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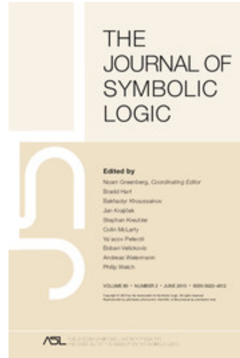

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ON THE EXPRESSIBILITY HIERARCHY OF MAGIDOR-MALITZ QUANTIFIERS

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Abstract. We prove that the logics of Magidor-Malitz and their generalization by Rubin are distinct even for *PC* classes.

Let $M \models Q^n x_1 \cdots x_n \varphi(x_1 \cdots x_n)$ mean that there is an uncountable subset A of $|M|$ such that for every $a_1, \ldots, a_n \in A$, $M \models \varphi[a_1, \ldots, a_n]$.

THEOREM 1.1 (SHELAH) ($\diamondsuit_{\aleph_1}$). *For every $n \in \omega$ the class $K_{n+1} = \{\langle A, R\rangle \mid \langle A, R\rangle \models \neg Q^{n+1} x_1 \cdots x_{n+1} R(x_1, \ldots, x_{n+1})\}$ is not an $\aleph_0$-PC-class in the logic $\mathscr{L}^n$, obtained by closing first order logic under $Q^1, \ldots, Q^n$. I.e. for no countable $\mathscr{L}^n$-theory T, is K_{n+1} the class of reducts of the models of T.*

THEOREM 1.2 (RUBIN) ($\diamondsuit_{\aleph_1}$).[3] *Let $M \models Q^E x\, y\, \varphi(x, y)$ mean that there is $A \subseteq |M|$ such that $E_{A,\varphi} = \{\langle a, b\rangle \mid a, b \in A$ and $M \models \varphi[a, b]\}$ is an equivalence relation on A with uncountably many equivalence classes, and such that each equivalence class is uncountable. Let $K^E = \{\langle A, R\rangle \mid \langle A, R\rangle \models \neg Q^E xyR(x, y)\}$. Then K^E is not an $\aleph_0$-PC-class in the logic gotten by closing first order logic under the set of quantifiers $\{Q^n \mid n \in \omega\}$ which were defined in Theorem* 1.1.

§1. Introduction. In [MM] Magidor and Malitz define for every $0 < n < \omega$ the quantifier Q^n. The κ-interpretation of Q^n is defined as follows: $M \models Q^n x_1 \cdots x_n \varphi(x_1, \ldots, x_n)$ iff there is $A \subseteq |M|$ such that $|A| = \kappa$ and for every $a_1, \ldots, a_n \in A$ $M \models \varphi[a_1, \ldots, a_n]$. Let $\mathscr{L}^n$ be the logic obtained by closing first order logic under Q^n, and $\mathscr{L}^{MM}$ be the logic obtained by closing first order logic under $\{Q^n \mid n \in \omega\}$.

Magidor and Malitz prove in [MM] that if $\diamondsuit_{\aleph_1}$ is assumed, then $\mathscr{L}^{MM}$ is countably compact in the $\aleph_1$-interpretation; i.e. if every finite subset of a countable set of sentences T of $\mathscr{L}^{MM}$ has a model, then T has a model. They also prove a completeness theorem for the $\aleph_1$-interpretation of $\mathscr{L}^{MM}$.

Also in [MM] the $<\kappa$ compactness for the κ-interpretation is proved when κ is weakly compact.

Shelah in [S1] proved, assuming $\diamondsuit_\kappa$ and $\diamondsuit_{\kappa^+}$, that $\mathscr{L}^{MM}$ is κ-compact in the κ^+-intepretation.

In two yet unpublished theorems Shelah proved: (1) It is consistent with ZFC and even with CH that $\mathscr{L}^{MM}$ is not countably compact in the $\aleph_1$-interpretation

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[3]A more general version of the theorem is proved in the paper.

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(this continues work of U. Avraham); (2) Consider $\mathscr{L}^{MM}$ in the $\aleph_1$-interpretation. If P is the partially ordered set for adding a Cohen real, then no matter what V is, $\mathscr{L}^{MM}$ is countably compact in V^P and it has the same validities as in any universe satisfying $\diamondsuit_{\aleph_1}$.

Clearly if $m > n$ then $\mathscr{L}^m$ is more expressible than $\mathscr{L}^n$; i.e. every $\mathscr{L}^n$ formula is logically equivalent to an $\mathscr{L}^m$ formula. J. Malitz asked whether $\mathscr{L}^m$ is indeed strictly more expressible than $\mathscr{L}^n$. More precisely, whether there is an $\mathscr{L}^m$-elementary class which is not an $\mathscr{L}_n$-$\aleph_0$-PC class. In Theorem 1.1 Shelah gives a positive answer to this question assuming $\diamondsuit_{\aleph_1}$.

Shelah (unpublished), and independently Garavaglia [G], proved in ZFC that for every $n \in \omega$ there are models M_1, M_2 which are $\mathscr{L}^n$ equivalent but not $\mathscr{L}^{n+1}$ equivalent.

Theorem 1.2 is a generalization of Theorem 1.1 for a larger hierarchy of quantifiers. $Q^E xy\varphi(x, y)$ can be expressed as a sentence using one of the new quantifiers to be defined in the sequel.

For $n \geq 0$ let $\bar{M}_n = \langle A, E_1, \ldots, E_n\rangle$, where $|A| = \aleph_1$, each E_i is an equivalence relation on A, for every $1 \leq i \leq n-1$ E_{i+1} refines E_i in such a way that every equivalence class of E_i is partitioned into $\aleph_1$ equivalence classes of E_{i+1}, E_1 has $\aleph_1$ equivalence classes, and every equivalence class of E_n has power $\aleph_1$.

Now we define the quantifier $Q^{n,m}$. Let $t_1, \ldots, t_r$ be the set of complete types with m variables in $\bar{M}_n$. $Q^{n,m}$ will bound m variables and will be applied to an r-tuple of formulas. $M \models Q^{n,m}x_1 \cdots x_m(\psi_1, \ldots, \psi_r)$ means that there is a function $h\colon |\bar{M}_n| \to |M|$ such that for every $\bar{a} \in |M_n|^m$: if $\bar{a}$ has the type t_i, then $M \models \psi_i[h(\bar{a})]$. Here we described the $\aleph_1$-interpretation of $Q^{n,m}$, the κ-interpretation is defined in a similar way. Note that $Q^{n,1}$ is just the cardinality quantifier and that $Q^{0,m}$ is just Q^m. Clearly

$$Q^E xy(x, y) \equiv Q^{1,2}xy(\varphi(x, y), x \neq y \wedge \varphi(x, y), x \neq y \wedge \neg\varphi(x, y))$$

where t_1, t_2, t_3 are the types $x = y$, $x \neq y \wedge E_1(x, y)$, and $\neg E_1(x, y)$ of $\bar{M}_1$, respectively. $Q^{1,2}$ is probably more expressive than Q^E in the sense defined in this work, but we do not know how to prove this. It seems that the class of models of $Q^{1,2}\, xy(R(x, y) \wedge S(x, y), R(x, y), S(x, y))$ is not an $\aleph_0$-PC class in $\mathscr{L}(Q^E)$. If $n \geq n'$ and $m \geq m'$, then $Q^{n',m'}$ can be expressed in terms of $Q^{n,m}$. Finally $Q^2xy\varphi(x, y)$ is PC expressible in terms of Q^E. For suppose $p(x, y)$ is a 1-1 pairing function and $\langle p_1(z), p_2(z)\rangle$ is p^{-1}; then

$$Q^2xy\varphi(x, y) \equiv Q^E xy(\varphi(p_1(x), p_1(y)) \wedge p_2(x) = p_2(y)).$$

Let $\bar{\mathscr{L}}$ be the logic obtained by closing first order logic under $\{Q^{n,m} | n, m \in \omega\}$. In [MR] assuming $\diamondsuit_{\aleph_1}$, Rubin proved that $\bar{\mathscr{L}}$ is countably compact in the $\aleph_1$-interpretation, and $< \kappa$ compact in the κ weakly compact interpretation. The proof for $\aleph_1$ yields a completeness theorem. The omitting type theorem of Shelah [S1] can be used to prove the λ-compactness of $\bar{\mathscr{L}}$ in the λ^+-interpretation, under the same set-theoretic assumptions, and in complete analogy to the proof of the $\mathscr{L}^{MM}$-compactness given there.

Let $\mathscr{L}^{n,m} = \mathscr{L}(Q^{n,m})$ and $\mathscr{L}^{[n,m]} = \mathscr{L}(\{Q^{n',m'} | n' < n \text{ or } m' < m\})$. Let $n \geq 0$

and $m \geq 2$. We define $M_{n,m} = \langle A, E_1, \ldots, E_n, R_{n,m}\rangle$, where $\langle A, E_1, \ldots, E_n\rangle = \bar{M}_n$ and $R_{n,m} = \{\langle a_1, \ldots, a_m\rangle \mid \text{for every } 1 \leq i \leq j \leq m\ a_i E_n a_j\}$. Let $\psi_{n,m}$ be the sentence with predicate symbols $E_1, \ldots, E_n$ and R saying that for some $B \subseteq |M|$ $\langle B, E_1 \restriction B, \ldots, E_n \restriction B, R \restriction B\rangle$ is isomorphic to $M_{n,m}$. $\psi_{n,m}$ can be written as a sentence in $\mathscr{L}^{n,m}$. $\psi_{1,2}$ can be written using Q^E only, not $Q^{1,2}$.

THEOREM 1.2 ($\diamondsuit_{\aleph_1}$). *Let $n \geq 0$ and $m > 1$, and let $K_{n,m} = \{M \mid M \models \neg\, \psi_{n,m}\}$. Then $K_{n,m}$ is not an $\mathscr{L}^{[n,m]}$-$\aleph_0$-PC-class.*

We did not deal with the problem whether K_m or $K_{n,m}$ can be PC-classes or $\aleph_1$-PC-classes in the weaker logics. In view of [Ra] and [Ma, Re] such questions might have a different nature.

Theorem 1.1 is a special case of Theorem 1.2. We bring it here because its combinatorial details are simpler, and so it might be helpful to understand the framework of both proofs using Theorem 1.1 as a model.

Lastly we want to mention a question about another possible generalization of Magidor-Malitz quantifiers. Let K be a class of models in the same finite similarity type. Let $Q_K\, x_1 \cdots x_n(\psi_1, \ldots, \psi_m)$ be the quantifier saying the following about a model M: $M \models Q_K x_1 \cdots x_n(\psi_1, \ldots, \psi_m)$ iff there is $N \in K$ and $B \subseteq |M|$ such that $\langle B, \psi_1^M | B, \ldots, \psi_m^M | B\rangle \cong N$. Note that n is the maximal number of places in a predicate in the language of K and m is the number of predicates in the language of K.

Investigate when Q_K is compact. E.g., we do not know whether Q_K is countably campact when $K = \{M\}$ and $M = \langle \boldsymbol{R}, <\,\rangle$ or M is the saturated linear ordering of power $\aleph_1$.

