary submodel of $(H(\chi),\in)$.

We can prove that when $i \leq j < k$, $(\forall b \in \mathcal{B}_{\alpha_j})[0 < b \leq s_j \rightarrow b \cap \bigcap_{\ell=i}^{j} s_\ell \neq 0]$. This is done by induction on j; when $j = i$ this is trivial. When $j > i$, let $b \in \mathcal{B}_{\alpha_j}$, $0 < b \leq s_j$, by the choice of s_j, $b \cap s_{j-1} \neq 0$, so $0 < b \cap s_{j-1} \leq s_{j-1}$ and clearly $b \cap s_{j-1} \in \mathcal{B}_{\alpha_{j-1}}$, so by the induction hypothesis on j, $(b \cap s_{j-1}) \cap \bigcap_{\ell=1}^{j-1} s_\ell \neq 0$ but $b \leq s_j$ so $b \cap \bigcap_{\ell=i}^{j} s_\ell \neq 0$.

Hence $\bigcap_{\ell<k} s_\ell \neq 0$, and also (when $0 \leq i < k$) that $(\forall b \in \mathcal{B}_{\alpha_j})[0 < b \leq s_j \rightarrow b \cap s_i \neq 0]$, now for $i = 0$ $s_i = z$, hence by definition of $G^{n-1}_{\alpha_j}(z,y)$, clearly $s_j \leq G^{n-1}_{\alpha_j}(z,y)$. So $0 \neq \bigcap_{\ell<k} s_\ell \leq \bigcap_{\ell<k} G^{n-1}_{\alpha_\ell}(z,y)$, so we have proved the induction step for $n > m$, hence Fact 2B:

2C Fact; If $\alpha \in \bigcup_{n<\omega} N^y_n$, $\alpha \in S$, $y \in \mathcal{B}$, $y \neq 0$, $\mathcal{D}$ an ultrafilter on $\mathcal{B}_\alpha$, and

$\Gamma = \{\tau(y) : \tau \in \mathcal{P}^{n,m}_y$ for some $m \leq n < \omega\}$ and $\Gamma \cap \mathcal{B}_\alpha \subseteq \mathcal{D}$

then $\mathcal{D} \cup \{\Gamma \cap \mathcal{B}_{\alpha+1}\})$ generates a proper filter.

Proof : Immediate, because :

2D Fact: When $m \le n < \omega$, $\{\tau(y) : \tau \in P_y^{n,m}\} \subseteq \{\tau(y) : y \in P_y^{n,0}\}$,

Proof: This can be proved by induction on n: for $n = m > 0$ choose $\alpha_0 > \cdots > \alpha_{m-1}$ in $S \cap N_0^y$ such that $y \in \mathcal{B}_{\alpha_{m-1}}$ and define $\tau_\ell \in P_y^{\ell,0}$ by induction on $\ell \le m$: $\tau_0 = \tau_1$, $\tau_{\ell+1} = G^\ell_{\alpha_\ell}(\tau_\ell, y)$; the other cases are trivial.

Continuation of the proof of Theorem 2:

Let E^y be any ultrafilter of $\mathcal{B} \cap (\bigcup_{n<\omega} N_y^n)$ which includes $\{\tau(y) : \tau \in P_y^{n,m}$ for some $m \le n < \omega\}$; by Fact 2B,2D it is proper. The rest of the proof is as in [Sh 1] 3.1. By Engelking and Karlowicz [EK] there are functions $f_\xi : \kappa \to \kappa$ (for $\xi < \kappa^+$) such that for every distinct $\xi_\beta (\beta < \beta_0 < \kappa)$ and $\gamma_\beta < \kappa$ $(\beta < \beta_0)$ for some $\varepsilon < \kappa$, $\bigwedge_{\beta<\beta_0} f_\xi(\varepsilon) = \gamma_\beta$. Let $g_\beta : \kappa^+ \to \kappa$ be defined by: $g_\beta(\xi) = f_\xi(\beta)$.

