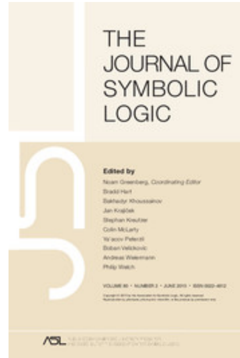
Sh:387



Full reflection of stationary sets below ω

Thomas Jech and Saharon Shelah

The Journal of Symbolic Logic / Volume 55 / Issue 02 / June 1990, pp 822 - 830
DOI: 10.2307/2274667, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 55, Number 2, June 1990

FULL REFLECTION OF STATIONARY SETS BELOW $\aleph_\omega$

THOMAS JECH AND SAHARON SHELAH

Abstract. It is consistent that, for every $n \geq 2$, every stationary subset of ω_n consisting of ordinals of cofinality ω_k, where $k = 0$ or $k \leq n - 3$, reflects fully in the set of ordinals of cofinality ω_{n-1}. We also show that this result is best possible.

1. Introduction. A stationary subset S of a regular uncountable cardinal κ *reflects at* $\gamma < \kappa$ if $S \cap \gamma$ is a stationary subset of γ. For stationary sets $S, A \subseteq \kappa$ let

$$S < A \quad \text{if } S \text{ reflects at almost all } \alpha \in A,$$

where "almost all" means modulo the closed unbounded filter on κ, i.e. with the exception of a nonstationary set of α's. If $S < A$ we say that *S reflects fully in A*. The *trace* of S, $\mathrm{Tr}(S)$, is the set of all $\gamma < \kappa$ at which S reflects. The relation $<$ is well-founded [1], and $o(S)$, the *order of* S, is the rank of S in this well-founded relation.

In this paper we investigate the question of which stationary subsets of ω_n reflect fully in which stationary sets; in other words, the structure of the well founded relation $<$. Clearly, $o(S) < o(A)$ is a necessary condition for $S < A$, and moreover, a set $S \subseteq \omega_n$ has order k just in case it has a stationary intersection with the set

$$S^n_k = \{\alpha < \omega_n \colon \mathrm{cf}\, \alpha = \omega_k\}.$$

Thus the problem reduces to the investigation of full reflection of stationary subsets of S^n_k in stationary subsets of S^n_m for $k < m < n$.

The problem for $n = 2$ is solved completely in Magidor's paper [2]: It is consistent that every stationary $S \subseteq S^2_0$ reflects fully in S^2_1. The problem for $n > 2$ is more complicated. It is tempting to try the obvious generalization, namely $S < A$ whenever $o(S) < o(A)$, but this is provably false:

PROPOSITION 1.1. *There exist stationary sets $S \subset S^3_0$ and $A \subset S^3_1$ such that S does not reflect at any $\gamma \in A$.*

PROOF. Let S_i, $i < \omega_2$, be any family of pairwise disjoint subsets of S^3_0, and let $\langle C_\gamma \colon \gamma \in S^3_1 \rangle$ be such that each C_γ is a closed unbounded subset of γ of order type ω_1.

Received June 23, 1989.
The first author's research was partially supported by the National Science Foundation and by a Fulbright grant at the Hebrew University of Jerusalem. The second author's research was supported in part by the U.S.-Israel Binational Science Foundation. This paper is number 387 in the master list of Professor Shelah's works.

0022-4812/90/5502-0030/$01.90

Clearly, at most $\aleph_1$ of the sets S_i can meet each C_γ, and so for each γ there is $i(\gamma) < \omega_2$ such that $C_\gamma \cap S_i = \emptyset$ for all $i \geq i(\gamma)$.

There is $i < \omega_2$ such that $i(\gamma) = i$ for a stationary set of γ's. Let $A \subset S^3_1$ be this stationary set and let $S = S_i$. Then $S \cap C_\gamma = \emptyset$ for all $\gamma \in A$, and so $S \cap \gamma$ is non-stationary. Hence S does not reflect at any $\gamma \in A$. □

There is of course nothing special in the proof about $\aleph_3$ (or about $\aleph_1$), and so we have the following generalization:

PROPOSITION 1.2. *Let $k < m < n - 1$. There exist stationary sets $S \subseteq S^n_k$ and $A \subseteq S^n_m$ such that S does not reflect at any $\gamma \in A$.*

Consequently, if $n > 2$ then full reflection in S^n_m is possible only if $m = n - 1$. This motivates our main theorem.