§§2 and 3 include results of S. Shelah. §4 includes results of Rubin.

§2. Description of the method of proof. The framework of the proof of both theorems is the same, so to be specific we choose to describe the proof of Theorem 1.1.

We will use forcing methods and then apply an absoluteness argument to get back to the ground model.

Assume $V \models \diamondsuit_{\aleph_1}$. Let us deal with a fixed but arbitrary $n \in \omega - \{0\}$. This n is fixed for §§2 and 3. Let R be an $n+1$-place relation symbol and $\psi \equiv Q^{n+1}\, x_1 \cdots x_{n+1} R(x_1, \ldots, x_{n+1})$.

We will construct a model M with the following properties:

(1) $M \models \neg\psi$.

(2) There is a set of forcing conditions P such that:

(a) In V^P $M \models \psi$;

(b) For every model $N \in V$ the $\mathscr{L}^n$-theory of N in V^P is equal to the $\mathscr{L}^n$-theory of N in V. (Note that ψ is in $\mathscr{L}^{n+1}$.)

Let us see why modulo the countable compactness, and a certain completeness theorem for $\mathscr{L}^{n+1}$, the existence of such a model M implies that $K \stackrel{\text{def}}{=} \{\langle A, R\rangle \mid \langle A, R\rangle \models \neg\, \psi\}$ is not an $\mathscr{L}^n$-$\aleph_0$-PC class.

We first describe the completeness theorem which is needed; it appears (at least implicitly) in [MM] and, for the case of $\mathscr{L}$, in [MR].

THEOREM (MAGIDOR AND MALITZ [MM]). *A recursively enumerable set of formulas $C \subseteq \mathscr{L}^{n+1}$ is presented, a proof from ZFC is given that $\forall\varphi(\varphi \in C \to \varphi$ is* *true in every model), and a proof from ZFC + $\diamondsuit_{\aleph_1}$ is given that $\forall\varphi(\varphi$ is true in every model $\rightarrow \varphi \in C)$.*

Suppose now by contradiction, K is an $\mathscr{L}^n$-$\aleph_0$-PC class, i.e. for some countable $\mathscr{L}^n$-theory $T \in V$, K is the class of reducts of models of T. So M can be expanded to a model M^* of T. Let P be the set of forcing conditions mentioned in (2). Then in V^P $M^* \models T \cup \{\psi\}$. We show that $T \cup \{\psi\}$ is finitely satisfiable in V. Suppose not; then for some finite $T_0 \subseteq T$, $T_0 \cup \{\psi\}$ does not have a model. So, since $\diamondsuit_{\aleph_1}$ holds in V by the second part of the completeness theorem $\neg(\bigwedge T_0 \wedge \psi) \in C$. Since belonging to C is a Δ_0-formula in Levi's Hierarchy it is absolute so

$$\neg(\bigwedge T_0 \wedge \psi) \in C \quad \text{in } V^P.$$

By the first part of the completeness theorem, $\bigwedge T_0 \wedge \psi$ does not have a model in V^P, a contradiction. So by the countable compactness of $\mathscr{L}^{n+1}$ $T \cup \{\psi\}$ has a model in V, and so K_{n+1} is not the class of reducts of T, a contradiction. This shows that it suffices to construct a model M with properties (1) and (2) mentioned above.

The next goal is to find a property S of forcing notions that will assure that the $\mathscr{L}^n$-theories of old models do not change after forcing with a forcing notion that satisfies S.

DEFINITION. P is an S_n-forcing if either $n = 1$ and P is c.c.c., or for every uncountable subset B of P, there is an uncountable subset B' of B, such that for every $q_1, \ldots, q_n \in B'$, there is $q \in P$, such that $q \geq q_i$ for every $i = 1, \ldots, n$. We call such a B' an n-compatible set.

S_n-forcings appear in [KT], where Suslin trees are shown to be preserved by them.

THEOREM 2.1. *Let P be an S_n-forcing and $M \in V$; then for every $\mathscr{L}^n$-formula φ and every $a_1, \ldots, a_m \in |M|$: $M \models \varphi[a_1, \ldots, a_m]$ in V iff $M \models \varphi[a_1, \ldots, a_m]$ in V^P.*

PROOF. The proof is trivial for $n = 1$; we thus assume that $n > 1$. Note that P satisfies the c.c.c. We prove by induction on the structure of φ that for every P-generic extension V^P of V and for every $a_1, \ldots, a_n \in |M|$ $M \models \varphi[a_1, \ldots, a_m]$ in V iff $M \models \varphi[a_1, \ldots, a_m]$ in V^P. The only less trivial case is when $\varphi \equiv Q^n x_1 \cdots x_n \chi(x_1, \ldots, x_n)$. ($\chi$ might contain parameters.) Suppose $M \models \varphi$ in V. Let A be an uncountable subset of $|M|$ such that for every $a_1, \ldots, a_n \in A$ $M \models \chi[a_1, \ldots, a_n]$ in V. Since P is c.c.c., A is uncountable in V^P, and by the induction hypothesis for every $a_1, \ldots, a_n \in A$ $M \models \chi[a_1, \ldots, a_n]$ in V^P, so $M \models \varphi$ in V^P.

Suppose $M \models \varphi$ in V^P. So there is a P-name τ and $q \in P$ such that $q \Vdash_P$ "τ is an uncountable subset of $|M|$ and for every $a_1, \ldots, a_n \in \tau$ $M \models \chi[a_1, \ldots, a_n]$". So $\{a | a \in |M| \wedge (\exists p \geq q)(p \Vdash \tilde{a} \in \tau)\}$ is uncountable in V. ($\tilde{a}$ is the standard name of a.) Let $\{\langle p_i, a_i\rangle | i < \aleph_1\}$ be a set such that: $q \leq p_i$, $p_i \Vdash \tilde{a}_i \in \tau$, and if $i \neq j$ then $a_i \neq a_j$. Let $B' \subseteq \{p_i | i < \aleph_1\}$ be an n-compatible subset of $\{p_i | i < \aleph_1\}$, and let $A' = \{a_i \mid p_i \in B'\}$. Let $a_{i_1}, \ldots, a_{i_n} \in A'$. Let $r \geq p_{i_1}, \ldots, p_{i_n}$. So $r \Vdash \tilde{a}_{i_1}, \ldots, \tilde{a}_{i_n} \in \tau$ and by the induction hypothesis $M \models \chi[a_{i_1}, \ldots, a_{i_n}]$ in V; so $M \models \varphi$ in V. Q.E.D.

DEFINITION. Let $M = \langle A, R\rangle$, R an $n + 1$-place relation; a subset B of A is called positive homogeneous (PH) if $B^{n+1} \subseteq R$.

For a model M as in the definition, let $P_M = \{\sigma | \sigma$ is a finite PH subset of $|M|\}$; P_M is partially ordered by inclusion.

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We now go back to the model M with properties (1) and (2) that we wish to construct. The P needed in condition (2) will be P_M. So we want that P_M will be an S_n-forcing.

The requirements from M are now clear, and it turns out that to construct such a model it is sufficient to assume CH.

THEOREM 2.2 (CH). *There is a model $M = \langle A, R\rangle$, R an $n+1$-place relation, with the following properties:*

(1) $M \models \neg \psi$;

(2) $\Vdash_{P_M} (M \models \psi)$;

(3) *P_M is an S_n-forcing.*

This concludes the description of the proof.

§3. **The construction of M.** We first deal with the case when $n > 1$. The case $n = 1$ is easier and is dealt with at the end of §3.

The universe of M will be $\aleph_1$. Since we want that in some generic extension M will satisfy ψ, and in order to prevent trivial problems, we decide that R will be symmetric, i.e. if $\langle a_1, \ldots, a_{n+1}\rangle = \bar{a} \in R$, then every permutation of $\bar{a}$ belongs to R. We also need that $R \supseteq T \stackrel{\text{def}}{=} \{\langle a_1, \ldots, a_{n+1}\rangle \mid a_1, \ldots, a_{n+1} \in \aleph_1$ and for some $i \neq j\ a_i = a_j\}$.

DEFINITION. A finitary system (FS) is a countable family of pairwise disjoint finite sets.

We make two lists: a list $\{G_i | i < \aleph_1\}$ of all subsets of $\aleph_1$ of power $\aleph_0$, and a list $\{D^i | i < \aleph_1\}$ of all FS's D such that $\bigcup D \subseteq \aleph_1$. For technical convenience we assume that for every i $G_i \subseteq i \cup \omega$ and $\bigcup D^i \subseteq i \cup \omega$. Let us denote $D^i = \{d^i_m | m \in \omega\}$.

We will arrange that R will have the following three properties.

(*) For every $\beta < \alpha < \aleph_1$ such that $\omega \leq \alpha$, there are $a_1, \ldots, a_n \in G_\beta$ such that $\langle a_1, \ldots, a_n, \alpha\rangle \notin R$.

(**) For every $\beta < \alpha < \aleph_1$ such that $\omega \leq \alpha$, and for every finite e_0 such that $d^\beta_0 \subseteq e_0 \subseteq \alpha - \bigcup_{0<m\in\omega} d^\beta_m$ there is a finite subset σ of ω such that: for every $a_1, \ldots, a_n \in \alpha$, if $\{a_1, \ldots, a_n\} \not\subseteq e_0$ and there are $i_1, \ldots, i_{n-1} \in \omega - \sigma$ such that $\{a_1, \ldots, a_n\} \subseteq e_0 \cup \bigcup_{j=1}^{n-1} d^\beta_{i_j}$, then $\langle a_1, \ldots, a_n, \alpha\rangle \in R$.

(***) For every finite $\sigma \subseteq \aleph_1$ the set $\{\alpha \mid$ for every $\{a_1, \ldots, a_n\} \subseteq \sigma \langle a_1, \ldots, a_n, \alpha\rangle \in R\}$ is uncountable.

We first show that a relation which satisfies (*), (**) and (***) is as required in Theorem 2.2.

LEMMA 3.1. *Let $R \subseteq \aleph_1^{n+1}$ be symmetric and $R \supseteq T$. Suppose R satisfies* (*), (**) *and* (***) *and let $M = \langle A, R\rangle$. Then*

(1) $M \models \neg \psi$;

(2) $\Vdash_{P_M} (M \models \psi)$;

(3) *P_M is an S_n-forcing.*

PROOF. (1) Let $A \subseteq \aleph_1$ be uncountable and let G be a subset of A of power $\aleph_0$. So, for some i $G = G_i$. Let $i < \alpha \in A$; then by (*) there are $a_1, \ldots, a_n \in G$ such that $\langle a_1, \ldots, a_n, \alpha\rangle \notin R$, so A is not PH.

It is well known that (2) follows from (***).

