Remarks on Squares.

The combinatorial principle square (and some variants) was introduced by Jensen [J]. We have been interested in deriving weak forms of it from ZFC, plus possibly restrictions on cardinal arithmetic, see [Sh 1], [Sh 2], Magidor and Shelah [MS] and Abraham, Shelah and Solovay [ASS]. The modest remarks appearing here were first intented to appear in [ASS]. I thank Shai Ben-David for deleting inaccurances here.

Convention: λ will be a fixed regular uncountable cardinal, δ vary on limit ordinals.

1. Definition : 1) We call $\bar{C} = \langle C_\delta : \delta \in S \rangle$ a square (or S-square) if:

(i) $S \subseteq \lambda$ is a stationary set.

(ii) for $\delta \in S$, C_δ is a closed unbounded subset of δ.

(iii) if γ is a limit point of C_δ, where $(\delta \in S)$ then $\gamma \in S$ and $C_\gamma = C_\delta \cap \gamma$.

2) We say there is a diamond on $\bar{C}$ for χ where $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a square, *if* there are $A_\delta \subseteq \delta$ for $\delta \in S$ such that for every $A \subseteq \lambda$:

$\{\delta \in S : C_\delta$ has order type $\geq \chi$ and for every limit point γ of $C_\delta \cup \{\delta\}$, $A \cap \gamma = A_\gamma\}$.

It may be interesting to note that we can find square sequences on some S from cardinality hypothesis only.

2. Lemma : 1) Suppose $\lambda = \mu^+$, $\mu^{<\chi} = \mu$. Then we can find $S_\xi (\xi < \mu)$ such that :

a) $\bigcup_{\xi<\mu} S_\xi = \{\delta < \lambda : cf\ \delta < \chi\}$.

b) for each $\xi < \mu$, there is an S_ξ-square sequence $\langle C_i : i \in S_\xi \rangle$ (so $C_i \subseteq S_\xi$ for each i, $otp(C_i) < \chi$).

2) Suppose $\lambda = \mu^+$, μ singular, $(\forall \vartheta < \mu)[\vartheta^{<\chi} < \mu]$.

Then we can find S_ξ $(\xi < \mu)$ such that :

a) $\bigcup_{\xi<\mu} S_\xi = \{\delta < \lambda : cf\ \delta < \chi, cf\ \delta \neq cf\ \mu\} - S^*(\lambda)$. ($S^*(\lambda)$-the bad set, see [Sh 1]) and called it S^+.

b) for each $\xi < \mu$ there is a weak ($< \chi$)-square sequence $\langle C_i^\xi : i \in S_\xi \rangle$

c) if $\delta \in S_\xi$, $cf\ \delta < cf\ \mu$ then $C_\delta^\xi \cap S^+ \subseteq S_\xi$.

d) if $\delta \in S^+$, $cf\ \delta > cf\ \mu$ then there are $\xi_\gamma < \mu(\gamma < cf\ \mu)$, such that $C_\delta^{\xi_\gamma} = C_\delta^{\xi_0}$, and $C_\delta^{\xi_0} \cap S^+ \subseteq \bigcup_\gamma S_{\xi_\gamma}$.

Proof : 1) By Engelking and Karlowicz [EK] there are functions $f_i : \mu \to \mu$ for $i < 2^\mu$ such that for any distinct $i_\gamma < 2^\mu(\gamma < \gamma^* < \chi)$ and $\xi_\gamma < \mu$, for some $\zeta < \mu$, $f_{i_\gamma}(\zeta) = \xi_\gamma$ (for $\gamma < \gamma^*$). For each $\delta < \mu^+$ let $\langle B_\xi^\delta : \xi < \mu \rangle$ be a list of all subsets of δ of power $< \chi$ (possible as $\mu = \mu^{<\chi}$). Now define a function $g_\zeta : \mu^+ \to \mu$, by $g_\zeta(i) = f_i(\zeta)$.

Now for each $\zeta < \mu$ we define S_ζ:

(*) S_ζ is the set of limit ordinals $\delta < \mu$ of cofinality $< \chi$ such that $B^\delta_{g_\zeta(\delta)}$ is a closed unbounded subset of δ, moreover for each accumulation point γ of $B^\delta_{g_\zeta(\delta)}$, $B^\gamma_{g_\zeta(\delta)} = B^\delta_{g_\zeta(\delta)} \cap \gamma$.

Clealy for every γ,δ as in (*) $\gamma \in S_\zeta$. So condition b) is satisfied: $\langle B^\delta_{g_\zeta(\delta)} : \delta \in S_\zeta \rangle$ exemplify it.

Why condition a) holds? If $\delta < \lambda$, $cf\ \delta < \chi$, let C_δ be a closed unbounded subset of it of cardinality $< \chi$. Let for $\gamma \in C_\delta \cup \{\delta\}$, $\xi_\gamma < \mu$ be such that $B^\gamma_{\xi_\gamma} = C_\delta \cap \gamma$ (possible by the choice of $\langle B^\gamma_\xi : \xi < \mu \rangle$). So by the choice of the functions f_i, there is $\zeta < \mu$ such that for every $\gamma \in C_\delta \cup \{\delta\}, f_\gamma(\zeta) = \xi_\gamma$ hence $g_\zeta(\gamma) = \xi_\gamma$. So easily $\delta \in S_\zeta$.

2) Left to the reader (just see what proof of the theorem from [EK gives).