Let $\mathcal{B}_{\xi+1}$ be generated by $\mathcal{B}_\xi \cup \{y^\xi_\beta : \beta < \kappa\}$ (and w.l.o.g. $\mathcal{B}_0 = \{0,1\}$, and w.l.o.g. $\langle\langle y^\xi_\beta, \xi, \beta\rangle : \xi < \kappa^+, \beta < \kappa\rangle$ belongs to every N_0^y). Let $\langle Y^\xi_\gamma : \gamma < \gamma\rangle$ list all subsets of $\{y^\xi_\beta : \beta < \kappa\}$ of cardinality $< \kappa$. We define by induction on $\xi < \kappa^+$ for each $\beta < \kappa$ an ultrafilter $\mathcal{D}^\xi_\beta$ of $\mathcal{B}_\beta$ such that:

(A) $\mathcal{D}^\xi_\beta$ is increasing continuous in ξ.

(B) if $\mathcal{D}^\xi_\beta \cup Y^\xi_{g_\xi(\beta)}$ generates a proper filter then $\mathcal{D}^\xi_\beta \cup Y^\xi_{g_\xi(\beta)} \subseteq \mathcal{D}^{\xi+1}_\beta$.

Clearly this can be done and each $\mathcal{D}_\beta = \mathcal{D}^{\kappa^+}_\beta$ is a (proper) ultrafilter of $\mathcal{B}$. Now if $y \in \mathcal{B}, y \ne 0$ then for each $\xi \in S \cap (\bigcup_{n<\omega} N_n^y)$ $(E_y \cap \{y^\xi_\alpha : \alpha < \kappa\}) \cup (E_y \cap \mathcal{B}_\xi)$ generates $E_y \cap \mathcal{B}_{\xi+1}$, [as $\mathcal{B}_\xi \cup \{y^\xi_\alpha : \alpha < \kappa\}$ generates $\mathcal{B}_{\xi+1}$, $\mathcal{B}_\xi \in N_n^y$, $\{y^\xi_\beta : \beta < \kappa\} \in N_n^y$, and $\mathcal{B}_\alpha \in N_n^y$ for every n such that $\alpha \in N_n^y$], so there is $\beta < \kappa$ such that for every $\xi \in \bigcup_{n<\omega} N_n^y$, $g_\beta(\xi) = \gamma_\xi$, and by Fact 2C, $E_y \subseteq \mathcal{D}_\beta$.

3 Claim; 1) In Theorem 2 we can replace κ^+ by 2^κ (its proof is written so that the changes are minimal, but the set $\{y^\xi_\beta : \beta < \kappa\}$ should still have cardinality κ.

2) In Theorem 2 (and Claim 3(1)) we really get that for every $Y \subseteq \mathcal{B}$ of cardinality $< \kappa$ which generates a proper filter, for some $\beta < \kappa$, $Y \subseteq \mathcal{D}_\beta$ (define $N_n^Y, \mathcal{P}_Y^{n,m}$ for any such Y, now Fact 2A, 2B have the same proof, and Fact 2C should be modified by having $\Gamma = \{\tau(y) : y \in Y, \ \ \tau \in \mathcal{P}_Y^{n,m}, m \leq n < \omega\}$.

4. Remark: We can go beyond 2^κ, see [Sh 4], Lemma 4.

5. Observation: Suppose $\lambda > \aleph_0$ is regular, $2^\lambda = \lambda^+$, I an ideal on λ, $\mathcal{B} = \mathcal{P}(\lambda)/I$. Suppose $\mathcal{B} = \bigcup_{i<\lambda^+} \mathcal{B}_i$, $\mathcal{B}_i$ increasing continuous. $\mathcal{B}_i$ of power $\leq \lambda$. Suppose further $S_{\mathcal{B}} = \{\xi < \lambda^+ : cf \ \ \xi = \lambda, \mathcal{B}_\xi \lessdot \mathcal{B}\}$ is stationary. Then some forcing notion Q of power λ^+, forcing by it does not add new subsets of λ, (so all relevant properties of I, are preserved), and in V^Q, $S_{\mathcal{B}} \cup \{\xi < \lambda^+ : cf \ \ \xi < \lambda\}$ contains a closed unbounded set.

