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FREE LIMITS OF FORCING AND MORE ON ARONSZAJN TREES†

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ABSTRACT

We prove that the Souslin Hypothesis does not imply "every Aron. (= Aronszajn) tree is special". For this end we introduce variants of the notion "special Aron. tree". We also introduce a limit of forcings bigger than the inverse limit, and prove it preserves properness and related notions not less than inverse limit, and the proof is easier in some respects.

§1. Free limits

1.1. Discussion and Definitions. For A a set of propositional variables, λ a regular cardinal, then: $L_\lambda(A)$ is the set of propositional sentences generated by A, by negation and conjunction and disjunctions on sets of power $<\lambda$. So $L_\mu(A) = \bigcup_{\lambda<\mu} L_\lambda(A)$ for μ limit cardinal ($>\aleph_0$) or ∞. Let φ, ψ, θ denote sentences; Φ, Ψ set of sentences.

We define (in $L_\infty(\lambda)$) $\vdash \psi$, or $\Phi \vdash \psi$ as usual (the rules of the finite case) and $\Phi \vdash \bigwedge \Phi$, from $\Phi \vdash \varphi_i$ for $i \in I$ deduce $\Phi \vdash \bigwedge_{i \in I} \varphi_i$, and let $\bigvee_i \varphi_i = \neg \bigwedge_i \neg \varphi_i$.

Always $\vdash$ means in $L_\infty(A)$ even if we deal with $L_\lambda(A)$.

The following is well known.

1.2. Theorem. *The following are equivalent for* Φ, φ:

(1) $\Phi \vdash \varphi$;

(2) *there is no model of* $\Phi \cup \{\neg \varphi\}$ *with truth values in a complete Boolean algebra*;

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(3) *if λ is such that $|\Phi|$, and the power of any set on which we make conjunction in some $\theta \in \Phi \cup \{\varphi\}$ is $\geqq \lambda$, P the collapsing of λ to ω by finite functions then*

$$\Vdash_P \text{"there is no model of } \Phi \cup \{\neg \varphi\}\text{"}.$$

REMARK. This can be proven by a small fragment of ZFC, I think admissibility axioms, at least when we prove only (1) $\Leftrightarrow$ (3). Hence (by proving not (1) implies not (3)).

1.3. CONCLUSION. If A is a transitive admissible set, $\Phi, \varphi \in A$ then "$\Phi \vdash \varphi$" has the same truth value in V and in A.

1.4. DEFINITION. For given A and $\theta \in L_\infty(A)$, let $\mathrm{FF}_\lambda(\theta)$ be $\{\psi : \psi \in L_\lambda(A)$, $\theta \nvdash \neg \psi\}$ partially ordered by $\psi_1 \leqq \psi_2$ if $\theta \wedge \psi_2 \vdash \psi_1$.

(FF denotes free-forcing; we can identify φ, ψ if $\varphi \leqq \psi \leqq \varphi$.)

Reversing the definition of $\leqq$ and adding a minimal element, we get a Boolean algebra in which every set of $< \lambda$ elements has a least upper bound provided we identify ψ_1, ψ_2 when $\theta \vdash \psi_1 \equiv \psi_2$.

CONVENTION. P, Q denote forcing notions, i.e., partially ordered sets with a minimal element 0, such that if $p \nleqq q$ then there is r, $p \leqq r$ and r incompatible with q.

1.5. DEFINITION. $P \lessdot Q$ for partial orders P, Q if ($P \subseteq Q$ and)

(a) for $p, q \in P, p, q$ are compatible in P iff they are compatible in Q; and $p \leqq q$ in R implies $p \leqq q$ in Q,

(b) every predense subset of P is a predense subset of Q.

REMARK (1). In BA terminology we would say "a complete Boolean subalgebra". Everything is dual.

(2) In fact (a) is not absolutely necessary.

1.6. DEFINITION. For any P let $\theta[P]$ be the following sentence: $\bigwedge \{c \to \neg d \wedge b \to a : a, b \in P,\ a \leqq b,\ c, d \in P,\ c, d \text{ incompatible}\} \wedge \bigwedge \{\bigvee_{a \in I} a : I \subseteq P \text{ a maximal set of pairwise incompatible elements}\}$.

1.6a. DEFINITION. Let P_i $(i < \delta)$ be $\lessdot$-increasing, δ an ordinal (λ an infinite regular cardinal). Then their λ-free limit ($\mathrm{F}\lim^\lambda_{i<\delta} P_i$) is $\mathrm{FF}_\lambda(\bigwedge_{i<\delta} \theta[P_i])$ (where the set of propositional variables is $\bigcup_{i<\delta} P_i$).

1.7. CLAIM. *$P \lessdot Q$ implies $\theta[Q] \vdash \theta[P]$, and $P \lessdot \mathrm{FF}_\lambda(\theta[P])$.*

PROOF. Trivial.

REMARK. Our notation may be confusing, as for conditions $p, q \in P$, $p \wedge q$ is "p and q", i.e., both are in the generic set; i.e., the same notation as in a BA.

1.8. CLAIM. *If as in Definition 1.6a, P_δ is the λ-free limit of P_i $(i < \delta)$ then $P_i \lessdot P_\delta$ for $i < \delta$.*

PROOF. Let us check the conditions.

(b) Let $I \subseteq P_i$ be a maximal set of pairwise incompatible elements of P_i. Suppose $\varphi \in \mathrm{F}\lim^\lambda_{i<\delta} P_i$ is incompatible with each $a \in I$. As $\varphi \in \mathrm{F}\lim^\lambda_{i<\delta} P_i$ by definition $\bigwedge_{i<\delta} \theta[P_i] \nvdash \neg\varphi$. So by 1.2, after some forcing there is a model of φ, $\bigwedge_{j<\delta} \theta[P_j]$. But $\bigvee_{a\in I} a$ is a conjunct of the second sentence, so in the model some $a \in I$ is true. So after some forcing, there is a model of $\varphi \wedge a$, $\bigwedge_{j<\delta} \theta[P_j]$, so by 1.2 $\bigwedge_{j<\delta} \theta[P_j] \nvdash \neg(\varphi \wedge a)$, so $\varphi \wedge a \in \mathrm{FF}_\lambda(\bigwedge_{j<\delta} \theta[P_j])$; so φ, a are compatible.

(a) Let $a, b \in P_i$, if they are compatible in P_i, for some $c \in P_i$, $a \leqq c$, $b \leqq c$, and this clearly holds in P_δ by its definition.

If they are incompatible then $a \to \neg b$ appears as a conjunct in $\theta[P_i]$ and we can finish. Similarly for $a, b \in P_i$, $a \leqq b$ in P_i implies $a \leqq b$ in P_δ.

§2. Preservation by free limit

DEFINITION. (1) If $N < (H(\lambda), \in)$, $|P| < \lambda$, $P \in N$ a forcing notion, $q \in P$, *then* q is (P, N)-generic if for every predense $I \subseteq P$, which belongs to N, $I \cap N$ is predense above p.

(2) P is called proper if for every λ big enough (any $\lambda > 2^{|P|}$ suffices, see [8]) $N < (H(\lambda), \in)$ is countable, $P \in N$, $p \in N \cap P$, *then* there is a (P, N)-generic $q \geqq p$ (in P).

2.1. NOTATION. If $P \lessdot Q$, "Q/P is..." is an abbreviation for: "for any generic set $G \subseteq P$, in $V[G]$, $Q_G = \{q \in Q$: q is compatible with every $p \in G\}$ is...".

2.2. THEOREM. *If each P_i $(i < \delta)$ is proper as well as P_j/P_i $(i < j < \delta)$ then their $\aleph_1$-free limit $P = \mathrm{F}\lim^{\aleph_1}_{i<\delta} P_i$ is proper. Also P/P_i is proper.*

REMARK. Similarly for μ-proper by [10] terminology if we take μ^+-free limit. We can restrict ourselves to non-limit i, j.

