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A DICHOTOMY THEOREM FOR REGULAR TYPES

Ehud HRUSHOVSKI*

Rutgers University, New Brunswick, NJ 08903, and Princeton University, Princeton, NJ 08540, USA

Saharon SHELAH**

Hebrew University in Jerusalem, Israel, and Rutgers University, New Brunswick, NJ 08903, USA

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Theorem 1. (a) Let T be a superstable theory without the omitting type order property. Then every regular type is either locally modular, or non-orthogonal to a strongly regular type. In the latter case, a realization of the strongly regular type can be found algebraically in any realization of the given one.

(b) Let T be a superstable theory with NOTOP and NDOP. Then every regular type is either locally modular or strongly regular.

Theorem 2. Let T be superstable.

(a) Let p be a nontrivial regular type. Then p-weight is continuous and definable inside some definable set D of positive p-weight. If p is non-orthogonal to B, then D can be chosen definable over B.

(b) Let p be a nontrivial regular type of depth 0. Let $\mathrm{stp}(a/B)$ be p-semi-regular. Then a lies in some $\mathrm{acl}(B)$-definable set D such that p-weight is continuous and definable inside D.

1. Introduction

It was shown in [10] that if the models of a theory do not encode second-order information, then the theory enjoys a number of structural properties: superstability, NDOP, NOTOP. If a theory has these properties, then any model of the theory is the prime model over an independent tree of countable submodels; and each model in the tree is roughly determined by a regular type over its predecessor. The notion of a regular type is the key to this analysis.

Not much is known about regular types in general. The only clearly understood ones are those whose geometry is locally modular, analyzed in [5]. Of those, the nontrivial ones are essentially the generic types of a definable abelian group, with a slightly distorted vector space structure, and no further relations 'near the generic'. The only further complication arises from the possible existence of an infinite chain of definable subgroups of finite index, creating a nontrivial type structure on the generics.

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A different kind of information is available about strongly regular types. There one knows nothing about the geometry, but at least the type structure is trivial; the type is isolated among the types non-orthogonal to it. Steve Buechler found a remarkable dichotomy in the rank 1 case: every weakly minimal type is either locally modular, or strongly regular. The two kinds of complexity cannot co-exist. In this paper we generalize Buechler's result to all regular types of depth 0. In 'classifiable' theories, this includes all nontrivial regular types.

In finite rank both U-rank and R^∞ enjoy many properties that fail in a more general context; for instance U-rank is fully additive in finite rank, while R^∞ has good definability properties [8, 1]. This makes their combined use a very powerful tool in finite rank; but both these properties fail in general. In [4] it was shown that p-weight can successfully replace U-rank near a regular type in a stable theory. However, p-weight need not have good definability properties, and so one seems to have no parallel to R^∞ in finite rank. We show in Theorem 2 that in fact the failure of definability of p-weight occurs only for trivial types; for nontrivial p p-weight has the maximal continuity and definability properties. Combined with the geome ric properties this gives an excellent technology, and it is this that we use to prove Theorem 1. The definitions of the terms used in Theorem 2 follow.

Definitions. Let p be a regular type.

(a) We will say that a regular type p is orthogonal to B if p is orthogonal to some conjugate of itself over B; this disagrees with standard usage ($p \perp B$ in the standard sense iff $p \perp \operatorname{acl}(B)$ in our sense).

(b) Let D be a definable set. Then $D^{eq} = \{f(b_1, \ldots, b_n) : f$ a 0-definable function in T^{eq}, $b_1, \ldots, b_n \in D^C\}$.

(c) A formula is said to have p-weight $\leq k$ if every type extending it has p-weight $\leq k$; it has p-weight k if it has p-weight $\leq k$, but not $\leq k-1$.

(d) Suppose p is nonorthogonal to B_0, and D is B_0-definable. We will say that p-weight is *continuous inside* D if whenever $B_0 \subseteq B$, $a \in D^{eq}$, and $w_p(a/B) \leq k$, there exists a formula in $\operatorname{tp}(a/B)$ of p-weight $\leq k$.

(e) p-weight is *definable* if for every formula $\varphi(x, y)$, $\{b : \varphi(x, b)$ has p-weight $k\}$ is a union of B_0-definable sets.

p weight is *definable inside* D if the same is true for every $\varphi(x, y)$ such that $\{x : \varphi(x, b)\} \subseteq D^{eq}$ for every b.

In fact we will prove a stronger statement than Theorem 2; see Proposition 2.4.

Here is a more precise statement of Theorem 1. We do not use the full power of NOTOP, but only a local consequence, p-PMOP, asserting more or less the existence of prime models over certain pairs [10, XII 4.3]. It states that $\operatorname{tp}(a/b_1b_2M)$ is isolated whenever the following hold:

(i) b_1, b_2 realize p-semi-regular types over M.

(ii) M is $\aleph_1$-saturated.

(iii) $b_1 \perp\!\!\!\perp b_2 \mid M$.

(iv) $\text{stp}(a/b_1b_2M)$ is almost orthogonal (over Mb_1b_2) to $q_1 \otimes q_2$ whenever q_1 is a strong type based on $M \cup \{b_1\}$, and q_2 one based on $M \cup \{b_2\}$.

In fact we will only need to know that under the conditions (i)–(iv), there exists a formula $\varphi(x) \in \text{tp}(a/b_1b_2M)$ such that $\varphi(x) \vdash \text{tp}(a/M)$. Let us call this weaker principle p-WIOP (weak isolation over pairs.)

Theorem 1(a). *Let T be superstable, $p = \text{stp}(a/B)$ regular, and assume p-WIOP. Then either p is locally modular, or there exists $a_1 \in \text{acl}(a, B)$ with $\text{tp}(a_1/B)$ strongly regular.*

Theorem 1(b). *Let T be superstable, p a regular type of depth 0 and with p-WIOP. Then p is locally modular or strongly regular.*

Even p-WIOP will only be needed in certain circumstances, in which $\text{stp}(a/M) = p$.