MAIN THEOREM 1.3. *Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$, and such that*

(a) *every stationary subset of S^2_0 reflects fully in S^2_1, and*

(b) *for every $n \geq 3$, every stationary subset of S^n_k, for all $k = 0, \ldots, n - 3$, reflects fully in S^n_{n-1}.*

We will show that the result of the main theorem is best possible. But first we prove a corollary:

COROLLARY 1.4. *In the model of the main theorem we have for all $n \geq 2$ and all m, $0 < m < n$:*

(a) *Any $\aleph_m$ stationary subsets of S^n_0 reflect simultaneously at some $\gamma \in S^n_m$.*

(b) *For every $k \leq m - 2$, any $\aleph_m$ stationary subsets of S^n_k reflect simultaneously at some $\gamma \in S^n_m$.*

PROOF. Let us prove (a), as (b) is similar. Let $m < n$ and let S_ξ, $\xi < \omega_m$, be stationary subsets of S^n_0. First, each S_ξ reflects fully in S^n_{n-1}, and so there exist club sets C_ξ, $\xi < \omega_m$, such that each S_ξ reflects at all $\alpha \in C_\xi \cap S^n_{n-1}$. As the club filter is ω_n-complete, there exists an $\alpha \in S^n_{n-1}$ such that $S_\xi \cap \alpha$ is stationary, for all $\xi < \omega_m$. Next we apply full reflection of subsets of S^{n-1}_0 in S^{n-1}_{n-2} (to the ordinal α of cofinality ω_{n-1} rather than to ω_{n-1} itself) and the ω_{n-1}-completeness of the club filter on ω_{n-1}, to find $\beta \in S^n_{n-2}$ such that $S_\xi \cap \beta$ is stationary for all $\xi < \omega_m$. This way we continue until we find a $\gamma \in S^n_m$ such that every $S_\xi \cap \gamma$ is stationary. □

Note that the amount of simultaneous reflection in 1.4 is best possible:

PROPOSITION 1.5. *If $\operatorname{cf} \gamma = \aleph_m$ and if S_ξ, $\xi < \omega_{m+1}$, are disjoint stationary sets, then some S_ξ does not reflect at γ.*

PROOF. γ has a club subset of size $\aleph_m$, and it can only meet $\aleph_m$ of the sets $S_\xi \cap \gamma$. □

By Corollary 1.4, the model of the main theorem has the property that whenever $2 \leq m < n$, every stationary subset of S^n_k reflects quite strongly in S^n_m, provided $k \leq m - 2$. This cannot be improved to include the case of $k = m - 1$, as the following proposition shows:

PROPOSITION 1.6. *Let $m \geq 2$. Either* (a) *for all $k < m - 1$ there exists a stationary set $S \subseteq S^m_k$ that does not reflect fully in S^m_{m-1}, or* (b) *for all $n > m$ there exists a stationary set $A \subseteq S^n_{m-1}$ that does not reflect at any $\delta \in S^n_m$.*

We shall given a proof of 1.6 in §3. In our model we have, for every $m \geq 2$, full reflection of subsets of S^m_0 in S^m_{m-1} (and of subsets of S^m_k for $k \leq m - 3$), and therefore 1.6(a) fails in the model. Thus the model necessarily satisfies 1.6(b), which shows that the consistency result is best possible.

2. Proof of the main theorem. Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of cardinals with the property that for each $n \geq 2$, κ_n is a $<\kappa_{n+1}$-supercompact cardinal, i.e. for every $\gamma < \kappa_{n+1}$ there exists an elementary embedding $j: V \to M$ with critical point κ_n such that $j(\kappa_n) > \gamma$ and $M^\gamma \subset M$.[1] We construct the generic extension by iterated forcing, an iteration of length ω with full support. The first stage of the iteration P_1 makes $\kappa_2 = \aleph_2$, and for each n, the nth stage P_n (a forcing notion in $V(P_1 * \cdots * P_{n-1})$) makes $\kappa_{n+1} = \aleph_{n+1}$. In the iteration, we repeatedly use three standard notions of forcing: $\mathrm{Col}(\kappa, \alpha)$, $\mathrm{C}(\kappa)$ and $\mathrm{CU}(\kappa, T)$.

DEFINITION. Let κ be a regular uncountable cardinal.

(a) $\mathrm{Col}(\kappa, \alpha)$ is the forcing that collapses $\alpha \geq \kappa$ with conditions of size $\langle \kappa$: A condition is a function p from a subset of κ of size $<\kappa$ into α; a condition q is stronger than p if $q \supseteq p$.

(b) $\mathrm{C}(\kappa)$ is the forcing that adds a Cohen subset of κ: A condition is a 0-1-function p on a subset of κ of size $<\kappa$; a condition q is stronger than p if $q \supseteq p$.

(c) $\mathrm{CU}(\kappa, T)$ is the forcing that shoots a club through a stationary set $T \subseteq \kappa$: A condition is a closed bounded subset of T; a condition q is stronger than p if q end-extends p.