This help us to show the consistency of "$\mathcal{P}(\lambda)/I$ is the union of λ ultrafilters" for a suitable ideal I.

Proof : The well known $Q = \{f \ : f$ and increasing continuous function from some $\alpha+1 < \lambda^+$ to λ^+, $[\beta \leq \alpha$ and $cf(\alpha) = \lambda \Longrightarrow f(\alpha) \in S_{\mathcal{B}}]\}$.

* * *

6. Theorem : If $2^{\aleph_0} < 2^{\aleph_1}$ then $\mathcal{D}_{\omega_1}$ is not $\aleph_1$-dense (which means the Boolean algebra $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$ is not $\aleph_1$-dense.)

This will follow from Conclusion 14.

7. Definition ; A Boolean algebra $\mathcal{B}$ is λ-dense if there is $B \subseteq \mathcal{B}$, $|B| \leq \lambda$ which is dense i.e., $(\forall x \in \mathcal{B})[x \neq 0 \rightarrow (\exists y \in B)(0 < y \leq x)]$.

Note in this connection the following two observations.

8. Observation: By [FMS] we can obtain a universe of set theory [starting with a model of ZFC + 'κ is supercompact') in which $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated and this is preserved by forcing satisfying the $\aleph_1$-chain condition, so if we add e.g. $\beth_{\omega_1}$ Cohen reals, still $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturate but $2^{\aleph_0} = \beth_{\omega_1} < \beth_{\omega_1+1} = 2^{\aleph_1}$.

We may be interested in using smaller large cardinals:

8A. Observation: 1) It is consistent with ZFC that $2^{\aleph_0} < 2^{\aleph_1}$ but $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated if we assume the consistency of ZFC + "κ is a suitable hypermeasurable as in [SW]."

2) If in V, $\mathcal{D}$ is a normal filter on ω_1, and $\mathcal{D}$ is $\aleph_2$-saturated.

Q is the forcing of adding λ-Cohen reals, *then* in V^Q;

a) $\mathcal{D}' = \{A \in V^Q : A \subseteq \omega_1$ and $(\exists B \in \mathcal{D})\ B \subseteq A\}$ is $\aleph_2$-saturated normal filter [so $\mathcal{D} = (\mathcal{D}_{\omega_1})^V \Rightarrow \mathcal{D}' = (\mathcal{D}_{\omega_1})^{V^Q}$].

b) $(2^{\aleph_0})^{V^Q} = (\lambda + \aleph_0)^{\aleph_0}$ (the second term is computed in V).

c) $(2^{\aleph_1})^{V^Q} = (\lambda + \aleph_1)^{\aleph_1}$ (the second term is computed in V.)

Proof : 1) By 2), starting with a universe of set theory in which $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated, from Shelah and Woodin [SW] .

Note that if in V, $\beth_{\omega_1+1}(\kappa) > \beth_{\omega_1}(\kappa)^{+\alpha}$, κ is supercompact, and P a forcing notion of cardinality κ, such that in V^P, $\kappa = \aleph_2, \mathcal{D}_{\omega_1}$ $\aleph_2$-saturated; choose in (2) $\lambda = \beth_{\omega_1}(\kappa)$, then in $V^{P*Q}, (2^{\aleph_0})^{+\alpha} < 2^{\aleph_1}$.

2) Straightforward.