The first part of the theorem can be considered as an approximation to ω-stability. This turned out not to be one of the structural properties; many model-theoretic questions become easy for ω-stable theories, but Morley's example of a vector space with a descending chain of subspaces of finite index shows that not all non-ω-stable theories encode second-order phenomena. The theorem can be thought of as saying that locally, the only obstructions to the ω-stability of classifiable theories are the variants of this example. To see this note:

Proposition 1.1. *Let T be countable and superstable. Suppose every regular type is non-orthogonal to a strongly regular one. Then T is ω-stable.*

Proof. SR abbreviates strongly regular. We first show by induction on the infinity rank of the SR formula D that every strongly regular formula has ordinal Morley rank. Let p be the SR type determined by D. Work inside D. Then every SR type except p has Morley rank $<\infty$. Hence whenever $q \neq p$ is a 1-type, as q is non-orthogonal to some SR type and $q \perp p$, q is non-orthogonal to a type with ordinal Morley rank. Thus if $\varphi(x)$ is any formula such that $\sim\varphi(x) \in p$, then by Proposition 2.1b of [6], φ has ordinal Morley rank $\alpha(\varphi)$. It follows that D itself has Morley rank $\sup\{\alpha(\alpha(\varphi) + 1 : \sim\varphi \in p\}$. This shows that every SR formula has Morley rank. Applying the same theorem from [6] once more, it follows that T is ω-stable. $\square$

An example in [7] shows that countability is necessary here.

The proof of Theorem 1 is a technical generalization of [2]; for motivation we direct the reader to the proof presented in [4]. The key is that if two generic 'curves' in $D \times D$ meet at all, then they must decide what set of strong types is realized in the intersection. If we know that the intersection is finite, this puts a limit on the possible number of strong types. In Theorem 1 we must work harder to get an intersection to which the hypothesis applies. A more straightforward generalization of the proof of [2] in [4] yields the following variant. (It also follows from Theorem 1, since p-WIOP with $\mathrm{tp}(a/M)=p$ follows readily from the assumptions.)

Proposition 1.2. *Let p be a regular type in a countable superstable theory, based on $A=\mathrm{acl}(A)$, A countable. Suppose every formula of smaller infinity-rank than p is consistent with only countably many types over A. Then p is either locally modular or strongly regular.*

This yields the following striking statement, suggested by A. Pillay.

Corollary. *Let T be countable and superstable, and assume that none of its regular types are locally modular. Then T is ω-stable.*

Proof. By induction on $R^\infty(D)$ we show that D has ordinal Morley rank. Suppose this is known for all formulas D' with $R^\infty(D')<R^\infty(D)$. Work inside D. Let p be a regular 1-type. Then the hypothesis of 1.2 is met. Since p is not locally modular, it is non-orthogonal to a strongly regular type. Since p is arbitrary in D, by Proposition 1.1, D has ordinal Morley rank. □

Theorem 1(b) goes beyond ω-stability; it is false in ω-stable theories with the DOP. Its proof uses stable groups (as does 2(b)).

We recall some of the facts we will need; however notations and results concerning regular types from [9] and [5] will be used rather freely. Let p be a stationary type p based on B, and let Q be an ∞-definable set (the solution set of a partial type.) $p \mid B$ denotes the restriction to B of the non-forking extension of p to $\mathbb{C}$. p is *foreign* to Q if p is orthogonal to every type q extending Q, over any set of parameters. p is *internal* to Q if for some set B, Q is defined over B, and there exist $a \Vdash p \mid B$ and $d_1, \ldots, d_n \Vdash Q$ with $a \in \mathrm{dcl}(B, d_1, \ldots, d_n)$. Assuming $|Q|>1$, p is Q-internal iff there exist n and a definable function f such that $\{c: \mathrm{stp}(c/B)=p\} \subseteq f[Q^n]$. The symbol $\boxdot$ is the symmetric version of $\trianglelefteq$. ($p \boxdot q$ iff $p \trianglelefteq q$ and $q \trianglelefteq p$.) $(d_p x)\,\varphi(x, y)$ denotes the p-definition of $\varphi(x, y)$ (read: 'for generic x realizing p, $\varphi(x, y)$'). We will also use the following results from [7]. Fact 1.3 and other basic facts concerning the notions 'internal' and 'foreign' (including the above equivalence) can be found in [7, §2]. 1.4 is essentially contained in [10, V.4–6]; it can be found explicitly in [5, §3, facts (4), (5)]. 1.5 is Theorem 2 in [6].

Fact 1.3. *If* $\mathrm{stp}(a/B)$ *is not foreign to Q, then there exists* $a_1 \in \mathrm{dcl}(\{a\} \cup \mathrm{acl}(B)) - \mathrm{acl}(B)$ *such that* $\mathrm{stp}(a_1/B)$ *is Q-internal.*

Fact 1.4. (a) *For all a, B in $\mathbb{C}^{eq}$, if* $\mathrm{stp}(a/B)$ *is p-simple and* $\hat{B}=\mathrm{dcl}(aB) \cap \mathrm{Cl}_p(B)$ *then* $\mathrm{stp}(a/\hat{B}) \boxdot p^k$ *for some k.*

(b) *If* $\operatorname{stp}(a/B)$ *is non-orthogonal to the regular type* p*, then there exists* $a_1 \in \operatorname{dcl}(B \cup \{a\})$ *such that* $\operatorname{stp}(a_1/B)$ *is* p*-simple of positive* p*-weight.*

Fact 1.5. ***Suppose*** q ***is a regular type based on*** B***,*** $r = \operatorname{stp}(a/B)$ ***is*** q***-semi-regular, and*** r ***is almost orthogonal to*** q^{ω} ***over*** B***. Then there exist*** $a' \in \operatorname{acl}(a, B) - \operatorname{acl}(B)$***,*** $r' = \operatorname{stp}(a'/B)$***, a definable group*** G***, a definable set*** D***, and a definable transitive action of*** G ***on*** D***, such that*** r' ***is a generic type of*** D ***with respect to this action.***

2. Definability properties of local weight

Terminology. Let P be a property of elements of $\mathbb{C}^{eq}$. We will say that $(a_1, \ldots, a_n)$ has P provably over B if there exists a formula $\varphi(x_1, \ldots, x_n)$ with parameters from B such that $\vDash \varphi(a_1, \ldots, a_n)$ and for all $a_1', \ldots, a_n'$, if $\vDash \varphi(a_1', \ldots, a_n')$ then P holds of $(a_1', \ldots, a_n')$. If B is understood it may be omitted. For example, '$\operatorname{tp}(a/c)$ has p-weight 0, provably over b' means that there exists a formula $\varphi(x, y, b) \in \operatorname{tp}(ac/b)$ such that if $\vDash \varphi(a', c', b)$ then $\operatorname{tp}(a'/c')$ has p-weight 0.

Remark 2.1 (T stable). *Let* P *be a property invariant under* $\operatorname{Aut}(\mathbb{C}/B)$*. Suppose* P *holds of* $\bar{a}$ *provably over* $B \cup C$*, and* $\bar{a} \perp\!\!\!\perp C \mid B$*. Then* P *holds of* $\bar{a}$ *provably over* B*.*

Proof. Let $\varphi(\bar{x}, B, C)$ demonstrate that P holds of $\bar{a}$ provably. Let $q = \operatorname{stp}(C/B)$. Then the q-definition of φ, $\varphi'(\bar{x}, B) \stackrel{\text{def}}{=} (d_q Z)\varphi(\bar{x}, B, Z)$, shows that P holds of $\bar{a}$ provably over the algebraic closure of B. Take the disjunction of all conjugates of φ' over B to see that P holds of $\bar{a}$ provably over B. □

As a further example of this usage, note

Fact 2.2. *Let* Q *be a* B*-definable set. If* $\operatorname{stp}(a/Bc)$ *is* Q*-internal, then* $\operatorname{stp}(a/Bc)$ *is* Q*-internal provably over* B*.*

Proof. If Q is finite, then $a \in \operatorname{acl}(B \cup \{c\})$ and the claim is obvious, so we may ignore this case. By the characterization of internality mentioned in Section 1, there exists a definable function $f = f(x, \bar{a})$ and n such that $f[Q^n]$ contains $\{a' : \operatorname{stp}(a'/Bc) = \operatorname{stp}(a/Bc)\}$. By compactness, there exists a formula $\alpha(x) \in \operatorname{tp}(a/\operatorname{acl}(Bc))$ such that $f[Q^n]$ contains $\{a' : \mathbb{C} \vDash \alpha(a')\}$. Let $\alpha_1, \ldots, \alpha_m$ be the conjugates of α over $B \cup \{c\}$; so $\alpha^* = \bigvee_i \alpha_i$ is definable over $B \cup \{c\}$. For each i, let f_i be the conjugate of f corresponding to α_i, so that $f_i[Q^n] \supseteq \alpha_i^{\mathbb{C}}$ and hence $\bigcup_i f_i[Q^n] \supseteq \alpha^{\mathbb{C}}$. It is now easy to get a single function f^* such that $f^*[Q^{nm+m}] \supseteq$

α^{*C}. Say $f^* = f^*(\bar{x}, a^*)$, and $\alpha^* = \beta(x, c)(\beta \in L(B))$. Then the formula:

$$\beta(x, y) \,\&\, (\exists u^*)(\forall x')(\beta(x', y) \Rightarrow (\exists z_1, \ldots, z_{nm+m})\Big(\bigwedge_j z_j \in Q \,\&\, x' = f^*(\bar{z}, u^*)\Big)\Big)$$

describes the situation, and shows that $\mathrm{tp}(a/Bc)$ is Q-internal provably over B. □

Proof of Theorem 2(a). T is superstable, and we work in $\mathbb{C}^{eq}$. Let p be a regular type. We will show in sequence:

(a) There exists a p-simple, weight-1 formula $\theta(x, b)$.

Assuming p is nontrivial, we show that θ can be chosen so that:

(b) If $\vDash \theta(a, b)$, $b \in \mathrm{dcl}(c)$. and $w_p(a/c) = 0$, then $w_p(a/c) = 0$ provably over b.

We then note

Lemma 2.3. *Any formula θ satisfying* (a) *and* (b) *also satisfies*:

(c) *If $a \in \mathrm{dcl}(a_1, \ldots, a_m, b)$ for some $a_1, \ldots, a_m$ such that $\vDash \theta(a_i, b)$ for each i, $b \in \mathrm{dcl}(c)$, and $w_p(a/c) = k$, then $w_p(a/c) \leqslant k$ provably over b.*

It then remains only to show that D can in fact be found definable over any set B such that p is non-orthogonal to B, and that p-weight is definable. This is proved in a considerably stronger form as Proposition 2.4.

(a) Let $\theta_0(x, b_0)$ be a formula of least possible R^∞ such that $p \not\perp \theta_0$. So if $\vDash \theta_0(a, b_0)$ and $a \not\downarrow B \mid b_0$, then $\mathrm{stp}(a/B \cup \{b_0\})$ is orthogonal to p. It follows that θ_0 is p-simple, of weight 1.

(b) From now on assume p is nontrivial. Hence there exist b, c_1, c_2, c_3 such that $b_0 \in \mathrm{dcl}(b)$, $\vDash \theta_0(c_i)$ $(i = 1, 2, 3)$, $w_p(c_i/bc_j) = 1$ if $i \neq j$, but $w_p(c_3/bc_1c_2) = 0$. Let $q = \mathrm{stp}(c_1/b)$. Choose $a \in \mathrm{Cb}(\mathrm{stp}(c_1c_2/bc_3))$ such that $c_1c_2 \downarrow c_3 \mid ab$. Then $c_1 \not\downarrow ac_2 \mid b$, so $\vDash \sim(d_q y)\, \varphi'(a, y, c_2) \,\&\, \varphi'(a, c_1, c_2)$ for some $\varphi' = \varphi'(x, y, z, b)$; without loss of generality $(d_q x)\, \varphi'$ is also over b. As $a \in \mathrm{acl}(bc_3)$, there exists a formula $\theta_1(x, b) \in \mathrm{tp}(a/b)$ such that θ_1 is p-simple of weight 1. Let (d_1, e_1), $(d_2, e_2), \ldots$ be a Morley sequence over ab, with $d_1 = c_1$, $e_1 = c_2$. So a is definable over $(d_1, e_1), \ldots, (d_n, e_n)$ for some n. Say $a = f(d_1, \ldots, d_n, e_1, \ldots, e_n)$, where f is a 0-definable function. Let

$$\varphi(x, y, z) = \theta_1(x) \,\&\, \theta_0(y) \,\&\, \theta_0(z) \,\&\, \sim(d_q y)\, \varphi'(x, y, z) \,\&\, \varphi'(x, y, z).$$

Let

$$\theta(x) = (d_q y_1) \cdots (d_q y_n)(\exists z_1 \cdots z_n)(\varphi(x, y_1, z_1) \,\&\, \cdots \,\&\, \varphi(x, y_n, z_n) \,\&\, x = f(y_1, \ldots, y_n, z_1, \ldots, z_n)).$$

θ is our desired formula. It is defined over $\bar{b} \in \mathrm{acl}(b)$, and we may again simplify the notation by adjusting so that $\bar{b} = b$.

Clearly $\models\theta(a)$, and θ is p-simple, of weight 1. To see that (b) holds, let $b\in\operatorname{dcl}(c)$, and suppose $\models\theta(a')$ and:

(i) $w_p(a'/c)=0$.

Choose $(d_1,\ldots,d_n)\models q^n\mid ca'$, and $(e_1,\ldots,e_n)$ such that $a'=f(\bar d,\bar e)$ and $\models\varphi(a',d_i,e_i)$ for $i\leq n$. By the choice of φ we have:

(ii) $w_p(e_i/c)\leq 1$, $w_p(d_i/c)\leq 1$, and all types occurring are p-simple.

(iii) It is not the case that $d_i\models q\mid ba'e_i$.

Hence $R^\infty(d_i/ca'e_i)\leq R^\infty(d_i/ba'e_i)<R^\infty(\theta_0)$. By the choice of θ_0, $\operatorname{stp}(d_i/ca'e_i)$ is orthogonal to p, so $w_p(d_i/ca'e_i)=0$. By (i) and the additivity of weight, $w_p(d_i/ce_i)=0$. Hence

(iv) It is not the case that $d_i\models q\mid ce_i$.

So for some $\alpha(y,u,z)$, $\models\sim(d_qy)\,\alpha(y,c,e_i)$ and $\models\alpha(d_i,c,e_i)$. Let $\varphi^*=\sim(d_y)\,\alpha(y,u,z)\ \&\ \alpha(y,u,z)$. As $b\in\operatorname{dcl}(c)$, $b=g(c)$ for some 0-definable function g. Let $\theta^*(x,u)$ be the formula:

$$\begin{aligned}(d_qy_1)\cdots(d_qy_n)(\exists z_1\cdots z_n)(\varphi(x,y_1,z_1)\ \&\cdots\&\ \varphi(x,y_n,z_n)\\ \&\ \varphi^*(y_1,u,z_1)\ \&\cdots\&\ \varphi^*(y_1,u,z_n)\\ \&\ b=g(u)\ \&\ x=f(y_1,\ldots,y_n,z_1,\ldots,z_n)).\end{aligned}$$

So $\models\theta^*(a',c)$. If $\models\theta^*(a^*,c^*)$, we have to show that $w_p(a^*/c^*)=0$. Choose $(d_1^*,\ldots,d_n^*)\models q^n\mid a^*c^*$, and $(e_1^*,\ldots,e_n^*)$ such that $\models\varphi(a^*,d_i^*,e_i^*)$ and $\varphi^*(d_i^*,c^*,e_i^*)$, and $x=f(d_1^*,\ldots,d_n^*,e_1^*,\ldots,e_n^*)$.

Since each $d_i^*\not\perp c^*e_i^*$, $w_p(\bar d^*/c^*\bar e^*)=0$. As θ_0 is p-simple of weight ≤ 1, $w_p(\bar e^*/c^*)\leq\sum w_p(e_i^*/c^*)\leq n$. So $w_p(\bar d^*\bar e^*/c^*)\leq n$; since $w_p(\bar d^*/c^*)=n$, it follows that $w_p(\bar e^*/c^*\bar d^*)=0$. Thus $w_p(a^*/\bar d^*c^*)=w_p(a^*/\bar e^*\bar d^*c^*)=0$ $(a^*\in\operatorname{dcl}(\bar e^*\bar d^*)$.) Since $a^*c^*\perp\bar d^*\mid b$, $a^*\perp\bar d^*\mid c^*$, so $w_p(a^*/c^*)=0$. This finishes the proof of (b). □

Proof of Lemma 2.3. Using 2.1 and the definition of p-simple types, one can reduce to the case $w_p(a_i/c,b)=0$ for each i. Let $J=\{i: w_p(a_i/\{c,a_1,\ldots a_{i-1}\})=1\}$. So $\operatorname{card}(J)=k$. As $w_p(a_i/\{c,a_1,\ldots,a_{i-1}\})=0$ provably over b if $i\notin J$ and $w_p(a_i/\{c,a_1,\ldots,a_{i-1}\})\leq 1$ provably over b if $i\in J$, one sees easily that $w_p((a_1,\ldots,a_m)/c)\leq k$ provably over b. Since $a\in\operatorname{dcl}(b,a_1,\ldots,a_m)$, it is clear that $w_p(a/c)\leq k$ provably over b. □

Proposition 2.4. *Let T be superstable, p be a nontrivial regular type, non-orthogonal to B. Suppose* $\operatorname{tp}(a/B)$ *is p-simple. Then there exists $a_1\in\operatorname{dcl}(a,B)$ and $\varphi\in\operatorname{tp}(a_1/B)$ such that φ is p-simple, and p-weight is continuous and definable inside φ; and $w_p(a/a_1B)=0$*

Proof. Let θ satisfy (a), (b) and (c) above. Without loss of generality p extends θ. By the definition of p-simplicity, there exists $B'\supseteq B$, $a\perp B'\mid B$, and $c_1,\ldots,c_k$ realizing p^k over B', with $w_p(a/B'c_1\cdots c_k)=0$. We may assume

$B' - B$ is a singleton $\{b\}$, and $\theta = \theta(x, b)$. Let $c = (c_1, \ldots, c_k)$. Find $a' \in \mathrm{Cb}(\mathrm{stp}(bc/Ba))$ such that $bc \mathrel{\unicode{x2AEB}} a \mid Ba'$. Then $w_p(a/Ba') = w_p(a/Ba'bc) = 0$. We have $a' \in \mathrm{acl}(Ba)$ rather than dcl; to fix this, let a_1 be an element of $\mathbb{C}^{\mathrm{eq}}$ coding the (finite) set of conjugates of a' over Ba. Then $a_1 \in \mathrm{dcl}(a, B)$, and $w_p(a/Ba_1) = 0$. Also, since a' is definable over a set of independent conjugates of bc over Ba, the same is true of each conjugate of a', and hence of a_1. Let $b^1c^1, \ldots, b^nc^n$ be independent conjugates of bc over Ba, such that $a_1 \in \mathrm{dcl}(b^1, c^1, \ldots, b^n, c^n)$, $\bar{b} = (b^1, \ldots, b^n)$. Let p_i be the conjugate of p corresponding to b_i (so $c^i \models p_i^k \mid Bb_i$). Note that p_i-simplicity is the same as p-simplicity, and $w_p = w_{p_i}$ for each i. (Because p is non-orthogonal to B.) Let $\varphi_1 = \theta(x, b^1) \vee \cdots \vee \theta(x, b^n)$. Then φ_1 is p-simple, of weight 1, and (by the choice of θ):

(i) If $\models \varphi_1(d)$, $\bar{b} \in \mathrm{dcl}(e)$, and $w_p(d/e) = 0$, then $w_p(d/e) = 0$ provably over $\bar{b}$.

Let $c_{i,j}$ be the jth co-ordinate of c_i. Then $a_1 \in \mathrm{dcl}(\bar{b}, c_{i,j} : i, j)$, and each $c_{i,j}$ satisfies φ_1. Thus by Lemma 2.3, there exists a formula $\varphi_2 \in \mathrm{tp}(a_1/B\bar{b})$ such that

(ii)$_a$ φ_2 is p-simple.

(ii)$_b$ If $\models \varphi_2(d)$, $\bar{b} \in \mathrm{dcl}(e)$, and $w_p(d/e) = k$, then $w_p(d/e) \leq k$ provably over $\bar{b}$.

Write $\varphi_2 = \varphi_2(x, \bar{b})$, let $r = \mathrm{stp}(\bar{b}/B)$, and let $\varphi_3(x) = (d_r\bar{y})\, \varphi_2(x, \bar{y})$. Since each $b_i \mathrel{\unicode{x2AEB}} a \mid B$ and the b_i's are independent over Ba, $\bar{b} \mathrel{\unicode{x2AEB}} a \mid B$, so $\bar{b} \mathrel{\unicode{x2AEB}} a_1 \mid B$. Thus $\models \varphi_3(a_1)$. Applying the idea of 2.1 we get:

(iii)$_a$ φ_3 is p-simple.

(iii)$_b$ If $\models \varphi_3(d)$, and $w_p(d/e) = k$, then $w_p(d/e) \leq k$ provably over $\mathrm{acl}(B)$.

Note that if $w_p(d/e) \leq k$ provably over $\mathrm{acl}(B)$, then the same is true over B (take the disjunction of the conjugates of the formula expressing the given fact). So if we let φ be the disjunction of the conjugates of φ_3 over B, then φ is p-simple, $\models \varphi(a_1)$, and (iii)$_b$ holds for φ in place of φ_3. Let $D = \{x : \varphi(x)\}$. By Lemma 2.3 again,

($*$) If $d \in D^{\mathrm{eq}}$ and $w_p(d/e) = k$, then $w_p(d/e) \leq k$ provably over B.

It follows immediately that p-weight is continuous inside φ. For definability, let $\psi(x, e)$ be such that $\{x : \psi(x, e)\} \subseteq D^{\mathrm{eq}}$, and $\psi(x, e)$ has p-weight i. We must show that $\psi(x, e)$ has p-weight i provably over B. By 2.1 we can harmlessly enlarge B by parameters independent from e. Hence we may assume that there exist $\bar{d}, d$ in D^{eq}, with $\bar{d} \models r^i \mid B \cup \{e\}$, r a regular type based on B, $r \not\perp p$, $B = \mathrm{acl}(B)$, $\models \psi(d, e)$, and $w_p(\bar{d}/ed) = 0$. By ($*$), $w_p(\bar{d}/ed) = 0$ provably over B, by virtue of some formula $\beta(\bar{y}, x, v) \in \mathrm{tp}(\bar{d}, e, d)$. Let $\alpha(x) = (d_r\bar{y})(\exists v)(\psi(v, x)\ \&\ \beta(\bar{y}, x, v))$; then α clearly shows that $\psi(x, e)$ has p-weight $\geq i$ provably over B. The other inequality follows directly from ($*$) and compactness. $\square$

Theorem 2(a) is now immediate, using Fact 1.4(b).

Proposition 2.5. *Let p be a regular type of depth 0 in a superstable theory. Suppose $\mathrm{stp}(a/B)$ is p-semi-regular, $a_1 \in \mathrm{acl}(aB)$, and $w_p(a/a_1B) = 0$. Then $\mathrm{tp}(a/a_1B)$ is p-simple of weight 0, provably over B.*

Lemma 2.6. *Assume the hypothesis of 2.5. Then there exist n and $a_2, \ldots, a_n$ such that:*

(i) *Each $a_i \in \mathrm{acl}(Ba)$.*

(ii) *For $i \geqslant 1$, there exists $e \perp\!\!\!\perp a \mid B$ and an e-definable set D such that $\mathrm{stp}(a_{i+1}/B \cup \{a_1, \ldots, a_i\})$ is D-internal, and p is foreign to D.*

(iii) *$a_n = a$.*

It is easy to see, using Fact 2.2, Remark 2.1 and induction, that if the conclusion of Lemma 2.6 is true of a, then there exists a formula $\varphi \in \mathrm{tp}(a/B)$ such that whenever $\models \varphi(a')$, the same conclusion is true of a'. Since it follows clearly that $\mathrm{stp}(a'/B)$ is p-simple of weight 0, the proposition follows.

Proof of Lemma 2.6. As long as $a \notin \mathrm{acl}(B \cup \{a_i\})$, we define by induction a_{i+1} satisfying (i) and (ii), and with $a_{i+1} \notin \mathrm{acl}(B \cup \{a_1, \ldots, a_i\})$. Since each $a_i \in \mathrm{acl}(a)$, it follows that $R^\infty(a/B \cup \{a_1, \ldots, a_i\})$ decreases with i. The chain must terminate, so $a \in \mathrm{acl}(a_i)$ for some i, and we can let $n = i + 1$.

Let $b = (a_1, \ldots, a_i)$. As $b \in \mathrm{acl}(B \cup \{a\}) - \mathrm{acl}(B)$, $\mathrm{stp}(b/B)$ is semi-regular, $\not\perp p$. p has depth 0; so $r_1 = \mathrm{stp}(a/Bb)$ is non-orthogonal to B. So there exists $C \supseteq B$, $a \perp\!\!\!\perp C \mid B$, and a regular type r based on C, with $r \not\perp r_1$. Choose r of least possible R^∞. Then by the theorem on existence of semi-regular types in [10, V.4], we may assume r_1 is r-semi-regular. Note that every extension of r_1 is orthogonal to p.

Claim. *r_1 is almost-orthogonal to r^ω over $C \cup \{b\}$.*

Proof. Suppose not. So there exists $\bar{c} \models r^m$ for some m, $\bar{c} \perp\!\!\!\perp b \mid C$, $a \not\perp\!\!\!\perp \bar{c} \mid Cb$. By transitivity of nonforking as $b \in \mathrm{acl}(a)$, $a \not\perp\!\!\!\perp \bar{c} \mid C$. But $r \perp p$ and $\mathrm{stp}(a/C)$ is p-semi-regular, a contradiction.

By Fact 1.5, there exist $a' \in \mathrm{acl}(a, B) - \mathrm{acl}(B, b)$, $r_2 = \mathrm{stp}(a'/Bb)$, a definable group G, a definable set D' and a definable transitive action of G on D', such that r_2 is a generic type of D' with respect to this action. Since every extension of r_2 is orthogonal to p, and every type inside D' is parallel to some extension of a translate of r_2, every type inside D' is orthogonal to p. Now the formula defining D may have parameters from $\mathrm{acl}(C \cup \{b\})$; call it $\delta(x, c)$, with $c \in C$ and $\delta(x, y) \in L(Bb)$. Let $t = \mathrm{stp}(c/Bb)$. Let $\delta_1(x) = (d_t y)\, \delta(x, y)$. Then $\models \delta_1(a')$, and p is foreign to δ_1. Let b' be a conjugate of b over $\mathrm{acl}(B)$, $a \perp\!\!\!\perp b' \mid B$, and let δ_2 be the corresponding conjugate of δ_1. Then p is foreign to δ_2, and δ_2 has parameters independent from a. Moreover, as r_2 is non-orthogonal to B, r_2 is not foreign to δ_2. (It is not orthogonal to a conjugate of itself extending δ_2.) By Fact 1.3, there exists $a'' \in \mathrm{acl}(a') - \mathrm{acl}(B \cup \{b\})$ with $\mathrm{stp}(a''/B \cup \{b\})$ δ_2-internal. Letting $a_{i+1} = a''$ and D the formula defined by δ_2 satisfies (ii), and hence proves the lemma. □

Proof of Theorem 2(b). Assume the hypothesis of 2(b). By 1(a) and Proposition 2.4, there exists $a_1 \in \mathrm{dcl}(a, \mathrm{acl}(B))$ with $w_p(a/a_1) = 0$, and a B-definable set D_1 with $a_1 \in D_1$ and p-weight is continuous and definable inside D_1. Say $a_1 = f(a)$, f an $\mathrm{acl}(B)$-definable function. By 2.5 there exists a formula $\varphi(x, y) \in L(B)$ such that $\vDash \varphi(a, a_1)$, and for any a_1', $\{x : \varphi(x, a_1')\}$ is p-simple of weight 0. Let $D = \{x : \varphi(x, f(x))\ \&\ D_1(f(x))\}$. It is easy to see that D works. $\square$

3. Proof of Theorem 1

Theorems 1(a) and 1(b) follow immediately from Proposition 2.4 and Theorem 2(b), respectively, and the Proposition 3.1 below. We will take the point of view of [5, Section 4], that local modularity can be best understood using imaginaries. Let p be based on $\emptyset$. Then p is locally modular iff any two p-closed sets (in $\mathbb{C}^{eq}$) are independent over their intersection. This statement turns out to be independent of the base; so 'modular in $\mathbb{C}^{eq}$' would be a better operational description than the equivalent 'locally modular'.

Proposition 3.1. *Let p be a regular type. Assume p is not locally modular, and p-WIOP holds. Suppose $D(x)$ is a p-simple formula, $D(x) \in p$, and p-weight is continuous and definable inside D. Then p is strongly regular.*

Proof. By elementary stability considerations, we may assume D is $x = x$, and p is based on $\emptyset$. The lemmas to the end of this section assume this hypothesis as well as the hypotheses of the proposition.

Lemma 3.2. *There exist a, b, C with the following properties.*

(a) $w_p(a/C) = w_p(b/C) = 2$; $w_p(a/Cb) = w_p(b/Ca) = 1$.

(b) $\mathrm{Cl}_p(a) \cap \mathrm{Cl}_p(b) = \mathrm{Cl}_p(C)$.

(c) *p is based on C, and $p \mid C$ is realized inside* $\mathrm{dcl}(a)$.

(d) $C = \mathrm{Cl}_p(C) \cap \mathrm{dcl}(a, b, C)$.

(e) $\mathrm{Cl}_p(a, C) \cap \mathrm{dcl}(a, b, C) \subseteq \mathrm{acl}(C, a)$.

(f) *If $\bar{C} \supseteq C$, $ab \perp\!\!\!\perp \bar{C} \mid C$, σ is an automorphism fixing* $\mathrm{acl}(\bar{C}, a)$, $\sigma b \perp\!\!\!\perp b \mid \bar{C}a$, *and $\bar{b} \in \mathrm{Cl}_p(b, \bar{C})$, then $a \perp\!\!\!\perp \{\bar{C}, \bar{b}, \sigma\bar{b}\} \mid \{C, b, \sigma b\}$.*

(g) $\mathrm{Cl}_p(b, C) \cap \mathrm{dcl}(a, b, C) \subseteq \mathrm{acl}(b, C)$.

The formulation and ordering of these clauses is intended to make the proof easy. To understand them, it may help to note that by Fact 1.4, (d) implies that $\mathrm{stp}(ab/C) \square p^k$ for some k. In view of (a), this is equivalent to: $\mathrm{stp}(ab/C) \square p^3$. In fact, (d) is equivalent to this statement together with the fact that $\mathrm{tp}(ab/C)$ is stationary. Similarly, (e) and (g) are equivalent to the statements: $\mathrm{stp}(b/Ca)$ and $\mathrm{stp}(a/Cb)$ are regular (respectively.)

Proof. For (a) and (b), find A, B p-closed with $A \not\perp B \mid (A \cap B)$. It is easy to see that if $w_p(A/A \cap B)$ and $w_p(B/A \cap B)$ are minimized, then both equal 2. Now by the definition of p-simplicity, there exists $C \supseteq A \cap B$, $AB \perp C \mid (A \cap B)$, such that $A = \mathrm{Cl}_p(Ca)$ and $B = \mathrm{Cl}_p(Cb)$ for some sequences a, b of independent realizations of $p \mid C$. This gives a, b, C satisfying (a), (b), (c).

Replacing C by $\mathrm{Cl}_p(C) \cap \mathrm{dcl}(a, b, C)$ does not hurt (a), (b) or (c), since $\mathrm{Cl}_p(C)$ is unchanged. By superstability, for every subset E of $\mathrm{dcl}(a, b, C)$ there is a finite $E_0 \subseteq E$ such that $E \subseteq \mathrm{acl}(E_0 \cup C)$ (consider $R^\infty(ab/E)$.) So there exists $a_1 \in \mathrm{dcl}(a, b, C) \cap \mathrm{Cl}_p(a, C)$ such that $(\mathrm{dcl}(a, b, C) \cap \mathrm{Cl}_p(a, C)) \subseteq \mathrm{acl}(C, a_1)$. As $\mathrm{Cl}_p(Ca) = \mathrm{Cl}_p(Caa_1)$ and $\mathrm{dcl}(Cab) = \mathrm{dcl}(Caa_1b)$, there is again no harm in replacing a by aa_1. Thus (d) and (e) can be met. Given a, b, C satisfying (a)–(e), let σ be an automorphism fixing $\mathrm{Cl}_p(Ca)$ such that $\sigma \perp b \mid Ca$, and let $\alpha(a, b, C) = R^\infty(a/b, \sigma b, C)$. Clearly $\mathrm{stp}(b, \sigma b/Ca)$ does not depend on σ, so α is well-defined. Choose a, b, C so that $\alpha(a, b, C)$ is minimized. Then (f) holds: let σ, $\bar{b}$, $\bar{C}$ be as in (f), and suppose $a \not\perp \{\bar{b}, \sigma\bar{b}\} \mid \{\bar{C}, b, \sigma b\}$. By (e) and Fact 1.4, $\mathrm{stp}(b/Ca)\, \underline{\square}\, p$. Hence $b \perp \mathrm{Cl}_p(\bar{C}a) \mid \bar{C}a$; applying σ, $\sigma b \perp \mathrm{Cl}_p(\bar{C}a) \mid \bar{C}a$. Thus there exist an automorphism σ' fixing $\mathrm{Cl}_p(\bar{C}a)$ pointwise and with $\sigma' b = \sigma b$. Let $b' = (\bar{b}, \sigma'^{-1}(\sigma\bar{b}))$, $C' = \mathrm{Cl}_p(\bar{C}) \cap \mathrm{dcl}(\bar{C}, a, b')$, and as above find a' such that $a' \in \mathrm{dcl}(a, C', b')$, $a \in \mathrm{dcl}(a')$, and $\mathrm{Cl}_p(\bar{C}, a) \cap \mathrm{dcl}(\bar{C}, a, b') \subseteq \mathrm{acl}(a')$. So a', b', C' satisfy (a)–(e); and

$$\begin{aligned}\alpha(a', b', C') = R^\infty(a'/b', \sigma' b', C') &\leq R^\infty(a/b', \sigma' b', C') \\ &\leq R^\infty(a/\bar{b}, \sigma\bar{b}, \bar{C}) < R^\infty(a/b, \sigma b, C) = \alpha(a, b, C),\end{aligned}$$

contradicting the minimality of $\alpha(a, b, C)$. (The first inequality is true because $a' \in \mathrm{dcl}(a, C', b')$; the second because $\sigma\bar{b} \in \mathrm{dcl}(\sigma' b')$; the third because $a \not\perp \{\bar{C}, \bar{b}, \sigma\bar{b}\} \mid \{C, b, \sigma b\}$.) Thus (f) holds. To satisfy (g) one replaces b by a somewhat bigger element inside $\mathrm{dcl}(a, b, C) \cap \mathrm{Cl}_p(C, b)$, as a was replaced in (e); this does not change $\mathrm{dcl}(a, b, C)$, so (e) and (f) remain valid. □

In the proof, we used the fact that if Ω is a subset of $\mathbb{C}$ invariant under $\mathrm{Aut}(\mathbb{C})$, then any two elements realizing the same type over Ω are $\mathrm{Aut}(\mathbb{C}/\Omega)$-conjugate. This is an easy exercise, using stability.

Lemma 3.3. *Let a, b, C be as in* 3.2, *and let $b' \models \mathrm{stp}(b/Ca)$, $b' \perp b \mid Ca$. Then*:

(a) $w_p(a/bb'C) = 0$,

(b) $b' \perp b \mid C$.

There exists a formula $\rho(x, y, y')$ over C such that for b' as above,

(c) $\rho(x, b, b') \vdash \mathrm{tp}(a/C)$.

Proof. (a) Since $b' \perp b \mid Ca$, $\mathrm{Cb}(\mathrm{stp}(b'/Cab)) \subseteq \mathrm{acl}(Ca)$. Suppose $b' \perp a \mid Cb$. Then $\mathrm{Cb}(\mathrm{stp}(b'/Cab)) \subseteq \mathrm{acl}(Cb)$. But $\mathrm{acl}(Ca) \cap \mathrm{acl}(Cb) \subseteq \mathrm{Cl}_p(C)$ by 3.2(b), so $b' \perp ab \mid \mathrm{Cl}_p(C)$, and in particular (as $b' \perp \mathrm{Cl}_p(C) \mid C$ by Fact 1.4 and 3.2(d)) $b' \perp a \mid C$. This is absurd. Thus $b' \not\perp a \mid Cb$. By Fact 1.4 and 3.2(g), $\mathrm{stp}(a/Cb)$ is regular; by symmetry, $a \not\perp b' \mid Cb$; so $w_p(a/Cbb') = 0$.

(b) $w_p(abb'/C) = w_p(a/C) + 2w_p(b/Ca) = 4$ while $w_p(a/Cbb') = 0$, so $w_p(bb'/C) = 4 = w_p(b/C) + w_p(b'/C)$. By 3.2(d), $\text{stp}(b/C) \square p^2$; hence $b \perp b' \mid C$.