The first stage P_1 of the iteration $P = \langle P_n: n = 1, 2, \ldots \rangle$ is a forcing of size κ_2 that is ω-closed,[2] satisfies the κ_2-chain condition and collapses each cardinal between $\aleph_1$ and κ_2 (it is essentially the Levy forcing with countable conditions). For each $n \geq 2$, we construct (in $V(P \mid n)$) the nth stage P_n such that

(2.1) (a) $|P_n| = \kappa_{n+1}$,

(b) P_n is $\aleph_{n-2}$ closed,

(c) P_n satisfies the κ_{n+1}-chain condition,

(d) P_n collapses each cardinal between $\aleph_n$ $(=\kappa_n)$ and κ_{n+1}, and

(e) P_n does not add any ω_{n-1}-sequences of ordinals,

and such that P_n guarantees the reflection of stationary subsets of $\aleph_n$ stated in the theorem.

It follows, by induction, that each κ_n becomes $\aleph_n$: Assuming that $\kappa_n = \aleph_n$ in $V(P \mid n)$, the nth stage P_n preserves $\aleph_n$ by (e), and the rest of the iteration $\langle P_{n+1}, P_{n+2}, \ldots \rangle$ also preserves $\aleph_n$ because it is $\aleph_{n-1}$-closed by (b); P_n makes κ_{n+1} the successor of κ_n by (c) and (d).

We first define the forcing P_1:

P_1 is an iteration, with countable support, $\langle Q_\alpha: \alpha < \kappa_2 \rangle$, where, for each α,

$$Q_\alpha = \mathrm{Col}(\aleph_1, \aleph_1 + \alpha) \times \mathrm{C}(\aleph_1).$$

It follows easily from well-known facts that P_1 is an ω-closed forcing of size κ_2, satisfies the κ_2-chain condition, and makes $\kappa_2 = \aleph_2$.

[1] We note in passing that the condition about the κ_n is equivalent to "every κ_n is $<\kappa_\omega$-supercompact", where $\kappa_\omega = \sup_{m<\omega} \kappa_m$.

[2] A forcing notion is *λ-closed* if every descending sequence of length $\leq \lambda$ has a lower bound.

Next we define the forcing P_2. (It is a modification of Magidor's forcing from [2], but the added collapsing of cardinals requires a stronger assumption on κ_2 than weak compactness. The iteration is padded up by the addition of Cohen forcing, which will make the main argument of the proof work more smoothly.) The definition of P_2 is inside the model $V(P_1)$, and so $\kappa_2 = \aleph_2$:

P_2 is an iteration, with $\aleph_1$-support, $\langle Q_\alpha: \alpha < \kappa_3\rangle$, where, for each α,

$$Q_\alpha = \mathrm{Col}(\aleph_2, \aleph_2 + \alpha) \times \mathrm{C}(\aleph_2) \times \mathrm{CU}(T_\alpha)$$

where T_α is, in $V(P_1 * P_2 \,|\, \alpha)$, some stationary subset of ω_2. We choose the T_α's so that each T_α contains all limit ordinals of cofinality ω. It follows easily that for each $\alpha < \kappa_3$, $P_2 \,|\, \alpha \Vdash Q_\alpha$ is ω-closed.

The crucial property of the forcing P_2 will be the following:

LEMMA 2.2. *P_2 does not add new ω_1-sequences of ordinals.*

One consequence of Lemma 2.2 is that the conditions $(p, q, s) \in Q_\alpha$ can be taken to be sets in $V(P_1)$ (rather than in $V(P_1 * P_2 \,|\, \alpha)$). Once we have Lemma 2.2, the properties (2.1)(a)–(e) follow easily.

It remains to specify the choice of the T_α's. By a standard argument using the κ_3-chain condition, we can enumerate all potential subsets of ω_2 by a sequence $\langle S_\alpha: \alpha < \kappa_3\rangle$ in such a way that each S_α is already in $V(P_1 * P_2 \,|\, \alpha)$. At stage α of the iteration we let $T_\alpha = \omega_2$, unless S_α is, in $V(P_1 * P_2 \,|\, \alpha)$, a stationary set of ordinals of cofinality ω. If that is the case, we let

$$T_\alpha = (\mathrm{Tr}(S_\alpha) \cap S^2_1) \cup S^2_0.$$

Assuming that Lemma 2.2 holds, we now show that in $V(P_1 * P_2)$ every stationary $S \subseteq S^2_0$ reflects fully in S^2_1.