(3) Let A be an uncountable subset of P_M. W.l.o.g. $A = \{e_i | i < \aleph_1\}$, where

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$e_i = \{\alpha_1, \ldots, \alpha_k\} \cup \{\alpha_{i,1}, \ldots, \alpha_{i,m}\}$ and for every $i < j < \aleph_1$: $\omega \leq \alpha_1 < \cdots < \alpha_k < \alpha_{i,1} < \cdots < \alpha_{i,m} < \alpha_{j,1}$. Denote $\{\alpha_1, \ldots, \alpha_k\} = e$. We define by induction a subsequence $\{e_{i_\nu} | \nu < \aleph_1\}$ of $\{e_i \mid i < \aleph_1\}$. Let $i_0 = 0$. Suppose $i_{\nu'}$ has been defined for every $\nu' < \nu$. Let $D = \{d_i | i \in \omega\}$, where $d_0 = e$ and $\{d_i | 0 < i < \omega\} = \{e_{i_{\nu'}} - e | \nu' < \nu\}$. So, for some $\beta < \aleph_1$, $D = D^\beta$. Let i_ν be the first ordinal i such that $\beta < \alpha_{i,1}$.

By renaming (denoting e_{i_ν} by e_ν), we can assume that $\{e_i | i \in \aleph_1\}$ has the following property: for every i there is β_i such that $d_0^{\beta_i} = e$, $\{d_l^{\beta_i} \mid 0 < l \in \omega\} = \{e_j - e | j < i\}$, and $\beta_i < \alpha_{i,1}$.

We now prove:

(****) For every i there is a finite subset $\sigma_i \subseteq i$ such that for $a_1, \ldots, a_{n+1}$ if $\{a_1, \ldots, a_{n+1}\} \cap (e_i - e) \neq \emptyset$, and there are $i_1, \ldots, i_{n-1} \in i - \sigma_i$ such that $\{a_1, \ldots, a_{n+1}\} \subseteq e_i \cup \bigcup_{j=1}^{n-1} e_{i_j}$, then $\langle a_1, \ldots, a_{n+1}\rangle \in R$. By (**), and since $\beta_i < \alpha_{i,1}$, for every $1 \leq l \leq m$, there is a finite subset σ^l of i such that for every $a_1, \ldots, a_n$, if $\{a_1, \ldots, a_n\} \subsetneq e \cup \{\alpha_{i,1}, \ldots, \alpha_{i,l-1}\}$ and there are $i_1, \ldots, i_{n-1} \in i - \sigma^l$ such that $\{a_1, \ldots, a_n\} \subseteq e \cup \{\alpha_{i,1}, \ldots, \alpha_{i,l-1}\} \cup \bigcup_{j=1}^{n-1} (e_{i_j} - e)$, then $\langle a_1, \ldots, a_n, \alpha_{i,l}\rangle \in R$. Let $\sigma_i = \bigcup_{l=1}^m \sigma^l$, let $\{a_1, \ldots, a_{n+1}\} \subseteq e_i \cup \bigcup_{j=1}^{n-1} e_{i_j}$, where $i_1, \ldots, i_{n-1} \in i - \sigma_i$ and $\{a_1, \ldots, a_{n+1}\} \cap (e_i - e) \neq \emptyset$. If for some $j \neq j'$ $a_j = a_{j'}$, then $\langle a_1, \ldots, a_{n+1}\rangle \in R$, since $R \supseteq T$. Hence assume that $a_j \neq a_{j'}$ whenever $j \neq j'$. Since $\{a_1, \ldots, a_{n+1}\} \cap (e_i - e) \neq \emptyset$, there is l such that $\alpha_{i,l} = \max(\{a_1, \ldots, a_{n+1}\})$. Since R is symmetric, w.l.o.g. we can assume that $a_{n+1} = \alpha_{i,l}$. Hence $a_1, \ldots, a_n < \alpha_{i,l}$. If $\{a_1, \ldots, a_n\} \subseteq e \bigcup\{\alpha_{i,1}, \ldots, \alpha_{i,l-1}\}$, then $\{a_1, \ldots, a_{n+1}\} \subseteq e_i \in P_M$, so $\langle a_1, \ldots, a_{n+1}\rangle \in R$. Finally, otherwise $\langle a_1, \ldots, a_n, \alpha_{i,l}\rangle \in R$ and (****) is proved.

By Fodor's theorem there is an uncountable $C \subseteq \aleph_1$ and a finite set σ such that for every $i \in C$, $\sigma_i = \sigma$. W.l.o.g. $\sigma \cap C = \emptyset$. Let $i_1, \ldots, i_n \in C$ and $i_1 < i_2 < \cdots < i_n$. We show that $\bigcup_{j=1}^n e_{i_j}$ is PH. Let $\{a_1, \ldots, a_{n+1}\} \subseteq \bigcup_{j=1}^n e_{i_j}$. If $\{a_1, \ldots, a_{n+1}\} \subseteq e$, then $\langle a_1, \ldots, a_{n+1}\rangle \in R$ since $e \subseteq e_{i_j}$ is PH. Otherwise let $p = \max(\{j | \{a_1, \ldots, a_{n+1}\} \cap (e_{i_j} - e) \neq \emptyset\})$; then since $i_1, \ldots, i_p \in C$, $\{i_1, \ldots, i_{p-1}\} \subseteq i_p - \sigma_p$. So by (****) $\langle a_1, \ldots, a_{n+1}\rangle \in R$. This shows that C is n-compatible. We thus proved (3).

To construct M we need two lemmas. Let $\exists^\infty$ mean "there are infinitely many".

DEFINITION. A set of m-tuples A is called small if $\neg\exists^\infty a_1 \exists^\infty a_2 \cdots \exists^\infty a_m (\langle a_1, \ldots, a_m\rangle \in A)$.

Note that a subset of a small set is small, and that if B is infinite then B^m is not small.

If $\vec{a}$ and $\vec{b}$ are finite sequences, then $\vec{a} \frown \vec{b}$ will denote their concatenation. If A is a set of finite sequences and $\vec{a}$ is a finite sequence then $A_{\vec{a}} \overset{\text{def}}{=} \{\vec{b} | \vec{a} \frown \vec{b} \in A\}$; if a is an element then A_a abbreviates $A_{\langle a\rangle}$.

LEMMA 3.2. (a) *Let A be a set of m-tuples and $k < m$. Then A is small iff $\{\vec{b} | \vec{b}$ is a k-tuple and $A_{\vec{b}}$ is not small$\}$ is small.*

(b) *A finite union of small sets of m-tuples is small.*

(c) *$\{\langle a_1, \ldots, a_m\rangle \mid \{a_1, \ldots, a_m\} \subseteq A$ and for some $i \neq j$ $a_i = a_j\}$ is small.*

PROOF. (a) is trivial from the definition.

(b): We prove (b) by induction on m. For $m = 1$, "small" means "finite" so the claim is trivially true. Assume (b) for m, let $A^1, \ldots, A^k$ be small sets of

$m + 1$-tuples, and let $A = \bigcup_{i=1}^k A^i$. By (a) A is small iff $B \overset{\text{def}}{=} \{a | A_a$ is not small$\}$ is finite. But for every a $A_a = \bigcup_{i=1}^k A_a^i$, and so by the induction hypothesis $B = \bigcup_{i=1}^k \{a | A_a^i$ is not small$\}$. Again by (a), for each i $\{a | A_a^i$ is not small$\}$ is finite, hence B is finite and hence A is small. Q.E.D.

The proof of (c) is left to the reader.

Let $E = \{e_i | i \in \omega\}$ be an FS and let $0 < m \in \omega$, let $E[m] = \{\langle a_1, \ldots, a_m \rangle |$ there are $i_1, \ldots, i_{m-1}$ such that $\{a_1, \ldots, a_m\} \subseteq e_0 \cup \bigcup_{j=1}^{m-1} e_{i_j}\}$.

LEMMA 3.3. *If E is an FS and $0 < m < \omega$, then $E[m]$ is small.*

PROOF. We prove by induction on m that for every FS E, $E[m]$ is small. $m = 1$. Let $E = \{e_i | i \in \omega\}$ be an FS. Then $E[1] = \{\langle a \rangle | a \in e_0\}$, so $E[1]$ is small. Assume the induction hypothesis for m. Let $E = \{e_i | i \in \omega\}$ be an FS. We prove that $\{a | E[m+1]_a$ is not small$\} \subseteq e_0$. (In fact if $|\bigcup E| = \aleph_0$, then equality holds.) Suppose $a \notin e_0$. If $a \notin \bigcup_{i \in \omega} e_i$, then $E[m+1]_a = \emptyset$ so it is small. Otherwise let $a \in e_{i_1}$; then $E[m+1]_a = \{\langle a_1, \ldots, a_m \rangle |$ there are $i_2, \ldots, i_m$ such that $\{a_1, \ldots, a_m\} \subseteq (e_0 \cup e_{i_1}) \cup \bigcup_{j=2}^m e_{i_j}\}$. So if e_i' is defined to be e_i when $i \neq 0, i_1$, $e_0' = e_0 \cup e_{i_1}$, $e_{i_1}' = \emptyset$, and if E' is defined to be $\{e_1' | i \in \omega\}$ then $E[m+1]_a = E'[m]$. Hence by the induction hypothesis $E[m+1]_a$ is small, hence $a \notin e_0$ implies $E[m+1]_a$ is small. Since e_0 is finite by 3.2(a) $E[m+1]$ is small.

The definition of M. Let $\{\tau_i | i < \aleph_1\}$ be an enumeration of all finite subsets of $\aleph_1$ such that for every $i < \aleph_1$ $\{j | \tau_j = \tau_i\}$ is uncountable and such that for every $i < \aleph_1$ $\tau_i \subseteq i$. If $S \subseteq \aleph_1^{n+1}$ and $\alpha < \aleph_1$, let $S^\alpha = \{\langle a_1, \ldots, a_n \rangle | \{a_1, \ldots, a_n\} \subseteq \alpha$ and $\langle a_1, \ldots, a_n, \alpha \rangle \in S\}$.

We define R^α for the desired relation R. If $\alpha < \omega$ let $R^\alpha = \alpha^n$.

The definition of R^α for $\alpha \geq \omega$. Let $\{G^i | i \in \omega\}$ be an enumeration of $\{G_\beta | \beta < \alpha\}$. Let $\{E^i | i \in \omega\}$ be an enumeration of all FS's $E = \{e_m | m \in \omega\}$ such that for some $\beta < \alpha$, $d_0^\beta \subseteq e_0 \subseteq \alpha$ and $e_m = d_m^\beta$ for every $0 < m \in \omega$. (Recall that we denoted $D^\beta = \{d_m^\beta | m \in \omega\}$, where $\{D^\beta | \beta < \aleph_1\}$ is the list of all FS's whose union is a subset of $\aleph_1$.)

