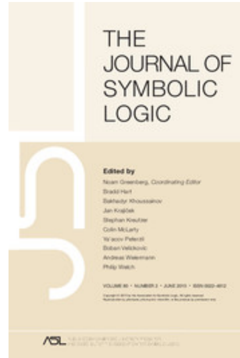

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STRONG MEASURE ZERO SETS WITHOUT COHEN REALS

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Abstract. If ZFC is consistent, then each of the following is consistent with ZFC + $2^{\aleph_0} = \aleph_2$:

(1) $X \subseteq \mathbb{R}$ is of strong measure zero iff $|X| \leq \aleph_1$ + there is a generalized Sierpinski set.

(2) *The union of $\aleph_1$ many strong measure zero sets is a strong measure zero set + there is a strong* measure zero set of size $\aleph_2$ + there is no Cohen real over L.

§0. Introduction. In this paper we continue the study of the structure of strong measure zero sets. Strong measure zero sets have been studied from the beginning of this century. They were discovered by Borel. Luzin, Sierpinski, Rothberger, and others turned their attention to the structure of these sets and proved very interesting mathematical theorems about them. Most of the constructions of strong measure zero sets involve Luzin sets, which have a strong connection with Cohen reals (see [10]). In this paper we will show that this connection is only apparent; namely, we will build models where there are strong measure zero sets of size $\mathfrak{c}$ without adding Cohen reals over the ground model.

Throughout this work we will investigate questions about strong measure zero sets under the assumption that $\mathfrak{c} = 2^{\aleph_0} = \aleph_2$. The reason is that CH makes many of the questions we investigate trivial, and there is no good technology available to deal with most of our problems when $2^{\aleph_0} > \aleph_2$. For example, if we want to obtain a model of $2^{\aleph_0} > \aleph_2 + \mathscr{S} = [\mathbb{R}]^{<\mathfrak{c}}$ (see below for definitions), we cannot use a finite support iteration, since the Cohen reals produced along the way will guarantee $\mathfrak{d} = \mathfrak{c}$, which makes $\mathscr{S} = [\mathbb{R}]^{<\mathfrak{c}}$ impossible (see 0.11). This disqualifies the approaches in [1], [21], [24]. We also have to be able to deal with more than $\aleph_1$ many requirements, so it is also unlikely that the method of [7] would succeed.

0.1. DEFINITION. A set $X \subseteq \mathbb{R}$ of reals has strong measure zero if for every sequence $\langle \varepsilon_i : i < \omega \rangle$ of positive real numbers there is a sequence $\langle x_i : i < \omega \rangle$ of real numbers such that

$$X \subseteq \bigcup_{i<\omega} (x_i - \varepsilon_i, x_i + \varepsilon_i).$$

We let $\mathscr{S} \subset \mathfrak{P}(\mathbb{R})$ be the ideal of strong measure zero sets.

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0.2. REMARK. (a) If we work in ${}^{\omega}2$ then $X \subseteq {}^{\omega}2$ has strong measure zero if

$$(\forall h \in {}^{\omega}\omega)\left(\exists g \in \prod_n {}^{h(n)}2\right)(\forall x \in X)(\exists^{\infty} n)(g(n) = x \upharpoonright h(n)),$$

or equivalently

$$(*) \qquad (\forall h \in {}^{\omega}\omega)\left(\exists g \in \prod_n {}^{h(n)}2\right)(\forall x \in X)(\exists n)(g(n) = x \upharpoonright h(n)).$$

(b) To every question about strong measure zero sets in $\mathbb{R}$ there is a corresponding question about strong measure zero sets of ${}^{\omega}2$, and for all the questions we consider the corresponding answers are the same. So we will work sometimes in $\mathbb{R}$, sometimes in ${}^{\omega}2$.

0.3. DEFINITION. Assume that $\mathscr{H} \subseteq {}^{\omega}\omega$. We say that $\bar{v}$ has index $\mathscr{H}$ if $\bar{v} = \langle v^h : h \in \mathscr{H} \rangle$ and, for all $h \in \mathscr{H}$, v^h is a function with domain ω and $\forall n\ v^h(n) \in {}^{h(n)}2$. We let

$$X_{\bar{v}} := \bigcap_{h \in \mathscr{H}} \bigcup_{k \in \omega} [v^h(k)]$$

(where we let $[\eta] := \{f \in {}^{\omega}2 : \eta \subseteq f\}$).

We say that $X_{\bar{v}}$ is the set "defined" by $\bar{v}$.

0.4. *Fact. Assume $\mathscr{H} \subseteq {}^{\omega}\omega$ is a dominating family, i.e., for all $f \in {}^{\omega}\omega$ there is $h \in \mathscr{H}$ such that $\forall n\ f(n) < h(n)$. Then*

(1) *If $\bar{v}$ has index $\mathscr{H}$, then $X_{\bar{v}}$ is a strong measure zero set.*

(2) *If X is a strong measure zero set, then there is a sequence $\bar{v}$ with index $\mathscr{H}$ such that $X \subseteq X_{\bar{v}}$.*

0.5. DEFINITION. A set of reals $X \subseteq \mathbb{R}$ is a GLuzin (generalized Luzin) set if for every meager set $M \subseteq \mathbb{R}$, $X \cap M$ has cardinality less than $\mathfrak{c}$. X is a GSierpinski set if for every set $M \subseteq \mathbb{R}$ of Lebesgue measure 0, $X \cap M$ has cardinality less than $\mathfrak{c}$.

0.6. *Fact.* (a) *If $\mathfrak{c}$ is regular and X is GLuzin, then X has strong measure zero.*

(b) *A set of mutually independent Cohen reals over a model M is a GLuzin set.*

(c) *If $\mathfrak{c} > \aleph_1$ is regular and X is a GLuzin set, then X contains Cohen reals over L.*

PROOF. See [10].

0.7. THEOREM [10]. Con(ZF) *implies* Con(ZFC + *there is a GLuzin set which is not strong measure zero*).

0.8. THEOREM [10]. Con(ZF) *implies* Con(ZFC + $\mathfrak{c} > \aleph_1$ + $\exists X \in [\mathbb{R}]^{\mathfrak{c}}$, *$X$ a strong measure zero set + there are no GLuzin sets*).

In Theorem 0.16 *we will show a stronger form of* 0.8.

0.9. DEFINITION. (1) Let **Unif**($\mathscr{S}$) be the following statement. "Every set of reals of size less than $\mathfrak{c}$ is a strong measure zero set."

(2) We say that the ideal of strong measure zero sets is $\mathfrak{c}$-additive, or **Add**($\mathscr{S}$), if for every $\kappa < \mathfrak{c}$ the union of κ many strong measure zero sets is a strong measure zero set. (So **Add**($\mathscr{S}$) $\Rightarrow$ **Unif**($\mathscr{S}$).)

0.10. REMARK. Rothberger ([18] and [17]) proved that the following are equivalent:

(i) **Unif**($\mathscr{S}$),

(ii) for every $h: \omega \to \omega$, for every $F \in [\prod_n h(n)]^{<\mathfrak{c}}$, there exists $f^* \in {}^{\omega}\omega$ such that for every $f \in F$ there are infinitely many n satisfying $f(n) = f^*(n)$.

Miller [14] noted that this implies the following:

$$\mathbf{Add}(\mathscr{M}) \quad \text{iff} \quad \mathbf{Unif}(\mathscr{S}) \text{ and } \mathfrak{b} = \mathfrak{c}.$$

(See 0.17 for definitions.)

Rothberger proved interesting results about the existence of strong measure zero sets, namely,

If $\mathfrak{b} = \aleph_1$, then there is a strong measure zero set of size $\aleph_1$. (See [8].)

In this spirit, we first prove the following result.

0.11. THEOREM. *If* $\mathbf{Unif}(\mathscr{S})$ *and* $\mathfrak{d} = \mathfrak{c}$, *then there exists a strong measure zero set of size* $\mathfrak{c}$.

We start the proof by proving the following

0.12. *Fact. If* $\mathfrak{d} = \mathfrak{c}$, *then there is a set* $\{f_i: i < \mathfrak{c}\}$ *of functions in* ${}^\omega\omega$ *such that for every* $g \in {}^\omega\omega$, *the set*

$$\{i < \mathfrak{c}: f_i \leq^* g\}$$

has cardinality less than $\mathfrak{c}$.

Proof of the fact. We build $\langle f_i: i < \mathfrak{c}\rangle$ by transfinite induction. Let ${}^\omega\omega = \{g_j: j < \mathfrak{c}\}$. We will ensure that for $j < i$, $f_i \not\leq^* g_j$. This will be sufficient.

But this is easy to achieve, as for any i, the family $\{g_j: j < i\}$ is not dominating, so there exists a function f_i such that for all $j < i$, for finitely many n, $f_i(n) > g_j(n)$.

This completes the proof of 0.12.

0.13. PROOF OF 0.11. Using $\mathfrak{d} = \mathfrak{c}$, let $\langle f_i: i < \mathfrak{c}\rangle$ be a sequence as in 0.12. Let $F: {}^\omega\omega \to [0,1] - \mathbf{Q}$ be a homeomorphism. ($\mathbf{Q}$ is the set of rational numbers.) We will show that $X := \{F(f_i): i < \mathfrak{c}\}$ is a strong measure zero set.

Let $\langle \varepsilon_n: n < \omega\rangle$ be a sequence of positive numbers. Let $\{r_n: n \in \omega\} = \mathbf{Q}$. Then $U_1 := \bigcup_{n\in\omega}(r_n - \varepsilon_{2n}, r_n + \varepsilon_{2n})$ is an open set. So $K := [0,1] - U_1$ is closed and, hence, is compact. As $K \subseteq \operatorname{rng}(F)$, also $F^{-1}(K) \subseteq {}^\omega\omega$ is a compact set. So for all n the projection of $F^{-1}(K)$ to the nth coordinate is a compact (hence bounded) subset of ω, say $\subseteq g(n)$. So

$$F^{-1}K \subseteq \{f \in {}^\omega\omega: f \leq^* g\}.$$

Let $Y := X - U_1 \subseteq K$. Then $Y \subseteq F(F^{-1}(K)) \subseteq \{F(f_i): f_i \leq^* g\}$ is (by asssumption on $\langle f_i: i < \mathfrak{c}\rangle$) a set of size $< \mathfrak{c}$ and, hence, has strong measure zero. So there exists a sequence of real numbers $\langle x_n: n < \omega\rangle$ such that $Y \subseteq U_2$, where

$$U_2 := \bigcup_{n\in\omega}(x_n - \varepsilon_{2n+1}, x_n + \varepsilon_{2n+1})$$

and $X \subseteq U_1 \cup U_2$. So X is indeed a strong measure zero set.