PROOF. Let $N \prec (H(\chi), \in)$, $\langle P_i : i < \delta \rangle \in N$, χ big enough (see [10] §3). Let $\{I_n : n < \omega\}$ be a list of all predense subsets of P which belong to N. We have to prove $p \wedge \bigwedge_n (\bigvee_{a \in N \cap I_n} a) \in P$ (in BA terms: is not zero) (p any member of $P \cap N$).

Now assume w.l.o.g. that everything is in some countable transitive model M. In the true world V we can find $\alpha_n < \alpha_{n+1}$, $\delta \cap N = \bigcup_n \alpha_n$.

Let $\langle \Phi_n : n < \omega \rangle$ be a list of all countable (in M) subsets of P which belongs to N.

We now define by induction on n, in V, G_n, p_n such that:

(1) $G_n \subseteq P_{\alpha_n}$, $G_n \subseteq G_{n+1}$,

(2) G_n is P_{α_n}-generic for M and $G_n \cap N$ is $(P_{\alpha_n} \cap N)$-generic for N,

(3) $p_n \leqq p_{n+1}$, $p = p_0$, $p_n \in N \cap P$,

(4) p_n is compatible (in P) with every member of G_n,

(5) p_{2n+1} is $\geqq q_n$ for some $q_n \in I_n$,

(6) either $p_{2n+2} \vdash \bigwedge \Phi_n$ or $p_{2n+2} \vdash \neg r_n$ for some $r_n \in \Phi_n$.

The proof is trivial (provided you know about the composition forcings).

In the end $G = \bigcup_n G_n$ gives us a model of $\bigwedge_{j<\delta} \theta[P_j]$ (by: members of G are true, members of $\bigcup_{j<\delta} P_j - G$ are false).

For $r \in P \cap N$, r is true in the model iff $p_n \geqq r$ for some n (this is proved by induction on the complexity of r, (see conditions (4) and (6)). In the model q_n is true, hence $\bigvee_{a \in I_n \cap N} a$ is true, hence $p \wedge \bigwedge_n (\bigvee_{a \in I_n \cap N} a)$ is true there (p true as $p_0 = p$).

So in V there is a model of $\bigwedge_{j<\delta} \theta[P_j]$, $p \wedge \bigwedge_n (\bigvee_{a \in I_n \cap N} a)$ so $p \wedge \bigwedge_n (\bigvee_{a \in I_n \cap N} a) \in P$ as required.

REMARK. Part of the proof is essentially a repetition of the completeness theorem for $L_{\omega_1, \omega}$ (propositional calculus). But note that in this proof there was no need (as in the ones for inverse limit) to use names. Also, almost all previous theorems on preservation hold.

2.3. DEFINITION. P_i, Q_i or $(\langle P_i, Q_i : i \leqq i_0 \rangle)$ is an ω_1-free iteration *if* (a) P_i is $\lessdot$-increasing, (b) $P_{i+1} = P_i * Q_i = \{\langle p, q \rangle : p \in P_i, \Vdash_{P_i} \text{“}q \in Q_i\text{”}\}$, $\langle p, q \rangle \leqq \langle p', q' \rangle \Leftrightarrow p \leqq p' \wedge p' \Vdash_P q \leqq q'$; and we identify $p \in P_i$ with $\langle p, 0 \rangle$, (c) for limit δ, P_δ is the $\aleph_1$-free limit. [So Q_{i_0} is not defined.]

2.4. DEFINITION. We say that $\aleph_1$-free iteration preserves a property if whenever each Q_i (in V^{P_i}) has it, then so does P_{i_0}.

2.5. THEOREM. *Properness is preserved by $\aleph_1$-free iteration.*

PROOF. See 2.2; and prove by induction on α that for $\beta < \alpha$,

(*) *If* $\langle P_i : i < i_0 \rangle \in N < (H(\lambda), \in)$, $\|N\| = \aleph_0$, $p \in P_\beta$, and for every predense $I \subseteq P_\beta$, $I \in N$, $I \cap N$ is predense above p, *then* for some $q \in P_\alpha$, for every predense $I \subseteq P_\alpha$, $I \in N$, $I \cap N$ is predense above q, and every p', $p \leqq p' \in P_\beta$ is compatible with q (and similarly if we restrict ourselves to forcing conditions $\geqq r$, $r \in N \cap P$).

The following Definition and Theorem are not really necessary for the rest of the paper, but will help in understanding §4.

2.6. DEFINITION. P is strongly proper if for large enough λ (i.e. $\lambda > (2^{|P|})^+$), $P \in N < (H(\lambda), \in)$, $\|N\| = \aleph_0$, $p \in P \cap N$ and $I_n \subseteq N$ predense in $N \cap P$ (but we do not ask $I_n \in N$), *then* for some q, $p \leqq q \in P$, each I_n is predense above q.

2.7. THEOREM. *Strong properness is preserved by $\aleph_1$-free iteration.*

PROOF. Let $\langle P_i, Q_i : i \leqq i_0 \rangle$ be an $\aleph_1$-free iteration. We prove by induction on $\alpha \leqq i_0$ that for any $\beta < \alpha$:

(*) Let $\langle P_i, Q_i : i \leqq i_0 \rangle \in N < (H(\lambda), \in)$, $\lambda > (2^{|P|})^+$, $\|N\| = \aleph_0$, C a family of $\aleph_0$ predense subsets of $P_\alpha \cap N$, closed under the operation listed below. Suppose $\beta < \alpha$, $p \in P_\alpha \cap N$, $\alpha \in N$, $\beta \in N$, $q \in P_\beta$, no q', $q \leqq q' \in P_\beta$ is incompatible with p, and $I \subseteq P_\beta \wedge I \in C \Rightarrow I$ predense above q. *Then* there is q_α, $p \leqq q_\alpha \in P_\alpha$, $q \leqq q_\alpha$, no q', $q \leqq q' \in P_\beta$ is incompatible with q_α and $I \in C \Rightarrow I$ predense above q_α.

The operation under which C is closed is

(Op 1) $\quad \mathrm{Op}_1(I, \gamma, p) = \{r : r \in P_\gamma$, and either for some $r^* \in I$, $r^* \geqq p$ and no r',

$r \leqq r' \in P_\gamma$ is incompatible with r^* *or* r is incompatible with $p\}$

for $\gamma \in N$, $I \in C$, $p \in P_\alpha$. (Note that for $p = 0$ the last phrase is vacuous.)

For $\alpha = 0$. Totally trivial.

For $\alpha = \gamma + 1$. So $\beta \leqq \gamma$, hence $\gamma \in N$ and by the induction hypothesis, for γ (*) holds, so w.l.o.g. $\beta = \gamma$. So we want to use the hypothesis $P_\alpha = P_\gamma * Q_\gamma$, Q_γ is strongly proper; then we use $\langle q, \underset{\sim}{r}\rangle$, $\underset{\sim}{r} \in Q_\gamma$ a name of an appropriate element of Q_γ. But as C is closed under (Op 1) this is easy.

For α limit. Let $\alpha_n \in N$, $\bigcup \alpha_n$ is α or at least $[\bigcup_n \alpha_n, \alpha) \cap N = \emptyset$ ($[\alpha', \alpha)$ interval of ordinals).

We work as in 2.2 using the induction hypothesis.

2.8. Claim. *If we iterate ω-proper, ω^ω-bounding forcings it does not matter whether we use $\aleph_1$-free iteration or countable support one (in the latter we get a dense subset of the first).*

See [8, July]. The parallel of 2.7 for countable support was noted by Harrington and the author.

By the way we note that unlike $\aleph_1$-c.c. forcing

2.9. Example. There are proper forcings P, Q such that $P \lessdot Q$ but Q/P is not proper.

Proof. We let P_0 = adding a subset $\underset{\sim}{r}$ of ω_1 with a condition being a countable characteristic function.

Let $Q_0 \in V^{P_0}$, $Q_0 = \{f : \mathrm{Dom} f = \alpha < \omega_1$, $\mathrm{Range} f = \{0, 1\}$, $f^{-1}(\{0\})$ is a closed set of ordinals included in $r\}$. (r denotes the generic subset of ω_1 which P_0 produces.)