We define by induction on $m \in \omega$ R_m^α and $\bar{R}_m^\alpha$ with the purpose that R^α will be defined as $\bigcup_{m \in \omega} R_m^\alpha$. Our induction hypotheses are: (1) R_m^α, $\bar{R}_m^\alpha$ are symmetric disjoint subsets of α^n; (2) R_m^α is small and $\bar{R}_m^\alpha$ is finite; (3) both R_m^α and $\bar{R}_m^\alpha$ are increasing with m. Let $\bar{R}_0^\alpha = \emptyset$ and $R_0^\alpha = (\tau_\alpha)^n \cup \{\langle a_1, \ldots, a_n \rangle | \{a_1, \ldots, a_n\} \subseteq \alpha$ and for some $i \neq j$, $a_i = a_j\}$. So, the induction hypotheses hold. Suppose R_m^α, $\bar{R}_m^\alpha$ have been defined. Since R_m^α is small and G^m is infinite, there is $\langle a_1, \ldots, a_n \rangle \in (G^m)^n - R_m^\alpha$. Let $\bar{R}_{m+1}^\alpha = \bar{R}_m^\alpha \cup \{\langle a_{\pi(1)}, \ldots, a_{\pi(n)} \rangle | \pi$ a permutation of $\{1, \ldots, n\}\}$. Let $R_{m+1}^\alpha = R_m^\alpha \cup (E^m[n] - \bar{R}_{m+1}^\alpha)$. It is easy to check that the induction hypotheses hold. This completes the definition of R_m^α and $\bar{R}_m^\alpha$.

Let $R^\alpha = \bigcup_{m \in \omega} R_m^\alpha$, let $\tilde{R}^\alpha$ be the symmetric closure of $R^\alpha \times \{\alpha\}$ and let $R = T \cup \bigcup_{\alpha < \aleph_1} \tilde{R}^\alpha$. Recall that $T = \{\langle a_1, \ldots, a_{n+1} \rangle | \{a_1, \ldots, a_{n+1}\} \subseteq \aleph_1$ and for some $i \neq j$ $a_i = a_j\}$.

We prove that R satisfies the requirements of 3.1. Certainly $R \supseteq T$ and R is symmetric. By the definition of R_0^α (***) is fulfilled.

To prove (*), let $\beta < \alpha$. Let $G_\beta = G^m$ in the enumeration of $\{G_\gamma | \gamma < \alpha\}$; that was defined before defining R^α and $\bar{R}^\alpha$. So for some $b_1, \ldots, b_n \in G_\beta$, $\bar{R}_{m+1}^\alpha \ni \langle b_1, \ldots, b_n \rangle$. We prove that $\langle b_1, \ldots, b_n, \alpha \rangle \notin R$. Since $R_{m+1}^\alpha \supseteq R_0^\alpha \supseteq \{\langle a_1, \ldots, a_n \rangle | a_1, \ldots, a_n \in \alpha$ and for some $i \neq j$ $a_i = a_j\}$ and since $\bar{R}_{m+1} \cap R_{m+1}^\alpha = \emptyset$,

$\bigwedge_{i \neq j}(b_i \neq b_j \neq \alpha)$. So $\langle b_1, \ldots, b_n, \alpha\rangle \notin T$. Certainly $\langle b_1, \ldots, b_n, \alpha\rangle \notin \tilde{R}^\gamma$ when $\gamma \neq \alpha$. Now $\langle b_1, \ldots, b_n, \alpha\rangle \notin \tilde{R}^\alpha$ since this would imply that $\langle b_1, \ldots, b_n, \alpha\rangle$ is a permutation of some element of $R^\alpha \times \{\alpha\}$, this in turn implies that for some $k \geq m + 1$: $\langle b_1, \ldots, b_n\rangle$ is a permutation of an element of R^α_k, and since R^α_k is symmetric, this implies that $\langle b_1, \ldots, b_n\rangle \in R^\alpha_k$. However $\langle b_1, \ldots, b_n\rangle \in \bar{R}_k$ and $R^\alpha_k \cap \bar{R}^\alpha_k = \emptyset$, and this is impossible. So (*) is satisfied.

To prove (**) let $\beta < \alpha$, $d^\beta_0 \subseteq e_0 \subseteq \alpha - \bigcup_{0<m\in\omega} d^\beta_m$. Let $e_i = d^\beta_i$ for $i > 0$, and let $\{e_i | i \in \omega\} = E^m$ where $\{E^j | j \in \omega\}$ is the enumeration of FS's defined before the construction of R^α and $\bar{R}^\alpha$. Let $\sigma = \{i \mid$ there is $\langle a_1, \ldots, a_n\rangle \in \bar{R}^\alpha_{m+1}$ such that $\{a_1, \ldots, a_n\} \cap e_i \neq \emptyset\}$, σ is finite since $\bar{R}^\alpha_{m+1}$ is. Let $\langle b_1, \ldots, b_n\rangle$ be a sequence such that $\{b_1, \ldots, b_n\} \not\subseteq e_0$ and there are $i_1, \ldots, i_{n-1} \in \omega - \sigma$ such that $\{b_1, \ldots, b_n\} \subseteq e_0 \cup \bigcup_{j=1}^{n-1} d^\beta_{i_j}$; so $\langle b_1, \ldots, b_n\rangle \in E^m[n]$; on the other hand $\langle b_1, \ldots, b_n\rangle \notin \bar{R}^\alpha_{m+1}$, so $\langle b_1, \ldots, b_n\rangle \in R^\alpha_{m+1}$, so $\langle b_1, \ldots, b_n, \alpha\rangle \in R$. Q.E.D.

This concludes the proof of Theorem 1.1 for the case $n > 1$.

The case $n = 1$. The construction of M in this case is a simplified version of the construction for $n > 1$. Let $\{G_i | i < \aleph_1\}$, $\{D^i | i < \aleph_1\}$ and $\{d^i_m | m \in \omega\}$ be as in the previous case. We construct R so that it will satisfy (*), (***) of the previous case and the following modification of (**).

Let Q be a binary relation on $\aleph_1$, $\omega \leq \alpha < \aleph_1$ and D an FS such that $\bigcup D \subseteq \alpha$. We say that D is $Q - \alpha$-definable, if for some $\beta < \alpha$ and some finite $\sigma \subseteq \alpha$ $D = \{d \in D^\beta | d \times \sigma \subseteq Q\}$.

(**)′ For every $\omega \leq \alpha < \aleph_1$ and every infinite $R - \alpha$-definable FS D there are infinitely many $d \in D$ such that $d \times \{\alpha\} \subseteq R$.

We first prove the analogue of 3.1.

LEMMA 3.4. *Let $R \subseteq \aleph_1^2$ be symmetric and $R \supseteq T$. Suppose R satisfies* (*), (**)′ *and* (***). *Then*

(1) $M \models \neg\psi$;

(2) P_M *is an* S_1*-forcing* (*i.e.* P_M *is c.c.c.*);

(3) $\Vdash_{P_M} (M \models \psi)$.

PROOF. (1) follows from (*). (3) follows from (***) and the fact that P_M is c.c.c. Since (2) means that P_M is c.c.c., the proof will be concluded once we prove (2).

Let $\{\sigma_i | i < \aleph_1\} \subset P_M$. W.l.o.g. $\{\sigma_i | i < \aleph_1\}$ is a Δ-system and by the nature of P_M it is sufficient to deal with the case when for every $i < j < \aleph_1$ $\sigma_i \cap \sigma_j = \emptyset$. Let β be such that $D^\beta = \{\sigma_i | i < \omega\}$, and let $\{\alpha_0, \ldots, \alpha_{r-1}\} = \sigma_j$ be such that $\beta < \alpha_0 < \cdots < \alpha_{r-1}$. For every $k \leq r$ let $D_k = \{d \in D^\beta | d \times \{\alpha_0, \ldots, \alpha_{k-1}\} \subseteq R\}$. Hence for every $k < r$ D_k is $R - \alpha_k$-definable. Using (**)′ it follows by induction on $k \leq r$ that D_k is infinite. Hence D_r is infinite, and this means that for some $n \in \omega$ $\sigma_j \cup \sigma_n \in P_M$. We have thus proved that P_M is c.c.c..

The construction of a model M which satisfies (*), (**)′ and (***) is along the same lines as the construction of M in the previous case. However, here we do not need any combinatorial facts. We thus leave the very easy construction to the reader.

REMARK. Since the completeness of $L(Q)$ [Kr] follows just from ZFC, in the case of $n = 1$ we obtain that Theorem 1.1 follows from CH rather than from $\diamondsuit_{\aleph_1}$.

This concludes the proof of Theorem 1.1.

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§4. The hierarchy theorem for $\mathscr{L}^{n,m}$. To prove Theorem 1.2 we first translate §2 in an obvious way. We have to modify, however, the combinatorial details. We assume $\diamondsuit_{\aleph_1}$. Later we shall fix $n \geq 0$ and $m > 1$, then we shall construct a model M with properties ($\bar{1}$), ($\bar{2}$) analogous to properties (1) and (2) of M. The argument why this suffices is the same as the one given in §2.

Our first goal is to find a property $S_{n,m}$ of forcing notions such that forcing over V with an $S_{n,m}$ forcing notion preserves $\mathscr{L}^{n,m}$-theories of all $N \in V$.

Let "$\exists^\lambda \ldots$" mean "there are at least λ elements ...".

DEFINITION. Let $k > 0$, λ an infinite cardinal and A a set of k-tuples. A is λ-large if $\exists^\lambda a_1 \cdots \exists^\lambda a_k(\langle a_1, \ldots, a_k\rangle \in A)$. Otherwise we say that A is λ-small.

DEFINITION. Let P be a forcing notion, $k \geq 0$ and $l \geq 1$; P is called an $S_{k,l}$ forcing if either $l = 1$ and P is c.c.c., or $l > 1$ and for every $h\colon \aleph_1^{k+1} \to P$, there is an $\aleph_1$-large $B \subseteq \aleph_1^{k+1}$ such that $h(B)$ is l-compatible.

P is called an $S_{k,\infty}$ forcing if for every $h\colon \aleph_1^{k+1} \to P$, there is an $\aleph_1$-large $B \subseteq \aleph_1^{k+1}$ such that $h(B)$ is l-compatible for every $l \in \omega$.

REMARK. It is possible to define the property $S_{\infty,l}$ or $S_{\infty,\infty}$ of forcing notions. Let T be the tree of finite sequences of countable ordinals with the partial ordering of being an initial segment. A subset T' of T is called a large subtree of T if it is closed under initial segments and is isomorphic to T.

P is an $S_{\infty,l}$ forcing if for every $h\colon T \to P$ such that $h(b)$ is l-compatible for every branch b of T, there is a large subtree T' of T such that $h(T')$ is l-compatible.

Let us define a canonical representation of the isomorphism type of $\bar{M}_k$. Let $\bar{M}_k = \langle \aleph_1^{k+1}, E_1, \ldots, E_k\rangle$ where $E_i = \{\langle\vec{\alpha}, \vec{\beta}\rangle \mid \vec{\alpha} \restriction i = \vec{\beta} \restriction i\}$. In what follows $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ always denote finite sequences of countable ordinals.