The set S appears as S_α at some stage α, and because it is stationary in $V(P_1 * P_2)$, it is stationary in the smaller model $V(P_1 * P_2 \,|\, \alpha)$. The forcing Q_α creates a closed unbounded set C such that $C \cap S^2_1 \subseteq \mathrm{Tr}(S)$ (note that because P_2 does not add ω_1-sequences, the meaning of $\mathrm{Tr}(S)$ or of S^2_1 does not change).

Thus in $V(P_1 * P_2)$ we have full reflection of subsets of S^2_0 in S^2_1. The later stages of the iteration do not add new subsets of ω_2, and so this full reflection remains true in $V(P)$.

We postpone the proof of Lemma 2.2 until after the definition of the rest of the iteration.

We now define P_n for $n \geq 3$. We work in $V(P_1 * \cdots * P_{n-1})$. By the induction hypothesis we have $\kappa_n = \aleph_n$.

P_n is an iteration with $\aleph_{n-1}$-support, $\langle Q_\alpha: \alpha < \kappa_{n+1}\rangle$, where for each α,

$$Q_\alpha = \mathrm{Col}(\aleph_n, \aleph_n + \alpha) \times \mathrm{C}(\aleph_n) \times \mathrm{CU}(T_\alpha)$$

where T_α is a $P_n \,|\, \alpha$-name for a subset of ω_n. To specify the T_α's, let $\langle S_\alpha: \alpha < \kappa_{n+1}\rangle$ be an enumeration of all potential subsets of ω_n such that each S_α is a $P_n \,|\, \alpha$-name. At stage α, let $T_\alpha = \omega_n$ unless S_α is a stationary set of ordinals and $S_\alpha \subseteq S^n_k$ for some $k = 0, \ldots, n-3$, in which case let

$$T_\alpha = (\mathrm{Tr}(S_\alpha) \cap S^n_{n-1}) \cup (S^n_0 \cup \cdots \cup S^n_{n-2})$$
$$= \{\gamma < \omega_n: \mathrm{cf}\,\gamma \leq \omega_{n-2} \text{ or } S_\alpha \cap \gamma \text{ is stationary}\}.$$

Due to the selection of the T_α's, Q_α is ω_{n-2}-closed, and so is P_n. The crucial property of the forcing is the analog of Lemma 2.2:

LEMMA 2.3. *P_n does not add new ω_{n-1}-sequences of ordinals.*

Given this lemma, properties (2.1)(a)–(e) follow easily. The same argument as given above for P_2 shows that in $V(P_1 * \cdots * P_n)$, and therefore in $V(P)$ as well, every stationary subset of S^n_k, $k = 0, \ldots, n-3$, reflects fully in S^n_{n-1}.

It remains to prove Lemmas 2.2 and 2.3. We prove Lemma 2.3, as 2.2 is an easy modification.

PROOF OF LEMMA 2.3. Let $n \geq 3$, and let us give the argument for a specific n, say $n = 4$. We want to show that P_4 does not add ω_3-sequences of ordinals.

We will work in $V(P_1 * P_2)$ (and so consider the forcing $P_3 * P_4$). As $P_1 * P_2$ has size κ_3, κ_4 is a $< \kappa_5$-supercompact cardinal in $V(P_1 * P_2)$, and $\kappa_3 = \aleph_3$. The forcing P_3 is an iteration of length κ_4 that makes $\kappa_4 = \aleph_4$ and is $\aleph_1$-closed; then P_4 is an iteration of length κ_5. By induction on $\alpha < \kappa_5$ we show

(2.4) $\quad P_4 \mid \alpha$ does not add ω_3-sequences of ordinals.

As P_4 has the $\aleph_5$-chain condition, (2.4) is certainly enough for Lemma 2.3. Let $\alpha < \kappa_5$.

Let j be an elementary embedding $j\colon V \to M$ (as we work in $V(P_1 * P_2)$, V means $V(P_1 * P_2)$) such that $j(\kappa_4) > \beta$ and $M^\beta \subset M$, for some inaccessible cardinal $\beta > \alpha$. Consider the forcing $j(P_3)$ in M. It is an iteration of which P_3 is an initial segment. By a standard argument, the elementary embedding $j\colon V \to M$ can be extended to an elementary embedding $j\colon V(P_3) \to M(j(P_3))$. We claim that every β-sequence of ordinals in $V(P_3)$ belongs to $M(j(P_3))$: the name for such a set has size $\leq \beta$ and so it belongs to M, and since $P_3 \in M$ and $M(P_3) \subseteq M(j(P_3))$, the claim follows. In particular, $P_4 \mid \alpha \in M(j(P_3))$.