In §3 we will build models where $\mathbf{Add}(\mathscr{S})$ holds and the continuum is bigger than $\aleph_1$ without adding Cohen reals. First, we will show in 3.7:

0.14. THEOREM. *If ZFC is consistent, then*

$$ZFC + \mathfrak{c} = \aleph_2 + \mathscr{S} = [\mathbb{R}]^{\leq\aleph_1} + \textit{there are no Cohen reals over } L$$

is consistent.

Note that $\mathfrak{c} = \aleph_2$ and $\mathscr{S} = [\mathbb{R}]^{\leq\aleph_1}$ implies

(1) $\mathbf{Add}(\mathscr{S})$ (trivially).

(2) $\mathfrak{b} = \mathfrak{d} = \aleph_1$ (by 0.11).

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The same result was previously obtained by Corazza [5]. In his model the nonexistence of strong measure zero sets of size $\mathfrak{c}$ is shown by proving that every set of size $\mathfrak{c}$ can be mapped uniformly continuously onto the unit interval (which is impossible for a strong measure zero set). Thus, the question arises whether it is possible to get a model of $\mathcal{S} = [\mathbb{R}]^{<\mathfrak{c}} + \mathfrak{c} = \aleph_2$ + "not all sets of size $\mathfrak{c}$ can be continuously mapped onto $[0, 1]$".

By adding random reals to our construction, we answer this question in the following stronger theorem.

0.15. THEOREM. *If ZFC is consistent, then*

$$ZFC + \mathfrak{c} = \aleph_2 + \mathcal{S} = [\mathbb{R}]^{\leq\aleph_1} + \textit{there are no Cohen reals over } L$$
$$+ \textit{ there is a GSierpinski set}$$

is consistent. (*See* 0.5.)

By a remark of Miller [12, §2] a GSierpinski set cannot be mapped continuously onto $[0, 1]$ (not even with a Borel function).

What is the strength of the hypothesis **Add**($\mathcal{S}$)? Carlson [4] showed that **Add**($\mathcal{S}$) is implied by **Add**($\mathcal{N}$) (= additivity of the Lebesgue null sets), and Judah and Shelah [9] showed it is not implied by **Add**($\mathcal{M}$). For the converse, Pawlikowski [15] showed that **Add**($\mathcal{S}$) does not imply **Add**($\mathcal{M}$). He uses a finite support iteration, so he adds many Cohen reals and in the final model **Cov**($\mathcal{M}$) holds (i.e., $\mathbb{R}$ cannot be written as the union of less than $\mathfrak{c}$ many meager sets). We will improve this result in the next theorem.

0.16. THEOREM. *If ZFC is consistent, then*

$$ZFC + \mathfrak{c} = \mathfrak{d} = \aleph_2 > \mathfrak{b} + \mathbf{Add}(\mathcal{S}) + \textit{no real is Cohen over } L$$

is consistent.

(Note that by 0.11, $\mathfrak{d} = \mathfrak{c}$ + **Add**($\mathcal{S}$) implies that there is a strong measure zero set of size $\mathfrak{c}$.)

0.17. *Notation.* We use standard set-theoretical notation. We identify natural numbers n with their set of predecessors, $n = \{0, \ldots, n-1\}$. ${}^AB$ is the set of functions from A into B, $A^{<\omega} := \bigcup_{n<\omega} {}^nA$. $|A|$ denotes the cardinality of a set A. $\mathfrak{P}(A)$ is the power set of a set A, $A \subset B$ means $A \subseteq B$ & $A \neq B$. $A - B$ is the set-theoretic difference of A and B. $[A]^\kappa := \{X \subseteq A: |X| = \kappa\}$. $[A]^{<\kappa}$ and $[A]^{\leq\kappa}$ are defined similarly. (We write $A := B$ or $B =: A$ to mean: the expression B defines the term (or constant) A.)

Ord is the set of ordinals. $\mathrm{cf}(\alpha)$ is the cofinality of an ordinal α.

$$S^\beta_\alpha := \{\delta \in \omega_\beta: \mathrm{cf}(\delta) = \omega_\alpha\}.$$

In particular, S^2_1 is the set of all ordinals $<\omega_2$ of uncountable cofinality.

$\mathbb{R}$ is the set of real numbers. $\mathfrak{c} = |\mathbb{R}|$ is the size of the continuum. For $f, g \in {}^\omega\omega$ we let $f < g$ iff for all n $f(n) < g(n)$, and $f <^* g$ if for some $n_0 \in \omega$, $\forall n \geq n_0$ $f(n) < g(n)$. The "bounding number" $\mathfrak{b}$ and the "dominating number" $\mathfrak{d}$ are defined as

$$\mathfrak{b} := \min\{|\mathcal{H}|: \mathcal{H} \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega\, \exists h \in \mathcal{H}\, \neg(h <^* g)\},$$
$$\mathfrak{d} := \min\{|\mathcal{H}|: \mathcal{H} \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega\, \exists h \in \mathcal{H}\, g < h\}$$
$$= \min\{|\mathcal{H}|: \mathcal{H} \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega\, \exists h \in \mathcal{H}\, g <^* h\}.$$

(It is easy to see that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.)

We call a set $\mathscr{H} \subseteq {}^\omega\omega$ dominating if $\forall g \in {}^\omega\omega\, \exists h \in \mathscr{H}\ g < h$. $\mathscr{M}$ is the ideal of meager subsets of $\mathbb{R}$ (or of ${}^\omega 2$), and $\mathscr{N}$ is the ideal of Lebesgue measure 0 sets. $\mathscr{S}$ is the ideal of strong measure zero sets. For any ideal $\mathscr{J} \subset \mathfrak{P}(\mathbb{R})$, **Add**($\mathscr{J}$) abbreviates the statement: "The union of less than $\mathfrak{c}$ many sets in $\mathscr{J}$ is in $\mathscr{J}$." **Cov**($\mathscr{J}$) means that the reals cannot be covered by less than $\mathfrak{c}$ many sets in $\mathscr{J}$.

If f is a function, $\mathrm{dom}(f)$ is the domain of f, and $\mathrm{rng}(f)$ is the range of f. For $A \subseteq \mathrm{dom}(f)$, $f \restriction A$ is the restriction of f to A. For $\eta \in 2^{<\omega}$, $[\eta] := \{f \in {}^\omega 2: \eta \subseteq f\}$.

0.18. *More notation.* If Q is a forcing notion, G_Q is the canonical name for the generic filter on Q. We interpret $p \leq q$ as: q is stronger (or "has more information") than p. (So $p \leq q \Rightarrow q \Vdash p \in G_Q$.)

When we deal with a (countable support) iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ (where P_α is the intermediate stage reached in the αth stage and Q_α is a P_α name for the next iteration step), then we write G_α for the canonical name of the generic filter on P_α and $G(\alpha)$ for the generic filter on Q_α. If there is a natural way to associate a "generic" real to the generic filter on Q_α, we write g_α for the real given by $G(\alpha)$. We write $\Vdash_\alpha$ for the forcing relation of P_α. If $\beta < \alpha$, G_β always stands for $G_\alpha \cap P_\beta$. $V = V_0$ is the ground model and $V_\alpha = V[G_\alpha]$. P_ε is the countable support limit of $\langle P_\alpha : \alpha < \varepsilon\rangle$. $\varnothing_\alpha$ is the weakest condition of P_α, and $\varnothing_\alpha \Vdash_\alpha \varphi$ is usually abbreviated to $\Vdash_\alpha \varphi$.

0.19. *Even more notation.* The following notation is used when we deal with trees of finite sequences.

For $\eta \in V^{<\omega}$, $i \in V$, $\eta^\frown i$ is the function $\eta \cup \{\langle |\eta|, i\rangle\} \in V^{<\omega}$.

$p \subseteq \omega^{<\omega}$ is a tree if $p \neq \varnothing$, and for all $\eta \in p$, all $k < |\eta|$, $\eta \restriction k \in p$. Elements of a tree are often called "nodes". We call $|\eta|$ the "length" of η. *We reserve the word "height" for the notion defined in* 2.2.

For $p \subseteq \omega^{<\omega}$, $\eta \in p$, we let $\mathrm{succ}_p(\eta) := \{i: \eta^\frown i \in p\}$ (= the set of immediate successors of η).

If p is a tree, $\eta \in p$, let $p^{[\eta]} := \{\nu \in p: \eta \subseteq \nu \text{ or } \nu \subseteq \eta\}$.

If $p \subseteq \omega^{<\omega}$ is a tree, $b \subseteq p$ is called a *branch*, if b is a maximal subset of p that is linearly ordered by $\subseteq$.

Clearly, if $\forall \eta \in p\, \mathrm{succ}_p(\eta) \neq \varnothing$, then a subset $b \subseteq p$ is a branch iff b is of the form $b = \{f \restriction n: n \in \omega\}$ for some $f \in {}^\omega\omega$.

We let $\mathrm{stem}(p)$ be the least element of p which has more than one successor.

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§1. A few well-known facts. We collect a few more or less well-known facts about forcing for later reference.

1.1. Definition. An ultrafilter $\mathscr{U}$ on ω is called a **P-point**, if for any sequence $\langle A_n: n \in \omega\rangle$ of sets in $\mathscr{U}$ there is a set A in $\mathscr{U}$ that is almost contained in every A_n (i.e., $\forall n\, A - A_n$ is finite).

1.2. Definition. For any ultrafilter $\mathscr{U}$ on ω, we define the P-point game $G(\mathscr{U})$ as follows:

> There are two players, "IN" and "NOTIN". The game consists of ω many moves.
> In the nth move, player NOTIN picks a set $A_n \in \mathscr{U}$, and player IN picks a finite set $a_n \subseteq A_n$.

Player IN wins if after ω many moves, $\bigcup_n a_n \in \mathscr{U}$.

We write a play (or run) of $G(\mathscr{U})$ as

$$\langle A_0; a_0 \to A_1; a_1 \to A_2; \ldots \rangle.$$

It is well known that an ultrafilter $\mathscr{U}$ is a P-point iff player NOTIN does not have a winning strategy in the P-point game. For the sake of completeness, we give a proof of the nontrivial implication "$\Rightarrow$" (which is all we will need later).