THEOREM 4.1. *If P is an $S_{k,l}$-forcing and $N \in V$ then for every P-generic extension W of V, for every $\varphi \in \mathscr{L}^{k,l}$ and $a_1, \ldots, a_m \in N$: $N \models \varphi[a_1, \ldots, a_m]$ in V iff $N \models \varphi[a_1, \ldots, a_m]$ in W.*

PROOF. The proof is by induction on the structure of φ, and the only less trivial case is when $\varphi \equiv Q^{k,l}x_1 \cdots x_l(\psi_1, \ldots, \psi_r)$. Since P is c.c.c. if $N \models \varphi$ in V then $N \models \varphi$ in W. Suppose $N \models \varphi$ in W, and let p force this fact. Let τ be a name of a function from $\bar{M}_k$ to $|N|$ which is forced by p to exemplify the fact that $N \models \varphi$. For every $\vec{\alpha} \in \aleph_1^{k+1}$ let $p_{\vec{\alpha}} \in P$ and $a_{\vec{\alpha}} \in |N|$ be such that $p_{\vec{\alpha}} \geq p$ and $p_{\vec{\alpha}} \Vdash \tau(\vec{\alpha}) = a_{\vec{\alpha}}$. Let $A \subseteq \aleph_1^{k+1}$ be an $\aleph_1$-large set such that $\{P_{\vec{\alpha}} \mid \vec{\alpha} \in A\}$ is l-compatible. By the l-compatibility and the induction hypothesis (used for $\psi_1, \ldots, \psi_p$), $h \restriction A$ exemplifies the fact that $N \models \varphi$ in V. Q.E.D.

For the rest of this section we fix $n \geq 0$ and $m > 1$. Let ψ denote $\psi_{n,m}$. We shall construct a model M of the form $\langle \aleph_1^{n+1}, E_1, \ldots, E_n, R\rangle$ where $\langle \aleph_1^{n+1}, E_1, \ldots, E_n\rangle = \bar{M}_n$, but R differs from $R_{n,m}$. (Recall that $R_{n,m}$ was defined before Theorem 1.2.) Let M be as above, $A \subseteq |M|$ is *correct* if $R \restriction A = R_{n,m} \restriction A$. Let $P_M = \langle\{\sigma \subseteq |M| \mid \sigma$ is finite and correct$\}, \subseteq\rangle$.

Lemma 4.2 is a generalization of Lemma 3.1.

LEMMA 4.2 (CH). *There is a model $M = \langle \aleph_1^{n+1}, E_1, \ldots, E_n, R\rangle$ with the following properties*:

(1) $M \models \neg\, \psi$;

(2) *P_M is an $S_{n',m'}$ forcing for every $\langle n', m'\rangle$ such that either $n' < n$ or $m' < m$*;

(3) $\Vdash_{P_M} (M \models \psi)$.

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REMARK. It is trivial that in P_M m-compatibility implies l-compatibility for every l, so in fact condition (2) implies that P_M is an $S_{n'\infty}$ forcing for every $n' < n$. We do not know how to assure that P_M will be an $S_{\infty,m'}$ for every $m' < m$.

Our next goal is to define three requirements analogous to (*), (**), (***) of §3.

Let $A^{[k]} \overset{\text{def}}{=} \{\sigma | \sigma \subseteq A \text{ and } |\sigma| = k\}$. Let $I_k(A) = \{B \subseteq A^{[k]} | \neg(\exists C \subseteq A)(C \text{ is infinite and } B \supseteq C^{[k]})\}$, and let $I^l(A) = \{B \subseteq A^l | B \text{ is } \aleph_0\text{-small}\}$. Let I_k^l be the ideal on $(\aleph_1^l)^{[k]}$ generated by $I_k(\aleph_1^l) \cup \{B^{[k]} | B \in I^l(\aleph_1)\}$.

LEMMA 4.3. (a) *$I_k(A)$ is an ideal.*

(b) *$I^l(A)$ is an ideal.*

(c) *If $B \subseteq \aleph_1^l$ and B is $\aleph_0$-large, then $B^{[k]} \notin I_k^l$.*

(d) *Let $\sigma \subseteq \aleph_1^l$ be finite and $B \in (\aleph_1^l)^{[k]}$. We define $B^*\sigma = \{\tau | |\tau| = k$ and $(\exists \tau' \in B)(\tau \subseteq \tau' \cup \sigma)\}$; then if $B \in I_k^l$ then $B^*\sigma \in I_k^l$.*

PROOF. We leave the easy proofs of (a), (b) and (d) to the reader.

(c) Let $B \subseteq \aleph_1$ be $\aleph_0$-large, and suppose by contradiction that $B^{[k]} \in I_k^l$, so there are $C \in I^l(\aleph_1)$ and $D \in I_k(\aleph_1^l)$ such that $B^{[k]} \subseteq C^{[k]} \cup D$. By (b), $B - C$ is $\aleph_0$-large, and hence it is infinite. But $D \supseteq (B - C)^{[k]}$, contradicting the fact that $D \in I_k(\aleph_1^l)$.

We have to deal separately with the case when $n = 0$ and $m = 2$. However, Theorem 1.1 when applied to $n = 1$ is just the same as Theorem 1.2 when applied to $\langle n, m\rangle = \langle 0, 2\rangle$. We have already remarked on this case, so from now on *we assume that $\langle n, m\rangle \neq \langle 0, 2\rangle$.*

Let I denote I_{m-1}^{n+1}. Let $\{\vec{\alpha}^i | i < \aleph_1\}$ be a 1-1 enumeration of $\aleph_1^{n+1}$, and let $\{G^i | i < \aleph_1\}$ be an enumeration of all countable (including finite) subsets of $(\aleph_1^{n+1})^{[m-1]}$ such that for every $i < \aleph_1$ $G^i \subseteq \{\vec{\alpha}^j | j < i\}^{[m-1]}$.

We shall construct R symmetric in such a way that the following three conditions will hold:

(*) If $i < j$ and $G^i \notin I$, then there is $\sigma \in G^i$ such that $\sigma \cup \{\vec{\alpha}^j\}$ is incorrect.

(**) (a) If $i < j$, $G^i \in I$ and τ is a finite subset of $\{\vec{\alpha}^l | l < j\}$, then $\{\sigma \in G^i * \tau | \sigma \cup \{\vec{\alpha}^j\} \text{ is incorrect}\}$ is finite. (b) For every $\sigma \in (\aleph_1^{n+1})^{[m]}$ if $\sigma \cap \{\vec{\alpha}^0, \ldots, \vec{\alpha}^{m-1}\} \neq \emptyset$, then σ is correct. (*Remark.* (b) is not really needed, we assume it just for technical convenience. Also note that (b) implies that for every $\sigma \subseteq \aleph_1^{n+1}$ if $|\sigma| < m$ then σ is correct.)

(***) For every finite correct subset $\sigma \subseteq \aleph_1^{n+1}$ and for every $\beta \in \aleph_1^n$ $|\{\gamma | \sigma \cup \{\vec{\beta}^\frown\langle\gamma\rangle\} \text{ is correct}\}| = \aleph_1$.

LEMMA 4.4. *Let $M = \langle \aleph_1^{n+1}, E_1, \ldots, E_n, R\rangle$ satisfy* (*), (**) *and* (***). *Then M satisfies* (1), (2) *and* (3) *of* 4.2.

PROOF. (1) follows trivially from (*) and 4.3(c).

The main novelty appears in the proof that (**) implies (2). We postpone this proof for a while.

(***) $\Rightarrow$ (3). Let τ be the name of the union of all conditions (which are finite sets) in the generic set. Certainly $\Vdash_{P_M}$ "τ is correct". So it remains to show that $\Vdash_{P_M}$ "τ is $\aleph_1$-large". But clearly (***) implies that

$$\Vdash_{P_M} (\forall \vec{\beta} \in \aleph_1^n)(|\{\gamma \in \aleph_1 | \vec{\beta}^\frown\langle\gamma\rangle \in \tau\}| = \aleph_1),$$

so the claim follows.

We now prove a sequence of lemmas that will lead to the proof of $(**) \Rightarrow (2)$.

Notations. For finite sequences $\vec{\alpha}$ and $\vec{\beta}$, let $\vec{\alpha} \leqslant \vec{\beta}$ mean that $\vec{\alpha}$ is an initial segment of $\vec{\beta}$, and $\vec{\alpha} < \vec{\beta}$ mean that $\vec{\alpha} \leqslant \vec{\beta}$ and $\vec{\alpha} \neq \vec{\beta}$. Let $\vec{\alpha} \wedge \vec{\beta}$ denote the maximal common initial segment of $\vec{\alpha}$ and $\vec{\beta}$. Let Λ denote the empty sequence. For every A, A^0 means $\{\Lambda\}$, $A^{<k} = \bigcup_{i<k} A^i$ and $A^{\leq k} = \bigcup_{i\leq k} A^i$. A set of finite sequences which is closed under initial segments is called a hereditary set. If A is a set of finite sequences, then $H(A)$ denotes the hereditary closure of A, and $H(\vec{\alpha})$ abbreviates $H(\{\vec{\alpha}\})$. $S_\omega(A) \overset{\text{def}}{=} \{\sigma \subseteq A \mid \sigma \text{ is finite}\}$. A set function is a function such that every element of its range is a set. A set function g is disjointed if for every $a, b \in \mathrm{Dom}(g)$ $g(a) \cap g(b) = \varnothing$.

We now generalize the notion of a Δ-system.

DEFINITION. Let h be a set function whose domain is an $\aleph_1$-large subset of $\aleph_1^k$, and denote $\mathrm{Dom}(h) = A$. h is called a $k - \Delta$-system, or in short, a Δ-system if h can be extended to a function $\bar{h}$ on $H(A)$ such that for every $\vec{\alpha}, \vec{\beta} \in A$ $\bar{h}(\vec{\alpha}) \cap \bar{h}(\vec{\beta}) = \bar{h}(\vec{\alpha} \wedge \vec{\beta})$. We call $\bar{h}$ the kernel of h.

If h is a Δ-system and $\bar{h}$ is its kernel, let h' be the function such that $\mathrm{Dom}(h') = \mathrm{Dom}(\bar{h})$ and for every $\vec{\alpha} \in \mathrm{Dom}(\bar{h})$ $h'(\vec{\alpha}) = h(\vec{\alpha}) - \bigcup \{h(\vec{\beta}) \mid \vec{\beta} < \vec{\alpha}\}$. We call h' the remainder function of h. Note that h' is disjointed.

LEMMA 4.5. *If h: $\aleph_1^k \to S_\omega(B)$, then there is $h_1 \subseteq h$ such that h_1 is a Δ-system.*

PROOF. The proof is by induction on k. For $k = 1$ it is well known. Suppose true for k, and let h: $\aleph_1^{k+1} \to S_\omega(B)$. For every $\alpha \in \aleph_1$, let h_α: $\aleph_1^k \to S_\omega(B)$ be defined as follows: $h_\alpha(\vec{\beta}) = h(\langle\alpha\rangle^\frown\vec{\beta})$. By the induction hypothesis and by renaming we can assume that for every α h_α is a Δ-system. Let $\bar{h}_\alpha$ and h'_α be, respectively, the kernel and the remainder function of h_α. By the induction hypothesis for $k = 1$ we can assume that $\{\bar{h}_\alpha(\Lambda) \mid \alpha < \aleph_1\}$ is a Δ-system. We define a function g' on $\aleph_1^{\leq k+1}$. $g'(\Lambda) = \bar{h}_\alpha(\Lambda) \cap \bar{h}_\beta(\Lambda)$ where $\alpha \neq \beta$, $g'(\langle\alpha\rangle) = \bar{h}_\alpha(\Lambda) - g'(\Lambda)$; and for every $\vec{\beta} \in \aleph^{\leq k} - \{\Lambda\}$ and $\alpha \in \aleph_1$ $g'(\langle\alpha\rangle^\frown\vec{\beta}) = h'_\alpha(\vec{\beta})$. Note that for every $\vec{\beta} \in \aleph_1^{\leq k}$ $\{g'(\beta^\frown\langle\gamma\rangle) \mid \gamma \in \aleph_1\}$ is a family of pairwise disjoint sets.