Let $p, \dot{F} \in V(P_3)$ be such that $p \in P_4 \mid \alpha$ and $\dot{F}$ is a $(P_4 \mid \alpha)$-name for an ω_3-sequence of ordinals. We shall find a stronger condition that decides all the values of $\dot{F}$. By the elementarity of j, it suffices to prove that

(2.5) $\quad \exists \bar{p} \leq j(p)$ in $j(P_4 \mid \alpha)$ that decides $j(\dot{F})$.

The rest of the proof is devoted to the proof of (2.5).

Let G be an M-generic filter on $j(P_3)$.

LEMMA 2.6. *In $M[G]$ there is a generic filter H on $P_4 \mid \alpha$ over $M[G \cap P_3]$ such that $M[G]$ is a generic extension of $M[G \cap P_3][H]$ by an $\aleph_1$-closed forcing, and such that $p \in H$.*

PROOF. There is an $\eta < j(\kappa_4)$ such that $P_4 \mid \alpha$ has size $\aleph_3$ in $M_\eta = M[G \cap (j(P_3) \mid \eta)]$. Since $P_4 \mid \alpha$ is $\aleph_2$-closed, it is isomorphic in M_η to the Cohen forcing $\mathrm{C}(\aleph_3)$. But $Q_\eta = (j(P_3))(\eta) = \mathrm{Col}(\aleph_3, \aleph_3 + \eta) \times \mathrm{C}(\aleph_3) \times \mathrm{CU}(T_\eta)$, so $G \mid Q_\eta = G_{\mathrm{Col}} \times G_{\mathrm{C}} \times G_{\mathrm{CU}}$, and using G_{C} and the isomorphism between $P_4 \mid \alpha$ and $\mathrm{C}(\aleph_3)$ we obtain H. Since the quotient forcing $j(P_3)/(P_3 \times \mathrm{C}(\aleph_3))$ is an iteration of $\aleph_1$-closed forcings, it is $\aleph_1$-closed. □

LEMMA 2.7. *In $M[G]$ there is a condition $\bar{p} \in j(P_4 \mid \alpha)$ that extends p, and extends every member of $j''H$.*

Sh:387

Lemma 2.7 will complete the proof of (2.5): since every value of $\dot{F}$ is decided by some condition in H, every value of $j(\dot{F})$ is decided by some condition in $j''H$, and therefore by $\bar{p}$.

PROOF OF LEMMA 2.7. Working in $M[G]$, we construct $\bar{p} \in j(P_4 | \alpha)$, a sequence $\langle p_\xi : \xi < j(\alpha) \rangle$ of length $j(\alpha)$, by induction. When ξ is not in the range of j, we let p_ξ be the trivial condition; that guarantees that the support of $\bar{p}$ has size $|\alpha|$, which is $\aleph_3$ (because $\alpha < j(\kappa_4) = \aleph_4$ in $M[G]$). So let $\xi < \alpha$ be such that $\bar{p} | j(\xi)$ has been defined, and construct $p_{j(\xi)}$.

The condition $p_{j(\xi)}$ has three parts u, v, s, where $u \in \mathrm{Col}(j(\kappa_4), j(\kappa_4) + j(\xi))$, $v \in \mathrm{C}((\kappa_4))$ and $s \in \mathrm{CU}(T_{j(\xi)})$. It is easy to construct the u-part and the v-part, as follows: The filter $H | P_4(\xi)$ has three parts; a collapsing function f of κ_4 onto $\kappa_4 + \xi$, a 0-1-function g on κ_4, and a club subset C of T_ξ. We let $u = j''f$ and $v = j''g$; these are functions of size $\aleph_3$ and therefore members of Col and C respectively. For the s-part, let $s = j''C \cup \{\kappa_4\}$. In order that this set be a condition in $\mathrm{CU}(T_{j(\xi)})$, we have to verify that $\kappa_4 \in T_{j(\xi)}$.

This is a nontrivial requirement if $S_{j(\xi)}$ is in $M(j(P_3) * (j(P_4) | j(\xi)))$, a stationary subset of $j(\kappa_4)$, and is a subset of either S^4_0 or of S^4_1 (of S^n_k for $n = 4$ and $k \le n - 3$). Then κ_4 has to be a reflecting point of $S_{j(\xi)}$, i.e. we have to show that $S_{j(\xi)} \cap \kappa_4$ is stationary, in $M(j(P_3) * (j(P_4) | j(\xi)))$.

By the assumption and by elementarity of j, S_ξ is a stationary subset of κ_4 in $V(P_3 * P_4 | \xi)$, and $S_\xi \subseteq S^4_0$ or $S_\xi \subseteq S^4_1$, i.e. consists of ordinals of cofinality $\le \omega_1$. Since $S_{j(\xi)} \cap \kappa_4 = j(S_\xi) \cap \kappa_4 = S_\xi$, it suffices to show that S_ξ is stationary not only in $V(P_3 * P_4 | \xi)$ but also in $M(j(P_3) * (j(P_4) | j(\xi)))$.