Now P_0, $P_0 * Q_0$ are proper but in V^{P_0}, $Q_0 \cong P_0 * Q_0/P_0$ is not proper. See [8, Sept., §5].

§3. Aron. trees: various ways to specialize

We introduce variants of the notion "special Aron. tree" and prove some known theorems and some easy ones. See Baumgartner, Malitz and Reinhart [3] Baumgartner [1] and also Devlin and Shelah [4].

3.1. Definition. (1) An ω_1-tree $T = (|T|, <_T)$ is a partially ordered set, such that (when no confusion arise, we write $<$ instead of $<_T$ and T instead of $|T|$):

(a) for every $x \in T$, $\{y \in T : y < x\}$ is well-ordered, and its order type which is denoted by $rk(x) = rk_T(x)$, is countable,

(b) $T_\alpha = \{x \in T : rk(x) = \alpha\}$ is countable, $\neq \emptyset$,

(c) if $rk(x) = rk(y)$ is a limit ordinal then $x = y \Leftrightarrow \{z : z < x\} = \{z : z < y\}$,

(d) if $x \in T_\alpha$, $\alpha < \beta$, then for some $y \in T_\beta$, $x < y$, in fact there are at least two distinct such y's.

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If we wave (c) and (d) we call it an almost ω_1-tree; similarly for the other definitions.

(2) A set $B \subseteq T$ is a *branch* if it is totally ordered (hence well ordered); it is an α-branch if it has order type α.

(3) An Aron. tree is an ω_1-tree with no ω_1-branch.

(4) An ω_1-tree is Souslin (or ω_1-Souslin tree) if there is no uncountable antichain (= set of pairwise incomparable elements).

REMARK. Condition 1(d) is not essential, except to make every Souslin tree an Aron. tree.

3.2. DEFINITION. (1) For a set $S \subseteq \omega_1$ which is unbounded, we call an ω_1-tree S-*special* if there is a monotonic increasing function f from $\bigcup_{\alpha \in S} T_\alpha$ to Q (the rationals), i.e., $x < y \Rightarrow f(x) < f(y)$.

(2) A special ω_1-tree is an ω_1-special ω_1-tree (this is the classical notion).

(3) r-special, S-r-special are defined similarly when the function is to $\mathbf{R}$ (the reals).

(4) We say f specializes (S-specialize, etc.) T. We can replace S by a function h, $\operatorname{Dom} h = \omega_1$, $\operatorname{Range} h = S$, h increasing.

3.3. DEFINITION. For a stationary set $S \subseteq \omega_1$, we call an ω_1-tree S-*st*-special if there is a function f, $\operatorname{Dom} f = \bigcup_{\alpha \in S-\{0\}} T_\alpha$, and $x \in T_\alpha \Rightarrow f(x) \in \alpha \times \omega$ (cartesian product) such that $x < y \Rightarrow f(x) \neq f(y)$ when defined. If S is a set of limit ordinals we can assume $x \in T_\alpha \Rightarrow f(x) \in \alpha$.

3.4. CLAIM. (1) *If T is S-special or S-r-special ($S \subseteq \omega_1$ unbounded) or S-st-special ($S \subseteq \omega_1$ stationary) ω_1-tree* then *T is an Aron. tree but not Souslin. Any ω_1-Souslin tree is an Aron. tree.*

(2) *The following implications among properties of ω_1-tree holds (where $S_2 \subseteq S_1 \subseteq \omega_1$, $\alpha(i) \in S_1$ increasing, $S_1 = \{\alpha(i) : i < \omega_1\}$):*

(a) $$\begin{array}{ccc} S_1\text{-special} & \Rightarrow & S_2\text{-special} \\ \Downarrow & & \Downarrow \\ S_1\text{-}r\text{-special} & \Rightarrow & S_2\text{-}r\text{-special} \\ \Downarrow & & \\ S_1 \cap \{\alpha(i+1) : i < \omega_1\}\text{-special,} & & \end{array}$$

(b) *for S_1 stationary S_1-special $\Rightarrow$ S_1-st-special,*

(c) *S_1-st-special $\Rightarrow$ S_2-st-special,*

(d) *for $C \subseteq \omega_1$ closed unbounded $S_1 \cap C$-st-special $\Rightarrow$ S_1-st-special,*

(e) *if $(\forall i)$ $h_1(i) \leqq h_2(i)$, T is h_1-special then T is h_2-special.*

PROOF. Trivial: (1) for S-special S-r-special — well known for S-st-special by the Fodour theorem.

(2) Trivial — check.

REMARK. By 2(d) dealing with S-st-special we can assume all members of S are limit, and so Range $f \subseteq \omega_1$ in the Definition.

3.5. CLAIM. (1) *T is S-special* iff *$S \subseteq \omega_1$ is unbounded and there is* $f : \bigcup_{\alpha \in S} T_\alpha \to \omega$, $x < y \wedge rk(x) \in S \wedge rk(y) \in S \Rightarrow f(x) \neq f(y)$.

(2) *T is ω_1-st-special* iff *T is special.*

REMARK. See Claim 3.11.

PROOF. (1) Well known.

(2) The "if" part is trivial.

So suppose f ω_1-st-specialize T. For every $x \in T$, let $K_x = \{t \in (rk(x)+1) \times \omega :$ for no $y \leqq x$ is $f(y) = t\}$. We now define by induction on $\alpha < \omega_1$, g_α and $A_{x,t}$ ($t \in K_x$, $x \in \bigcup_{\beta < \alpha} T_\beta$) such that

(a) g_α is a function from $T_{<\alpha} = \bigcup_{\beta<\alpha} T_\beta$ to ω,

(b) $x < y$, $x \in T_{<\alpha}$, $y \in T_{<\alpha} \Rightarrow g_\alpha(x) \neq g_\alpha(y)$,

(c) $\beta < \alpha \Rightarrow g_\beta \subseteq g_\alpha$,

(d) $A_{x,t}$ (for $t \in K_x$, $x \in T_{<\alpha}$) is an infinite subset of ω,

(e) for every $x \in T_{<\alpha}$, $t \neq s \in K_x \Rightarrow A_{x,t} \cap A_{x,s} = \emptyset$,

(f) $t \in K_x$, $x \in T_{<\alpha} \Rightarrow A_{x,t} \cap \{g_\alpha(y) : y \leqq x\} = \emptyset$,

(g) if $x < y \wedge x \in T_{<\alpha} \wedge y \in T_{<\alpha}$, $t \in K_x \cap K_y$ then $A_{x,t} = A_{y,t}$.

For $\alpha = 0$, *α limit* — no problem.

For $\alpha + 1$ — let $x \in T_\alpha \subseteq T_{<(\alpha+1)}$, $s = f(x)$, so by K's definition for some $y < x$, $s \in K_y$. We choose $g_{\alpha+1}(x) \in A_{y,s}$ ($= A_{z,t}$ for every $y \leqq z < x$) and $A_{x,t} = A_{z,t}$ if $z < x \wedge t \in K_z \cap K_x$ and $g_{\alpha+1} \restriction T_{<\alpha} = g_\alpha$.

If $t \in K_x - \bigcup_{z<x} K_z$ (there are $\aleph_0$ such t's) we choose $A_{x,t} \subseteq A_{y,s}$ infinite pairwise disjoint and $g_{\alpha+1}(x) \notin A_{x,t}$.

Now by 3.4(1) $g = \bigcup_{\alpha<\omega_1} g_\alpha$ shows T is special.

3.6. CLAIM. (1) *Let $S \subseteq \omega_1$ be unbounded. If every Aron. tree is S-special* then *every Aron. tree is special.*

(2) *if every Aron. tree is S-r-special then every Aron. tree is special.*

PROOF. (1) Let T be an Aron. tree, $S = \{\alpha(i) : i < \omega_1\}$, $\alpha(i)$ increasing.