Suppose $Q = \{f : f$ a finite function from λ to $\{0,1\}\}$, and $q \in Q$, $q \Vdash_Q$ " $\langle \underset{\sim}{S}_\alpha : \alpha < \omega_2 \rangle$ is a counterexample: Let for $\alpha < \omega_2$, $S^0_\alpha = \{\delta < \omega_1$: there is q', $q \le q' \in Q$, $q' \Vdash$" $\delta \in \underset{\sim}{S}_\alpha$ "$\}$, and for $\delta \in S^0_\alpha$ choose $q^\alpha_\delta \in Q$, $q \le q^\alpha_\delta$, $q^\alpha_\delta \Vdash$ " $\delta \in \underset{\sim}{S}_\alpha$ ", (so $\langle \langle q^\alpha_\delta : \delta \in S^0_\alpha \rangle : \alpha < \omega_2 \rangle$ is in V) Clearly $S^0_\alpha \ne \phi \ mod\ \mathcal{D}$, hence for each $\alpha < \omega_2$ for some $k_\alpha < \omega$, $S'_\alpha = \{\delta \in S^0_\alpha :$ Dom q^α_δ has cardinality $k_\alpha\} \ne \phi \ mod\ \mathcal{D}$ hence for some k , $W = \{\alpha < \omega_2 : k_\alpha < k\}$ has cardinality $\aleph_2$. Let m be a natural number such that $m \to (3)^2_{2^{k^2}}$.

As $\mathcal{D}$ is $\aleph_2$-saturated there are distinct $\alpha_1, \ldots, \alpha_m \in W$ suc that $S \overset{def}{=} \bigcap_{\ell=1}^{m} S^1_{\alpha_\ell} \ne \phi \ mod\ \mathcal{D}$. For every $\delta \in S$ for some distinct

$\ell(1),\ell(2) \in \{1,\dots,m\}$, $q_\delta^{\alpha_{\ell(1)}}, q_\delta^{\alpha_{\ell(2)}}$, are compatible. Hence there are distinct $\ell(1),\ell(2) \in \{1,\dots,m\}$ such that $\{\delta \in \omega_1 : \delta \in S$, and $q^{\alpha_{\ell(1)}}, q_\delta^{\alpha_{\ell(2)}}$ are compatible$\} \neq \phi \bmod \mathcal{D}$. Now it is easy to show that for some $q', q \subseteq q' \in Q$, $q' \Vdash \{\delta \in S : q_\delta^{\alpha_{\ell(1)}} \cup q_\delta^{\alpha_{\ell(2)}} \in \underset{\sim Q}{G}\} \neq \bmod \mathcal{D}$ contradiction.

Remark: The inaccessible f needed in 8A(8) is $\{\kappa : \kappa$ strongly inaccessible with $Pr_2(\kappa)\}$ is stationary is not in the weak compactness ideal) "$\mathcal{D}_{\omega_1}$ is indestructible by $\aleph_1$-c.c. forcing big hyperinaccessible like in"

9. Observation: If $\mathcal{D}$ is a normal filter on a regular $\mu > \aleph_0, 2^\mu = \mu^+$ then the following are equivalent:

(a) $\mathcal{D}$ is μ-dense.

(b) there are normal filters $\mathcal{D}_i$ $(i < \mu)$, $\mathcal{D} \subseteq \mathcal{D}_i$, and $[A \neq \phi \bmod \mathcal{D} \Longrightarrow A \in \bigcup_{i<\mu} \mathcal{D}_i]$.

(c) for every $A_i \subseteq \lambda$, $A_i \neq \phi \bmod \mathcal{D}$ for $i < \mu^+$, *there is* $S \subseteq \mu^+, |S| = \mu^+$, such that for any distinct $i(\alpha) \in S$ $(\alpha < \lambda)$ the diagonal intersection of $A_{i(\alpha)}(\alpha < \lambda)$ (i.e. $\{\gamma < \lambda : \gamma \in \bigcap_{\alpha<\gamma} A_{i(\alpha)}\}$) is $\neq \phi \bmod \mathcal{D}$.

Proof : (a) $\Longrightarrow$ (b). Suppose $\{A_i/\mathcal{D} : i < \mu\}$ is a dense subset of $\mathcal{P}(\lambda)/\mathcal{D}$. Let (for $i < \mu$), $\mathcal{D}_i \overset{\text{def}}{=} \mathcal{D} + A_i = \{X \subseteq \lambda : X \cup (\lambda - A_i) \in \mathcal{D}\}$, then the $\mathcal{D}_i$'s exemplify that (b) holds.