Firstly $M(P_3 * P_4 | \xi) \subseteq V(P_3 * P_4 | \xi)$, and so S_ξ is stationary in $M(P_3 * P_4 | \xi)$. Secondly, $j(P_4)$ is $\aleph_1$-closed, and by Lemma 2.6, $M(j(P_3))$ is an $\aleph_1$-closed forcing extension of $M(P_3 * P_4 | \xi)$, and so the proof is completed by application of the following lemma (taking $\kappa = \aleph_0$ or $\aleph_1$ and $\lambda = \aleph_4$).

LEMMA 2.8. *Let $\kappa < \lambda$ be regular cardinals and assume that for all $\alpha < \lambda$ and all $\beta < \kappa$, $\alpha^\beta < \lambda$. Let Q be a κ-closed forcing and S a stationary subset of λ of ordinals of cofinality κ. Then $Q \Vdash S$ is stationary.*

This lemma is due to Baumgartner; we include the proof for lack of reference.

PROOF OF LEMMA 2.8. Let q be a condition and let $\dot{C}$ be a Q-name for a closed unbounded subset of λ. We shall find $\bar{q} \le q$ and $\gamma \in S$ such that $\bar{q} \Vdash \gamma \in \dot{C}$. Let M be a transitive set such that M is a model of enough set theory, is closed under $<\kappa$-sequences, and is such that $M \supseteq \lambda$, $q \in M$, $Q \in M$, $\dot{C} \in M$. Let $\langle N_\gamma : \gamma < \lambda \rangle$ be an elementary chain of submodels of M such that each N_γ has size $<\lambda$, contains q, Q and $\dot{C}$, $N_\gamma \cap \lambda$ is an ordinal, and $N_{\gamma+1}$ contains all $<\kappa$-sequences in N_γ. Since S is stationary, there exists a $\gamma \in S$ such that $N_\gamma \cap \lambda = \gamma$. As cf $\gamma = \kappa$, $N = N_\gamma$ is closed under $<\kappa$-sequences.

Let $\{\gamma_\xi : \xi < \kappa\}$ be an increasing sequence with limit γ. We construct a descending sequence $\{q_\xi : \xi < \kappa\}$ of conditions such that $q_0 = q$, such that, for all $\xi < \kappa$, $q_\xi \in N$, and, for some $\beta_\xi \in N$ greater than γ_ξ, $q_{\xi+1} \Vdash \beta_\xi \in \dot{C}$. At successor stages, $q_{\xi+1}$ exists because in N, q_ξ forces that $\dot{C}$ is unbounded. At limit stages $\eta < \kappa$, the η-sequence $\langle q_\xi : \xi < \eta \rangle$ is in N and has a lower bound in N because $N \models Q$ is κ-closed.

Since Q is κ-closed, the sequence $\langle q_\xi : \xi < \kappa \rangle$ has a lower bound $\bar{q}$, and because of the β's, $\bar{q}$ forces that $\dot{C}$ is unbounded in γ. Therefore $\bar{q} \Vdash \gamma \in \dot{C}$. □

3. Negative results. We shall now present several negative results on the structure of the relation $S < T$ below $\aleph_\omega$. With the exception of the proof of Proposition 1.6, we state the results for the particular case of reflection of subsets of S^3_0 in S^3_1, but the results generalize easily to other cardinalities and other cofinalities.

The first result uses a simple calculation (as in Proposition 1.1):

PROPOSITION 3.1. *For any $\aleph_3$ stationary sets $A_\alpha \subseteq S^3_1$, $\alpha < \omega_3$, there exists a stationary set $S \subseteq S^3_0$ such that $S \not< A_\alpha$ for all α.*

PROOF. Let A_α, $\alpha < \omega_3$, be stationary subsets of S^3_1. By [3], there exist $\aleph_4$ almost disjoint stationary subsets of S^3_0; let S_i, $i < \omega_4$, be such sets. Assuming that each S_i reflects fully in some $A_{\alpha(i)}$, we can find $\aleph_4$ of them that reflect fully in the same A_α. Take any $\aleph_2$ of them and reduce each by a nonstationary set to get $\aleph_2$ pairwise disjoint stationary subsets $\{T_\xi: \xi < \omega_2\}$ of S^3_0, such that each of them reflects fully in A_α. Hence there are clubs C_ξ, $\xi < \omega_2$, such that $\mathrm{Tr}(T_\xi) \supseteq A_\alpha \cap C_\xi$ for every ξ. Let $\gamma \in \bigcap_{\xi<\omega_2} C_\xi \cap A_\alpha$. Then every T_ξ reflects at γ, and so γ has $\aleph_2$ pairwise disjoint stationary subsets $\{T_\xi \cap \gamma: \xi < \omega_2\}$. This is a contradiction because γ has a closed unbounded subset of size cf $\gamma = \aleph_1$. □

The next result uses the fact that under GCH there exists a $\diamondsuit$-sequence for S^3_1.