2. Conclusion: If for simplicity G.C.H., χ regular, $\mu > \chi^*, \lambda > \mu^+$ then there is a χ-square S with diamond on it. (see [ASS])

3. Question: Let $\lambda = \mu^+$, μ regular, $\diamondsuit_{\{\delta<\lambda: cf\ \delta=\mu\}}$, and assume G.C.H. Is there a μ-square with diamond on it.

4. Lemma: Let λ be regular uncountable cardinal, R a set of regular cardinals $< \lambda$, such that $|R| < \lambda$, and $(\forall \kappa \in R)\ \kappa^+ < \lambda$. Then we can find $S_\kappa (\kappa \in R)$ such that :

a) S_κ is a stationary subsets of λ.

b) for every $\delta \in S_\kappa$, $cf\ \delta = \kappa$.

c) if $\delta \in S_{\kappa_1}$, $\kappa_1 \neq \kappa_2$ then $S_{\kappa_2} \cap \delta$ is not stationary in δ.

Remark: In (d) only the case $\kappa_2 < \kappa_1$ is relevant.

Proof : For every κ choose pairwise disjoint stationary subsets $\{S(\kappa,i) : i < \lambda\}$ of $\{\delta < \lambda : cf\ \delta = \kappa\}$, such that $\kappa, i < Min\ S(\kappa,i)$ (exists by Solovay [So]). Suppose the lemma fails Now we define by induction on $\xi < \lambda$, $\kappa_\xi \in S$ and $\langle S^\xi_\kappa : \kappa < \kappa_\xi, \kappa \in R \rangle$, and $\gamma(\xi,\kappa)\ \gamma^\xi_\kappa$ such that

(i) $S^\xi_\kappa \subseteq S(\kappa, \gamma^\xi_\kappa)$ for $\kappa \in \kappa_\xi \cap R$) (i.e. $\kappa < \kappa_\xi$, $\kappa \in R$

(ii) $\gamma^\xi_\kappa \neq \gamma^\zeta_\kappa$ for $\zeta < \xi$ (when both are defined).

(iii) if $\vartheta < \sigma < \kappa_\xi$, $\kappa \in R$, $\sigma \in R, \delta \in S^\xi_\sigma$ then $S^\xi_\vartheta \cap \delta$ is not stationary in δ.

(iv) the set $T_\xi = \{\delta : \delta \in \bigcup \{S(\kappa_\xi, i) : i \not\in \{\gamma^\zeta_{\kappa_\xi} : \zeta < \xi\}\}$, and no $S^\xi_\kappa (\kappa \in R \cap \kappa_\xi)$ is stationary in $\delta\}$ is not stationary and so disjoint to some club C^ξ of λ.

There is no problem is the definition: for each ξ we define γ^ξ_κ, S^ξ_κ by indunction on $\kappa \in R$. If it impossible to choose S^ξ_κ then the set defined in (iv) for κ cannot be stationary (as then the lemma's conclusion holds - remember $\kappa, i < Min\ S(\kappa, i)$ and by Fodour Lemma for some γ, $S(\kappa,\gamma) \cap T$ is stationary and we could have choose $S^\xi_{\kappa_\xi} = S(\kappa_\xi,\gamma) \cap T$, $\gamma^\xi_\kappa = \gamma$, but we have assumed this is impossible.

Now as $|R| < \lambda$ for some κ_α, $A = \{\xi < \lambda : \kappa_\xi = \kappa_\alpha\}$ has power λ, and choose $B \subseteq A$, $|B| = \kappa^+_\alpha$ so $|B| < \lambda$. Let $B = \{\xi_\varepsilon : \varepsilon < \kappa^+\}$ and so $\xi^* = \bigcup_\varepsilon \xi_\varepsilon < \lambda$. Hence there is $\gamma < \lambda$ such that $\gamma \notin \{\gamma^\zeta_{\kappa_\alpha} : \zeta < \xi^*\}$, and there is $\delta \in S(\kappa_\alpha,\gamma) \cap \bigcap_{\varepsilon < \kappa_0^+} C^{\xi_\varepsilon}$. Working carefully with the choice of C^{ξ_ε} we see that for each $\varepsilon < \kappa^+_\alpha$, $\delta \cap (\bigcup_{\kappa < \kappa_\alpha} S^{\xi_\varepsilon}_\kappa)$ is stationay in δ. So an ordinal of cofinality κ_α has κ^+_α pairwise disjoint stationary subsets, contradiction.

References.

[ASS]
U. Avraham, S. Shelah and R. M. Solovay, Squares with diamonds and Souslin trees with special squares for successor of singulars, *Fund Math*, submitted.

[MS] M. Magidor and S. Shelah, Compactness in successor of singular, in preparation, see *Abstracts of A.M.S.* Aug (or. Oct.) 1983.

[Sh1]
S. Shelah, Successor of singular cardinals, Proc. of the ASL meeting Mons. 8/1978, *Logic Colloquium 78.* ed M. Boffa, D. van Dalem and K. McAlon, Studies in Logic and the Foundation of Math, Vol. 97, North Holland Pub. Co. 1979, 357-380.

[Sh2]
S. Shelah, Diamonds and uniformization, *J.Symb. Logic, in press.*

[So] R. M. Solovay. Real valued measurable cardinals, *Proc. Symp. in Pur Math.* Vol XIII 1 (ed. D. Scott) Amer. Math. Soc. (1971), 397-428.