Define T^* (a partial order): The set of elements is $\{\langle x, \gamma\rangle : x \in T,\ \gamma \leqq \alpha(rk_T(x))$ and $y < x \Rightarrow \alpha(rk_T(y)) < \gamma\}$; the order in T^* is: $\langle x, \gamma\rangle <_{T^*} \langle x', \gamma'\rangle$ *if* $x < x'$ or $x = x',\ \gamma < \gamma'$.

Now T^* is almost an Aron. tree; the only missing part is in Definition 3.1, part (d) (in fact there are at least two distinct such y's). We can add more elements so that it becomes an Aron. tree, T^{**} and if g^{**} S-specializes it, $g : T \to Q$, $g(x) = g^{**}(\langle x, \alpha(rk_T(x))\rangle)$ specializes T.

(2) By 3.5(1) and 3.4(2)(a).

3.7. Lemma. (1) ($\diamondsuit_{\omega_1}$) *There is an r-special Aron. tree which is not special.*

(2) *Moreover there is no antichain* I, s.t. $rk(I) = \{rk(x) : x \in I\}$ *contains a closed unbounded subset of* ω_1.

(3) ($\diamondsuit^*_{\omega_1}$) *There is an r-special Aron. tree, such that for no antichain* $I \subseteq T$ *is* $rk(I) = \{rk(x) : x \in I\}$ *stationary.*

Remark. (1) was proved by Baumgartner [2].

Proof. We define by induction on $\alpha < \omega_1$ the tree $(T_{<\alpha}, <_T \restriction T_{<\alpha})$ and $f : T_{<\alpha} \to R$, $x < y \Rightarrow f(x) < f(y)$ such that if $\beta < \gamma < \alpha$, $x \in T_\beta$, ε a real positive number (>0), then for some y, $x < y \in T_\gamma$, $f(y) < f(x) + \varepsilon$; and $x \in T_{\alpha+1} \Leftrightarrow f(x) \in Q$.

For $\alpha = 0$, α-successor of successor or α limit, no problem.

For $\alpha + 1$, α limit, we are given antichains $I^\alpha_n \subseteq T_{<\alpha}$ $(n < \omega)$ (by $\diamondsuit_{\aleph_1}$ or $\diamondsuit^*_{\aleph_1}$) and we can define $T_{<\alpha+1}$ such that

(*) if $x \in T_\alpha$, $n < \omega$ and $\{y \in T_{<\alpha} : y < x\} \cap I^\alpha_n = \emptyset$ *then* for some $y < x$, and $\varepsilon > 0$, $f(y) < f(x) < f(y) + \varepsilon$, and there is no z, $z \in I^\alpha_n$, $y < z \in T_{<\alpha}$, $f(y) < f(z) < f(y) + \varepsilon$.

Now suppose $I \subseteq T$ is an antichain ($T = \bigcup_{\alpha<\omega_1} T_{<\alpha}$ defined in the end). Let $C = \{\alpha < \omega_1$: α limit, and if $x \in T_{<\alpha}$, $\varepsilon > 0$, and there is $y \in I$, $x < y$, $f(x) < f(y) < f(x) + \varepsilon$ then there is such $y \in T_{<\alpha}\}$ be closed unbounded (note it suffices to consider $\varepsilon \in \{1/n$: n positive natural number$\}$).

Now if $\alpha \in C$, $I \cap T_{<\alpha} = I^\alpha_{n(0)} \in \{I^\alpha_n : n < \omega\}$, $\alpha \in rk(I)$ then by (*) we get $I \cap T_\alpha = \emptyset$ (if $y \in I$, $y \in T_\alpha$, by (*) $\{z : z < y\} \cap I^\alpha_{n(0)} \neq \emptyset$; let z be in it, then $z < y$ both in I, but I is an antichain).

Now by defining I^α_n using $\diamondsuit_{\omega_1}$ or $\diamondsuit^*_{\omega_1}$ we get (1), (2) and (3).

3.8. Conclusion. Let h be a function from ω_1 to ω_1. ($\diamondsuit^*_{\aleph_1}$) There is a tree T which is h_1-special iff $\{i : h(i) < h_1(i)\}$ contains a closed unbounded subset of ω_1 (see Definition 3.2(4)).

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3.9. LEMMA. *Let $S \subseteq \omega_1$ be stationary, $(\diamondsuit^*_{\omega_1 - S})$. There is an S-st-special tree which is S_1-st-special iff $S_1 - S$ is not stationary; moreover there is no antichain I, $rk(I) - S$ stationary. (If $S = \omega_1$ we do not need any hypothesis, $\diamondsuit^*_\varnothing$ is meaningless anyhow and this is the classical theorem on the existence of special Aron. trees of Aronszajn himself.) Also we can make the tree such that it is not h-special for any h.*

PROOF. (1) We define by induction on $\alpha < \omega_1$, $(T_{<\alpha}, <_T \restriction T_{<\alpha})$, $f: T_{<\alpha} \to \alpha \times \omega_1$; $x \in T_{<\alpha} - T_0$, $rk(x) \in S \Rightarrow f(x) \in rk(x) \times \omega$; $x \in T_0 \Rightarrow f(x) \in \{0\} \times \omega = 1 \times \omega$, $rk(x) \in \omega_1 - S \Rightarrow f(x) \in [rk(x)+1] \times \omega$ and $x < y \Rightarrow f(x) \neq f(y)$, such that

(a) $x \in T_\beta \Rightarrow |\beta \times \omega - \{f(y): y < x\}| = \aleph_0$,

(b) if $x \in T_\beta$, $\beta < \gamma < \alpha$, $\{\langle \xi, n\rangle\} \cup A \subseteq ((\beta+1) \times \omega - \{f(z): z < x\})$, A finite, then there is $y \in T_\gamma$, $x < y$, $\{f(z): z < y\} \cap A = \varnothing$ but $\langle \xi, n \rangle \in \{f(z): z < y\}$.

We can demand

(c) if α is limit, $\alpha \notin S$ then $|\{\alpha \times \omega - \{f(z): z < x\}| < \aleph_0$.

There is no special problem.

3.10. LEMMA. $(\diamondsuit_{\omega_1})$ *There is a special Aron. tree T, such that for no antichain $I \subseteq T$ is $rk(I)$ closed unbounded.* (For stationary: there is necessarily: this is mentioned in [4] p. 25.)

REMARK. E.g., $MA + 2^{\aleph_0} > \aleph_1$ implies that this fails.

PROOF. We define by induction on α, $(T_{<\alpha}, <_T \restriction T_{<\alpha})$ $f: T_{<\alpha} \to Q$ monotonic, so that $\beta < \gamma < \alpha$, $x \in T_\beta$, $\varepsilon > 0$ implies for some $y \in T_\gamma$, $x < y \wedge f(x) < f(y) < f(x) + \varepsilon$. For limit $\delta < \omega_1$ we are given an antichain $I^\alpha \subseteq T_{<\alpha}$ (by $\diamondsuit_{\aleph_1}$) and for $x \in T_\alpha$,

either

$$(\exists y \in I^\alpha) y < x$$

or

$$(\exists y < x)\,[\text{there is no } z, y < z \in I^\alpha, f(z) < f(x) \in Q].$$

The checking is easy.

3.11. LEMMA. *T is $\{\alpha + 1: \alpha < \omega_1\}$-special* iff *$T$ is r-special.*

REMARK. Proved by Baumgartner [2].

PROOF. The direction $\Leftarrow$ already appears.

For $\Rightarrow$ let f $\{\alpha+1:\alpha<\omega_1\}$-specialize T.

Let $g:Q\to Q$, $\varepsilon:Q\to\{1/n:n>0 \text{ natural}\}$ be such that the intervals $[g(q)-\varepsilon(q), g(q)+\varepsilon(q)]$ are pairwise disjoint (possible: Let $Q=\{q_n:n<\omega\}$ and define by induction).