(b) $\Longrightarrow$ (c): Let $\mathcal{D}_i(i < \mu)$ exemplify (b), and let $A_i \subseteq \mu$, $A_i \neq \phi \bmod \mathcal{D}$ for $i < \mu^+$. For each $i < \mu^+$ for some $\gamma(i) < \mu^+$, $A_i \in \mathcal{D}_{\gamma(i)}$. So for some γ $S = \{i : \gamma(i) = \gamma\}$ has power μ^+. Clearly $\{\gamma(i) : i \in S\}$ is as required.

(c) $\Longrightarrow$ (a): Assume (a) fails. Let $\{A \subseteq \mu : A \neq \phi \bmod \mathcal{D}\}$ be listed as $\{A_\alpha : \alpha < \mu^+\}$. As for $\xi < \mu^+$ $\{A_\alpha : \alpha < \xi\}$ cannot exemplify "$\mathcal{D}$ is μ-dense" there is $\alpha(\xi) < \mu^+$ such that for no $\beta < \xi$, $A_{\alpha(\xi)} \subseteq A_\beta \bmod \mathcal{D}$. By (c) there is $S \subseteq \mu^+$ of cardinality μ^+ such that for any $\alpha_i \in S$ $(i < \mu^+)$, $\{\gamma < \lambda : \gamma \in A_{\xi(\alpha_i)}$ for every $i < \gamma\} \neq \phi \bmod \mathcal{D}$. Let for $\zeta < \mu^+$, B_ζ be the diagonal intersection of $\{A_{\alpha(\xi)} : \xi < \zeta\}$. Note that B_ζ is not uniquely determined as a set (it depends on the enumeration of ζ) but *mod* $\mathcal{D}$ (and even *mod* $\mathcal{D}_\lambda$) it is uniquely determined. Clearly $\zeta_1 < \zeta_2 \Longrightarrow B_{\zeta_1} \supseteq B_{\zeta_2}$ *mod* $\mathcal{D}$. Now necessarily for some $\zeta(*)$ for every $\zeta \geq \zeta(*)$ (but $< \mu^+$), $B_\zeta = B_{\zeta(*)}$ *mod* $\mathcal{D}$, as otherwise there is an increasing sequence $\zeta(i)$ for $i < \mu^+$, such that $B_{\zeta(i+1)} \neq B_{\zeta(i)}$ *mod* $\mathcal{D}$, so $\{B_{\zeta(i+1)} - B_{\zeta(i)} : i < \mu^+\}$ show $\mathcal{D}$ is not μ^+-saturated and clearly contradict (c) which we are assuming.

Now as $B_{\zeta(*)} \neq \phi$ *mod* $\mathcal{D}$, for some $\gamma(*) < \mu^+$, $B_{\zeta(*)} = A_{\gamma(*)}$. Choose $\beta < \mu^+$, $\beta > \gamma(*)$, $\beta > \zeta(*)$. So by the choice of $\zeta(*)$ $B_{\beta+1} = B_{\zeta(*)}$ *mod* $\mathcal{D}$ but by the choice of $B_{\beta+1}$, $B_{\beta+1} \subseteq A_{\xi(\beta)}$ *mod* $\mathcal{D}$ hence $B_{\zeta(*)} \subseteq A_{\xi(\beta)}$ *mod* $\mathcal{D}$ but $B_{\zeta(*)} = A_{\gamma(*)}$ so $A_{\gamma(*)} \subseteq A_{\xi(\beta)}$ *mod* $\mathcal{D}$. But remember the choice of $\xi(\beta)$, as $\beta > \gamma(*)$ it implies $A_{\gamma(*)} \not\subseteq A_{\xi(\beta)}$ *mod* $\mathcal{D}$. Contradiction.

10. Definition: 1) For a regular uncountable λ and $\mu < 2^\lambda$ let

(a) Dom $(\lambda,\mu) = \{f : f$ a function with domain ${}^{\omega>}\alpha - \{\Lambda\}$ for some ordinal $\alpha < \lambda, f(\eta) < \mu$, for $\eta \in {}^{\omega\geq}\alpha - \{\Lambda\}$, where Λ is the empty sequence.