Let $\{\vec{\alpha}^i \mid i < \aleph_1\}$ be an enumeration of $\aleph_1^{\leq k}$ such that for every $i < \aleph_1$ $|\{j \mid \vec{\alpha}^j = \vec{\alpha}^i\}| = \aleph_1$. We define by induction on $i < \aleph_1$ a sequence $\{\vec{\alpha}_i \mid i < \aleph_1\} \subseteq \aleph_1^{\leq k+1}$ such that for every $i \leq \aleph_1$ $A_i \overset{\text{def}}{=} \{\vec{\alpha}_j \mid j < i\}$ is hereditary. Suppose $\vec{\alpha}_j$ has been defined for every $j < i$. If $\vec{\alpha}^i \notin A_i$, let $\vec{\alpha}_i = \Lambda$. Suppose $\vec{\alpha}^i \in A_i$, and let $B_i = \bigcup\{g'(\vec{\alpha}_j) \mid j < i\}$. B_i is countable and $\{g'(\vec{\alpha}^{i\frown}\langle\gamma\rangle) \mid \gamma \in \aleph_1\}$ is a family of pairwise disjoint sets, so for some $\gamma \in \aleph_1$ $g'(\vec{\alpha}^{i\frown}\langle\gamma\rangle) \cup B_i = \varnothing$. Let $\vec{\alpha}_i = \vec{\alpha}^{i\frown}\langle\gamma\rangle$.

By the construction it is clear that $A \overset{\text{def}}{=} A_{\aleph_1} \cap \aleph_1^{k+1}$ is $\aleph_1$-large. We prove that $h \restriction A$ is a Λ-system. Let f be defined on $A_{\aleph_1}$ as follows: $f(\Lambda) = g'(\Lambda)$, and $f(\langle\alpha\rangle^\frown\vec{\beta}) = \bar{h}_\alpha(\vec{\beta})$. We verify that f is the kernel of $h \restriction A$. Clearly f extends $h \restriction A$. It is also trivial that for every $\vec{\beta} \in A_{\aleph_1}$ $f(\vec{\beta}) = \bigcup\{g'(\vec{\alpha}) \mid \vec{\alpha} \leqslant \vec{\beta}\}$. By the construction $g' \restriction A_{\aleph_1}$ is disjointed, hence for every $\vec{\alpha}, \vec{\beta} \in A_{\aleph_1}$

$$f(\vec{\alpha}) \cap f(\vec{\beta}) = \bigcup\{g'(\vec{\gamma}) \mid \vec{\gamma} \leq \vec{\alpha} \text{ and } \vec{\gamma} \leq \vec{\beta}\} = f(\vec{\alpha} \wedge \vec{\beta}).$$

Q.E.D.

REMARK. See [RS] for a generalization of this lemma.

DEFINITION. Let h: $B \to S_\omega(\aleph_1^{\leq m})$, h is hereditary if for every $b \in B$ $h(b)$ is hereditary.

LEMMA 4.6. *Let* $0 < k < m$ *and let* $h: \aleph_1^k \to S_\omega(\aleph_1^{\leq m})$ *be a hereditary* Δ*-system; let* $H_h = \aleph_1^m \cap \bigcup\{h(\vec{\alpha}) \mid \vec{\alpha} \in \aleph_1^k\}$, *then* H_h *is* $\aleph_0$*-small.*

PROOF. For k, m, h as above and $\vec{\beta} \in \aleph_1^{\leq m}$, let $A_{\vec{\beta}}^h = \{\vec{\alpha} \mid \vec{\beta}\,\hat{}\,\vec{\alpha} \in H_h\}$. We prove the following claim by induction on k: let k, m, h be as above, and $\tilde{h}$ be the kernel of h. Then if $\vec{\beta} \in \aleph_1^{m-k}$ and $A_{\vec{\beta}}^h$ is $\aleph_0$-large, then $\vec{\beta} \in \tilde{h}(\Lambda)$.

4.6 follows from this claim since $\tilde{h}(\Lambda)$ is finite.

The case when $k = 1$. Let $\vec{\beta} \in \aleph_1^{m-1} - \tilde{h}(\Lambda)$. Then there is at most one $\alpha \in \aleph_1$ such that $\vec{\beta} \in h(\langle\alpha\rangle)$. If there is no such α, then $A_{\vec{\beta}}^h = \varnothing$, so it is $\aleph_0$-small. Otherwise let $\vec{\beta} \in h(\langle\alpha_0\rangle)$; since h is a hereditary Δ-system $\vec{\beta}\,\hat{}\,A_{\vec{\beta}}^h \subseteq h(\langle\alpha_0\rangle)$, and since $h(\langle\alpha_0\rangle)$ is finite $A_{\vec{\beta}}^h$ is finite, and so it is $\aleph_0$-small.

We now assume the induction hypothesis for every $k' \leq k$, and we prove it for $k + 1$. If $m \leq k + 1$ there is nothing to prove, so we assume $k + 1 < m$.

First by the induction hypothesis (1): if $h: \aleph_1^k \to S_\omega(\aleph^{\leq m})$ is a hereditary Δ-system, $i < m-k$ and $\vec{\beta} \in \aleph_1^i$, then $A_{\vec{\beta}}^h$ is $\aleph_0$-small.

Let $h: \aleph_1^{k+1} \to S_\omega(\aleph^{\leq m})$ be a hereditary Δ-system, and let $\tilde{h}$, h' be, respectively, the kernel of h and the remainder function of h. Let $\vec{\beta} \in \aleph_1^{m-(k+1)} - \tilde{h}(\Lambda)$. If $\vec{\beta} \notin \bigcup\{h(\vec{\alpha}) \mid \vec{\alpha} \in \aleph_1^{k+1}\}$, then $A_{\vec{\beta}}^h = \varnothing$, and hence it is $\aleph_0$-small. Otherwise let l and $\vec{\alpha}_0$ be such that $\vec{\alpha}_0 \in \aleph_1^l$ and $\vec{\beta} \in h'(\vec{\alpha}_0)$. Clearly since $\vec{\beta} \notin \tilde{h}(\Lambda)$, $l > 0$. By the disjointedness of h' $\vec{\alpha}_0$ is unique and (2): $\vec{\beta}\,\hat{}\,A_{\vec{\beta}}^h \subseteq \bigcup\{h(\vec{\alpha}) \mid \vec{\alpha}_0 \lessdot \vec{\alpha}\}$. Let g be defined on $\aleph_1^{k+1-l}$ in the following way: $g(\vec{\gamma}) = h(\vec{\alpha}_0\,\hat{}\,\vec{\gamma})$. By (2) $A_{\vec{\beta}}^h = A_{\vec{\beta}}^g$. The induction hypothesis can be applied to $k + 1 - l \leq k$. Hence (3): if $i < m - (k + 1 - l)$, and $\vec{\gamma} \in \aleph_1^i$, then $A_{\vec{\gamma}}^g$ is $\aleph_0$-small. We can apply (3) to $i = m - (k + 1)$ and $\vec{\beta}$ because $m - (k + 1) < m - (k + 1 - l)$. Hence $A_{\vec{\beta}}^g$ is $\aleph_0$-small, i.e., $A_{\vec{\beta}}^h$ is $\aleph_0$-small. Q.E.D.

COROLLARY 4.7. *If* $k < m$ *and* $h: \aleph_1^k \to S_\omega(\aleph_1^m)$, *then there is an* $\aleph_1$*-large* $A \subseteq \aleph_1^k$ *such that* $\bigcup\{h(\vec{\alpha}) \mid \vec{\alpha} \in A\}$ *is* $\aleph_0$*-small.*

PROOF. Let $h_1(\vec{\alpha}) = H(h(\vec{\alpha}))$, let $A \subseteq \aleph_1^k$ be such that $h_1 \restriction A$ is a Δ-system; then A satisfies the requirements of the corollary.

DEFINITION. Let $g: \aleph_1^k \to S_\omega(\aleph_1^{\leq k})$, and let $A \subseteq \aleph_1^k$ be called f-free if for every $\vec{\alpha}$, $\vec{\beta} \in A$ $g(\vec{\alpha}) \cap H(\vec{\beta}) = \varnothing$.

LEMMA 4.8 (*A generalization of a theorem of Fodor and Hajnal* [F], [H] *for the case of* $\lambda = \aleph_1$). *Let* $g: \aleph_1^k \to S_\omega(\aleph_1^{\leq k})$ *be such that for every* $\vec{\alpha} \in \aleph_1^k$ $g(\vec{\alpha}) \cap H(\vec{\alpha}) = \varnothing$. *Then there is an* $\aleph_1$*-large* g*-free subset of* $\aleph_1^k$.

PROOF. W.l.o.g. g is a Δ-system (though now its domain is only an $\aleph_1$-large subset of $\aleph_1^k$); let $\tilde{g}$ and g' denote, respectively, the kernel and the remainder function of g. Let $\{\vec{\alpha}^i \mid i < \aleph_1\}$ be an enumeration of $\aleph_1^{<k}$ such that for every $i < \aleph_1$ $|\{j \mid \vec{\alpha}^j = \vec{\alpha}^i\}| = \aleph_1$. We define by induction a sequence $\{\vec{\alpha}_i \mid i < \aleph_1\} \subseteq \aleph_1^{\leq k}$ such that for every $i \leq \aleph_1$ $A_i \stackrel{\text{def}}{=} \{\vec{\alpha}_j \mid j < i\}$ is hereditary. Suppose $\vec{\alpha}_j$ has been defined for every $j < i$. If $\vec{\alpha}^i \notin A_i$ let $\vec{\alpha}_i = \Lambda$. Otherwise let $\gamma \in \aleph_1$ be such that $g'(\vec{\alpha}^i\,\hat{}\,\langle\gamma\rangle) \cap A_i = \varnothing$ and $\alpha^i\,\hat{}\,\gamma \notin \bigcup\tilde{g}[A_i]$. Such γ exists by the disjointedness of g' and the countability of A_i. Let $A = A_{\aleph_1} \cap \aleph_1^k$. It is easy to see that A is as required in the lemma.

REMARK. See a generalization of 4.8 in [RS].

LEMMA 4.9. *Let* $h: \aleph_1^l \to S_\omega(A)$ *be a* Δ*-system, let*

$$E(h, k) = \{\sigma \in A^{[k]} \mid \exists\, \vec{\alpha}_1, \ldots, \vec{\alpha}_{k-1}(\sigma \subseteq \bigcup_{i=1}^{k-1} h(\vec{\alpha}_i))\}.$$

Then $E(h, k) \in I_k(A)$.

PROOF. Trivial.

PROOF OF 4.4 (**) $\Rightarrow$ (2). *Part* I. Let M satisfy (**) and we prove that P_M is an $S_{n-1,\infty}$ forcing notion.

Let $g\colon \aleph_1^n \to P_M$. By Corollary 4.7 we can assume that g is a Δ-system, and that $\bigcup\{g(\vec\beta) \mid \vec\beta \in \aleph_1^n\}$ is $\aleph_0$-small. By (**)(b) we can also assume that for every $\vec\beta \in \aleph_1^n$ $g(\vec\beta) \supseteq \{\vec\alpha^0, \ldots, \vec\alpha^{m-1}\}$. Let g' be the remainder function of g. Let $\{\vec\beta^i \mid i < \aleph_1\}$ be an enumeration of $\aleph_1^{<n}$ such that for every i $|\{j \mid \vec\beta^j = \vec\beta^i\}| = \aleph_1$. Recall that in (**) we had an enumeration $\{\vec\alpha^i \mid i < \aleph_1\}$ of $\aleph_1^{n+1}$ and an enumeration $\{G^i \mid i < \aleph_1\}$ of all countable subsets of $(\aleph_1^{n+1})^{[m-1]}$.

We define by induction a sequence $\{\vec\beta_i \mid i < \aleph_1\} \subseteq \aleph_1^{\le n}$ such that for every $i \le \aleph_1$ $B_i \stackrel{\text{def}}{=} \{\vec\beta_j \mid j < i\}$ is hereditary. Let $\vec\beta_0 = \Lambda$. Suppose $\vec\beta_j$ has been defined for every $j < i$, and we define $\vec\beta_i$. If $\vec\beta^i \notin B_i$, let $\vec\beta_i = \Lambda$. Suppose $\vec\beta^i \in B_i$. Let $A_i = \bigcup\{g'(\vec\beta_j) \mid j < i\}$. For some ν_i $A_i^{[m-1]} = G^{\nu_i}$. Since g' is disjointed, there is $\gamma \in \aleph_1$ such that $\vec\beta^{i\frown}\langle\gamma\rangle \notin B_i$ and $g'(\vec\beta^{i\frown}\langle\gamma\rangle) = \{\vec\alpha^{\eta(i,1)}, \ldots, \vec\alpha^{\eta(i,l_i)}\}$ where $\nu_i < \eta(i, 1) < \cdots < \eta(i, l_i)$. Let $\vec\beta_i = \vec\beta^{i\frown}\langle\gamma\rangle$.

By the construction $\langle B_{\aleph_1}, \leqslant\rangle \cong \langle \aleph_1^{\le n}, \leqslant\rangle$, so w.l.o.g. we can assume that $B_{\aleph_1} = \aleph_1^{\le n}$. (This assumption is just for notational simplicity.)

We shall define now a function $h_1\colon \aleph_1^{\le n} \to S_\omega(\aleph_1^{\le n})$. Since $\aleph_1^{\le n} = \{\vec\beta_i \mid i < \aleph_1\}$ it suffices to define $h_1(\vec\beta_i)$ for every $i < \aleph_1$. Let $i < \aleph_1$. For every $l \le l_i$ let $G^{i,l} = G^{\nu_i} * \{\vec\alpha^{\eta(i,1)}, \ldots, \vec\alpha^{\eta(i,l-1)}\}$ (cf. Lemma 4.3), and $G_{i,l} = \{\sigma \in G^{i,l} \mid \sigma \cup \{\vec\alpha^{\eta(i,l)}\}$ is incorrect$\}$. $A_i \subseteq \bigcup\{g(\vec\beta) \mid \vec\beta \in \aleph_1^n\}$, so A_i is an $\aleph_0$-small subset of $\aleph_1^{n+1}$. Hence $G^{\nu_i} = A_i^{[m-1]} \in I$. By (**) (substituting j of (**) by $\eta(i, l)$ and i of (**) by ν_i), $G_{i,l}$ is finite. Let $G_i = \bigcup\{\sigma \mid \sigma \in \bigcup_{l \le l_i} G_{i,l}\}$, hence G_i is a finite subset of $\aleph_1^{n+1}$. Let $h_1(\vec\beta_i) = \{\vec\beta_t \mid g'(\vec\beta_t) \cap G_i \neq \varnothing$ and $\vec\beta_t \not\leqslant \vec\beta_i\}$. By the finiteness of G_i and the disjointedness of g', $h_1(\vec\beta_i)$ is finite. Note that since $G_i \subseteq A_i \cup g'(\vec\beta_i)$, for every t, if $\vec\beta_t \in h_1(\vec\beta_i)$, then $t < i$. This concludes the definition of h_1.

Let $h\colon \aleph_1^n \to S_\omega(\aleph_1^{\le n})$ be defined as follows: $h(\vec\beta) = \bigcup\{h_1(\vec\gamma) \mid \vec\gamma \leqslant \vec\beta\}$. By the definition of h_1 and h for every $\vec\beta \in \aleph_1^n$ $h(\vec\beta) \cap H(\vec\beta) = \varnothing$, so by 4.8 there is an $\aleph_1$-large subset B of $\aleph_1^n$ such that B is h-free.

We prove that $g(B)$ is m-compatible, and hence it is also l-compatible for every l. It suffices to show that every $\sigma \in (\bigcup g(B))^{[m]}$ is correct. Suppose by contradiction σ is incorrect. Let $\sigma = \{\vec\alpha^{j_1}, \ldots, \vec\alpha^{j_m}\}$ where $j_1 < j_2 < \cdots < j_m$. Let $\vec\beta$ be such that $\vec\alpha^{j_m} \in g(\vec\beta)$, and $\vec\beta_i$ be such that $\vec\alpha^{j_m} \in g'(\vec\beta_i)$. By disjointness of g' $\vec\beta_i \leqslant \vec\beta$. Let $l \le l_i$ be such that $j_m = \eta(i, l)$, and let $\sigma' = \sigma - \{\vec\alpha^{j_m}\}$. We shall prove that $\sigma' \in G_{i,l}$. Let $\sigma_1 = \sigma' - g'(\vec\beta_i)$; since $j_1, \ldots, j_{m-1} < j_m$, it follows that $\sigma_1 \subseteq A_i$. Since by our assumption on g $|g'(\Lambda)| \ge m$ and since $\Lambda = \vec\beta_0$, then $G^{\nu_i} \neq \varnothing$ so there is $\sigma_1' \in G^{\nu_i}$ such that $\sigma_1 \subseteq \sigma_1'$. Hence $\sigma' \subseteq \sigma_1' \cup \{\vec\alpha^{\eta(i,1)}, \ldots, \vec\alpha^{\eta(i,l-1)}\}$, so $\sigma' \in G^{i,l}$. Since $\sigma' \cup \{\vec\alpha^{\eta(i,l)}\}$ is incorrect, $\sigma' \in G_{i,l}$. Hence $\sigma' \subseteq G_i$. There is some t such that $\vec\beta_t \not\leqslant \vec\beta_i$ but $g'(\vec\beta_t) \cap \sigma' \neq \varnothing$, for otherwise $\sigma \subseteq g(\vec\beta)$, hence it cannot be incorrect. So $g'(\vec\beta_t) \cap G_i \neq \varnothing$, hence $\vec\beta_t \in h_1(\vec\beta_i) \subseteq h(\vec\beta)$. But by disjointness of g', for some $\vec\gamma \in B$ $\vec\beta_t \leqslant \vec\gamma$. Hence $H(\vec\gamma) \cap h(\vec\beta) \neq \varnothing$, contradicting the h-freeness of B. This proves that $g(B)$ is m-compatible, and we have thus proved that P_M is an $S_{n-1,\infty}$ forcing notion.

Part II. We prove that P_M is an $S_{l,m-1}$ forcing notion for every l. If $m = 2$ then we just have to prove that P_M is c.c.c. But since $\langle n, m\rangle \neq \langle 0, 2\rangle$, n

must be greater than 0, so by Part I P_M is c.c.c. We can thus assume that $m > 2$.

Let $g: \aleph_1^l \to P_M$. W.l.o.g. g is a Δ-system. By(**)(b) we can assume that for every $\vec{\beta} \in \aleph_1^l$ $g(\vec{\beta}) \supseteq \{\vec{\alpha}^0, \ldots, \vec{\alpha}^{m-1}\}$. Let g' be the remainder function of g. The proof is very similar to the proof of Part I. Let $\{\vec{\beta}^i \mid i < \aleph_1\}$ be an enumeration of $\aleph_1^{\le l}$ such that for every $i < \aleph_1$ $|\{j \mid \vec{\beta}^j = \vec{\beta}^i\}| = \aleph_1$. We define by induction a sequence $\{\vec{\beta}_i \mid i < \aleph_1\} \subseteq \aleph_1^{\le l}$ such that for every $i \le \aleph_1$ $B_i \stackrel{\text{def}}{=} \{\vec{\beta}_j \mid j < i\}$ is hereditary.

Suppose $\vec{\beta}_j$ has been defined for every $j < i$. If $\vec{\beta}^i \notin B_i$ let $\vec{\beta}_i = \Lambda$. Suppose $\vec{\beta}^i \in B_i$. Let $A_i = \bigcup\{g'(\vec{\beta}_j) \mid j < i\}$. The set $E(g, m-1) \cap A_i^{[m-1]}$ is a countable subset of $(\aleph_1^{n+1})^{[m-1]}$, so it appears in the list $\{G^i \mid i < \aleph_1\}$ given in (**); suppose $E(g, m-1) \cap A_i^{[m-1]} = G^{\nu_i}$. By the disjointedness of g' there is $\gamma \in \aleph_1$ such that $\vec{\beta}^i{}^\frown\langle\gamma\rangle \notin B_i$ and $g'(\vec{\beta}^i{}^\frown\langle\gamma\rangle) = \{\vec{\alpha}^{\eta(i,1)}, \ldots, \vec{\alpha}^{\eta(i,l_i)}\}$ where $\nu_i < \eta(i, 1) < \cdots < \eta(i, l_i)$. We define $\vec{\beta}_i = \vec{\beta}^i{}^\frown\langle\gamma\rangle$. By the construction $\langle B_{\aleph_1}, \leqslant\rangle \cong \langle \aleph_1^{\le l}, \leqslant\rangle$, so we assume that $B_{\aleph_1} = \aleph_1^{\le l}$.

We now define a function $h_1 : \aleph_1^{\le l} \to S_\omega(\aleph_1^{\le l})$. It suffices to define $h_1(\vec{\beta}_i)$ for every $i < \aleph_1$. For every $1 \le k \le l_i$ let

$$G^{i,k} = G^{\nu_i} * \Big(\Big(\bigcup_{\vec{\gamma} < \vec{\beta}_i} g'(\vec{\gamma})\Big) \cup \{\vec{\alpha}^{\eta(i,1)}, \ldots, \vec{\alpha}^{\eta(i,k-1)}\}\Big),$$

and let $G_{i,k} = \{\sigma \in G^{i,k} \mid \sigma \cup \{\vec{\alpha}^{\eta(i,k)}\}$ is incorrect$\}$. We prove that $G_{i,k}$ is finite. $G^{\nu_i} \subseteq E(g, m-1)$, hence by Lemma 4.9, $G^{\nu_i} \in I$.

$$\tau \stackrel{\text{def}}{=} \Big(\bigcup_{\vec{\gamma} < \vec{\beta}_i} g'(\vec{\gamma})\Big) \cup \{\vec{\alpha}^{\eta(i,1)}, \ldots, \vec{\alpha}^{\eta(i,k-1)}\}$$

is a finite subset of $\{\vec{\alpha}^t \mid t < \eta(i, k)\}$, hence by (**) (substituting j of (**) by $\eta(i, k)$ and i of (**) by ν_i) we conclude that $G_{i,k}$ is finite. Let $G_i = \bigcup\{\sigma \mid \sigma \in \bigcup_{k \le l_i} G_{i,k}\}$. So G_i is a finite subset of $\aleph_1^{n+1}$. Let $h_1(\vec{\beta}_i) = \{\vec{\beta}_t \mid g'(\vec{\beta}_t) \cap G_i \ne \emptyset$ and $\vec{\beta}_t \not\leqslant \vec{\beta}_i\}$. By the finiteness of G_i and the disjointedness of g' $h_1(\vec{\beta}_i)$ is finite.

Let $h: \aleph_1^l \to S_\omega(\aleph_1^{\le l})$ be defined as follows: $h(\vec{\beta}) = \bigcup\{h_1(\vec{\gamma}) \mid \vec{\gamma} \leqslant \vec{\beta}\}$. By the definitions of h_1 and h for every $\vec{\beta} \in \aleph_1^l$ $h(\vec{\beta}) \cap H(\vec{\beta}) = \emptyset$, so by 4.8 there is an h-free $\aleph_1$-large subset of $\aleph_1^l$, which we denote by B.

We shall prove that $g(B)$ is $m-1$-compatible. Suppose by contradiction it is not, hence, there are $\tau \in B^{[m-1]}$ and $\sigma \in (\aleph_1^{n+1})^{[m]}$ such that $\sigma \subseteq \bigcup\{g(\vec{\beta}) \mid \vec{\beta} \in \tau\}$ and σ is incorrect. Let $\sigma = \{\vec{\alpha}^{j_1}, \ldots, \vec{\alpha}^{j_m}\}$ where $j_1 < \cdots < j_m$. Let $\vec{\beta} \in \tau$ be such that $\vec{\alpha}^{j_m} \in g(\vec{\beta})$, and let i be such that $\vec{\alpha}^{j_m} \in g'(\vec{\beta}_i)$. Clearly $0 < i$ and $\vec{\beta}_1 \leqslant \vec{\beta}$. There is $k \le l_i$ such that $j_m = \eta(i, k)$. Let $\sigma' = \sigma - \{\vec{\alpha}^{j_m}\}$. We shall prove that $\sigma' \in G_{i,k}$. Let $\sigma_1 = \sigma' - g(\vec{\beta})$, clearly $\sigma_1 \subseteq A_i$. Moreover $\sigma_1 \subseteq \bigcup\{g(\vec{\gamma}) \mid \vec{\gamma} \in \tau - \{\vec{\beta}\}\}$ and $|\tau - \{\vec{\beta}\}| = m - 2$. By our assumption on g: $|g'(\Lambda)| \ge m$; so there is $\sigma_1' \supseteq \sigma_1$ such that $\sigma_1' \in (\bigcup\{g(\vec{\gamma}) \mid \vec{\gamma} \in \tau - \{\vec{\beta}\}\})^{[m-1]}$. Hence $\sigma_1' \in E(g, m-1) \cap A_i^{[m-1]} = G^{\nu_i}$. Let $\sigma_2 = \sigma' \cap g(\vec{\beta})$, hence $\sigma_2 \subseteq (\bigcup_{\vec{\gamma} < \vec{\beta}_i} g'(\vec{\gamma})) \cup \{\vec{\alpha}^{\eta(i,1)}, \ldots, \vec{\alpha}^{\eta(i,k-1)}\}$. It follows that

$$\sigma' = \sigma_1 \cup \sigma_2 \in G^{\nu_i} * \Big(\Big(\bigcup_{\vec{\gamma} < \vec{\beta}_i} g'(\vec{\gamma})\Big) \cup \{\vec{\alpha}^{\eta(i,1)}, \ldots, \vec{\alpha}^{\eta(i,k-1)}\}\Big) = G^{i,k}.$$

Since $\sigma' \cup \{\vec{\alpha}^{\eta(i,k)}\}$ is incorrect, $\sigma' \in G_{i,k}$ and, hence, $\sigma' \subseteq G_i$. There is some t such that $\vec{\beta}_t \not\leqslant \vec{\beta}_i$ and $g'(\vec{\beta}_t) \cap \sigma' \ne \emptyset$, for otherwise $\sigma \subseteq g(\vec{\beta})$, and hence it is correct.

So $\vec{\beta}_t \in h_1(\vec{\beta}_i) \subseteq h(\vec{\beta})$. Let $\vec{\gamma} \in B$ be such that $\vec{\beta}_t \leqslant \vec{\gamma}$. So $h(\vec{\beta}) \cap H(\vec{\gamma}) \neq \emptyset$, contradicting the h-freeness of B. We have thus proved that $g(B)$ is $m-1$-compatible, so P_M is an $S_{l,m-1}$ forcing notion.

The construction of M. Let $\{\vec{\alpha}^i \mid i < \aleph_1\}$ and $\{G^i \mid i < \aleph_1\}$ be as in (*), (**), (***), and let $A^i = \{\vec{\alpha}^j \mid j < i\}$. Let $\{\sigma_i \mid i < \aleph_1\}$ be an enumeration of $S_\omega(\aleph_1^{n+1})$ such that for every $i, j < \aleph_1 \mid \{k \mid \sigma_k = \sigma_i$ and $\vec{\alpha}^k \upharpoonright n = \vec{\alpha}^j \upharpoonright n\} \mid = \aleph_1$. Since $R_{n,m}$ is a symmetric relation we regard it by abuse of notation as a subset of $\bigcup_{1 \le l \le m} (\aleph_1^{n+1})^{[l]}$. Since R is going to be defined as a symmetric relation we make the same convention for R. If A is a set let $A^{\le [k]} \stackrel{\text{def}}{=} \bigcup_{1 \le l \le k} A^{[l]}$. For every $i < \aleph_1$ we shall define a set $R^i \subseteq (A^i)^{[m-1]}$ with the purpose that

$$R = (R_{n,m} \cap (\aleph_1^{n+1})^{\le [m-1]}) \cup \bigcup_{i < \aleph_1} \{\sigma \cup \{\vec{\alpha}^i\} \mid i < \aleph_1, \sigma \in R_i\}.$$

The construction of R^i. For every $i < \omega$ let $R^i = \{\sigma \in (A^i)^{[m-1]} \mid \sigma \cup \{\vec{\alpha}^i\} \in R_{n,m}\}$. Suppose $i \ge \omega$. Let $\{G_j^i \mid j < \omega\}$ be an enumeration of $\{G^j * \tau \mid j < i$ and $\tau \in S_\omega(A^i)\}$. We now define by induction on $k \in \omega$ sets $R_k^i, \bar{R}_k^i \subseteq (A^i)^{[m-1]}$ with the purpose that

$$R^i = \{\sigma \in R_k^i \mid \sigma \cup \{\vec{\alpha}^i\} \in R_{n,m}\} \cup \{\sigma \in \bar{R}_k^i \mid \sigma \cup \{\vec{\alpha}^i\} \notin R_{n,m}\}.$$

Our induction hypotheses on the R_k^i's and $\bar{R}_k^i$'s are the following:

(1) $\{R_k^i \mid k \in \omega\}$ and $\{\bar{R}_k^i \mid k \in \omega\}$ are increasing with k;
(2) $R_k^i \cap \bar{R}_k^i = \emptyset$;
(3) $R_k^i \in I$; and
(4) $\bar{R}_k^i$ is finite.

Let $R_0^i = \sigma_i^{[m-1]} \cup \{\sigma \in (A^i)^{[m-1]} \mid \sigma \cap \{\vec{\alpha}^0, \ldots, \vec{\alpha}^{m-1}\} \neq \emptyset\}$, and let $\bar{R}_0^i = \emptyset$. It is easy to check that $R_0^i \in I$, so the induction hypotheses hold. Suppose R_k^i and $\bar{R}_k^i$ have been defined. If $G_k^i \in I$, let $R_{k+1}^i = R_k^i \cup (G_k^i - \bar{R}_k^i)$. If $G_k^i \notin I$, then since $R_k^i \in I$ it follows that there is $\sigma \in G_k^i - R_k^i$; let $R_{k+1}^i = R_k^i$ and $\bar{R}_{k+1}^i = \bar{R}_k^i \cup \{\sigma\}$. It is trivial to check that the induction hypotheses hold. This concludes the definition of R_k^i and $\bar{R}_k^i$, and hence R^i is defined, so R is also defined.

It is easy to check that R satisfies requirements (*), (**) and (***). This concludes the proof of Theorem 1.2.

REFERENCES

[F] G. Fodor, *Proof of a conjecture of P. Erdos*, ***Acta Universitatis Szegediensis, Acta Scientiarum Mathematicarum***, vol. 14 (1952), pp. 219–227.

[G] S. Garavaglia, *Relative strength of Malitz quantifiers*, ***Notre Dame Journal of Formal Logic***, vol. 19 (1978), pp. 495–503.

[H] A. Hajnal, *Proof of a conjecture of S. Ruziewicz*, ***Fundamenta Mathematicae***, vol. 50 (1961), pp. 123–128.

[Kr] H. J. Keisler, *Logic with the quantifier "there exist uncountably many"*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 1–93.

[KT] K. Kunen and F. Tall

[Ma, Re] J. Malitz and W. Reinhardt, *Maximal models in the language with quantifier "there exist uncountably many"*, ***Pacific Journal of Mathematics***, vol. 40 (1972), pp. 139–155.

[MM] M. Magidor and J. Malitz, *Compact extensions of L(Q)(part Ia)*, ***Annals of Mathematical Logic***, vol. 2 (1977), pp. 217–261.

[MR] J. MALITZ and M. RUBIN, *Fragments of higher order logic, Mathematical Logic in Latin America*, ***North-Holland Studies in Logic***, vol. 99, North-Holland, Amsterdam, 1980, pp. 219–238.

[Ra] M.O. RABIN, *Arithmetical extensions with prescribed cardinality*, ***Koninklijke Nederlandse Akademie von Wetenschappen, Proceedings, Series A, Mathematical Sciences***, vol. 21 (1959), pp. 439–446.

[RS] M. RUBIN and S. SHELAH, *Combinatorial problems on trees. Partitions, Δ-systems and large free subsets*, ***Annals of Mathematical Logic*** (to appear).

[Sl] S. SHELAH, *Models with second order properties*. III, *Omitting types in λ^+ for $L(Q)$*, ***Proceedings of the "Berlin Workshop in Logic July 1977", Archiv fur Mathematische Logik und Grundlagenforschung***, vol. 21 (1980), pp. 1–11.

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