PROPOSITION 3.2 (GCH). *There exists a stationary set $A \subseteq S^3_1$ that is not the trace of any $S \in S^3_0$; precisely: for every $S \subseteq S^3_0$ the set $A \triangle (\mathrm{Tr}(S) \cap S^3_1)$ is stationary.*

PROOF. Let $\langle S_\gamma: \gamma \in S^3_1\rangle$ be a $\diamondsuit$-sequence for S^3_1; it has the property that for every set $S \subseteq \omega_3$, the set $D(S) = \{\gamma \in S^3_1: S \cap \gamma = S_\gamma\}$ is stationary. Let

$$A = \{\gamma \in S^3_1: S_\gamma \text{ is nonstationary}\}.$$

The set A is stationary because $A \supseteq D(\varnothing)$. If S is any stationary subset of S^3_0, then for every γ in the stationary set $D(S)$, $\gamma \in A$ iff $\gamma \notin \mathrm{Tr}(S)$, and so $D(S) \subseteq A \triangle \mathrm{Tr}(S)$. □

The remaining negative results use the following theorem of Shelah which proves the existence of sets with the "square property".

THEOREM ([4], LEMMA 4.2). *Let $1 \le k \le n-2$. The set S^n_k is the union of $\aleph_{n-1}$ stationary sets A, each having the following property. There exists a collection $\{C_\gamma: \gamma \in A\}$ (a "square sequence for A") such that for each $\gamma \in A$, C_γ is a club subset of γ of order type ω_k, consisting of limit ordinals of cofinality $<\omega_k$, and such that for all $\gamma_1, \gamma_2 \in A$ and all α, if $\alpha \in C_{\gamma_1} \cap C_{\gamma_2}$ then $C_{\gamma_1} \cap \alpha = C_{\gamma_2} \cap \alpha$.*

Square sequences can be used to construct a number of counterexamples. For instance, if S_n, $n < \omega$, are $\aleph_0$ stationary subsets of S^3_0, then $\mathrm{Tr}(\bigcup_{n=0}^\infty S_n) = \bigcup_{n=0}^\infty S_n$. Using a square sequence, we get:

PROPOSITION 3.3. *There is a stationary set $A \subseteq S^3_1$ and stationary subsets S_i, $i < \omega_1$, of S^3_0 such that $\mathrm{Tr}(S_i) \cap A = \varnothing$ for each i but $\mathrm{Tr}(\bigcup_{i<\omega_1} S_i) \supseteq A$.*

PROOF. Let A be a stationary subset of S^3_1 with a square sequence $\{C_\gamma: \gamma \in A\}$, and let $S = \bigcup_{\gamma \in A} C_\gamma$. Clearly, $S \subseteq S^3_0$ is stationary, and $\mathrm{Tr}(S) \supseteq A$. For each $\xi < \omega_1$, let

$$S_\xi = \{\alpha \in S: \text{order type}(C_\gamma \cap \alpha) = \xi\}$$

(this is independent of the choice of $\gamma \in A$). For every $\gamma \in S$ and every $\xi < \omega_1$, the set $S_\xi \cap C_\gamma$ has exactly one element, and so S_ξ does not reflect at γ. It is easy to see that $\aleph_1$ of the sets S_ξ are stationary. [The definition of S_ξ is a well-known trick.] □

The argument used in the above proof establishes the following:

PROPOSITION 3.4. *If a stationary set $A \subseteq S^n_m$ has a square sequence and if $k < m$, then there exists a stationary $S \subseteq S^n_k$ that does not reflect at any $\gamma \in A$.* □

PROOF OF PROPOSITION 1.6. Let $2 \leq m < n$, and let us assume that (b) fails, i.e. that every stationary set $A \subseteq S^n_{m-1}$ reflects at some δ of cofinality $\aleph_m$. We shall prove that (a) holds. For each $k < m - 1$ we want a stationary set $S \subseteq S^m_k$ that does not reflect fully in S^m_{m-1}. Let $k < m - 1$.

Let A be a stationary subset of S^n_{m-1} that has a square sequence $\{C_\gamma: \gamma \in A\}$. The set A reflects at some δ of cofinality ω_m. Let C be a club subset of δ of order type ω_m. Using the isomorphism between C and ω_m, the sequence $\{C_\gamma \cap C: \gamma \in A\}$ becomes a square sequence for a stationary subset B of S^m_{m-1}. It follows that there is a stationary subset of S^m_k that does not reflect at any $\gamma \in B$. □

The last counterexample also uses a square sequence.