(b) Dom $^+(\lambda,\mu) = \{f : f$ a function from ${}^{\omega>}\lambda - \{\Lambda\}$ to $\mu\}$.

(c) Let $I_{\lambda,\mu}$ be the set of $A \subseteq \lambda$ such that :

for some function F from Dom (λ,μ) to $\{0,1\}$, for every $h : A \to \{0,1\}$ there is $f \in$ Dom $^+(\lambda,\mu)$ such that for some $C \in \mathcal{D}_\lambda$ $(\forall \delta \in A \cap C)\,[h(\delta) = F(f \restriction \delta)]$.

2) For λ,μ as above and function F from Dom (λ,μ) to $\{0,1\}$ let $I^F_{\lambda,\mu}$ be the set of $A \subseteq \lambda$ such that ; for every $B \subseteq A$, there is $f \in$ Dom (λ,μ) such that for some $C \in \mathcal{D}_\lambda$

$$(\forall \delta \in C)[\delta \in B \quad iff \quad F(f \restriction \delta) = 1]$$

3) For λ,μ, F as above let $J^F_{\lambda,\mu}$ be the normal ideal on λ which $I^F_{\lambda,\mu}$ generates.

Remark: This is close by related with the weak diamond, see Devlin and Shelah [SD] and Shelah [Sh , Ch. XIV , §1].

11. Lemma : 1) $I_{\lambda,\mu}$ is a normal ideal on λ (but it may be $\mathcal{P}(\lambda)$) and we could have in the definition of Dom (λ,μ) replace ${}^{\omega>}\alpha$ by α .

2) If $\kappa < \lambda$, $2^\kappa = 2^{<\lambda}$, $\mu = \mu^{<\lambda} < 2^\lambda$, $\mu < \lambda^{+\lambda}$ (i.e. $\mu < \aleph_{\alpha+\lambda}$ where $\lambda = \aleph_\alpha$) (or even a weaker restriction) then $\lambda \notin I_{\lambda,\mu}$.

3) $I^F_{\lambda,\mu} \subseteq J^F_{\lambda,\mu} \subseteq I_{\lambda,\mu}$, and $I_{\lambda,\mu} = \bigcup \{I^F_{\lambda,\mu} : F$ a function from Dom (λ,μ) to $\{0,1\}\}$.

4) For every function $F : \text{Dom}\ (\lambda,\mu) \to \{0,1\}$, there is a function $F^* : \text{Dom}\ (\lambda,\mu) \to \{0,1\}$ such that

$$J^{F^*}_{\lambda,\mu} = I^{F^*}_{\lambda,\mu} = J^F_{\lambda,\mu}$$

5) For any function $F : \text{Dom}\ (\lambda,\mu) \to \{0,1\}$, for every $C \in \mathcal{D}_\lambda$, $\lambda - C \in I^F_{\lambda,\mu}$.

Proof : Part 1) is straightforward. For 2) see [Sh 2 , Ch. XIV §1]. Now (3), (5) are trivial and for (4), note that in Definition 10(2) we demand $(\forall \delta \in C)[\delta \in B \Longrightarrow F(f \restriction \delta) = 1]$ and not just $(\forall \delta \in C \cap A)[\delta \in B \Longleftrightarrow F(f \restriction \delta) = 1]$.

12. Lemma : Suppose λ is regular and uncountable, $\mu < 2^\lambda$, and $\lambda \notin I_{\lambda,\mu}$.

Then for no F is $J^F_{\lambda,\mu}$ μ-dense, λ^+-saturated.

Proof : Suppose F is a counterexample and let $\{A_i / J^F_{\lambda,\mu} : i < \mu\}$ be a dense subset of $\mathcal{P}(\lambda) / J^F_{\lambda,\mu}$. We now define a function H from Dom $(\lambda,\mu) = \bigcup\{f : f$ a function from some ${}^{\omega>}\delta - \{\Lambda\}$ into μ where $\delta < \lambda\}$ to $\{0,1\}$.