Now define $f^*: x\in T_{\alpha+1}\Rightarrow f^*(x)=g(f(x))$

$$x\in T\alpha,\quad \alpha \text{ limit} \Rightarrow f^*(x)=\sup\{g(f(y)):y<x,\, y\in T_{\beta+1},\, \beta<\alpha\}.$$

Now f^* r-specializes T; the only point to check is:

$$x\in T_{\alpha+1},\quad \alpha \text{ limit} \Rightarrow g(f(x))>\sup\{g(f(y)):y<x,\, y\in T_{\beta+1},\, \beta+1<\alpha\}$$

which follows by g's definition (the sup is $\leqq g(f(x))-\varepsilon(f(x))$ as for every $y<x$, $g(f(x))$ is smaller than it).

§4. Independence results

It is well known that

4.1. CLAIM. *If T is an $\aleph_1$-Souslin tree, $\lambda>\aleph_1$, $N<(H(\lambda),\in)$, $T\in N$, $x\in T_\delta$, $\delta=\omega_1\cap N$ then $B_T(x)=\{y\in T_{<\delta}:y<x\}$ is generic for (T,N), i.e., for every $I\in N$, $I\subseteq T$ which is predense*

$$I\cap B_T(x)=I\cap N\cap B_T(x)\neq\emptyset.$$

4.2. DEFINITION. For an Aron. tree T,

$$\begin{aligned} Q(T)=\ \{(h,f):\ & h \text{ is a finite function from } \omega_1 \text{ to } \omega_1;\\ & \alpha<\beta\in\operatorname{Dom} h\Rightarrow \alpha\leqq h(\alpha)<\beta\leqq h(\beta);\\ & f \text{ is a finite function,}\\ & \operatorname{Dom} f\subseteq \bigcup_{\alpha\in\operatorname{Dom} h} T_{h(\alpha)};\ x\in T_{h(\alpha)}\Rightarrow f(x)\in\alpha\times\omega;\\ & x<y\wedge x,y\in\operatorname{Dom} f\Rightarrow f(x)\neq f(y)\},\end{aligned}$$

$$(h,f)\leqq(h',f')\qquad \textit{iff}\quad h\subseteq h',\quad f\subseteq f';$$

we let $(h,f)\cup(h',f')=(h\cup h',f\cup f')$, $(h,f)\cup h'=(h\cup h',f)$,
$(h,f)\cup f'=(h,f\cup f')$.

4.3. DEFINITION. For an Aron. tree T and stationary set S,

$Q(T,S)=\{(h,f)\colon (h,f)\in Q(T)$, and $\alpha\in(\operatorname{Dom} h)\cap(S-\{0\})$ implies $h(\alpha)=\alpha\}$,

order — as before.

EXPLANATION. Our aim is to get a universe in which SH (Souslin Hypothesis) holds (i.e., there is no Souslin tree) but not every Aron. tree is special. The question was raised by U. Avraham, and is natural as, until now, the consistency of SH was proved by making every Aron. tree special; see the proof of Solovay and Tennenbaum [12], Martin and Solovay [7], Baumgartner, Malitz and Reinhart [3] without CH, and Jensen proofs in Devlin and Johnsbraten [5] with CH. For this aim we introduce in §3 various notions of specializations (each implying the tree is not Souslin). So the program is to make every tree special in some weaker than usual sense. The notion r-special which had been introduced by Kurepe [6] is not suitable, as if every Aron. tree is r-special then every Aron. tree is $\{\alpha+1\colon\alpha<\omega_1\}$-special (see 3.4(a)), hence every Aron. tree is special (see 3.6). Similarly "h-special" for any increasing $h\colon\omega_1\to\omega$ is not suitable by 3.6 (see Definition 3.2(4)).

So a natural candidate is "h-special for some h" (i.e., for every tree there is an h for which it is h-special). Forcing by $Q(T)$ does the job for T — we take generic h and f. (It would be more natural to let f go to Q and be monotonically increasing, but by 3.5(2) the forcing $Q(T)$ makes T h-special for some h, and this way we have more uniformity with Definition 4.3.) So we should iterate such forcings, but retain some T as not special.

A second way is to make each T S-st-special for some fixed stationary S; for this $Q(T,S)$ is tailored. (Note that the f we get from a generic set of $Q(T,S)$ has domain $\bigcup_{\alpha\in S_1}T_\alpha$ where $S_1\subseteq S$, $S-S_1$ non-stationary.) For $S=\varnothing$ we get the previous case, so we shall ignore $Q(T)$.

This leads to a secondary problem: Can every Aron. tree be S_1-st-special, but some Aron. trees are not S_2-st-special (S_2-S_1 stationary of course)? We answer positively.

4.4. CLAIM. *For T an Aron. tree, $S\subseteq\omega_1$, $Q(T,S)$ is proper.*

PROOF. We can assume w.l.o.g. $|T_0|=\aleph_0$.

Let $\lambda>(2^{\aleph_1})$, $N\prec(H(\lambda),\in)$, $T,S,\in N$, $p_0=(h,f)\in Q(T,S)\cap N$, and let $\delta=N\cap\omega_1$.

Then $p_1=(h\cup\{\langle\delta,\delta\rangle\},f)\in Q(T,S)$ exemplifies what is required.

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For checking, we really repeat the proof that the standard forcing for specializing an Aron. tree satisfies the $\aleph_1$-c.c.

4.5. DEFINITION. We call a forcing P, (T^*, S)-preserving (do you have a better name?), where T^* is Aron. tree, *iff* for every $\lambda > (2^{|P|+\aleph_1})^+$, $\langle P, T^*, S\rangle \in N < (H(\lambda), \in)$, N countable, $\delta =^{df} N \cap \omega_1 \notin S$, $p \in N \cap P$, *there is* p_1 which is preserving for (p, P, T^*, S); i.e.,

(i) $p \leqq p_1 \in P$ and $p \leqq p' \in P \cap N \Rightarrow p_1, p'$ are compatible,

(ii) p_1 is (P, N)-generic ($\equiv$ generic for (P, N)), i.e., for every predense $I \subseteq P$, $I \in N$, $I \cap N$ is predense above p_1,

(iii) for every $x \in T^*_\delta$, *if*

(*) $\quad x \in A \rightarrow (\exists y < x)(y \in A)$ hold for every $A \subseteq T^*$, $A \in N$,

then

(**) $\quad$ for every P-name $\underset{\sim}{A}$, $\underset{\sim}{A} \in N \Vdash_P \underset{\sim}{A} \subseteq T^*$ the following holds:

$$p_1 \Vdash \text{“} x \in \underset{\sim}{A} \rightarrow (\exists y < x) y \in \underset{\sim}{A} \text{”}.$$

4.6. LEMMA. *If T^*, T are Aron. trees, $S \subseteq \omega_1$, then $Q(T, S)$ is (T^*, S)-preserving.*

REMARK. If T^* is Souslin tree then (*) from Definition 4.5 is satisfied by every $x \in T^*_\delta$ (this follows by 4.1).

PROOF. Let $N < (H(\lambda), \in)$, $\delta =^{df} N \cap \omega_1 \notin S$, $\|N\| = \aleph_0$, $\langle T^*, T, S\rangle \in N$, $p = (h_0, f_0) \in P \cap N$ (as in Definition 4.5), and (remembering $\delta = N \cap \omega_1$) let

$$\delta^* = \sup\{f(\delta) + 1 : f \in N, f(\delta) \text{ an ordinal } < \omega_1\}.$$

Define $p_1 = (h_0 \cup \{\langle \delta, \delta^*\rangle\}, f_0)$ and suppose $x \in T^*_\delta$ and

(*) if $A \subseteq T^*$, $A \in N$, $x \in A$ then $(\exists y)(y < x \wedge y \in A)$.

Let $\underset{\sim}{A}$ be a $Q(T, S)$-name of a subset of T^*, $\underset{\sim}{A} \in N$.

We shall prove that for every p_2, $p_1 \leqq p_2 \in Q(T, S)$, for some p_3, $p_2 \leqq p_3 \in Q(T, S)$, and $p_3 \Vdash x \notin \underset{\sim}{A}$ or $p_2 \Vdash y \in \underset{\sim}{A}$ for some $y <_{T^*} x$.