Let $\mathscr{U}$ be a P-point, and let σ be a strategy for player NOTIN. We will construct a run of the game in which player NOTIN followed σ, but IN won. Let A_0 be the first move according to σ. For each n, let $\mathscr{A}_n$ be the set of all responses of player NOTIN according to σ in an initial segment of a play of length $\leq n$ in which player IN has played only subsets of n.

$$\mathscr{A}_n := \{A_k : k \leq n, \langle A_0; a_0 \to A_1; \ldots; a_{k-1} \to A_k \rangle \text{ is an initial segment of a play in which NOTIN obeyed } \sigma \text{ and } a_0, \ldots, a_{k-1} \subseteq n\}.$$

Note that $\mathscr{A}_0 = \{A_0\}$, and for all n, $\mathscr{A}_n$ is a finite subset of $\mathscr{U}$. As $\mathscr{U}$ is a P-point, there is a set $X \in \mathscr{U}$ such that for all $A \in \bigcup_n \mathscr{A}_n$, $X - A$ is finite. Let $X \subseteq A_0 \cup n_0$, and for $k > 0$ let n_k satisfy

$$n_k > n_{k-1} \quad \text{and} \quad \forall A \in \mathscr{A}_{n_{k-1}}\, X \subseteq A \cup n_k.$$

Either $\bigcup_{k\in\omega}[n_{2k}, n_{2k+1}) \in \mathscr{U}$, or $\bigcup_{k\in\omega}[n_{2k+1}, n_{2k+2}) \in \mathscr{U}$. Without loss of generality we assume $\bigcup_{k\in\omega}[n_{2k}, n_{2k+1}) \in \mathscr{U}$. Now define a play $\langle A_0; a_0 \to A_1; a_1 \to A_2; \ldots \rangle$ of the game $G(\mathscr{U})$ by induction as follows.

A_0 is given.

Given A_j, let $a_j := A_j \cap [n_{2j}, n_{2j+1})$ and let A_{j+1} be σ's response to a_j.

Then as $a_0, \ldots, a_{j-1} \subseteq n_{2j}$, we have $A_j \in \mathscr{A}_{n_{2j}}$, so $X \subseteq A_j \cup n_{2j}$ for all j. Therefore, for all j we have

$$X \cap [n_{2j}, n_{2j+1}) \subseteq (A_j \cup n_{2j}) \cap [n_{2j}, n_{2j+1}) = A_j \cap [n_{2j}, n_{2j+1}) = a_j.$$

So $\bigcup_{j\in\omega} a_j \supseteq X \cap \bigcup_{j\in\omega}[n_{2j}, n_{2j+1}) \in \mathscr{U}$.

Thus, player IN wins the play $\langle A_0; a_0 \to A_1; a_1 \to A_2; \ldots \rangle$ in which player NOTIN obeyed σ.

1.3. Definition. We say that a forcing notion Q preserves P-points, if for every P-point ultrafilter $\mathscr{U}$ on ω, $\Vdash_Q$ "$\mathscr{U}$ generates an ultrafilter", i.e., $\Vdash_Q$ "$\forall x \in \mathfrak{P}(\omega) \exists u \in \mathscr{U}$ $(u \subseteq x$ or $u \subseteq \omega - x)$".

[13] defined the following forcing notion.

1.4. Definition. "Rational perfect set forcing", RP, is defined as the set of trees $p \subseteq \omega^{<\omega}$ satisfying

(1) for all $\eta \in p$, $|\mathrm{succ}_p(\eta)| \in \{1, \aleph_0\}$ (see 0.19),

(2) for all $\eta \in p$ there is $\nu \in p$ with $\eta \subseteq \nu$ and $|\mathrm{succ}_p(\eta)| = \aleph_0$.

We let $p \geq q$ iff $p \subseteq q$.

Then the following hold:

1.5. Lemma. (1) *RP preserves P-points* [13, 4.1].

(2) *RP adds an unbounded function* [13, §2].

(3) *RP is proper.* (*This is implicit in* [13]. *See also* 2.16.)

The next lemma is easy (see, e.g., [11, Chapter VII, 6.8, 6.9, and Exercise H2].

1.6. *Fact. If Q is a forcing notion satisfying the $\aleph_2$-cc (chain condition) and not collapsing $\aleph_1$, then*

(1) *If* $\Vdash_Q \underset{\sim}{c}: \omega_2^V \to \omega_2^V$, *then there is a function* $c: \omega_2 \to \omega_2$ *such that* $\Vdash_Q \forall\alpha < \omega_2$: $\underset{\sim}{c}(\alpha) < c(\alpha)$.

(2) $\Vdash_Q \aleph_2^V = \aleph_2$.

(3) *For every stationary* $S \subseteq \aleph_2$, $\Vdash_Q$ *"S is stationary on $\aleph_2$".*

The following fact is from [19, V 4.4].

1.7. *Fact. Assume* $\langle P_\alpha, Q_\alpha: \alpha < \omega_2\rangle$ *is a countable support iteration of proper forcing notions* Q_α. *Then for every* $\delta \le \omega_2$ *of cofinality* $> \omega$, $\Vdash_\delta {}^\omega\omega \cap V_\delta = {}^\omega\omega \cap \bigcup_{\alpha<\delta} V_\alpha$, *or in other words: "no new reals appear in limit stages of cofinality* $> \omega$".

As a consequence $\Vdash_{\omega_2}$ *"If* $X \subseteq {}^\omega\omega$, $|X| \le \aleph_1$, *then there is* $\delta < \omega_2$ *such that* $X \in V_\delta$", *provided that* P_{ω_2} *satisfies the* $\aleph_2$-*cc.*

We also recall the following facts about iterations of proper forcing notions.

1.8. LEMMA. *Assume CH, and let* $\langle P_\alpha, Q_\alpha: \alpha < \omega_2\rangle$ *be a countable support iteration such that for all* $\alpha < \omega_2$, $\Vdash_\alpha$ *"Q_α is a proper forcing notion of size* $\le \mathfrak{c}$."

Then

(1) $\forall\alpha < \omega_2$: $\Vdash_\alpha \mathfrak{c} = \aleph_1$ (*see* [19, III 4.1]).

(2) $\Vdash_{\omega_2} \mathfrak{c} \le \aleph_2$. (*This follows from* 1.7 *and* (1).)

(3) *For all* $\alpha \le \omega_2$, P_α *is proper* [19, III 3.2] *and satisfies the* $\aleph_2$-*cc.* (*See* [19, III 4.1].)

(4) $\Vdash_{\omega_2} \aleph_1^V = \aleph_1$. (*See* [19, III 1.6].)

In [3, 4.1] the following is proved.

1.9. LEMMA. *Assume* $\langle P_\alpha, Q_\alpha: \alpha < \omega_2\rangle$ *is as in* 1.8 *and for all* $\alpha < \omega_2$,

$$\Vdash_\alpha \text{"}Q_\alpha \textit{ preserves P-points}\text{"}.$$

Then for all $\alpha \le \omega_2$, P_α *preserves P-points.*

1.10. DEFINITION. We say that a forcing notion Q is ${}^\omega\omega$-bounding, if the set of "old" functions is a dominating family in the generic extension by Q, or equivalently,

$$\Vdash_Q \forall f \in {}^\omega\omega\, \exists g \in {}^\omega\omega \cap V\, \forall n\, f(n) < g(n).$$

[19, V 4.3] proves

1.11. LEMMA. *Assume* $\langle P_\alpha, Q_\alpha: \alpha < \omega_2\rangle$ *is as in* 1.8 *and for all* $\alpha < \omega_2$,

$$\Vdash_\alpha \text{"}Q_\alpha \textit{ is } {}^\omega\omega\textit{-bounding and } \omega\textit{-proper}\text{"}.$$

Then for all $\alpha \le \omega_2$, P_α *is* ${}^\omega\omega$-*bounding.* (*We may even replace* ω-*proper by "proper"; see* [6], [19].)

The following is trivial to check.

1.12. *Fact. Assume Q is a forcing notion that preserves P-points or is* ${}^\omega\omega$-*bounding. Then*

$$\Vdash_Q \text{"}\textit{There are no Cohen reals over } V\text{"}.$$

1.13. DEFINITION. A forcing notion P is **strongly ${}^\omega\omega$-bounding**, if there is a sequence $\langle \le_n: n \in \omega\rangle$ of binary reflexive relations on P such that for all $n \in \omega$

(1) $p \le_n q \Rightarrow p \le q$.

(2) $p \le_{n+1} q \Rightarrow p \le_n q$.

(3) If $p_0 \leq_0 p_1 \leq_1 p_2 \leq_3 \cdots$, then there is a q such that $\forall n\, p_{n+1} \leq_n q$.

(4) If $p \Vdash$ "$\underset{\sim}{\alpha}$ is an ordinal", and $n \in \omega$, then there exists $q \geq_n p$ and a finite set $A \subseteq Ord$ such that $q \Vdash \underset{\sim}{\alpha} \in A$.

1.14. DEFINITION. (1) If $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon \rangle$ is an iteration of strongly ${}^\omega\omega$-bounding forcing notions, $F \subseteq \varepsilon$ finite, $n \in \omega$, $p, q \in P_\varepsilon$, we say that $p \leq_{F,n} q$ iff $p \leq q$ and $\forall \alpha \in F\, q \restriction \alpha \Vdash p(\alpha) \leq_n q(\alpha)$.

(2) A sequence $\langle \langle p_n, F_n \rangle : n \in \omega \rangle$ is called a fusion sequence if $\langle F_n : n \in \omega \rangle$ is an increasing family of finite subsets of ε, $\langle p_n : n \in \omega \rangle$ is an increasing family of conditions in P_ε, $\forall n\, p_n \leq_{F_n, n} p_{n+1}$ and $\bigcup_n \mathrm{dom}(p_n) \subseteq \bigcup_n F_n$.

Note that 1.13 is not literally a strengthening of Baumgartner's "Axiom A" (see [2]), as we do not require that the relations $\leq_n$ are transitive, and in (2) we only require $p_{n+1} \leq_n q$ rather than $p_{n+1} \leq_{n+1} q$. (This makes it possible also for random forcing to satisfy our demands.) Nevertheless, the same proof as in [2] shows the following fact.