PROPOSITION 3.5 (GCH). *There is a stationary set $A \subseteq S^3_1$ and $\aleph_4$ stationary sets $S_i \subseteq S^3_0$ such that the sets $\{\mathrm{Tr}(S_i) \cap A: i < \omega_4\}$ are stationary and pairwise almost disjoint.*

PROOF. Let A be a stationary subset of S^3_1 with a square sequence $\langle C_\gamma: \gamma \in A\rangle$, and let $S = \bigcup_{\gamma \in A} C_\gamma$. Let $\{f_i: i < \omega_4\}$ be regressive functions on $S^3_0 \cup S^3_1$ with the property that for any two f_i, f_j, the set $\{\alpha: f_i(\alpha) = f_j(\alpha)\}$ is nonstationary (such a family exists by [3]). For each i and each $\gamma \in A$, the function f_i is regressive on C_γ and so there is some $\eta = \eta(i, \gamma) < \gamma$ such that $\{\alpha \in C_\gamma: f_i(\alpha) < \eta\}$ is stationary. Let $T_{i,\gamma} \subseteq \omega_1$ be the stationary set $\{\text{o.t.}(C_\gamma \cap \alpha): f_i(\alpha) < \eta\}$ and let $H_{i,\gamma}$ be the function on $T_{i,\gamma}$ (with values $< \eta$) defined by $H(\xi) = f_i(\xi\text{th element of } C_\gamma)$. For each i, the function on A that to each γ assigns $(T_{i,\gamma}, H_{i,\gamma})$ is regressive, and so constant $= (T_i, H_i)$ on a stationary set. By a counting argument, (T_i, H_i) is the same for $\aleph_4$ i's; so without loss of generality we assume that they are the same (T, H) for all i.

Now we let, for each i, $A_i = \{\gamma \in A: (\forall \alpha \in C_\gamma) \text{ if } \xi = \text{o.t.}(C_\gamma \cap \alpha) \in T \text{ then } f_i(\alpha) = H(\xi)\}$ and $S_i = \{\alpha \in S: \text{o.t.}(C_\gamma \cap \alpha) \in T \text{ and } (\forall \beta \leq \alpha, \beta \in C_\gamma) \text{ if } \xi = \text{o.t.}(C_\gamma \cap \beta) \in T \text{ then } f_i(\beta) = H(\xi)\}$. By the definition of T and H, each A_i is a stationary set, and each S_i reflects at every point of A_i. We claim that if $\gamma \in A$ and $S_i \cap \gamma$ is stationary, then $\gamma \in A_i$. So let $\gamma \in A$ be such that $S_i \cap \gamma$ is stationary. Let $\xi \in T$ and let α be the ξth element of C_γ; we need to show that $f_i(\alpha) = H(\xi)$. As $S_i \cap \gamma$ is stationary, there exists a $\beta \in S_i \cap C_\gamma$ greater than α. By the definition of S_i, $f_i(\alpha) = H(\xi)$. Thus $\gamma \in A_i$, and $A_i = A \cap \mathrm{Tr}(S_i)$.

Finally, we show that the sets A_i are pairwise almost disjoint. Let C be a club disjoint from the set $\{\alpha: f_i(\alpha) = f_j(\alpha)\}$. We claim that the set C' of all limit points of C is disjoint from $A_i \cap A_j$. If $\gamma \in C'$ then $C \cap \gamma$ is a club in γ, and so is $C \cap C_\gamma$. Since T is stationary in ω_1, there is a $\xi \in T$ such that the ξth element α of C_γ is in C, and therefore $f_i(\alpha) \neq f_j(\alpha)$; it follows that γ cannot be both in A_i and in A_j. □

REFERENCES

[1] T. JECH, *Stationary subsets of inaccessible cardinals*, ***Axiomatic set theory*** (J. Baumgartner et al., editors), Contemporary Mathematics, vol. 31, American Mathematical Society, Providence, Rhode Island, 1984, pp. 115–142.

[2] M. MAGIDOR, *Reflecting stationary sets*, this JOURNAL, vol. 47 (1982), pp. 755–771.

[3] S. Shelah [Sh 247], *More on stationary coding*, ***Around classification theory of models***, Lecture Notes in Mathematics, vol. 1182, Springer-Verlag, Berlin, 1986, pp. 224-246.

[4] ——— [Sh 351], *Reflecting stationary sets and successors of singular cardinals* (to appear).

DEPARTMENT OF MATHEMATICS
THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL