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WHITEHEAD GROUPS MAY BE NOT FREE, EVEN ASSUMING CH, I

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ABSTRACT
We prove the consistency with ZFC + G.C.H. of an assertion, which implies several consequences of $MA + 2^{\aleph_0} > \aleph_1$, which $\diamondsuit_{\aleph_1}$ implies their negation.

§0. Introduction

The author has advocated for several years the problem of finding an assertion (X) consistent with ZFC + G.C.H., but still similar to $MA + 2^{\aleph_0} > \aleph_1$, and far from $V = L$ (or even $\diamondsuit_{\aleph_1}$). The reason was a hope it will imply

(A) there are non-free Whitehead groups of cardinality $\aleph_1$ (see [9] or the presentation of Eklof in [5]).

Remember that by [9], $V = L$ (or even $\diamondsuit^*$) implies not (A), whereas $MA + 2^{\aleph_0} > \aleph_1$ implies (A). There are many assertions which are in a similar situation (i.e., are implied by $MA + 2^{\aleph_0} > \aleph_1$ but contradicted by $V = L$) and it is natural to try to replace $\diamondsuit_{\aleph_1}$ by CH (two preprints do it for (A)) or find a suitable (X) as mentioned above.

It seemed that the right (X) should solve other problems, and natural candidates seemed

(B) for every stationary $S \subseteq \omega_1$,

$(B)_S$ every graph G of the following form has cromatic number $\aleph_0$: its set of vertices is ω_1, and there are increasing ω-sequences with limit δ, η_δ, for each limit $\delta \in S$ such that the set of edges of G is

$$\{(\eta_\delta(n), \delta) \colon n < \omega,\ \delta \in S,\ \delta \text{ limit}\}.$$

Hajnal and Mate prove $MA + 2^{\aleph_0} > \aleph_1 \Rightarrow (B)$, $\diamondsuit_S \Rightarrow \neg (B)_S$ and asked what is the situation assuming CH (see [6]).

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Another problem (see [10] but the part with MA was omitted there):

(C) Is there a graph G of cardinality $\aleph_1$, with colouring number $\aleph_1$ such that $\aleph_1 \to (G)^2_2$? (see Definition 2.1).

The most famous of those problems is, of course:

(D) (1) Are there Suslin trees?

(2) Is every Aronszajn tree special? (see, e.g., [7]).

U. Avraham and the author tried to work on this on the thesis that the right way is to solve the equation

consistency proof of (X)/Jensen proof of Consis (ZFC + G.C.H. + SH)

= consistency proof of MA/Solovay–Tennenbaum proof of Consis (ZFC + SH)

(see [3] for Jensen proof, and [14] on Solovay–Tennenbaum proof and [8] on Martin axiom).

As a result they (see [1]) found an (X) consistent with G.C.H., and derived from Jensen's proof, but it implies (B) only.

The author translated (A) to a set-theoretic assertion, Devlin looked at the following variant of a disjunct of that assertion (equivalent to it):

(E) for some stationary $S \subseteq \omega_1$,

$(\mathrm{E})_S$ if η_δ is an increasing sequence of ordinals with limit $\delta \in S$ and $h_\delta \in {}^\omega 2$ for $\delta \in S$ *then* there is $f\colon \omega_1 \to \{0,1\}$ such that for each limit $\delta \in S$ for every n big enough $h_\delta(n) = f(\eta_\delta(n))$.

A great surprise was that Devlin proved CH $\Rightarrow$ not $(\mathrm{E})_{\omega_1}$ (for this and more, see Devlin and Shelah [4]).

From this we see that as Jensen's proof does not discriminate ω_1 from any other stationary subset of ω_1, it cannot be used to prove that for some stationary $S \subseteq \omega_1$ $(\mathrm{E})_S$ which would imply there is a non-free Whitehead group.

However in §1, we show the consistency of ZFC + G.C.H. + "$(\mathrm{E})_S$ for some stationary S". In §2 we mention stronger assertions whose consistency (variants of) the proof in §1 shows, and show they implied (A), (C) and $(\mathrm{B})_S$, and even more. We naturally hope more applications will be found.

A nice feature of our proof is that it generalized easily to higher cardinals, unlike MA. Hence it is consistent with ZFC + G.C.H. that the first non-free Whitehead group has a large power.

Let us mention related results. Avraham, Devlin and Shelah [2] deal with what can be sucked from Jensen's proof. In [12] we show that if $(\mathrm{E})_S$ holds for one $\langle \eta_\delta\colon \delta \in S\rangle$, it does not necessarily hold for another $\langle \eta'_\delta\colon \delta \in S\rangle$; and $(\mathrm{E})_{S_1} \wedge (\mathrm{E})_{S_2} \not\Rightarrow (\mathrm{E})_{S_1 \cup S_2}$; hence the question whether G is Whitehead is delicate;

e.g., in Theorem 2.4's notation, the knowledge of S is not sufficient. We also show (in the notation of [9]) that a Whitehead group can be in case I, by showing the consistency with $\mathrm{ZFC} + 2^{\aleph_1} = 2^{\aleph_0}$, of an assertion similar to $(\mathrm{E})_S$, where $\mathrm{Range}(\eta_\delta) \subseteq \omega$. Note that this contradicts MA, as by [11] not MA implies every $\aleph_1$-free (abelian) group of cardinality $\aleph_1$ is Whitehead iff it is in case II or III (see [9]).

The author would like to thank Uri Avraham for writing up §1. The main result was announced in [13].

Added in proof, April 1977.

1) We find another application; it is consistent with ZFC + G.C.H. that there is a non-metrizable normal Moore space of cardinality $\aleph_1$. In the model constructed in Theorem 2.1 take, e.g., $X = {}^{\omega>}\omega_1 \cup \{\eta_\delta \cdot \delta \in S\}$ as the space, with the topology generated by $\{\{\eta\} : \eta \in {}^{\omega>}\omega_1\} \cup \{\{\eta_\delta \restriction \alpha : n \leqq \alpha \leqq \omega\} : n < \omega, \delta \in S\}$. We can get as an example a special Aronszajn tree in which we refine the usual topology by making some limit points into isolated points. For background and details, see Devlin and Shelah [4a].

2) Devlin pointed out that $\diamondsuit_{\omega_1} \rightarrow (\forall \text{ stationary } S \leqq \omega_1)\diamondsuit_S$ was an open question and is solved by this paper. (The answer is not, as in the model constructed in Theorem 2.1, $\diamondsuit_{\omega_1}$ holds by Theorem 2.4 by $\diamondsuit_S$ fail as $\diamondsuit_S \Rightarrow \neg(\mathrm{E})_S$ of course (or as $\diamondsuit_S \Rightarrow \neg(\mathrm{B})_S$ by [6] and Conclusion 2.6).)

3) Notice that for any regular λ, $\{S \subseteq \lambda : \diamondsuit_S \text{ holds}\}$ is a normal ideal.

§1. Negation of the $\diamondsuit$ consistent with CH

We saw in [4] that if $2^{\aleph_0} < 2^{\aleph_1}$ a closed unbounded subset of ω_1 cannot be small. Can a stationary set be small? In $V = L$ the answer is no, however a consistency result shows this is possible (with G.C.H.).

THEOREM 1.1. *Suppose* $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$; $S \subseteq \omega_1$ *and* $\omega_1 - S$ *are stationary. For* $\delta < \omega_1$, η_δ *is an increasing* ω*-sequence of ordinals with limit* δ. *Then there is a set of forcing conditions* $(P, \leqq)$ *such that*:

1) $|P| = \aleph_2$, *P satisfies the* $\aleph_2$*-CC and adds no new sequences of length* ω, *so if V satisfies G.C.H., then also* V^P *satisfies it.*

2) *Every stationary set remains stationary* (*in* V^P) (*in particular S itself, this is the point, for if S becomes non-stationary then* 3), *which is our aim, holds trivially*).

3) *In* V^P *the following holds*: *For every* $\langle c_\delta : \delta \in S\rangle$ $(c_\delta \in {}^{\omega}2)$ *there is* $f : \omega_1 \rightarrow 2$ *such that*

$S \subseteq \{\alpha < \omega_1$: *there is* $n_\alpha < \omega$ *such that for every* $n \geqq n_\alpha \quad f(\eta_\alpha(n)) = c_\alpha(n)\}$.

Proof. We describe at first the basic step, the iteration of which will give the final set of conditions. Let $\bar{c} = \langle c_\delta: \delta \in S \rangle$ ($c_\delta \in {}^\omega 2$) be given, we define $P_{\bar{c}}$ to be the set of functions f such that: $\operatorname{Dom} f$ is some ordinal $\alpha < \omega_1$ and $(\forall \delta \leqq \alpha)$ $(\delta \in S \Rightarrow$ from some n onward $f(\eta_\delta(n)) = c_\delta(n))$. The order is inclusion. It is easy to see that for $\alpha < \omega_1$ $E_\alpha = \{f: \alpha \subseteq \operatorname{Dom} f\}$ is dense, hence a generic filter will give us the desired unifying f.

To see that $P_{\bar{c}}$ does not add a new sequence of length ω, we take D_n, $n < \omega$, dense open subsets and have to show that $\bigcap_{n<\omega} D_n$ is dense. Let $f \in P_{\bar{c}}$. Look at the model $N = (H(\omega_2), \Vdash, \in, P_{\bar{c}}, D_n)_{n<\omega}$. We can find an elementary chain (increasing and continuous) $N_\alpha < N$ $(\alpha < \omega_1)$, $N_\alpha \supseteq \alpha$, N_α is countable, $f \in N_\alpha$. As $C = \{\alpha: N_\alpha \cap \omega_1 = \alpha\}$ is closed unbounded choose $\alpha \in C - S$ and let $\alpha = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$. We choose by induction on $n < \omega$ $f_n \in N_\alpha$, $f_{n+1} \geqq f_n$, $\alpha_n \subseteq \operatorname{Dom} f_{n+1}$, $f_{n+1} \in D_n$, $f = f_0$. Now as $\alpha \notin S$, $\bigcup_{n<\omega} f_n \in P_{\bar{c}}$ and $\bigcup_{n<\omega} f_n \in \bigcap_{n<\omega} D_n$.

Now we show that a stationary $S^* \subseteq S$ remains stationary (for $S^* \subseteq \omega_1 - S$ it is easier). Suppose $f \Vdash$ "τ is a closed unbounded subset of ω_1". Define N_α, C as before and choose $\delta \in (C') \cap S^*$ (C' is the set of limit points of C). So there are $\alpha_n \in C$, $n < \omega$, increasing with limit δ. We shall define $f_n \in N_{\alpha_n}$, $f_0 = f$, $f_n \leqq f_{n+1}$, $\alpha_n \subseteq \operatorname{Dom} f_{n+1} \subseteq \alpha_{n+1}$ such that:

1) for any $k < \omega$ if $\alpha_n \leqq \eta_\delta(k) < \alpha_{n+1}$ then $f_{n+1}(\eta_\delta(k)) = c_\delta(k)$ (note that only a finite number of k's satisfy the requirement for each n);

2) $f_{n+1} \Vdash$ "there is some $\zeta \in \tau$, $\alpha_n < \zeta < \alpha_{n+1}$".

Now $\bigcup_{n<\omega} f_n \in P$ because of 1) and $\bigcup_{n<\omega} f_n \Vdash$ "$\delta \in \tau \cap S^*$ because of 2). We define f_n by induction on n, $f_0 = f$; and for a given f_n, we first find $f'_{n+1} \in N_{\alpha_{n+1}}$ satisfying 1), $f_n \leqq f'_{n+1}$, and then $f_{n+1} \in N_{\alpha_{n+1}}$ satisfying 2), $f'_{n+1} \leqq f_{n+1}$.

Remark. Actually the second proof shows that $P_{\bar{c}}$ does not introduce new ω sequences, so here we don't have to assume that $\omega_1 - S$ is stationary. But the assumption will be needed in the iteration and we wanted to present the ideas in a simple form.

We now iterate $P_{\bar{c}}$ extensions ω_2 times taking inverse limit at stages of cofinality ω. More explicitly we define by induction sets of forcing conditions P_α for $\alpha \leqq \omega_2$ and carefully chosen $\bar{c}^\alpha$ names in P_α (with boolean value 1) of a sequence $\langle \underset{\sim}{c}^\alpha_\delta \in {}^\omega 2: \delta \in S \rangle$. The elements of P_α are all the functions p with $\operatorname{Dom} p \subseteq \alpha$, $\operatorname{Dom} p$ countable and for $\zeta \in \operatorname{Dom} p$, $p(\zeta)$ is a function (in V) such that: $p \restriction \zeta \in P_\zeta$ and $p \restriction \zeta \Vdash^{P_\zeta}$ "$p(\zeta) \in P_{\bar{c}^\zeta}$", the ordering of P_α is $p \geqq q$ iff $\operatorname{Dom} q \subseteq \operatorname{Dom} p$ and for $\zeta \in \operatorname{Dom} q$, $p(\zeta)$ extends $q(\zeta)$. Note that $p(\zeta)$ is a function in V (not a name in P_ζ) but this is okay since we will show that P_ζ does not add new ω-sequences. Now $P_{\omega_2}=P$ is the desired set of conditions.

LEMMA 1.2. *P satisfies the $\aleph_2$-C.C.*

PROOF. Let $p_i \in P$, $i < \omega_2$; as $\mathrm{Dom}\, p_i$ is countable and $2^{\aleph_0} = \aleph_1$ we can find $I \subseteq \omega_2$, $|I| = \aleph_2$ and A such that $\beta < \alpha \in I \Rightarrow \mathrm{Dom}(p_\alpha) \cap \mathrm{Dom}(p_\beta) = A$ and $p_\alpha \restriction A = p_\beta \restriction A$ hold too (remember P_{ξ^ξ} has cardinality $\aleph_1$) hence p_α, p_β are compatible by $(p_\alpha \cup p_\beta)$.

DEFINITION 1.1. If p, q are functions, $p \vee q$ is the function defined on $\mathrm{Dom}\, p \cup \mathrm{Dom}\, q$ such that

$$\zeta \in \mathrm{Dom}\, p - \mathrm{Dom}\, q \Rightarrow [p \vee q](\zeta) = p(\zeta),$$

$$\zeta \in \mathrm{Dom}\, q - \mathrm{Dom}\, p \Rightarrow [p \vee q](\zeta) = q(\zeta),$$

$$\zeta \in \mathrm{Dom}\, q \cap \mathrm{Dom}\, p \Rightarrow [p \vee q](\zeta) = p(\zeta) \cup q(\zeta).$$

FACT 1.3. If $p \in P_\alpha$, $q \in P_\beta$, $\alpha \leqq \beta$, $p \geqq q \restriction \alpha$ then $p \vee q \in P_\beta$.

DEFINITION 1.2. Let t be a function defined on a finite subset of $\alpha \leqq \omega_2$ such that $\zeta \in \mathrm{Dom}\, t \Rightarrow t(\zeta)$ is a finite function from ω_1 into $\{0, 1\}$. A condition $p \in P_\alpha$ *induces* t iff $\zeta \in \mathrm{Dom}\, t \Rightarrow \zeta \in \mathrm{Dom}\, p$ and $t(\zeta) \subseteq p(\zeta)$. We say p is *consistent* with t iff for $\zeta \subseteq \mathrm{Dom}\, t \cap \mathrm{Dom}\, p$, $p(\zeta) \cup t(\zeta)$ is a function.

The following Lemmas 1.4–1.6 are proved simultaneously by induction on α,

LEMMA 1.4. *If $p \in P_\alpha$ is consistent with t then for some q, $p \leqq q$, $q \in P_\alpha$ and q induce t.*

PROOF. Let $\mathrm{Dom}\, t = \{\beta_1, \cdots, \beta_k\}$, we define by induction $p_i \in P_\alpha$, $i \leqq k$, $p_0 = p$, $p_{n+1} \geqq p_n$, $\beta_i \in \mathrm{Dom}\, p_i$, $0 < i \leqq k$ and $p_i(\beta_i) \supseteq t(\beta_i)$, and p_n is consistent with t.

Suppose p_{i-1} is defined. By Lemma 1.6 P_{β_i} does not introduce new ω-sequences hence we can find q, $p_{i-1} \restriction \beta_i \leqq q \in P_{\beta_i}$ such that q "describes" $\underset{\sim}{c}{}^{\beta_i}_\delta$, for $\delta \leqq \sup(\mathrm{Dom}\, t(\beta_i))$. Now we can extend $p_i(\beta_i)$ and using Fact 1.3 find p_i as required. Set $q = p_k$ to end the proof.

LEMMA 1.5. *Every $p \in P_\alpha$ has an extension $p^* \in P_\alpha$ such that for some $\delta \notin S$ for every $\beta \in \mathrm{Dom}\, p^*$, $\delta = \mathrm{Dom}\, p^*(\beta)$.*

PROOF. Let $N = \langle H(\omega_2), \in, P_\alpha, \Vdash \rangle$ and taking $N_\delta < N$ $(\delta < \omega_1)$ a continuous chain of countable elementary submodels such that $p \in N_\delta$, we find as before a closed unbounded $C \subseteq \omega_1$, $\delta \in C \Rightarrow N_\delta \cap \omega_1 = \delta$. Now for $\delta \in [C' \cap \omega_1 - S]$ we take $\delta = \bigcup_{n<\omega}\delta_n$, $\delta_n \in C$ and define $p_n \in N_{\delta_n}$, $p_n \leqq p_{n+1}$, $p_0 = p$ and $\beta_n \in \mathrm{Dom}\, p_n$, such that each $\beta \in \mathrm{Dom}\, p_n$ is β_m for infinitely many m' and $\delta_n \subseteq \mathrm{Dom}\,(p_{n+1}(\beta_n))$; hence $p^* = \bigcup_{n<\omega} p_n$ will satisfy the claim of the lemma.

We can define p_{n+1} as in the proof of Lemma 1.4, using Lemma 1.6, and can choose appropriate β_n because $\mathrm{Dom}\,(p_n)$ is countable. $p^* \in P_\alpha$ because $\delta \notin S$.

LEMMA 1.6. *P_α does not add new ω-sequences.*

PROOF. As the proof for $P_{\bar{c}}$. We use Lemma 1.5 to ensure that our conditions have even height.

In order to see that S remains stationary we need the following lemma, where the fact that $\omega_1 - S$ is stationary is used. This lemma and Lemma 1.8 are the heart of the proof.

LEMMA 1.7. *Suppose $\{\beta_i : i < \gamma\}$ is an increasing sequence of ordinals, $\gamma < \omega_1$, $\beta_i < \omega_2$. Suppose $\delta \in S$ and for every sequence $\bar{c} = \langle c_i \mid i < \gamma \rangle$, $c_i \in {}^\omega 2$ we have a function $p_{\bar{c}}$, $\mathrm{Dom}\, p_{\bar{c}} = \{\beta_i : i < \gamma\}$, and $\xi \in \mathrm{Dom}\, p_{\bar{c}} \Rightarrow p_{\bar{c}}(\xi)$ is a function from δ to 2 such that:*

(i) *for $i < \gamma$, $\bar{c} \restriction i = \bar{c}^* \restriction i \Rightarrow p_{\bar{c}} \restriction \beta_i = p_{\bar{c}^*} \restriction \beta_i$ (and we name this common value by $p_{\bar{c}\restriction i}$),*

(ii) *for $i < \gamma$, $[p_{\bar{c}}(\beta_i)](\eta_\delta(n)) = c_i(n)$ from some n onward,*

(iii) *each $p_{\bar{c}}$ is the union of an increasing ω-sequence of members of P.*

Then for some $\bar{c} = \langle c_i : i < \gamma \rangle$ and $q \in P$, $p_{\bar{c}} \leqq q$ (this is not well defined as maybe $p_{\bar{c}} \notin P$, but the meaning is

$$\xi \in \mathrm{Dom}\, p_{\bar{c}} \Rightarrow p_{\bar{c}}(\xi) = q(\xi) \restriction \mathrm{Dom}\, p_{\bar{c}}(\xi)).$$

PROOF. Note that if γ satisfies the assumptions of the lemma then so does each $\gamma' < \gamma$. We prove by induction on γ the following stronger claim:

$(*)_\gamma$ If $\gamma(0) < \gamma$, $\bar{c}_0 = \langle c_i : i < \gamma(0) \rangle$ and $P_{\bar{c}_0} \leqq r \in P_{\beta_{\gamma(0)}}$ then for some extension $\bar{c} = \langle c_i : i < \gamma \rangle$ of $\bar{c}_0$ and $q \in P_{\beta_\gamma}$, $q \geqq p_{\bar{c}} \vee r$.

$\gamma = \zeta + 1$. By induction hypothesis we can assume $\gamma(0) = \zeta$. Given $\bar{c}_0$ and $p_{\bar{c}_0} \leqq r \in P_{\beta_\zeta}$ we can find by Lemma 1.6 $r' \geqq r$, $r' \in P_{\beta_\zeta}$ such that $r' \Vdash \underset{\sim}{c}{}^\zeta_\delta = c_\zeta$ for some $c_\zeta \in {}^\omega 2$. Now let $\bar{c} = \langle c_i : i < \zeta + 1 \rangle$ extend $\bar{c}_0$, then $p_{\bar{c}} \vee r' \in P_{\beta_{\zeta+1}}$ is as required.

γ limit. Let $\gamma = \bigcup_{n<\omega} \gamma_n$, $\gamma_n < \gamma_{n+1}$. Using again the argument of elementary submodels we can find

$$N < (H(\omega_2), \Vdash, \in, \delta, \langle \gamma(n) : n < \omega \rangle, \{(\bar{c}, p_{\bar{c}}) : \bar{c} \in {}^\gamma({}^\omega 2)\}, \{\beta_i : i < \gamma\})$$

such that $N \cap \omega_1 = \rho \in \omega_1 - S$. Now we construct in N an increasing sequence $p_n \in P_{\beta_{\gamma(n)}}$ and c_i $(i < \gamma(n))$ such that, letting $\bar{c}_n = \langle c_i : i < \gamma(n)\rangle$, $p_n \geqq p_{\bar{c}_n}$ and $p_0 \geqq r$. The induction step is by (*). Moreover, by Lemma 1.5 we can ensure that if $\zeta \in \operatorname{Dom} p_n$ for some n then $\bigcup_{k \geqq n} p_k(\zeta)$ is defined on ρ. As $\rho \in S$ we have $q = \bigcup p_n \in P$ as required.

LEMMA 1.8. *Every stationary subset remains so in V^P.*

PROOF. Let $S^* \subseteq S$ be stationary (for $S^* \subseteq \omega_1 - S$ it is easier), τ a name of a closed unbounded set, $p \in P$ a condition; we want an extension of it forcing $\delta \in \tau$ for some $\delta \in S^*$.

Again we can find N_k, $k < \omega$, countable elementary submodels of $N = \langle H(\omega_2), \in, p, \Vdash, \tau, P, S^*\rangle$, such that $N_k \cap \omega_1 = \alpha_k$, $\alpha_k < \alpha_{k+1}$, $N_k \prec N_{k+1}$, $\bigcup_{k<\omega} \alpha_k = \delta \in S^*$.

By W we shall denote finite functions, $\operatorname{Dom} W \subseteq \omega_2$ and $W(\zeta) \in \omega$ for $\zeta \in \operatorname{Dom} W$. For such W and $k < \omega$ we define $Q(W, k)$ to be the set of all functions t such that $\operatorname{Dom} t$ is an initial segment of $\operatorname{Dom} W$ and $t(\zeta)$ is a function from $\{\eta_\delta(i) : i < \omega,\ \alpha_{W(\zeta)} \leqq \eta_\delta(i) < \alpha_k\}$ whose Range $\subseteq \{0, 1\}$.

We call $T = \{T(t) : t \in Q(W, k)\}$ a $Q(W, k)$-*tree* if the following hold:

1) $T \in N_k$, $T(t) \in P$,

2) $T(t)$ is consistent with t,

3) For any $\gamma \in \operatorname{Dom} W$, $T(t \restriction \gamma) = T(t) \restriction \gamma$.

Let T_l be $Q(W_l, k_l)$-trees, $l = 0, 1$. We say $T_0 \leqq T_1$ if: (a) $W_0 = W_1 \restriction \operatorname{Dom} W_0$, $k_0 \leqq k_1$ and (b) for any $t \in Q(W_1, k_1)$, $T_0(t \restriction (W_0, k_0)) \leqq T_1(t)$ except possibly when $\operatorname{Dom} W_0 \subseteq \operatorname{Dom} t \neq \operatorname{Dom} W_0$, where $t' = t \restriction (W_0, k_1)$ is the unique function with domain $\operatorname{Dom} t \cap \operatorname{Dom} W_0$ and $t'(\zeta) = t(\zeta) \restriction \alpha_{k_0}$.

We now define by induction on $k < \omega$ functions W_k, and $Q(W_k, k)$-trees $T_k = \{T_k(t) : t \in Q(W_k, k)\}$ such that:

i) $W_0 = \emptyset$ $(Q(W_0, 0) = \{\emptyset\})$, $T_0 = \{T_0(\emptyset)\}$ where $T_0(\emptyset) = p$ (the condition we started from); $W_{k+1} \supseteq W_k$, $T_{k+1} \geqq T_k$;

ii) $T_k(t)$ induce t for every $t \in Q(W_k, k)$;

iii) for every $t \in Q(W_{k+1}, k+1)$ such that $\operatorname{Dom} t = \operatorname{Dom} W_{k+1}$ (we will say that t is of maximal length)

$$T_{k+1}(t) \Vdash \text{"for some } \zeta,\ \zeta \in \tau \text{ and } \alpha_{k+1} > \zeta \geqq \alpha_k\text{"};$$

iv) for every $t \in Q(W_{k+1}, k+1)$ and $\zeta \in \operatorname{Dom} t$

$$\alpha_k \subseteq \operatorname{Dom}[T_{k+1}(t)](\zeta);$$

v) for every $t \in Q(W_k, k)$ and $\zeta \in \operatorname{Dom} T_k(t)$ there is $k^* \geqq k$ such that $\zeta \in \operatorname{Dom} W_{k^*}$.

Suppose W_k, T_k are defined.

To obtain W_{k+1}. We add one element σ to $\operatorname{Dom} W_k$ and set $W_{k+1}(\sigma) = k+1$. We choose σ such that v) will eventually be satisfied. Let $t_1, \cdots, t_l$ be the elements of $Q(W_{k+1}, k+1)$ of maximal length. We construct $Q(W_{k+1}, k+1)$-trees $S_0 \leqq S_1, \cdots, \leqq S_l$ such that $S_l = T_{k+1}$ will be the required tree, and $S_0 = T_k$, i.e., for $i = 1, \cdots, l$ $S_0(t_i) = T_k(t_i \restriction (W_k, k))$; S_0 is a $Q(W_{k+1}, k+1)$-tree by the choice of $W_{k+1}(\sigma)$. We will require that:

a) $S_j(t_j)$ induce t_j, $l \geqq j \geqq 1$, and $S_i(t_j)$ is consistent with t_j,

b) $\alpha_k \subseteq \operatorname{Dom}[S_j(t_j)](\zeta)$ for $\zeta \in \operatorname{Dom} t_j$,

c) $S_j(t_j) \Vdash$ "$\zeta \in \tau$ for some ζ, $\alpha_k \leqq \zeta < \alpha_{k+1}$".

Suppose S_i is defined, we define S_{i+1} in N_{k+1}. $S_i(t_{i+1})$ is consistent with t_{i+1}. From Lemma 1.4 we can enlarge $S_i(t_{i+1})$ and find a condition that induces t_{i+1}. Enlarging it further by Lemma 1.6 we take care of b) and enlarging once more we get $S_{i+1}(t_{i+1})$, so that c) holds too. Now for any $t \in Q(W_{k+1}, k+1)$ for some γ, $t \restriction \gamma = t_{i+1} \restriction \gamma$ (e.g. $\gamma = 0$), take the maximal such γ (it always exists); then $S_i(t) \restriction \gamma = S_i(t_{i+1}) \restriction \gamma$, hence $S_i(t) \vee (S_{i+1}(t_{i+1}) \restriction \gamma) \in P$. We define $S_{i+1}(t) = S_i(t) \vee (S_{i+1}(t_{i+1}) \restriction \gamma)$. One can check that S_{i+1} is a $Q(W_{k+1}, k+1)$-tree and S_l satisfies i)–iv).

The next stage is to get the conditions of Lemma 1.7.

Let $\{\beta_i : i < \gamma\} = \bigcup_{k<\omega} \operatorname{Dom} W_k$. Given a sequence $\bar{c} = \langle c_i : i < \gamma \rangle$ we construct the sequence $t_k \in (W_k, k)$. If $\alpha_{W_k(\zeta)} \leqq \eta_\delta(l) < \alpha_k$, $\zeta = \beta_i \in \operatorname{Dom} W_k$ then $[t_k(\zeta)](\eta_\delta(l)) = c_i(l)$. Now, $T_k(t_k)$ is an increasing sequence of conditions in P, and we set $p_{\bar{c}} = \bigvee_{k<\omega} T_k(t_k)$ (i.e. for every β_i $p_{\bar{c}}(\beta_i) = \bigcup_{k<\omega}([(T_k(t_k)](\beta_i))$. (Note $p_{\bar{c}}$ is not necessarily a condition.) It is easy to check that the conditions of Lemma 1.7 hold, hence for some $\bar{c}$ and $q \in P$, $p_{\bar{c}} \leqq q$. Now $p \leqq p_{\bar{c}}$ and $q \Vdash$ "$\delta \in \tau$" because of condition iii) is as required.

§2. Generalization and applications

By changing somewhat the proof of Theorem 1.1, we can get by a similar forcing (for proof of Theorems 2.1–2.4 see [12])

THEOREM 2.1. *Suppose $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, D is an $\aleph_1$-complete normal filter over ω_1.*

There is a set of forcing conditions $(P, \leqq)$ satisfying 1) *and* 2) *of Theorem* 1.1 *and in V^P the following holds: G.C.H. and*

3)

(*): *for every* $S \subseteq \omega_1$, $\omega_1 - S \in D$, *and* $\langle \eta_\delta : \delta \in S \rangle$ *where* η_δ *is an increasing* ω-*sequence converging to* δ *and* $\bar{c} = \langle c_\delta : \delta \in S \rangle$ *where* $c_\delta \in {}^\omega\omega$ *there is a function* $f: \omega_1 \to \omega$ *such that for each* $\delta \in S$ *for every* $n < \omega$ *big enough* $f(\eta_\delta(n)) = c_\delta(n)$.

THEOREM 2.2. *In Theorem* 2.1, *instead of* (*) *we can demand*

(**) *Let* $S \subseteq \omega_1$ $\omega_1 - S \in D$, *T a tree of height* ω_1, $T = \bigcup_{\alpha<\omega_1} T_\alpha$, T_α *the* α-*th level.*

Then T has an ω_1-*branch provided that the following conditions hold*:

a) $T_0 \neq \emptyset$, *and every element of T has at most* $\aleph_1$ *immediate successors, and at least one successor.*

b) *For each limit* $\delta \notin S$, *if* $\delta = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$, $a_n \in T_{\alpha_n}$, $a_n \leqq a_{n+1}$ (*in the tree*) *then for some* $a \in T_\delta$, $a_n \leqq a$ *for every n.*

c) *Let* $T = \bigcup_{\alpha<\omega_1} T^\alpha$, T^α *increasing continuous, and each* T^α *is countable, then for some closed unbounded* $C \subseteq \omega_1$, *for each* $\delta \in C \cap S$, *and* $a \in T_\alpha \cap T^\delta$, $\alpha < \delta$ *there is a subtree* $T^* \subseteq \bigcup_{i \leqq \delta} T_i$ *of T such that* (*see mainly* (iv))

(i) $a \in T^*$ (*and* $b < c$, $c \in T^* \Rightarrow b \in T^*$ *of course*), *and* $T^* - T_\delta \subseteq T^\delta$,

(ii) *every element b of* $T^* - T_\delta$ *has an immediate successor, in* T^*,

(iii) *for every* $\delta' \in C$, $\delta' < \delta$ *and* $a \in T^* \cap T_\alpha \cap T^\delta$, $\alpha < \delta'$, *there is* $b \in T^* \cap T_\beta \cap T^\delta$, $a < b$ *for some* $\beta < \delta'$ *such that* $b < c \wedge c \in \bigcup_{\gamma<\delta} T_\gamma \cap T^\delta \Rightarrow c \in T^*$,

(iv) *if* $a_n \in T^* \cap T_{\alpha_n}$, $a_n \leqq a_{n+1}$, $\delta = \bigcup_{n<\omega} \alpha_n$, *then for some* $a \in T^* \cap T_\delta$, $a_n \leqq a$ *for every* n;

or even

(**)′ *We can replace* (b) *by*

(b′) *There is a function* $f: T \to T$, $a \leqq f(a)$ *such that for any limit ordinal* $\delta \notin S$, $\delta < \omega_1$, *if* $\delta = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$, $a_n \in T_{\alpha_n}$, $a_n \leqq f(a_n) \leqq a_{n+1}$ *then for some* $a \in T_\delta$, $a_n \leqq a$ *for every n.*

THEOREM 2.3. *In Theorems* 2.1, 2.2 (*and also* 2.4–2.7) *we can replace* $\aleph_0$ *by any regular cardinality* λ, *provided that we make the obvious changes, and* $\{\delta < \lambda^+ : cf\delta = \lambda\} \in D$ (*so, e.g., in Theorem* 2.1(3), η_δ *is a* λ-*sequence*).

THEOREM 2.4. *In the models* (*of set theory*) *we construct in Theorems* 2.1 *and* 2.2 *if* $V = L$ (*i.e., we start with the constructible universe*) $\diamondsuit_{\aleph_1}$ *holds. Moreover, if* $S \in V$, $S \subseteq \omega_1$, $\omega_1 - S \notin D$, *then* $\diamondsuit_S$ *holds. In fact for each* $\alpha < \omega_1$, *there is a countable family* $\underline{S}_\alpha$ *of subsets of* α, *and there is a normal* $\aleph_1$-*complete filter* D^* *over* ω_1 *such that for every* $S \subseteq \omega_1$, $\{\alpha < \omega_1 : S \cap \alpha \in \underline{S}_\alpha\} \in D^*$.

Now we turn to applications. In all of them suppose we are in the model of set theory satisfying (**)′ from Theorem 2.2.

CONCLUSION 2.5. If $G = \bigcup_{i<\omega_1} G_i$, G, G_i abelian groups, G_i free and countable, G/G_{i+1} is $\aleph_1$-free, and $S = \{i: G/G_i \text{ is not } \aleph_1\text{-free}\}$, $\omega_1 - S \in D$. Then G is a Whitehead group.

PROOF. Use (**)′ from Theorem 2.2 and [9].

We suppose $f: H \to G$ is a homomorphism onto G, with kernel $Z \subseteq H$ (= the integers), let $H_i = f^{-1}(G_i)$, and let $T_\alpha = \{g: g: G_{i+1} \to H_{i+1}$ a homomorphism, $(f \restriction H_{i+1})g = 1_{G_{i+1}}\}$.

CONCLUSION 2.6. Suppose G is a graph whose set of vertices is ω_1, for every α, $A_\alpha = \{\beta: \beta$ is connected to infinitely many $\gamma < \alpha\}$ is countable, and $S = \{\delta < \omega_1$: some $\beta \geqq \delta$ is connected to infinitely many $\gamma < \delta\}$. If $\omega_1 - S \in D$ then G has cromatic number $\aleph_0$.

PROOF. By renaming we can assume $A_\alpha \subseteq \alpha + \omega$ for each α, and $A_{\omega(\alpha+1)} = \emptyset$. Let T_α be the set of functions f from $\omega(\alpha + 1)$ to ω, such that for β, γ connected in G, $f(\alpha) \neq f(\beta)$. Now apply (**)′ (in fact, (**)).

DEFINITION 2.1. 1) For a graph G let $\mathrm{cl}(G)$, the colouring number of G, be the minimal cardinal G such that we can enumerate the vertices of G by $\{v_i: i < \alpha\}$ such that for every i, $|\{j < i: (v_i, v_j) \in G\}| < \lambda$.

2) $\aleph_1 \to (G)^2_2$ means that for every 2-colouring of the (unordered) pairs of ω_1, (i.e. $f: [\omega_1]^2 = \{\{i,j\}: i < j < \omega_1\} \to \{0,1\}$) there is a one-to-one function F from G to ω_1, and $i < 2$ such that $(\forall (a,b) \in G)\ (a \neq b \to i = f(F(a), F(b))$.

CONCLUSION 2.7. 1) There are graphs G with colouring number $\aleph_1$, such that $\aleph_1 \to (G)^2_2$

2) Suppose G is a graph whose set of vertices is ω_1, for $\alpha \geqq \delta + \omega$ α is connected only to finitely many $\gamma < \delta$, for $\alpha > \beta > \delta$ only finitely many $\gamma < \delta$ are connected to α and β, and $S = \{\alpha < \omega_1$: α limit and some $\beta \geqq \alpha$ is connected to infinitely many $\gamma < \alpha\}$ is a set of limit ordinals and $\omega_1 - S \in D$. Then $\aleph_1 \to (G)^2_2$, and when S is stationary, $\mathrm{cl}(G) = \aleph_1$.

REMARK. By [10] if $\diamondsuit_S$ then $\aleph_1 \not\to (G)^2_2$.

PROOF. The part on the colouring number is immediate. So suppose f is a 2-colouring of ω_1.

Let E be a uniform ultrafilter over ω_1, for each α let $i_\alpha \in \{0,1\}$ be such that

$A_\alpha = \{\beta < \omega_1 : f(\alpha, \beta) = i_\alpha\} \in E$ (as E is an ultrafilter, i_α exists). Now for some $i \in \{0, 1\}$, $A = \{\alpha : i_\alpha = i\} \in E$, and w.l.o.g. $i = 0$.

Case I. There are n, $\alpha_{(0)}, \cdots, \alpha_{(n)} < \alpha$ such that for every $\beta > \alpha$ for some $\gamma > \beta$:

$$\gamma \in A \cap \bigcap_{l=0}^{n} A_{\alpha(l)},$$

and

$$(\forall \xi)\ (\alpha \leqq \xi < \beta \wedge \xi \in A \cap \bigcap_{l=0}^{n} A_{\alpha(l)} \rightarrow f(\xi, \gamma) = 1).$$

For each $\beta < \alpha$ let us call the γ we assure its existence $g(\beta)$. Now define $\gamma_j (j < \omega_1)$ inductively: $\gamma_0 = g(\alpha + 1)$, $\gamma_{j+1} = g(\gamma_j)$ and for limit δ, $\gamma_\delta = g(\bigcup_{j<\delta} \gamma_j)$. Clearly for $j(1) < j(2)$, $f(\gamma_{j(1)}, \gamma_{j(2)}) = 1$, so the mapping $j \rightarrow \gamma_j$ is as required.

Case II. Not I.

So for every n, $\alpha(0), \cdots, \alpha(n)$, α there is a β contradicting I. As for a fixed α, there are only countable many n, $\alpha(0), \cdots, \alpha(n)$. We can choose a β depending only on α and call it $g(\alpha)$. So we can define β_j $(j < \omega_1)$ increasing, such that $g(\beta_j) < \beta_{j+1}$, and for every $\alpha(0), \cdots, \alpha(n) \in \beta_{j+1} \cap A$,

$$\bigcap_{l=0}^{n} A_{\alpha(l)} \cap A \cap \beta_{j+1} \neq \emptyset.$$

Now let T_α be the set of functions F, $\mathrm{Dom}\, F = \omega(\alpha + 1)$, $\beta_i \leqq F(i) < \beta_{i+1}$, $\mathrm{Range}\, F \subseteq \beta_{\omega(\alpha+1)} \cap A$, and $(a, b) \in G$, $a, b < \omega(\alpha + 1)$ implies $f(F(a), F(b)) = 0$. Now use $(**)'$.

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