(c) We will show that the conditions listed in p-WIOP hold. Let $\bar{C} \supset C$, $B \supseteq \bar{C} \cup \{b\}$, $B' \supseteq \bar{C} \cup \{b'\}$, $abb' \perp \bar{C} \mid C$, $B \perp B' \mid \bar{C}$; we must show that $a \perp BB' \mid Cbb'$. Let $B_0 = \text{dcl}(B) \cap \text{Cl}_p(\bar{C}b)$, $B_0' = \text{dcl}(B') \cap \text{Cl}_p(\bar{C}b')$. It follows from 3.2(f) that $a \perp B_0B_0' \mid Cbb'$. But (because $x = x$ is p-simple, and by Fact 1.4), $\text{stp}(B/B_0)$ and $\text{stp}(B'/B_0')$ are both domination-equivalent to powers of p. As $B \perp B' \mid \bar{C}$ and $\bar{C} \subseteq (B_0 \cup B_0') \subseteq B \cup B'$, $B \perp B' \mid (B_0 \cup B_0')$; so $\text{stp}(BB'/B_0B_0')$ is also domination-equivalent to a power of p. But $w_p(a/B_0B_0') = 0$, so $a \perp BB' \mid B_0B_0'$. By transitivity, $a \perp BB' \mid Cbb'$, as required. □

Lemma 3.4. *The following facts are true of a provably over Cb.*

(1) $w_p(a/C) \leq 2$, $w_p(a/Cb) \leq 1$, $w_p(b/Ca) \leq 1$.

(2) *For all* b',

either (i) $\text{stp}(b'/Ca) \neq \text{stp}(b/Ca)$,

or (ii) $w_p(b'/Cab) = 0$,

or (iii) $\models \rho(a, b, b')$, *and* $w_p(a/b, b') = 0$.

Proof. (1) is immediate from 3.2 and the definability hypothesis we are working under.

(2) First note that (2) is true: by 3.2(e), $\text{stp}(b/Ca) \square p$, so if $\text{stp}(b'/Ca) = \text{stp}(b/Ca)$ and $w_p(b'/Cab) \neq 0$, then $b' \perp b \mid Ca$, and 3.3 applies giving (iii). Now for any particular b' realizing $\text{stp}(b/Ca)$, the statement (i) ∨ (ii) ∨ (iii) is true of a, b' provably over Cb. By compactness, only finitely many formulas are involved, so we can use the universal quantifier to get that $(\forall b')((\text{i}) \vee (\text{ii}) \vee (\text{iii}))$ is true of a provably over Cb. □

Lemma 3.5. *If* (1) *and* (2) *of Lemma* 3.4 *hold of* a' *in place of* a, *and* $w_p(a'/Cb) \neq 0$ *then* $\text{tp}(a'/C) = \text{tp}(a/C)$.

Proof. Let a' be such an element, and choose b' such that $\text{stp}(b'/Ca') = \text{stp}(b/Ca')$ and $b' \perp b \mid Ca'$. First compute that

$$w_p(b/Ca') = w_p(b/C) + w_p(a'/Cb) - w_p(a'/C) = 2 + (\geq 1) - (\leq 2) \geq 1.$$

But by 3.4(1), $w_p(a'/Cb) \leq 1$ and $w_p(b/Ca') \leq 1$. So

(a) $w_p(b/Ca') = 1$,

(b) $w_p(a'/C) = 2$,

(c) $w_p(b'/Ca'b) = w_p(b'/Ca') = 1$ (from (a)).

Thus neither (i) nor (ii) of 3.4(2) hold, so (iii) must: $\models \rho(a', b, b')$, and $w_p(a'/Cbb') = 0$. Using the last equality we compute again:

$$w_p(bb'/C) = w_p(a/C) + w_p(b/Ca) + w_p(b'/Cab) - w_p(a'/bb'C) = 2 + 1 + 1 - 0 = 4.$$

As $\text{stp}(b/C) = \text{stp}(b'/C) \square p^2$, it follows that $b \perp b' \mid C$. Thus by the choice of ρ, $\text{tp}(a/C) = \text{tp}(a'/C)$, as required. □

Proof of Proposition 3.1. By the last two lemmas, there exists a formula $\alpha(x) \in \text{tp}(a/Cb)$ such that if $\vDash \alpha(a')$ and $w_p(a'/Cb) \neq 0$ then $\text{tp}(a'/C) = \text{tp}(a/C)$. By 3.2(c) there exists a 0-definable function h such that $h(a) \vDash p \mid C$. Let $d = h(a)$. By 3.2(b), $d \notin \text{Cl}_p(Cb)$, so $d \vDash p \mid C \cup \{b\}$. It follows in particular that $w_p(a/C, b, h(a)) = 0$; as p-weight is continuous and definable we may assume that $\alpha(a')$ implies $w_p(a'/C, b, h(a')) = 0$. Let $\alpha^*(x) = (\exists x')(x = h(x') \,\&\, \alpha(x'))$. Then $p \mid Cb$ is strongly regular via α^*. For suppose $\vDash \alpha^*(d')$ and $w_p(d'/Cb) \neq 0$. Pick a' such that $d' = h(a')$ and $\vDash \alpha(a')$. Then $w_p(a'/Cb) \neq 0$, so $\text{tp}(a'/C) = \text{tp}(a/C)$. Hence $\text{tp}(d'/C) = \text{tp}(d/C) = p \mid C$. As p is regular and $w_p(d'/Cb) > 0$, $d' \perp b \mid C$. As $p \mid C$ is stationary, $d' \vDash p \mid Cb$. This shows that $p \mid Cb$ is strongly regular via α^*; hence p is strongly regular.

References

[1] J.T. Baldwin, α_T is finite for $\aleph_1$-categorical T, Trans. Amer. Math. Soc. 181 (1973) 37–51.

[2] S. Buechler, The geometry of weakly minimal types, J. Symbolic Logic 50 (1985) 1044–1053.

[3] G. Cherlin, $\aleph_0$-stable, $\aleph_0$-categorical structures of modular type, Preprint.

[4] E. Hrushovski, Unidimensional theories: an introduction to geometric stability theory, in: M. Garrido et al., eds., Logic Colloquium '87 (North-Holland, Amsterdam, 1989).

[5] E. Hrushovski, Locally modular regular types, in: Proceedings of the U.S.–Israel Workshop in Model Theory (Springer, Berlin, 1988).

[6] E. Hrushovski, Unidimensional theories are superstable, ∞.

[7] E. Hrushovski, Kueker's conjecture for stable theories, J. Symbolic Logic 54 (1989) 207–221.

[8] D. Lascar, Ranks and definability in superstable theories, Israel J. Math. 23 (1976) 53–87.

[9] M. Makkai, A survey of basic stability theory with particular emphasis on orthogonality and regular types, Israel J. Math. 49 (1984) 181–238.

[10] S. Shelah, Classification Theory and the Number of Non-Isomorphic Models: completed for countable theories (North-Holland, Amsterdam, 1987).