1.15. *Fact.* (1) *If the sequence $\langle \langle p_n, F_n \rangle : n \in \omega \rangle$ is a fusion sequence, then there exists a condition $q \in P_\varepsilon$ such that for all $n \in \omega$, $p_{n+1} \geq_{F_n, n} q$.*

(2) *If $\underset{\sim}{\alpha}$ is a P_ε-name of an ordinal, $n \in \omega$, $F \subseteq P_\varepsilon$ finite, then for all p there exists a condition $q \geq_{F,n} p$ and a finite set A of ordinals such that $q \Vdash \underset{\sim}{\alpha} \in A$.*

(3) *If $\underset{\sim}{X}$ is a P_ε-name of a countable set of ordinals, $n \in \omega$, $F \subseteq P_\varepsilon$ finite, then for all p there exists a condition $q \geq_{F,n} p$ and a countable set A of ordinals such that $q \Vdash \underset{\sim}{X} \subseteq A$.*

The next fact is also well known.

1.16. *Fact. Let B be the random real forcing. Then B is strongly ${}^\omega\omega$-bounding.*

PROOF. Note that though random forcing is ccc and hence trivially satisfies axiom A, witnessed by $p \leq_n q \leftrightarrow p = q$, these relations clearly do not work to show strong ${}^\omega\omega$-bounding. Rather, we define $\leq_n$ as follows.

Conditions in B are Borel subsets of $[0, 1]$ of positive measure, $p \leq q$ iff $p \supseteq q$. We let $p \leq_n q$ iff $p \leq q$ and $\mu(p - q) \leq 10^{-n-1}\mu(p)$, where μ is the Lebesgue measure. Then if $p_0 \geq_0 p_1 \geq_1 \cdots$, letting $q := \bigcap_n p_n$ we have for all n, all $k \geq n$, $\mu(p_k - p_{k+1}) \leq 10^{-k-1}\mu(p_k) \leq 10^{-k-1}\mu(p_n)$, so $\mu(p_n - q) \leq 10^{-n-1} + 10^{-n-2} + \cdots \leq 2 * 10^{-n-1}\mu(p_n)$. In particular, $\mu(q) \geq 0.8 * \mu(p_0)$, so q is a condition and $q \geq_{n-1} p_n$ for all $n > 0$.

Given a name $\underset{\sim}{\alpha}$, an integer n, and a condition p such that $p \Vdash$ "$\underset{\sim}{\alpha}$ is an ordinal", let A be the set of all ordinals β such that $[\![\underset{\sim}{\alpha} = \beta]\!] \cap p$ has positive measure ($[\![\varphi]\!]$ is the boolean value of the statement φ, i.e., the weakest condition forcing φ). Since $\sum_{\beta \in A} \mu([\![\underset{\sim}{\alpha} = \beta]\!] \cap p) = \mu(p)$ there is a finite subset $F \subseteq A$ such that letting $q := p \cap \bigcup_{\beta \in F} [\![\underset{\sim}{\alpha} = \beta]\!]$ we have $\mu(q) \geq (1 - 10^{-n-1})\mu(p)$. So $q \geq_n p$ and $q \Vdash \underset{\sim}{\alpha} \in F$.

We will also need the following lemma from [23, §5, Theorem 9]:

1.17. LEMMA. *Every stationary $S \subseteq \aleph_2$ can be written as a union of $\aleph_2$ many disjoint stationary sets.*

Finally, we will need the following easy fact (which is true for any forcing notion Q).

1.18. *Fact. If $\underset{\sim}{f}$ is a Q-name for a function from ω to ω, $\Vdash_Q \underset{\sim}{f} \notin V$, and r_0, r_1 are any two conditions in Q, then there are $l \in \omega$, $j_0 \neq j_1$, $r'_0 \geq r_0$, $r'_1 \geq r_1$ such that $r'_0 \Vdash \underset{\sim}{f}(l) = j_0$, $r'_1 \Vdash \underset{\sim}{f}(l) = j_1$.*

Proof. There are a function f_0 and a sequence $r_0 = r^0 \leq r^1 \leq \cdots$ of conditions in Q such that for all n, $r^n \Vdash \underset{\sim}{f} \restriction n = f_0 \restriction n$. Since $r_1 \Vdash \underset{\sim}{f} \notin V$, $r_1 \Vdash \exists l\, \underset{\sim}{f}(l) \neq f_0(l)$.

There is a condition $r_1' \geq r_1$ such that for some $l \in \omega$ and some $j_1 \neq f_0(l)$, $r_1' \Vdash \underset{\sim}{f}(l) = j_1$. Let $j_0 := f_0(l)$, and let $r_0' := r^{l+1}$.

§2. *H*-Perfect trees. In this section we describe a forcing notion PT_H that we will use in an iteration in the next section. We will prove the following properties of PT_H:

(a) PT_H is proper and ${}^\omega\omega$-bounding.

(b) PT_H preserves P-points.

(c) PT_H does not "increase" strong measure zero sets defined in the ground model.

(d) PT_H makes the reals of the ground model (and hence, by (c), the union of all strong measure zero sets defined in the ground model) a strong measure zero set.

2.1. Definition. For each function H with domain ω satisfying $\forall n \in \omega\ 1 < |H(n)| < \omega$, we define the forcing PT_H, the set of H-perfect trees, to be the set of all p satisfying

(A) $p \subseteq \omega^{<\omega}$ is a tree.

(B) $\forall \eta \in p\, \forall l \in \operatorname{dom}(\eta)$: $\eta(l) \in H(l)$.

(C) $\forall \eta \in p$: $|\operatorname{succ}_p(\eta)| \in \{1, |H(|\eta|)|\}$.

(D) $\forall \eta \in p\, \exists \nu \in p$: $\eta \subseteq \nu$, $|\operatorname{succ}_p(\nu)| = |H(|\nu|)|$.

2.2. Definition. (1) For $p \in PT_H$, we let the set of "splitting nodes" of p be

$$\operatorname{split}(p) := \{\eta \in p \colon |\operatorname{succ}_p(\eta)| > 1\}.$$

(2) The **height** of a node $\eta \in p \in PT_H$ is the number of splitting nodes strictly below η:

$$\operatorname{ht}_p(\eta) := |\{\nu \subset \eta \colon \nu \in \operatorname{split}(p)\}|.$$

(Note that $\operatorname{ht}_p(\eta) \leq |\eta|$.)

(3) For $p \in PT_H$, $k \in \omega$, we let the kth splitting level of p be the set of splitting nodes of height k.

$$\operatorname{split}_k(p) := \{\eta \in \operatorname{split}(p) \colon \operatorname{ht}_p(\eta) = k\}.$$

(Note that $\operatorname{split}_0(p) = \{\operatorname{stem}(p)\}$.)

(4) For $u \subseteq \omega$, we let

$$\operatorname{split}^u(p) := \bigcup_{k \in u} \operatorname{split}_k(p).$$

2.3. Remarks. (i) Since $H(n)$ is finite, (3) just means that either η has a unique successor $\eta^\frown i$, or $\operatorname{succ}_p(\eta) = H(|\eta|)$.)

(ii) Letting $H'(n) = |H(n)|$, clearly PT_H is isomorphic to $PT_{H'}$ (and the obvious isomorphism respects the functions $\eta \mapsto \operatorname{ht}_p(\eta)$, $\langle p, k\rangle \mapsto \operatorname{split}_k(p)$, etc.).

2.4. Remark. If we let $H(n) = \omega$ for all n, then 2.1(A)–(D) define RP, rational perfect set forcing. The definitions in 2.2 make sense also for this forcing. Since we will not use the fact that $H(n)$ is finite before 2.12, 2.5–2.11 will be true also for RP.

2.5. *Fact. Let* $p, q \in PT_H$, $n \in \omega$, $\eta, \nu \in \omega^{<\omega}$. *Then*

(1) *If* $\eta \subset \nu \in p$, *then* $\operatorname{ht}_p(\eta) \leq \operatorname{ht}_p(\nu)$. *If, moreover,* $\eta \in \operatorname{split}(p)$, *then* $\operatorname{ht}_p(\eta) < \operatorname{ht}_p(\nu)$.

(2) *If* $b \subseteq p$ *is a branch, then* $b \cap \operatorname{split}_n(p) \neq \varnothing$.

(3) *If* $p \supseteq q$, *then for all* n, $q \cap \operatorname{split}_n(p) \neq \varnothing$.

(4) *If* $\eta \in p$ *and* $\mathrm{ht}_p(\eta) \leq n$, *then* $\exists \nu \in p, \eta \subseteq \nu$, *and* $\nu \in \mathrm{split}_n(p)$.

(5) *If* $\eta_0 \neq \eta_1$ *are elements of* $\mathrm{split}_n(p)$, *then* $\eta_0 \not\subseteq \eta_1$ *and* $\eta_1 \not\subseteq \eta_0$.

Proof. (1) is immediate from the definition of ht.

For (2), it is enough to see that $b \cap \mathrm{split}(p)$ is infinite. (Then ordering b by inclusion, the nth element of $b \cap \mathrm{split}(p)$ will be in $\mathrm{split}_{n-1}(p)$.)

So assume that $b \cap \mathrm{split}(p)$ is finite. Recall that each $\eta \in b - \mathrm{split}(p)$ has a unique successor in p. By 2.1(C), b cannot have a last element, so b is infinite. Hence, there is $\eta_0 \in b$ such that

$$\forall \nu \in b\colon \eta_0 \subseteq \nu \Rightarrow |\mathrm{succ}_p(\nu)| = 1.$$

A trivial induction on $|\nu|$ shows that this implies

$$\forall \nu \in p\colon \eta_0 \subseteq \nu \Rightarrow \nu \in b.$$

Hence,

$$\forall \nu \in p\colon \eta_0 \subseteq \nu \Rightarrow |\mathrm{succ}_p(\nu)| = 1.$$

This contradicts 2.1(D).

To prove (3), let b be any branch of q. b is also a branch of p, so (2) shows that $q \cap \mathrm{split}_n(p) \supseteq b \cap \mathrm{split}_n(p) \neq \varnothing$.

To prove (4) let b be a branch of p containing η. By (2) there is $\nu \in b \cap \mathrm{split}_n(p)$. If $\nu \subset \eta$, then $\mathrm{ht}_p(\eta) > \mathrm{ht}_p(\nu) = n$, which is impossible. Hence, $\eta \subseteq \nu$.

(5) follows easily from (1).

2.6. Definition. For $p, q \in PT_H$, $n \in \omega$, we let

(1) $p \leq q$ ("q is stronger than p") iff $q \subseteq p$.

(2) $p \leq_n q$ iff $p \leq q$ and $\mathrm{split}_n(p) \subseteq q$. (So also $\mathrm{split}_k(p) \subseteq q$ for all $k < n$.)

2.7. *Fact. If* $p \leq_n q$, $n > 0$, *then* $\mathrm{stem}(p) = \mathrm{stem}(q)$.

2.8. *Fact. Assume* $p, q \in PT_H$, $n \in \omega$, $p \leq_n q$.

(0) *For all* $\eta \in q$, $\mathrm{ht}_q(\eta) \leq \mathrm{ht}_p(\eta)$.

(1) *For all* $k \leq n$, $\mathrm{split}_k(p) \subseteq q$.

(2) *For all* $k < n$, $\mathrm{split}_k(p) = \mathrm{split}_k(q)$.

(3) *If* $p \leq_n q \leq_n r$, *then* $p \leq_n r$.

Proof. (0) is clear.

(1) Let $\eta \in \mathrm{split}_k(p)$ for some $k < n$; then by 2.5(4) there is a $\nu, \eta \subseteq \nu \in \mathrm{split}_n(p) \subseteq q$, so $\eta \in q$.

(2) Let $\eta \in \mathrm{split}_k(p)$. Each $\nu \in \mathrm{succ}_p(\eta)$ has an extension $\nu' \supseteq \nu$, $\nu' \in \mathrm{split}_n(p) \subseteq q$. So $\mathrm{succ}_p(\eta) \subseteq q$, and hence $\eta \in \mathrm{split}(q)$. Clearly, $\mathrm{ht}_q(\eta) \leq \mathrm{ht}_p(\eta) = k$. Using (1) inductively, we also get $\mathrm{ht}_q(\eta) \geq k$.

(3) Let $\eta \in \mathrm{split}_n(p)$. So $\eta \in q$, $\mathrm{ht}_q(\eta) \leq \mathrm{ht}_p(\eta) = n$. By 2.5(4), there is $\nu \in \mathrm{split}_n(q)$, $\eta \subseteq \nu$. Also $\nu \in r$, $\eta \in r$.

2.9. *Definition and Fact. If* $p_0 \leq_1 p_1 \leq_2 p_2 \leq_3 \cdots$ *are conditions in* PT_H, *then we call the sequence* $\langle p_n\colon n < \omega\rangle$ *a "fusion sequence". If* $\langle p_n\colon n < \omega\rangle$ *is a fusion sequence, then*

(1) $p_\infty := \bigcap_{n\in\omega} p_n$ *is in* PT_H.

(2) *For all* n, $p_n \leq_{n+1} p_\infty$.

2.10. *Fact.* (1) *If* $\eta \in p \in PT_H$, *then* $p^{[\eta]} \in PT_H$ *and* $p \leq p^{[\eta]}$. (*See* 0.19.)

(2) *If* $p \leq q$ *are conditions in* PT_H, $\eta \in q$, *then* $p^{[\eta]} \leq q^{[\eta]}$.

2.11. *Fact. If for all* $\eta \in \text{split}_n(p)$, $q_\eta \geq p^{[\eta]}$ *is a condition in* PT_H, *then*

(1) $q := \bigcup_{\eta \in \text{split}_n(p)} q_\eta$ *is in* PT_H,

(2) $q \geq_n p$,

(3) *for all* $\eta \in \text{split}_n(p)$, $q^{[\eta]} = q_\eta$.

2.12. *Fact. If* $n \in \omega$, $p \in PT_H$, *then* $\text{split}_n(p)$ *is finite.*

Proof. This is the first time that we use the fact that each $H(n)$ is a finite set. Assume that the conclusion is not true, so for some n and p, $\text{split}_n(p)$ is infinite. Then also

$$T := \{\eta \restriction k : \eta \in \text{split}_n(p), k \leq |\eta|\} \subseteq p$$

is infinite. As T is a finitely splitting tree, there has to be an infinite branch $b \subseteq T$. By 2.5(2), there is $\nu \in b \subseteq T$, $\text{ht}_p(\nu) > n$. This is a contradiction to 2.5(1).

2.13. *Fact.* PT_H *is strongly* ${}^\omega\omega$*-bounding, i.e., if* $\underset{\sim}{\alpha}$ *is a* PT_H*-name for an ordinal,* $p \in PT_H$, $n \in \omega$, *then there exists a finite set* A *of ordinals and a condition* $q \in PT_H$, $p \leq_n q$, *and* $q \Vdash \alpha \in A$.

Proof. Let $C := \text{split}_n(p)$. C is finite. For each node $\eta \in C$, let $q_\eta \geq p^{[\eta]}$ be a condition such that for some ordinal α_η, $q_\eta \Vdash \underset{\sim}{\alpha} = \alpha_\eta$. Now let

$$q := \bigcup_{\eta \in C} q_\eta \quad \text{and} \quad A := \{\alpha_\eta : \eta \in C\}.$$

Since any extension of q must be compatible with some $q^{[\eta]}$ (for some $\eta \in C$), $q \Vdash \underset{\sim}{\alpha} \in A$.

2.14. COROLLARY. PT_H *is proper (and indeed satisfies axiom A, so is* α*-proper for any* $\alpha < \omega_1$) *and* ${}^\omega\omega$*-bounding. Moreover, if* $n \in \omega$, $p \in PT_H$, $\underset{\sim}{\tau}$ *a name for a set of ordinals, then there exists a condition* $q \geq_n p$ *such that*

(1) *if* $p \Vdash$ "$\underset{\sim}{\tau}$ *is finite", then there is a finite set* A *such that* $q \Vdash$ "$\underset{\sim}{\tau} \subseteq A$";

(2) *if* $p \Vdash$ "$\underset{\sim}{\tau}$ *is countable", then there is a countable set* A *such that* $q \Vdash$ "$\underset{\sim}{\tau} \subseteq A$".

PROOF. Use 2.13 and 2.9.

Similarly to 2.13 we can show

2.15. *Fact. Assume that* $\underset{\sim}{\alpha}$ *is a* RP*-name for an ordinal,* $p \in RP$, $n \in \omega$. *Then there exists a countable set* A *of ordinals and a condition* $q \in PT_H$, $p \leq_n q$, *and* $q \Vdash \alpha \in A$.

Proof. Same as the proof of 2.13, except that now the set C, and hence also the set A, may be countable.

2.16. *Fact.* RP *is proper (and satisfies axiom A).*

Proof. By 2.15 and 2.9.

2.17. DEFINITION. Let $G \subseteq PT_H$ be a V-generic filter. Then we let $\underset{\sim}{g}$ be the PT_H-name defined by

$$\underset{\sim}{g} := \bigcup\left(\bigcap_{p \in G} p\right).$$

We may write $\underset{\sim}{g}_H$ or $\underset{\sim}{g}_{PT_H}$ for this name $\underset{\sim}{g}$. If PT_H is the αth iterand Q_α in an iteration, we write $\underset{\sim}{g}_\alpha$ for $\underset{\sim}{g}_H$.

2.18. *Fact.* $\varnothing_{PT_H}$ *forces that*

(0) $\underset{\sim}{g}$ *is a function with domain* ω.

(1) $\forall n\, \underset{\sim}{g}(n) \in H(n)$.

(2) *For all* $f \in V$, *if* $\forall n\, f(n) \in H(n)$, *then* $\exists^\infty n\, f(n) = \underset{\sim}{g}(n)$.

Furthermore, for all $p \in PT_H$,

(3) $p \Vdash$ "$\{\underset{\sim}{g} \upharpoonright n \colon n \in \omega\}$ *is a branch through* p".

(4) $p \Vdash \forall k \exists n\, \underset{\sim}{g} \upharpoonright n \in \mathrm{split}_k(p)$.

Proof. (0) and (2) are straightforward density arguments. (1) and (3) follow immediately from the definition of $\underset{\sim}{g}$. (4) follows from (3) and 2.5(2), applied in V^{PT_H}.

2.19. REMARK. Since **Unif**$(\mathscr{S})$ is equivalent to

> for every $H\colon \omega \to \omega$, for every $F \in [\prod_n H(n)]^{<\mathfrak{c}}$, there exists $f^* \in {}^\omega\omega$ such that for every $f \in F$ there are infinitely many n satisfying $f(n) = f^*(n)$,

2.18(2) shows that if we have $\mathfrak{c} = \aleph_2$ and Martin's Axiom for the forcing notions PT_H (for all H), then we also have **Unif**$(\mathscr{S})$. (In fact the "easy" implication "$\Leftarrow$" of this equivalence is sufficient.) This can be achieved by a countable support iteration of length $\aleph_2$ of forcing notions PT_H, with the usual bookkeeping argument (as in [22]).

We will show a stronger result in 3.3. If $P := P_{\omega_2}$ is the limit of a countable support iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha \colon \alpha < \omega_2 \rangle$, where "many" $\underset{\sim}{Q}_\alpha$ are of the form PT_{H_α} for some H_α, then some bookkeeping argument can ensure that $V^P \models$ **Add**$(\mathscr{S})$.

Since PT_H is ${}^\omega\omega$-bounding, it does not add Cohen reals. The same is true for a countable support iteration of forcings of the form PT_H. However, in 3.8 we will have to consider a forcing iteration in which some forcing notions are of the form PT_H, but others do add an unbounded real. To establish that even these iterations do not add Cohen reals, we will need the fact that the forcing notion PT_H preserves many ultrafilters.

2.20. DEFINITION. Let Q be a forcing notion, $\underset{\sim}{x}$ a Q-name, $p \in Q$, $p \Vdash \underset{\sim}{x} \subseteq \omega$. We say that $x^* \subseteq \omega$ is an *interpretation* of $\underset{\sim}{x}$ (above p), if for all n there is a condition $p_n \geq p$ such that $p_n \Vdash \underset{\sim}{x} \cap n = x^* \cap n$.

2.21. *Fact. Assume* Q, p, $\underset{\sim}{x}$ *are as in* 2.20. *Then*

(1) *there exists* $x^* \subseteq \omega$ *such that* x^* *is an interpretation of* $\underset{\sim}{x}$ *above* p;

(2) *if* $p \leq p'$ *and* x^* *is an interpretation of* $\underset{\sim}{x}$ *above* p', *then* x^* *is an interpretation of* $\underset{\sim}{x}$ *above* p.

2.22. LEMMA. PT_H *preserves P-points, i.e., if* $\mathscr{U} \in V$ *is a P-point ultrafilter on* ω, *then*

$$\Vdash_{PT_H} \text{"}\mathscr{U} \text{ generates an ultrafilter"}.$$

PROOF. Assume that the conclusion is false. Then there is a PT_H-name $\underset{\sim}{\tau}$ for a subset of ω and a condition p_0 such that

$$p_0 \Vdash_{PT_H} \forall x \in \mathscr{U} \colon |x \cap \underset{\sim}{\tau}| = |(\omega - x) \cap \underset{\sim}{\tau}| = \aleph_0.$$

For each $p \in PT_H$ we choose a set $\tau(p)$ such that

- $\tau(p)$ is an interpretation of $\underset{\sim}{\tau}$ above p.
- If there is an interpretation of $\underset{\sim}{\tau}$ above p which is an element of $\mathscr{U}$, then $\tau(p) \in \mathscr{U}$.

Note that if $\tau(p) \in \mathscr{U}$ and $p \geq p'$, then also $\tau(p') \in \mathscr{U}$, since (by 2.21(2)) we could have chosen $\tau(p') := \tau(p)$. Hence, either for all p $\tau(p) \in \mathscr{U}$, or for some $p_1 \geq p_0$, all $p \geq p_1$, $\tau(p) \notin \mathscr{U}$. In the second case we let $\underset{\sim}{\sigma}$ be a name for the complement of $\underset{\sim}{\tau}$, and let $\sigma(p) = \omega - \tau(p)$. Then $\sigma(p) \in \mathscr{U}$ for all $p \geq p_1$. Also $\sigma(p)$ is an interpretation of $\underset{\sim}{\sigma}$ above p. So wlog for some $p_1 \in PT_H$, $p_1 \geq p_0$ we have: for all $p \geq p_1$, $\tau(p) \in \mathscr{U}$.

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We will show that there is a condition $q \geq p_1$ and a set $a \in \mathcal{U}$ such that $q \Vdash a \subseteq \underset{\sim}{\tau}$. Recall that as $\mathcal{U}$ is a P-point, player NOTIN does not have a winning strategy in the P-point game for $\mathcal{U}$ (see 1.2). We now define a strategy for player NOTIN. On the side, player NOTIN will construct a fusion sequence $\langle p_n : n < \omega \rangle$ and a sequence $\langle m_n : n < \omega \rangle$ of natural numbers.

p_0 is given. Given p_n, we let

$$A_n = \bigcap_{\eta \in \mathrm{split}_{n+1}(p_n)} \tau(p_n^{[\eta]}).$$

This set is in $\mathcal{U}$. Player IN responds with a finite set $a_n \subseteq A_n$. Let $m_n := 1 + \max(a_n)$. For each $\eta \in \mathrm{split}_{n+1}(p_n)$ there is a condition $q_\eta \geq p_n^{[\eta]}$ forcing $\underset{\sim}{\tau} \cap m_n = \tau(p_n^{[\eta]}) \cap m_n$, so in particular

$$q_\eta \Vdash a_n \subseteq \underset{\sim}{\tau} \cap m_n.$$

Let $p_{n+1} = \bigcup_{\eta \in \mathrm{split}_{n+1}(p_n)} q_\eta$.

Then

$$(*) \qquad p_{n+1} \geq_{n+1} p_n \quad \text{and} \quad p_{n+1} \Vdash a_n \subseteq \underset{\sim}{\tau}.$$

This is a well-defined strategy for player NOTIN. As it is not a winning strategy, there is a play in which IN wins. During this play, we have constructed a fusion sequence $\langle p_n : n < \omega \rangle$. Letting $a := \bigcup_n a_n$, $q := \bigcap_n p_n$, we have that $a \in \mathcal{U}$, $p_0 \leq q \in PT_H$ (by 2.9), and $q \Vdash a \subseteq \underset{\sim}{\tau}$ (by $(*)$), a contradiction to our assumption.

The following facts will be needed for the proof that if we iterate forcing notions PT_H with carefully chosen functions H, then we will get a model where the ideal of strong measure zero sets is $\mathfrak{c}$-additive.

2.23. *Fact and Definition. Assume $p \in PT_H$, $u \subseteq \omega$ is infinite, $v = \omega - u$. Then we can define a stronger condition q by "trimming" p at each node in* $\mathrm{split}^v(p)$. *(See 2.2(4).) Formally, let $\bar{\imath} = \langle i_\eta : \eta \in \mathrm{split}^v(p) \rangle$ be a sequence satisfying $i_\eta \in H(|\eta|)$ for all $\eta \in \mathrm{split}^v(p)$.*

Then

$$p_{\bar{\imath}} := \{\eta \in p : \forall n \in \mathrm{dom}(\eta) : \textit{If } \eta \restriction n \in \mathrm{split}^v(p), \textit{ then } \eta(n) = i_{\eta \restriction n}\}$$

is a condition in PT_H.

Proof. Let $q := p_{\bar{\imath}}$. q satisfies (A)–(B) of Definition 2.1 of PT_H. The definition of $p_{\bar{\imath}}$ immediately implies the following:

(1) if $\eta \in \mathrm{split}^v(p) \cap q$, then $\mathrm{succ}_q(\eta) = \{i_\eta\}$;

(2) if $\eta \in \mathrm{split}^u(p) \cap q$, then $\mathrm{succ}_q(\eta) = \mathrm{succ}_p(\eta) = H(|\eta|)$;

(3) if $\eta \in q - \mathrm{split}(p)$, then $\eta \in p - \mathrm{split}(p)$, so $\mathrm{succ}_q(\eta) = \mathrm{succ}_p(\eta)$ is a singleton.

Note that $\mathrm{split}(p) = \mathrm{split}^u(p) \cup \mathrm{split}^v(p)$, so (1)–(3) cover all possible cases for $\eta \in q$. So q also satisfies 2.1(C). From (1)–(3) we can also conclude

(4) for all $\eta \in q$: $\mathrm{succ}_q(\eta) \neq \emptyset$.

To show that $q \in PT_H$, we still have to check condition 2.1(D). So let $\eta \in q$. Since u is infinite, there is $k \in u$, $k > |\eta|$. By (4), there is an infinite branch $b \subseteq q$ containing η. By 2.5(2) there is $v \in b$, $\mathrm{ht}_p(v) = k$. Then $\eta \subseteq v$ and $v \in \mathrm{split}(q)$.

2.24. *Fact.* $p_{\bar{\imath}} \Vdash$ "$\eta \subseteq \underset{\sim}{g}$ & $\eta \in \mathrm{split}^v(p) \Rightarrow \underset{\sim}{g}(|\eta|) = i_\eta$" *(where $\underset{\sim}{g}$ is a name for the generic branch defined in* 2.18).

Proof. $p_{\bar{\imath}} \Vdash \underset{\sim}{g} \subseteq p_{\bar{\imath}}$ and $\mathrm{succ}_{p_{\bar{\imath}}}(\eta) = \{i_\eta\}$.

2.25. LEMMA. *If Q is a strongly ${}^\omega\omega$-bounding forcing notion, then Q does not increase strong measure zero sets defined in the ground model, i.e., whenever $\mathcal{H}$ is a* *dominating family in V and $\bar{v} = \langle v^h \colon h \in \mathscr{H} \rangle \in V$ has index $\mathscr{H}$, then*

$$\Vdash_Q \bigcap_{h \in \mathscr{H}} \bigcup_{k \in \omega} [v^h(k)] \subseteq V.$$

We will prove a stronger lemma (about iterations of strongly ${}^\omega\omega$-bounding forcing notions) in the next section. We do not know if in this lemma "strongly ${}^\omega\omega$-bounding" can be replaced by "proper and ${}^\omega\omega$-bounding".

Finally, we show how the generic real introduced by PT_H can be used to cover old strong measure zero sets.

2.26. *Fact. Assume $h^* \colon \omega \to \omega - \{0\}$, $H^*(n) = {}^{h^*(n)}2$. Let $\mathscr{H} \subseteq {}^\omega\omega$ be a dominating family, and let $\bar{v}$ have index $\mathscr{H}$. Let $\underset{\sim}{g}$ be the name of the generic function added by PT_{H^*}.*

Then

$$\Vdash_{PT_{H^*}} \exists h \in \mathscr{H} \bigcup_{k \in \omega} [v^h(k)] \subseteq \bigcup_{n \in \omega} [\underset{\sim}{g}(n)].$$

Proof. Assume not, then there is a condition p such that

$$(*) \qquad p \Vdash \forall h \in \mathscr{H} \bigcup_{k \in \omega} [v^h(k)] \not\subseteq \bigcup_{n \in \omega} [\underset{\sim}{g}(n)].$$

Let $h \in \mathscr{H}$ be a function such that $\forall k \in \omega \forall \eta \in \text{split}_{2k+1}(p)\, h^*(|\eta|) \leq h(k)$. This function h will be a witness contradicting $(*)$.

For $\eta \in \text{split}_{2k+1}(p)$ let $i_\eta \in \text{succ}_p(\eta) = H^*(|\eta|) = {}^{h^*(|\eta|)}2$ be defined by $i_\eta := v^h(k) \restriction h^*(|\eta|)$. (Note that $v^h(k) \in {}^{h(k)}2$ and $h(k) \geq h^*(|\eta|)$.) Let $\bar{i} := \langle i_\eta \colon \eta \in \text{split}_{2k+1}(p), k \in \omega \rangle$, and let $q := p_{\bar{i}}$. Then $q \Vdash \forall n \forall k (\underset{\sim}{g} \restriction n \in \text{split}_{2k+1}(p) \Rightarrow \underset{\sim}{g}(n) = i_{\underset{\sim}{g} \restriction n} \subseteq v^h(k))$ by 2.24. Since also $q \Vdash \forall k \exists n\, \underset{\sim}{g} \restriction n \in \text{split}_{2k+1}(p)$, we get $q \Vdash \forall k \exists n [v^k(k)] \subseteq [\underset{\sim}{g}(n)]$. This contradicts $(*)$.

§3. Two models of Add($\mathscr{S}$).

In this section we will construct two models of $\mathfrak{c} = \aleph_2$ where the ideal of strong measure zero sets is additive, i.e., closed under unions of less than $\mathfrak{c}$ many sets. Both models will be obtained by a countable support iteration of length ω_2, starting from a ground model satisfying CH.

Our first task is to characterize strong measure zero sets in iterated extensions and to show that if we force sufficiently often with PT_H, we get additivity of strong measure zero sets.

3.1. LEMMA. *Let $\langle P_\alpha, Q_\alpha \colon \alpha < \omega_2 \rangle$ be an iteration of proper forcing notions as in 1.8, $p \in P_{\omega_2}$, $\underset{\sim}{A}$ a P_{ω_2}-name. If $p \Vdash$ "$\underset{\sim}{A}$ is a strong measure zero set", then there is a closed unbounded set $C \subseteq \omega_2$ and a sequence $\langle \bar{v}_\delta \colon \delta \in C \cap S^2_1 \rangle$ such that each $\bar{v}_\delta$ is a P_δ-name, and for all $\delta \in C \cap S^2_1$,*

$$p \Vdash_{\omega_2} \bar{v}_\delta \text{ has index } {}^\omega\omega \cap V_\delta \text{ and } \underset{\sim}{A} \subseteq \bigcap_{h \in {}^\omega\omega \cap V_\delta} \bigcup_{n \in \omega} [v^h(n)].$$

PROOF. (Recall that $S^2_1 := \{\delta < \omega_2 \colon cf(\delta) = \omega_1\}$.) Let $\underset{\sim}{c}$ be a P_{ω_2}-name for a function from ω_2 to ω_2 such that for all $\alpha < \omega_2$,

$$\Vdash_{\omega_2} \forall h \in {}^\omega\omega \cap V_\alpha \exists v^h \in V_{\underset{\sim}{c}(\alpha)} \colon \forall n\, v^h(n) \in {}^{h(n)}2 \;\&\; \underset{\sim}{A} \subseteq \bigcup_n [v^h(n)].$$

(Why does $\underset{\sim}{c}$ exist? Working in $V[G_{\omega_2}]$, note that there are only $\aleph_1$ many functions in ${}^\omega\omega \cap V_\alpha$, and for each such h there is a v^h as required in $\bigcup_{\beta < \omega_2} V_\beta$, by 1.7.)

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As P_{ω_2} satisfies the $\aleph_2$-cc, by 1.6(1) we can find a function $c \in V$ such that $\Vdash_{\omega_2} \forall\alpha\, \underset{\sim}{c}(\alpha) < c(\alpha)$. Let

$$C := \{\delta : \forall\alpha < \delta\ c(\alpha) < \delta\}.$$

The set C is closed unbounded. In V we can assign to each P_α-name $\underset{\sim}{h}$ (for $\alpha < \delta \in C$) a P_δ-name $\underset{\sim}{v}^{\underset{\sim}{h}}$ such that

$$\Vdash_{\omega_2} \forall n\, \underset{\sim}{v}^{\underset{\sim}{h}}(n) \in {}^{\underset{\sim}{h}(n)}2\ \&\ \underset{\sim}{A} \subseteq \bigcup_n [\underset{\sim}{v}^{\underset{\sim}{h}}(n)].$$

Now in $V[G_\delta]$ we can choose for each $h \in {}^\omega\omega$ an $\alpha < \delta$ and a P_α-name $\underset{\sim}{h}$ such that $h = \underset{\sim}{h}[G_\delta]$. Then we let $v^h := (\underset{\sim}{v}^{\underset{\sim}{h}})[G_\delta]$. Thus, we found a sequence $\bar{v} = \langle v^h : h \in V_\delta\rangle \in V_\delta$, as required.

3.2. Lemma. *Assume $\langle P_\alpha, Q_\alpha : \alpha < \omega_2\rangle$ is a countable support iteration of proper forcing notions, where for each ordinal $\delta \in S^2_1 \Vdash_\delta Q_\delta = PT_{H_\delta}$ for some P_δ-name H_δ. We will write g_δ for the generic function added by Q_δ.*

Assume $\mathscr{H}$ is a name for a dominating family $(\subseteq {}^\omega(\omega - \{0\}))$ in V_{ω_2} and

$$\Vdash_{\omega_2} \text{“For all } h \in \mathscr{H}, S_h := \{\delta \in S^2_1 : Q_\delta = PT_H{}^{V_\delta}\} \text{ is stationary (where } H(n) = {}^{h(n)}2)\text{”}.$$

Let $G_{\omega_2} \subseteq P_{\omega_2}$ be V-generic; then in $V[G_{\omega_2}]$ a set $A \subseteq \mathbb{R}$ is a strong measure zero set iff there is a closed unbounded set $C \subseteq \omega_2$ such that for every $\delta \in C \cap S^2_1$, $A \subseteq \bigcup_n [g_\delta(n)]$.

Proof. First we prove the easy direction. Assume that for some club C, for all $\delta \in C \cap S^2_1$, $A \subseteq \bigcup_n [g_\delta(n)]$. Then for every $h \in V_{\omega_2} \cap {}^\omega(\omega - \{0\})$ there is a $\delta = \delta_h \in C \cap S_h \subseteq S^2_1$. So $Q_{\delta_h} = (PT_H)^{V_{\delta_h}}$, where $H(n) = {}^{h(n)}2$. Since $g_{\delta_h}(n) \in {}^{h(n)}2$ and $A \subseteq \bigcup [g_{\delta_h}(n)]$ for arbitrary h, A is a strong measure zero set.

Now for the reverse implication. In V_{ω_2} let A be a strong measure zero set. By the previous lemma, there is a club set $C \subseteq \omega_2$ and a sequence $\langle \bar{v}_\delta : \delta \in C \cap S^2_1\rangle$ such that each $\bar{v} \in V_\delta$ is a sequence with index ${}^\omega\omega \cap V_\delta$ and $V_{\omega_2} \Vdash A \subseteq X_{\bar{v}_\delta}$. By 2.26 we have for all $\delta \in C \cap S^2_1$ that

$$V_{\delta+1} \models \exists h \in V_\delta \bigcup_n [v^h_\delta(n)] \subseteq \bigcup_n [g_\delta(n)].$$

So fix $h_0 \in V_\delta$ witnessing this. This inclusion is absolute, so also

$$V_{\omega_2} \models \bigcup_n [v^{h_0}_\delta(n)] \subseteq \bigcup_n [g_\delta(n)].$$

Thus,

$$V_{\omega_2} \models A \subseteq X_{\bar{v}_\delta} \subseteq \bigcup_n [v^{h_0}_\delta(n)] \subseteq \bigcup_n [g_\delta(n)],$$

and we are done.

3.3. Corollary. *Assume P_{ω_2} is as above. Then $\Vdash_{P_{\omega_2}}$* **Add**$(\mathscr{S})$.

Proof. Let $\langle A_i : i \in \omega_1\rangle$ be a family of strong measure zero sets in V_{ω_2}. To each i we can associate a closed unbounded set C_i as in 3.2. Let $C := \bigcap_{i\in\omega_1} C_i$, then also C is closed unbounded and for all $\delta \in C \cap S^2_1$, $\bigcup_{i\in\omega_1} A_i \subseteq \bigcup_{n\in\omega} [g_\delta(n)]$. Again by 3.2, $\bigcup_{i\in\omega_1} A_i$ is a strong measure zero set.

Now we will show that if we iterate strongly ${}^\omega\omega$-bounding forcing notions then we do not increase strong measure zero sets which were defined in the ground model.

3.4. LEMMA. *Assume P_{ω_2} is an iteration of strongly ${}^{\omega}\omega$-bounding forcing notions. Let $\underset{\sim}{f}$ be a P_{ω_2}-name for a function, p a condition, $n \in \omega$, F a finite subset of ω_2, $p \Vdash \underset{\sim}{f} \notin V$. Then there exists a natural number k such that*

(*) *for all $\eta \in {}^{k}2$ there is a condition $q \geq_{F,n} p$, $q \Vdash \underset{\sim}{f} \notin [\eta]$.*

We will write $k_{p,F,n}$ or $k_{\underset{\sim}{f},p,F,n}$ for the least such k. Note that for any $k \geq k_{p,F,n}$, (*) will also hold.

PROOF. Assume that this is false. So for some $\underset{\sim}{f}$, n_0, p_0, F_0,

(★) $$\forall k \in \omega\, \exists \eta_k \in {}^{k}2 \colon \neg(\exists q \geq_{F_0,n_0} p_0 \colon q \Vdash \underset{\sim}{f} \notin [\eta_k]).$$

Let

$$T := \{\eta_k \restriction l \colon l \leq k,\, k \in \omega\}.$$

T is a finitely branching tree ($\subseteq {}^{\omega}2$) of infinite height, so it must have an infinite branch. Let $f^* \in {}^{\omega}2$ be such that $\{f^* \restriction j \colon j \in \omega\} \subseteq T$.

Since $f^* \in V$ but $p_0 \Vdash \underset{\sim}{f} \notin V$, there exists a name $\underset{\sim}{m}$ of a natural number such that $p_0 \Vdash f^* \restriction \underset{\sim}{m} \neq \underset{\sim}{f} \restriction \underset{\sim}{m}$. By 1.15(2) we can find $q \geq_{F_0,n_0} p_0$ such that for some $m^* \in \omega$, $q \Vdash \underset{\sim}{m} < m^*$.

Claim. For some k, $q \Vdash \underset{\sim}{f} \notin [\eta_k]$. This will contradict (★).

Proof of the claim. We have $q \Vdash \underset{\sim}{f} \restriction m^* \neq f^* \restriction m^*$. Since $f^* \restriction m^* \in T$, there is a $k \geq m^*$ such that $f^* \restriction m^* = \eta_k \restriction m^*$. Hence, $q \Vdash \underset{\sim}{f} \restriction m^* \neq f^* \restriction m^* = \eta_k \restriction m^*$, so $q \Vdash \underset{\sim}{f} \notin [\eta_k \restriction m^*]$. But then also $q \Vdash \underset{\sim}{f} \notin [\eta_k]$.

This finishes the proof of the claim and hence of the lemma.

3.5. LEMMA. *Assume that P_{ω_2} is an iteration of strongly ${}^{\omega}\omega$-bounding forcing notions, $\mathscr{H}$ is a dominating family in V, and $\bar{v} = \langle v^h \colon h \in \mathscr{H} \rangle$ has index $\mathscr{H}$. Then*

$$\Vdash \bigcap_{h \in \mathscr{H}} \bigcup_{k \in \omega} [v^h(k)] \subseteq V.$$

PROOF. Assume that for some condition p and some P_{ω_2}-name $\underset{\sim}{f}$,

$$p \Vdash \underset{\sim}{f} \notin V \;\&\; \underset{\sim}{f} \in \bigcap_{h \in \mathscr{H}} \bigcup_{n \in \omega} [v^h(n)].$$

We will define a tree of conditions such that along every branch we have a fusion sequence. Specifically, we will define an infinite sequence $\langle l_n \colon n \in \omega \rangle$ of natural numbers, an increasing sequence $\langle F_n \colon n \in \omega \rangle$ of finite subsets of ω_2, and for each n a finite family

$$\langle p(\eta_0, \ldots, \eta_{n-1}) \colon \eta_0 \in {}^{l_0}2, \ldots, \eta_{n-1} \in {}^{l_{n-1}}2 \rangle$$

of conditions satisfying

(0) $p(\,) = p$.

(1) For all n, $\forall \eta_0 \in {}^{l_0}2, \ldots, \eta_{n-1} \in {}^{l_{n-1}}2 \colon l_n \geq k_{p(\eta_0,\ldots,\eta_{n-1}),F_n,n}$.

(2) For all n, $\forall \eta_0 \in {}^{l_0}2, \ldots, \eta_{n-1} \in {}^{l_{n-1}}2\, \forall \eta_n \in {}^{l_n}2$,

 (a) $p(\eta_0, \ldots, \eta_{n-1}) \leq_{F_n,n} p(\eta_0, \ldots, \eta_{n-1}, \eta_n)$.

 (b) $p(\eta_0, \ldots, \eta_{n-1}, \eta_n) \Vdash \underset{\sim}{f} \notin [\eta_n]$.

(3) For all n, $\forall \eta_0 \in {}^{l_0}2, \ldots, \eta_{n-1} \in {}^{l_{n-1}}2$, $\operatorname{dom}(p(\eta_0, \ldots, \eta_{n-1})) \subseteq \bigcup_m F_m$.

Given $p(\eta_0, \ldots, \eta_{n-1})$, first define F_n using a bookkeeping strategy such that eventually condition (3) will be satisfied. Then going through all $\eta_0 \in {}^{l_0}2, \ldots, \eta_{n-1} \in {}^{l_{n-1}}2$, we can find l_n satisfying condition (1). By the definition of $k_{p(\eta_0,\ldots,\eta_{n-1}),F_n,n}$ we can (for all $\eta_n \in {}^{l_n}2$) find $p(\eta_0, \ldots, \eta_{n-1}, \eta_n)$ satisfying (2).

Now let $h \in \mathscr{H}$ be a function such that for all n, $h(n) > l_n$. Define a sequence $\langle \eta_n : n \in \omega \rangle$ by $\eta_n := \nu^h(n) \upharpoonright l_n$, and let

$$p_n := p(\eta_0, \ldots, \eta_n).$$

Then $p \leq p_0 \leq_{F_0,0} p_1 \leq_{F_1,1} \cdots$, so there exists a condition q extending all p_n. So for all n, $q \Vdash \underset{\sim}{f} \notin [\eta_n]$. But then also for all n, $q \Vdash \underset{\sim}{f} \notin [\nu^h(n)]$, a contradiction.

3.6. COROLLARY. *If P_{ω_2} is a countable support iteration of strongly ${}^{\omega}\omega$-bounding forcing notions, P_{ω_2} satisfies the $\aleph_2$-cc, and for all $\alpha < \omega_2$ we have $\Vdash_\alpha$ CH, then* $\Vdash_{\omega_2} \mathscr{S} \subseteq [\mathbb{R}]^{\leq \aleph_1}$.

So we get

3.7. THEOREM. *If ZFC is consistent, then*

$$ZFC + \mathfrak{c} = \aleph_2 + \mathscr{S} = [\mathbb{R}]^{\leq \aleph_1}$$
$$+ \textit{ no real is Cohen over } L$$
$$+ \textit{ there is a GSierpinski set}$$

is consistent.

PROOF. We will start with a ground model V_0 satisfying $V = L$. Let

$$\mathscr{H} := {}^{\omega}(\omega - \{0\}) \cap L = \{h_\alpha : \alpha < \omega_1\},$$

and let $H_\alpha(n) = {}^{h_\alpha(n)}2$.

Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a family of disjoint stationary sets $\subseteq \{\delta < \omega_2 : \mathrm{cf}(\delta) = \omega_1\}$. Construct a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ satisfying

(1) For all even $\alpha < \omega_2$,

$$\Vdash_{P_\alpha} \text{For some } h : \omega \to \omega - \{0\}, \text{ letting } H(n) = {}^{h(n)}2, Q_\alpha = PT_H.$$

(2) If $\delta \in S_\alpha$, then $\Vdash_\delta Q_\delta = PT_{H_\alpha}$.

(3) For all odd $\alpha < \omega_2$,

$$\Vdash_{P_\alpha} Q_\alpha = \text{random real forcing}.$$

By 1.11 (or as a consequence of 1.15), P_{ω_2} is ${}^{\omega}\omega$-bounding, so $\Vdash_{\omega_2}$ "$\mathscr{H}$ is a dominating family". By 1.8(3) and 1.6 the assumptions of 3.3 are satisfied, so $\Vdash_{\omega_2}$ **Add**$(\mathscr{S})$. Also $\Vdash_{\omega_2}$ "$\mathfrak{c} = \aleph_2$ and there are no Cohen reals over L". Letting X be the set of random reals added at odd stages, X is a GSierpinski set: any null set $H \in V_{\omega_2}$ is covered by some G_δ null set H' which is coded in some intermediate model. As coboundedly many elements of X are random over this model, $|H \cap X| \leq |H' \cap X| \leq \aleph_1$.

Finally, by 3.6 we get $\mathscr{S} \subseteq [\mathbb{R}]^{\leq \aleph_1}$, and hence by **Add**$(\mathscr{S}) = [\mathbb{R}]^{\aleph_1}$.

Our next model will satisfy

$$(*) \qquad \mathbf{Unif}(\mathscr{S}) + \mathfrak{d} = \mathfrak{c} = \aleph_2.$$

This in itself is very easy, as it is achieved by adding $\aleph_2$ Cohen reals to L. (Also Miller [14] showed that **Unif**$(\mathscr{S}) + \mathfrak{c} = \aleph_2 + \mathfrak{b} = \aleph_1$ is consistent.) Our result says that we can obtain a model for (*) (and indeed, satisfying **Add**$(\mathscr{S})$) without adding Cohen reals. In particular, (*) does not imply **Cov**$(\mathscr{M})$.

3.8. THEOREM. Con(ZFC) *implies*

$$\mathrm{Con}(ZFC + \mathfrak{c} = \mathfrak{d} = \aleph_2 > \mathfrak{b} + \mathbf{Add}(\mathscr{S}) + \textit{no real is Cohen over } L).$$

PROOF (sketch). We will build our model by a countable support iteration of length ω_2 where at each stage we either use a forcing of the form PT_H or rational perfect set forcing. A standard bookkeeping argument ensures that the hypothesis of 3.3 is satisfied, so we get $\Vdash_{\omega_2} \mathbf{Add}(\mathscr{S})$. Using rational perfect set forcing on a cofinal set yields $\Vdash_{\omega_2} \mathfrak{d} = \mathfrak{c} = \aleph_2$. Since all P-point ultrafilters from V_0 are preserved, no Cohen reals are added.

PROOF (detailed version). Let

$$\{\delta < \omega_2 : \operatorname{cf}(\delta) = \omega_1\} \supseteq \bigcup_{\gamma < \omega_2} S_\gamma,$$

where $\langle S_\gamma : \gamma < \omega_2 \rangle$ is a family of disjoint stationary sets. Let $\Gamma : \omega_2 \times \omega_1 \to \omega_2$ be a bijection. We may assume that $\delta \in S_{\Gamma(\alpha,\beta)} \Rightarrow \delta > \alpha$.

First we claim that there is a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ and a sequence of names $\langle \langle \underset{\sim}{h}_\alpha^\beta : \alpha < \omega_2 \rangle : \beta < \omega_1 \rangle$ such that

(1) For all $\alpha < \omega_2$, all $\beta < \omega_1$, $\underset{\sim}{h}_\alpha^\beta$ is a P_α-name.

(2) For all $\alpha < \omega_2$, $\Vdash_\alpha \{h_\alpha^\beta : \beta < \omega_1\} = {}^\omega(\omega - \{0,1\})$.

(3) For all $\alpha < \omega_2$, if $\alpha \notin \bigcup_{\gamma < \omega_2} S_\gamma$, then $\Vdash_\alpha Q_\alpha = RP$.

(4) For all $\alpha < \omega_2$, all $\beta < \omega_1$, all $\delta \in S_{\Gamma(\alpha,\beta)}$: $\Vdash_\delta Q_\delta = PT_{\underset{\sim}{H}_\alpha^\beta}$, where $H_\alpha^\beta(n) := {}^{h_\alpha^\beta(n)}2$.

Proof of the first claim. By induction on α we can first define P_α, then $\langle \underset{\sim}{h}_\alpha^\beta : \beta < \omega_1 \rangle$ (by 1.8(1)), then Q_α (by (3) or (4), depending on whether $\alpha \in \bigcup_{\gamma < \omega_2} S_\gamma$ or not).

Our second claim is that letting $\underset{\sim}{\mathscr{H}}$ be a name for all functions from ω to $\omega - \{0,1\}$ in $V[G_{\omega_2}]$, the assumptions of 3.3 are satisfied, namely,

(a) $\Vdash_{\omega_2}$ "$\forall h \in \underset{\sim}{\mathscr{H}}\ \exists \gamma < \omega_2\ S_\gamma \subseteq S_h$".

(b) $\Vdash_{\omega_2}$ "$\forall \gamma < \omega_2\ \underset{\sim}{S}_\gamma$ is stationary".

(b) follows from 1.8(3) and 1.6, and (a) follows from

$$\Vdash_{\omega_2} \text{"For all } h \in \underset{\sim}{\mathscr{H}} \text{ there is } \alpha < \omega_2 \text{ and } \beta < \omega_1 \text{ such that } h = \underset{\sim}{h}_\alpha^\beta\text{"}$$

which in turn is a consequence of 1.7. So by 3.3, $V_{\omega_2} \models \mathbf{Add}(\mathscr{S})$. Let $G_{\omega_2} \subseteq P_{\omega_2}$ be a generic filter, $V_{\omega_2} = V[G_{\omega_2}]$. Again by 1.7, every $\mathscr{H} \subseteq {}^\omega\omega \cap V_{\omega_2}$ of size $\leq \aleph_1$ is a subset of some V_α, $\alpha < \omega_2$, so $\mathscr{H}$ cannot be a dominating family, as rational perfect set forcing $Q_{\alpha+1}$ will introduce a real not bounded by any function in $\mathscr{H} \subseteq V_\alpha \subseteq V_{\alpha+1}$. Hence, $\mathfrak{d} = \mathfrak{c} = \aleph_2$.

Finally, by 1.5, 1.9, and 2.22, any P-point ultrafilter from V generates an ultrafilter in V_{ω_2}, so there are no Cohen reals over V.

This ends the proof of 3.8.

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