Suppose $\delta < \lambda$ is limit, $f : ({}^{\omega>}\delta - \{\Lambda\}) \to \mu$, for $\nu \in {}^{\omega>}\delta$ let f_ν be the function from ${}^{\omega>}\delta - \{\Lambda\}$ to $\{0,1\}$ defined by $f_\nu(\eta) = f(\nu^\frown\eta)$. We define $H(f)$ by cases:

Case I: For some $\alpha,\beta < \delta$, $F(f_{<0,\alpha,\beta>}) = 1$.

Then we let $H(f)$, be $F(f_{<1,\alpha,\beta>})$ for the minimal such α,β (lexicographically).

Case II: Not Case I, but for some $\alpha < \delta$, $\delta \in A_{<2,\alpha>}$.

Then $H(f) = f(<3,\alpha>)$ for the minimal such α.

Case III: Not Case I nor II.

Then $H(f) = 0$.

If $f : {}^{\omega>}\alpha - \{\Lambda\} \to \mu$, α not limit, let $H(f) = 0$.

Now we get contradiction by Fact 12A below (as $\lambda \notin I_{\lambda,\mu}$, $I_{\lambda,\mu}$ is normal and $J^H_{\lambda,\mu} \subseteq I_{\lambda,\mu}$).

12A Fact: $\lambda \in I^H_{\lambda,\mu}$.

Let $B \subseteq \lambda$ and we shall find $f \in \mathrm{Dom}^+(\lambda,\mu)$ such that for some $C \in \mathcal{D}_\lambda$, $(\forall \delta \in C)[\delta \in B \text{ iff } H(f \restriction \delta) = 1]$.

Let $\mathcal{P} \subseteq \{A_i : i < \mu\}$ be a maximal subset satisfying:

(a) for every $a \neq b \in \mathcal{P}$, $a \cap b \in J^F_{\lambda,\mu}$ (i.e. $\mathcal{P}$ is $J^F_{\lambda,\mu}$- disjoint.)

(b) for every $a \in \mathcal{P}$, $a \subseteq B \bmod J^F_{\lambda,\mu}$ or $a \cap B = \phi \bmod J^F_{\lambda,\mu}$.

As F is a counterexample, $\mathcal{P}(\lambda)/J^F_{\lambda,\mu}$ is λ^+-saturated hence $|\mathcal{P}| \leq \lambda$, so let $\mathcal{P} = \{A_{i(\alpha)} : \alpha < \alpha(*)\}$, $\alpha(*) \leq \lambda$. We shall assume $\alpha(*) = \lambda$ (the other case is easier). Let B^* be the diagonal union of the $A_{i(\alpha)}$ i.e. $\{\beta < \lambda : \beta \in \bigcup_{\alpha<\beta} A_{i(\alpha)}\}$, so clearly $a_0 \stackrel{\text{def}}{=} \lambda - B^* \in J^F_{\lambda,\mu}$. For each $\alpha < \lambda$ let $a_{1+\alpha}$ be $A_{i(\alpha)} - B$ if $A_{i(\alpha)} \subseteq B \bmod J^F_{\lambda,\mu}$ and $A_{i(\alpha)} \cap B$ if $A_{i(\alpha)} \cap B = \phi \bmod J^F_{\lambda,\mu}$. So in any case $a_\alpha \in J^F_{\lambda,\mu}$, so there are sets $a_{\alpha,\beta} \in I^F_{\lambda,\mu}$, (for $\beta < \lambda$) such that $a_\alpha = \{\gamma < \lambda : \gamma \in \bigcup_{\beta<\gamma} a_{\alpha,1+\beta}\}$. As $a_{\alpha,\beta} \in I^F_{\lambda,\mu}$ there are functions $f^0_{\alpha,\beta}, f^1_{\alpha,\beta}$ from ${}^{\omega>}\lambda - \{\Lambda\}$ to μ, such that for some $C_{\alpha,\beta} \in \mathcal{D}_\lambda$:

$$(\forall \delta \in C_{\alpha,\beta})[\delta \in a_{\alpha,\beta} \cap B \Longleftrightarrow F(f^1_{\alpha,\beta} \restriction \delta) = 1]$$

$$(\forall \delta \in C_{\alpha,\beta})[\delta \in a_{\alpha,\beta} \Longleftrightarrow F(f^0_{\alpha,\beta} \restriction \delta) = 1]$$

Now we can define $f^* : ({}^{\omega>}\lambda - \{\Lambda\}) \to \mu$

$$f^*(<0,\alpha,\beta> {}^\frown \eta) = f^0_{\alpha,\beta}(\eta)$$

$$f^*(<1,\alpha,\beta>) = f^1_{\alpha,\beta}(\eta)$$

$f^*(<2,\alpha>) = 1 \qquad$ if $\delta \in A_{i(\alpha)}$,

$f^*(<3,\alpha>) = 1 \qquad$ if $\delta \in A_{i(\alpha)} \subseteq B \bmod I_{\lambda,\mu}$

$f^*(\eta) = 0 \qquad$ *otherwise*.

It is easy to check that $\{\delta : H(f^* \restriction \delta) = 1 \iff \delta \in B\}$ belong to $\mathcal{D}_\lambda$. As B was any subset of λ this shows $\lambda \in I^H_{\lambda,\mu}$ but $I^H_{\lambda,\mu} \subseteq I_{\lambda,\mu}$, $\lambda \not\in I_{\lambda,\mu}$, contradiction.

13. Conclusion: Suppose λ is regular uncountable and $\lambda \not\in I_{\lambda,\mu}$ (see 11(1)). Then $\mathcal{D}_\lambda$ is not μ-dense, λ^+-saturated.

Proof: As $\mathcal{D}_\lambda$ is λ^+-saturated, and $I_{\lambda,\mu}$ a normal ideal on λ, it is known that for every appropriate F, for some $Y(F) \subseteq \lambda$ $Y(F) \neq \phi \bmod \mathcal{D}_\lambda$ and $J^F_{\lambda,\mu} = \{A \subseteq \lambda: (Y(F)-A) \cup (\lambda - Y(F)) \in \mathcal{D}_\lambda\}$ and so $J^F_{\lambda,\mu}$ is μ-dense λ^+-saturated too contradicting 12.

14. Conclusion: If $\lambda = \kappa^+, 2^\lambda > 2^\kappa$, $\mu = \mu^{<\lambda} < Min\{2^\lambda, \lambda^{+\lambda}\} < 2^\lambda$ then $\mathcal{D}_\lambda$ cannot be λ^+-saturated, μ-dense.

Proof : By 13 and 11(2) (so we could get a little more).

References

[BHM]

J. E. Baumgartner, A. Hajnal and A.Mate. Weak saturation propertes of ideals. *Infinite and Finite Sets*. Proc. of a Symp. for Erdos 60th Birthday, Budapest 1973. Colloq. Math. Soc. Jano Bolayi 10 ed. . Hajnal, R. Rado and T. Sos, North-Holland Publ. Co. Vol 11 (1975), 137-158.

[DS] K. Devlin and S. Shelah, A weak form of the diamond follows from $2^{\aleph_0} < 2^{\aleph_1}$, *Israel J. Math.* 29 (1978), 239-247.

[FMR]

M. Forman, M. Magidor and S. Shelah, In preparation.

[KE] R. Engelking and M. Karlowicz. Some theorems of set theory and their topological consequences. *Fund Math.* 57 (1965), 275-285.

[Sh1]
S. Shelah, Remarks on Boolean algebras, *Algebra Universalis*, 11 (1980) , 77-84.

[Sh2]
--------, Proper Forcing, *Springer Lecture Notes*, 940 (1982).

[Sh3]
---------, On saturation for a predicate, *Notre Dame J. of Formal Logic*, 22 (1981), 301-307.

[Sh4]
--------, A note on κ-freeness, *A Springer Lecture Notes*, volume here.

[Sh5]
--------, From supercompacts to special normal ideals on small cardinal, in preparation.

[SW] S. Shelah, and H. Woodin, Hypermeasurability cardinals implies every projective set is Lebesgue measurable, In preparation.