Let $p_2 = (h_2, f_2)$, if $p_2 \Vdash_p$ “$x \notin \underset{\sim}{A}$”, then we can choose $p_3 = p_2$. Otherwise there is $p_2' \in P$, such that

$$p_2 \leqq p_2' \quad \text{and} \quad p_2' \Vdash_p \text{“} x \in \underset{\sim}{A} \text{”}.$$

Let $p_2' = (h_2', f_2')$, $p_2 = p_2^a \cup p_2^b$, $p_2^a = (h_2^a, f_2^a)$, $p_2^b = (h_2^b, f_2^b)$ where $h_2^a = h_2' \restriction \delta$, $h_2^b = h_2' \restriction [\delta, \omega_1)$ (closed open interval) and

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$$f_2^a = f_2' \restriction T_{<\delta}, \qquad f_2^b = f_2' \restriction (T - T_{<\delta}).$$

Note that by the definition of $Q(T,S)$:

FACT. (1) $p_2^a \in P \cap N$,
(2) $z \in \mathrm{Dom}\, f_2^b \Rightarrow rk(z) \geqq \delta^*$.

Now we define (in $H(\lambda), \in$) $\alpha_0 = \mathrm{Max\,Range}\, h_2^a$ (which is $< \delta$) and we define a function F

$$\mathrm{Dom}\, F = \{y \in T^* : rk(y) > \alpha_0\},$$

$F(y) = \mathrm{Sup}\{\alpha^* < \omega_1$: there is $({}^*h_2^b, {}^*f_2^b)$ (in $Q(T,S)$) such that:

(1) $\mathrm{Min}(\mathrm{Dom}\, {}^*h_2^b) = rk(y)$,

(2) ${}^*h_2^b(rk(y)) = \alpha^*$,

(3) $(h_2^a \cup {}^*h_2^b, f_2^a \cup {}^*f_2^b) \Vdash_{Q(T,S)}$ "$y \in \underset{\sim}{A}$"$\}$

(so we demand also that $(h_2^a \cup {}^*h_2^b, f_2^a \cup {}^*f_2^b)$ is in $Q(T,S)$).

Now clearly $F \in N$ (as it is defined by a (first-order) formula in $(H(\lambda), \in)$ whose parameters are in N). Clearly $F(y) \leqq \omega_1$ (for $y \in T^* - T^*_{\leqq\alpha_0}$). Let $A^* = \{y \in T^*:\ rk(y) > \alpha_0,\ F(y) = \omega_1\}$. (Note that $A^* \subseteq T^*$ is a set, not a P-name of a set.)

Let F^* be a function from ω_1 to ω_1 defined by:

$$F^*(\alpha) = \mathrm{Sup}\{F(y)+1 : y \in T^*_{\leqq\alpha},\ rk(y) > \alpha_0,\ y \notin A^*, \text{ i.e., } F(y) < \omega_1\}.$$

As $|T^*_{\leqq\alpha}| \leqq \aleph_0$, $F^* : \omega_1 \to \omega_1$, and clearly $F^* \in N$ (same reason).

By the definition of δ^*, $F^*(\delta) < \delta^*$. But (h_2^b, f_2^b) exemplify $F(x) \geqq \delta^*$, so necessarily $F(x) = \omega_1$. So by definition $x \in A^*$. Hence by the hypothesis $(*)$ there is $y <_{T^*} x$, $y \in A^*$. So (in $H(\lambda)$, hence in N) we can define a sequence $\bar{p} = \langle (h_2^{b,i}, f_2^{b,i}) : i < \omega_1 \rangle$ such that

(1)′ $\mathrm{Min}(\mathrm{Dom}\, h_2^{b,i}) = rk(y)$,

(2)′ $h_2^{b,i}(rk(y)) \geqq \alpha_0 + i$,

(3)′ $(h_2^a \cup h_2^{b,i}, f_2^a \cup f_2^{b,i}) \Vdash_{Q(T,S)}$ "$y \in \underset{\sim}{A}$".

For $i < \delta$ let $p_3^i = (h_2^a \cup h_2 \cup h_2^{b,i}, f_2^a \cup f_2 \cup f_2^{b,i})$. If $p_3^i \in Q(T,S)$ then by (3)′ it is as required.

Why can p_3^i be not in $Q(T,S)$? The first coordinate $(h_2^a \cup h_2 \cup h_2^{b,i})$ is O.K., as $h_2^{b,i} \in N$.

What about the second? Note $f_2^a \cup f_2$, $f_2^a \cup f_2^{b,i}$ are O.K. as $p_2 \in Q(T,S)$ and by (3)′ above correspondingly. Hence the only danger is that there are $z_1 \in \mathrm{Dom}\, f_2^b$,

$z_2 \in \mathrm{Dom}\, f_2^{b,i}$, $z_2 <_T z_1$ (as $f_2^{b,i} \in N$, $rk\, z_1 \geqq h_2^b(\delta) = \delta^*$ this is the only bad possibility).

But remember (in $H(\lambda)$) $z \in \mathrm{Dom}\, f_2^{b,i} \Rightarrow rk\, z \geqq i$, so by a lemma on Aron. trees due to Baumgartner, Malitz and Reinhart (in their proof of MA $\vdash$ "every Aron. tree is special") there is a sequence $\langle i_n : n < \omega \rangle$ ($i_n < \omega_1$) such that

$$m \neq n \wedge z_1 \in \mathrm{Dom}\, f_2^{b,i_m} \wedge z_2 \in \mathrm{Dom}\, f_2^{b,i_n} \Rightarrow \begin{pmatrix} z_1 \not< z_2 \\ z_2 \not< z_1 \end{pmatrix}.$$

So again there is such a sequence in N, and all but at most $|\mathrm{Dom}\, f_2^b|$ are O.K., i.e., $p_3^i \in Q(T, S)$. So we finish.

4.7. Theorem. *Let T^* be a Souslin tree. Suppose P_α ($\alpha \leqq \alpha_0$), Q_α ($\alpha < \alpha_\delta$) form an $\aleph_1$-free iteration (i.e., $P_{\alpha+1} = P_\alpha * Q_\alpha$, $P_\delta = \mathrm{F}\lim_{\alpha<\delta}^{\aleph_1} P_\alpha$) and for every α at least one of the following holds:*

(α) Q_α is (in V^{P_α}) (T^*, S)-preserving,

(β) there is an antichain $I_\alpha \subseteq T^*$ (in V^{P_α}), $S_\alpha = rk(I_\alpha) = \{rk(x) : x \in I\} \subseteq \omega_1 - S$, and

$$Q_\alpha = Q_{\mathrm{club}}(\omega - S_\alpha) = \{g : \text{for some } i < \omega_1,\ \mathrm{Dom}\, g = i+1,\ \mathrm{Range}(g) = \{0,1\},\ \{j \leqq i : g(i) = 1\} \text{ is closed and is } \subseteq \omega_1 - S_\alpha\}.$$

Then P_{α_0} is (T^*, S)-preserving.

Remark. We can amalgamate conditions (α) and (β) but it has no use.

Proof. We prove by induction on α the following:

$(+)_\alpha$ *Suppose $\beta < \alpha$, $N < (H(\lambda), \in)$, $\beta \in N$, $\alpha \in N$, $\langle P_i : i \leqq \alpha \rangle \in N$, $\delta = N \cap \omega_1 \notin S$, $p \in P_\alpha \cap N$, $q_1 \in P_\beta$, $\langle I_i : i < \alpha$, I_i* defined$\rangle \in N$ and

(i) $p \restriction \beta \leqq q_1$ (meaning no q', $q_1 \leqq q' \in P_\beta$ is incompatible with p; if we deal with complete BA, $p \restriction \beta$ is the projection). Moreover if $p \restriction \beta \leqq p' \in P_\beta \cap N$, then q_1, p' are compatible;

(ii) q_1 is (P_β, N)-generic (see in Definition 2.1);

(iii) *if* $x \in T_\delta^*$ and $(\forall A \subseteq T^*)$ $(A \in N \wedge x \in A \rightarrow (\exists y < x) y \in A)$ *then* for every P_β-name $\underset{\sim}{A} \in N$, $q_1 \Vdash_{P_\beta}$ "$x \in \underset{\sim}{A} \rightarrow (\exists y <_{T^*} x) y \in \underset{\sim}{A}$".

Then there is $p_1 \in P_\alpha$ such that

(i)$'$ $p_1 \restriction \beta = q_1$ (natural meaning) and $p \leqq p' \in P_\alpha \cap N$, $q_1 \leqq q' \in P_\beta$; $q', p' \restriction \beta$ compatible *implies* p', p_1, q' are compatible ($=$ has an upper bound),

(ii)$'$ p_1 is (P_α, N)-generic,

(iii)$'$ the parallel of (iii) with $\beta \mapsto \alpha$, $q_1 \mapsto p_1$.

$\alpha = 0$. Trivial.

$\alpha + 1$. By the similarity between the assumptions on q_1 and the conclusion on p_1, we can assume w.l.o.g. $\beta = \alpha$. Let $G \subseteq P_\alpha$ be generic $q_1 \in G$. Then $N[G] < (H(\lambda)[G], \in)$ (see e.g. [8]).

Now in $V[G]$ (hence in $H(\lambda)[G]$) we can find $p_1' \geqq p(\alpha)$, $(N[G], Q_\alpha)$-generic, as in Definition 4.5. Why? We have two cases (α) and (β) from the theorem:

(α) Straightforward, by 4.6.

(β) By the choice of T^* (Souslin) by Claim 4.1, $x \in A \in N$ and $x \in T^*_\delta \Rightarrow\!|$ $(\exists y < x) y \in A$. So by the assumption on q_1 for every $A \in V[G]$, $A \subseteq T^*$, $A \in N[G]$, of course there is a P_α-name $\underset{\sim}{A}$ so $q_1 \Vdash_{P_\alpha}$ "$x \in \underset{\sim}{A} \rightarrow (\exists y < x) y \in \underset{\sim}{A}$", hence in $V[G]$, $x \in A \cap T^*_\delta \Rightarrow (\exists y < x) y \in A$.

In particular we can take $A = I_\alpha \in N[G]$ (remember $I_\alpha \in V^{P_\alpha} \cdots$). So clearly if $x \in T^*_\delta \cap I_\alpha$ then $x \in A$ implies I_α is not an antichain, contradiction. *So* $T^*_\delta \cap I_\alpha = \emptyset$, so $\delta \notin S_\alpha = rk(I_\alpha)$, and then the desired conclusion is immediate (remember Q_α's definition).

So we have p_1' as required. p_1' is in V^{P_α}, so in V we have a P_α-name p_1' for it, and let $p_1 = (q_1, p_1') \in P_\alpha * Q_\alpha$ which by the usual thing for composition of forcing, is as required.

α *limit*. Let $\alpha \cap N = \bigcup_n \alpha_n$, $\beta = \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$; $\alpha_n \in N$.

We define by induction on $n < \omega$, $n \geqq 1$, $q_n \in P_{\alpha_n}$, $q_{n+1} \restriction \alpha_n = q_n$, each q_n satisfies the hypothesis of the theorem with α_n replacing β.

For $\underset{\sim}{A}$ a P_α-name of a subset of T^*, $\underset{\sim}{A} \in N$, $r \in P_\alpha \cap N$, let

$$\begin{aligned}\underset{\sim}{A}[\alpha_n, r] = \{y \in T^* : \ & \text{if } r \text{ is compatible with every element of } \underset{\sim}{G}_{\alpha_n} \\ & (= \text{the generic set of } P_{\alpha_n}) \text{ } \textit{then} \text{ for some } r',\ r < r' \in P_\alpha, \\ & r' \text{ compatible with every element of } \underset{\sim}{G}_{\alpha_n} \text{ and} \\ & r' \Vdash_{P_\alpha} \text{"} y \in \underset{\sim}{A} \text{"}\}\end{aligned}$$

(we could have used P_α / P_{α_n}).

Let $\langle \underset{\sim}{A}_n, x_n \rangle : n < \omega \rangle$ be a list of all pairs $(\underset{\sim}{A}, x)$, $\underset{\sim}{A}$ a P_α-name of a subset of T^*, $x \in T_\delta$ and $\underset{\sim}{A}$, x are in N; $\langle I_n : n < \omega \rangle$ be a list of all predense subsets of P_α which belong to N. Let

$$\begin{aligned} p_1 = p \wedge \bigwedge_n q_n \wedge \bigwedge_n \Big(\bigwedge_{r \in I_n} r \Big) \wedge \bigwedge_{n<\omega} [\ & \bigvee \{p \in P_\alpha \cap N : p \Vdash y \in \underset{\sim}{A}_n \text{ for some } y <_{T^*} x_n\} \vee \\ & \vee \bigvee \{q_n \wedge \bigwedge_{r \in J} r : J \subseteq N \cap P_\alpha,\ J \text{ is definable in } (N, \{y : y < x\}) \text{ and} \\ & \qquad q_n \wedge \bigwedge_{r \in J} r \Vdash_{P_\alpha} \text{"} x_n \notin \underset{\sim}{A}_n \text{"}\}. \end{aligned}$$

There are two facts on p_1 we have to prove:

(A) $p_1 \in P_\alpha = \mathrm{F}\lim^{\aleph_1}_{i<\alpha} P_i$, i.e., $\bigwedge_{i<\alpha}\theta[P_i] \nvdash p_1$ (as clearly p_1 has the right form),

(B) (i)′, (ii)′, (iii)′ hold.

For proving both facts we do the following. We assume everything is in some countable transitive model M (or $M \Rightarrow V$, $V \Rightarrow V^*$, in V^* $|H(\lambda)^V|$ is countable which is easy by forcing).

Let p', q' be as in (i)′.

We let $G_{\alpha_0} \subseteq P_{\alpha_0} = P_\beta$ be generic (i.e., M-generic and $p' \restriction \beta'$, $q' \in G_{\alpha_0}$).

We shall find $G \subseteq P_{\alpha_n}$ such that $G \cap P_{\alpha_n}$ is generic (for M, P_{α_n}), and the truth values it gives to all $p \in \bigcup_{n<\omega} P_{\alpha_n}$ make $p_1 \wedge p'$ true (so we have, in V, a model of $\bigwedge_{i<\alpha}\theta[P_i] \vdash \neg p_1 \wedge p'$ (= fact (A)).

As for fact (B), fact (B)(ii)′ holds trivially by the definition of p_1 (i.e., $\bigwedge_n(\bigvee_{r\in I_n} r)$). Similarly the last conjunct takes care of (B)(iii)′.

The second phrase of (B)(i)′ holds by the free choice of p', q' (and the way G_{α_0}, G are chosen), hence $p_1 \restriction \beta = q_1$; the other inequality follows by p_1's definition.

We define by induction G_n, p_n s.t. (like 2.2)

(1) $G_n \subseteq P_{\alpha_n}$, $G_n \subseteq G_{n+1}$,

(2) G_n is P_{α_n}-generic for M,

(3) $p_n \leqq p_{n+1}$, $p = p'$, $p_n \in P_\alpha \cap N$,

(4) p_n is compatible (in P_α) with every member of G_n, $q_n \in G_n$,

(5) p_{3n+1} is $\geqq q'_n$ for some $q'_n \in I_n \cap N$,

(6) $p_{3n+2} \vdash \bigwedge \Phi_n$ or $p_{3n+2} \vdash \neg r_n$ for some $r_n \in \Phi_n$, where $\langle \Phi_n : n < \omega\rangle$ is a list of all countable $\Phi \subseteq P$, $\Phi \in N$,

(7) in $M[G_n]$ for every $A \in N[G_n]$, $A \subseteq T^*$, $x \in T^*_\delta$, $\bigwedge x \in A \rightarrow (\exists y < x)$ holds ($q_n \in G_n$ do the job),

(8) $p_{3n+3} \Vdash_{P_\alpha}$ "$(\exists y < x_k) y_\alpha \in \underset{\sim}{A}_n$" or $q_{3n+3} \wedge \bigwedge_{r\in J} r \Vdash$ "$x \notin \underset{\sim}{A}_n$", for some $J - \{p_{3n+3}\} \subseteq G_{n+1}$, J definable in $(N, \{y : y <_{T^*} x_n\})$.

As in the proof of 2.2, this suffices. The only non-trivial part in the definition is taking care of (8). So let $n = 3k + 2$, p_n, G_n, be defined, and we shall define p_{n+1}, G_{n+1}. We define

$\underset{\sim}{A}'_k = \{y \in T^*$: there is $r \in P_\alpha$, $r \geqq p_n$, which is compatible with every member of $\underset{\sim}{G}_{n+1}$ (= the name of the generic subset of $P_{\alpha_{n+1}}$) such that $r \Vdash_{P_\alpha}$ "$y \notin \underset{\sim}{A}_k$"$\}$.

Clearly $\underset{\sim}{A}'_k$ is a $P_{\alpha_{n+1}}$-name (as we use $\underset{\sim}{G}_{n+1}$ in the definition) and if $p_{n+1} \restriction \alpha_{n+1} \leqq r \in P_{\alpha_{n+1}}$ then

$$(*) \qquad r \Vdash_{P_{\alpha_{n+1}}} \text{“} y \notin \underset{\sim}{A}'_k \text{”} \text{ implies } r \Vdash_{P_{\alpha_{n+1}}} \text{“} y \notin \underset{\sim}{A}_k \text{”}.$$

However the inverse implication does not follow. Now *if* we can choose p_{n+1}, such that $p_n \leqq p_{n+1} \in P_\alpha \cap N$, p_{n+1} compatible with every member of G_n (equivalently of $G_n \cap N$) such that $p_{n+1} \Vdash_{P_\alpha}$ "$y \in \underset{\sim}{A}_k$" for some $y <_{T^*} x_n$, *then* we can proceed to define G_{n+1} with no problem.

We assume that there is no such p_{n+1} and let $p_{n+1} = p_n$. Let

$$J = \{p_n\} \cup \{\neg r : r \in P_{\alpha_{n+1}}, r \Vdash_{P_{\alpha_{n+1}}} \text{"} y \in \underset{\sim}{A}'_k \text{" for some } y <_{T^*} x_k\}.$$

Clearly J is definable in $(N, \{y : y <_{T^*} x_k\})$, $J \subseteq P_\alpha \cap N$, so it is enough to prove $q_{n+1} \wedge \bigwedge_{r \in J} r \Vdash_{P_\alpha}$ "$x_k \notin \underset{\sim}{A}_k$".

Now $\underset{\sim}{A}'_k$ is a $P_{\alpha_{n+1}}$-name of a subset of T^*, so by the choice of q_{n+1}

$$q_{n+1} \Vdash_{P_{\alpha_{n+1}}} \text{"} x_n \in \underset{\sim}{A}'_k \rightarrow (\exists y <_{T^*} x_k) y \in \underset{\sim}{A}'_k \text{"}.$$

However for each $y <_{T^*} x_k$,

$$I_y = \{r \in P_{\alpha_{n+1}} : r \Vdash_{P_{\alpha_{n+1}}} \text{"} y \in \underset{\sim}{A}'_k \text{" or } y \Vdash_{P_{\alpha_{n+1}}} \text{"} y \notin \underset{\sim}{A}'_k \text{"}\}$$

is a subset of $P_{\alpha_{n+1}}$ hence $I_y \cap N$ is predense above q_{n+1} (in $P_{\alpha_{n+1}}$) (as $y \in N$). So q_{n+1} forces that if $y \in \underset{\sim}{A}'_k (y <_{T^*} x_k)$ then some $r \in I_y \cap N$ is in the generic subset of $P_{\alpha_{n+1}}$. Hence $q_{n+1} \wedge p_n \in P_\alpha$ forces that: if $x_k \in \underset{\sim}{A}_k$, then necessarily $x_k \in \underset{\sim}{A}'_k$ (see (*)) hence some $y <_{T^*} x_k$ is in $\underset{\sim}{A}'_k$. Hence some $r \in I_y \cap N$ for which $r \Vdash_{P_{\alpha_{n+1}}}$ "$y \in \underset{\sim}{A}'_k$" is in the generic set, clearly $\neg r \in J$. So clearly (as $p_n \in J$) $q_{n+1} \wedge \bigwedge_{r \in J} r$ forces that $x_k \in \underset{\sim}{A}_k$ leads to a contradiction (as r and x are incompatible) so it forces $x_k \in \underset{\sim}{A}_k$.

As we have assumed there is no p_{n+1}, $p_n \leqq p_{n+1} \in P_\alpha \cap N$, p_{n+1} compatible with every member of G_n such that $p_{n+1} \Vdash_{P_{\alpha_n}}$ "$y \in \underset{\sim}{A}_k$" for some $y <_{T^*} x_k$, clearly if $G_{n+1} \subset P_{\alpha_{n+1}}$, generic for M, $G_n \subset G_{n+1}$, $q_{n+1}, p_n \restriction \alpha_{n+1} \in G_n$ *then* $r \in J \Rightarrow r \in G_{n+1}$. So finish proving (8) hence the theorem.

4.8. Conclusion. If ZFC is consistent so is: ZFC + every Aron. tree is S-*st*-special, but some Aron. tree T^* is not S^*-*st*-special for any $S^* \subseteq \omega_1 - S$ stationary (S is co-stationary — otherwise it is not interesting, but there is no other restriction).

Proof. Trivial by the previous Theorem 4.6, 4.7, but note that for ensuring T^* remains an Aron. tree we had better start the iterated forcing by $Q(T^*, S)$, as for the $\aleph_2$-chain condition, see [8]. Remember also that our forcings are proper and proper forcing preserves stationarity of subsets of ω_1 (see [8]).

Concluding Remarks. (1) We can ask: can we do it with G.C.H. and can we get independence of other variants of "every Aron. tree is non-Souslin, special, etc." but we have not tried. For G.C.H. it is natural to use a variant of the forcing used in [8] for the consistency of G.C.H. + SH with ZFC.

(2) By the definition of the forcing $Q(T, S)$; and by 3.5(2) (applied to an almost subtree), in 4.8 we get that every Aron. tree is S'-special for some S' (the range of the generic h). So for S empty, we get: every Aron. tree is S'-special for some S' (equivalently h-special for some $h:\omega_1 \to \omega_1$) but some tree is not S^*-st-special for any stationary $S^* \subseteq \omega_1$.

(3) If we use also case (β) in 4.7, we can strengthen the conclusion of 4.8 to: for no antichain $I \subseteq T^*$ is $rk(I) - S$ stationary (by adding a closed unbounded subset of ω_1 disjoint to any such $rk(I) - S$).

(4) Avraham noted that "T is h-special for some h" is equivalent to "T is S-r-special for some closed unbounded $S \subseteq \omega_1$". Note that we can define S-P-special for every partial order P, and if $\alpha_i \in P$ ($i < \omega_1$) implies ($\exists i < j < \omega_1$) $\alpha_i \leqq \alpha_j$ then "T S-P-special" implies "T is not Souslin". Note also that "S-r-special for some closed unbounded S" implies ω_1-$\mathbf{R} \times \mathbf{Q}$-special [$\mathbf{R}$ — reals, $\mathbf{Q}$ — rationals, the order — lexicographic). So we have proved, e.g., "every Aron. tree is ω_1-$R \times Q$ special" does not imply "every Aron. tree is special".

(5) We can also try to get a model of ZFC where, e.g., (1) (for some stationary co-stationary $S \subseteq \omega_1$) every Aron. tree is S-st-special, but some Aron. tree T^* is not h-special for any S; or (2) there is no Souslin tree but some Aron. tree is not h-special for any h. For (1) it is natural to define $Q(T, S) = \{(h, f)$: $(h, f) \in Q(T, S)$, $\operatorname{Dom} f \subseteq \bigcup_{h(\alpha)=\alpha} T_\alpha\}$. But T is the union of $\aleph_0$ disjoint copies of T^*, so $Q(T, S)$ makes T^* h-special for some S.

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