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SETS IN A EUCLIDEAN SPACE WHICH ARE NOT A COUNTABLE UNION OF CONVEX SUBSETS

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ABSTRACT

We prove that if a closed planar set S is not a countable union of convex subsets, then exactly one of the following holds:

(a) There is a perfect subset $P \subseteq S$ such that for every pair of distinct points $x, y \in P$, the convex closure of x, y is not contained in S.

(b) (a) does not hold and there is a perfect subset $P \subseteq S$ such that for every pair of points $x, y \in P$ the convex closure of $\{x, y\}$ is contained in S, but for every triple of distinct points $x, y, z \in P$ the convex closure of $\{x, y, z\}$ is not contained in S.

We show that an analogous theorem is impossible for dimension greater than 2. We give an example of a compact planar set with countable degree of visual independence which is not a countable union of convex subsets, and give a combinatorial criterion for a closed set in $\mathbf{R}^d$ not to be a countable union of convex sets. We also prove a conjecture of G. Kalai, namely, that a closed planar set with the property that each of its visually independent subsets has at most one accumulation point, is a countable union of convex sets. We also give examples of sets which possess a (small) finite degree of visual independence which are not a countable union of convex subsets.

0. Introduction

We wish to investigate here several decomposition properties of sets in a Euclidean space. S will always denote a set in $\mathbf{R}^d$ for some positive integer d.

0.1. Definition. (1) Two points $a, b \in S$ see each other in S iff $[a, b] \subseteq S$.

(2) A subset $A \subseteq S$ is called a *seeing subset* if any two of its points see each other in S.

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(3) A subset $I \subseteq S$ is *visually independent* in S if no two of its points see each other in S.

We are interested in the ways S is covered by convex subsets, and particularly in coverings of minimal size. We are also interested in coverings of S by seeing subsets. Every covering of S by convex subsets is, of course, also a covering of S by seeing subsets, but not conversely. Suppose there is a visually independent subset I of S of size Γ. As no two of the points of I can be in one seeing subset of S, every covering of S by seeing subsets is of size greater than or equal to Γ.

0.2. Definition. (1) $\gamma(S)$ is the minimal cardinality of a collection of convex subsets of S which covers S.

(2) $\beta(S)$ is the minimal cardinality of a collection of seeing subsets of S which covers S.

(3) $\alpha(S)$ is the supremum of cardinalities of all visually independent subsets of S.

By what we have noted before,

0.3. Fact. For every subset S of a Euclidean space, $\alpha(S) \leqq \beta(S) \leqq \gamma(S)$.

For closed S in the plane with a finite $\alpha(S) = m$ it is known that $\gamma(S) \leqq m^6$ (see [PS]). In this paper we deal with Ss whose α, β and γ are infinite, and at least one of them (namely at least γ) is uncountable. In the first section we prove that if $\beta(S)$, for a closed set S, is uncountable, then $\alpha(S) = \beta(S) = \aleph$. Then in section two we give an example of a compact $S \subseteq R^2$ with countable $\beta(S)$ and with $\gamma(S) = \aleph$. We conclude section two with a theorem that for closed planar sets an uncountable γ is always $\aleph$, due to the existence of a perfect subset with the property that the convex closure of no three of the subset's points is included in the set. This gives a more or less complete picture of the situation for closed planar sets, showing that for such sets α, β and γ respect the continuum hypothesis.

In section three we give a criterion for a closed set in $\mathbf{R}^d$ to have an uncountable γ. Then we give an example that shows that in $\mathbf{R}^3$, $\gamma(S)$ for a closed S may not respect the continuum hypothesis.

In section four we define more delicate restrictions on visually independent subsets and investigate them. Also, we prove a conjecture of G. Kalai, namely that a closed planar set, with the property that any of its visually independent sets has at most one accumulation point, is a countable union of convex sets.

In section five we give examples of non-closed sets with a small finite α and $\gamma = \aleph$.

The definition of α, β and γ is due to M. Perles, and so are Theorems 1.1 and 1.2 and Examples 2.1, 5.1, 5.2 and 5.3. Theorems 2.2, 3.1, and the independence result in section three were proved by S. Shelah, who also gave Examples 4.2 and 5.4. Theorem 4.3 was proved by M. Kojman and M. Perles. Shelah's original proof of 2.2, which made use of a partition theorem on trees, was simplified by M. Kojman and now uses only elementary methods.

Prerequisites: The reader should know that the space ${}^{\omega}n$ of all infinite sequences of $0, 1, \dots, n-1$ is a complete metric space when equipped with the metric $d(\eta, \nu) = 1/h$ where h is the first coordinate in which ν and η disagree. No other prerequisites are needed for all sections, except for the independence result, which requires a background in forcing.

§1. Uncountable $\beta(S)$

1.1. THEOREM. *If S is closed, then $\alpha(S) > \aleph_0 \Rightarrow \alpha(S) = \aleph$.*

1.2. THEOREM. *If S is closed, and $\beta(S) > \aleph_0$, then there is a perfect, non-empty, visually independent subset of S. In particular, $\alpha(S) = \beta(S) = \gamma(S) = \aleph$.*

REMARK. Of course the second theorem implies the first, as $\beta(S)$ is always $\geqq \alpha(S)$.

PROOF OF 1.2. Let us label as *good points* those points of S for which there is some neighborhood u such that $S \cap u$ is a union of countably many seeing subsets. The points which are not good are called *bad points*. Clearly, the good points form an open subset of S, thus the bad points form a closed subset of S. Denote the latter subset by T. Now $S - T$ can be covered by $\aleph_0$ seeing sets — because there is a countable base for the topology. So if T were countable, we could add to the cover all the sets $\{t\}$ for $t \in T$ (or leave it untouched, in case T is empty) to obtain a countable cover of S by seeing sets. This is a contradiction to our assumption that $\beta(S) > \aleph_0$. Note now the easy

1.3. FACT. Every $t \in T$ is a condensation point of T.

PROOF OF FACT. Suppose to the contrary that $t \in T$, but that there is a neighborhood u of T which contains only countably many members of T. As we have a countable base for the topology, we can cover the set of all good points in u with a countable cover of seeing sets; add to this cover the countably many seeing sets $\{x\}$ for $x \in T \cap u$ to get a countable cover of $S \cap u$, and thus a contradiction.

Now either for every point $t \in T$ and every neighborhood u of t there are two points t_0, $t_1 \in T$ distinct from each other and from t which do not see each other in S, or there is some $t^* \in T$ and a neighborhood u of t^* such that any two distinct points in $u \cap T$ see each other in S. The second alternative is impossible — as it implies that $T \cap u$ is a seeing set — which, via an argument similar to the one we used in the last fact, leads to a contradiction. So we know that near every point $t \in T$ we can find two other points of T which do not see each other.

Note that if two points $a, b \in S$ do not see each other, there must be a point $c \in (\mathbf{R}^d - S) \cap [a, b]$. As the complement of S is open, there is an open ball $B(c, \varepsilon)$ which is contained in the complement. This ball is a sight obstacle which does not let any point in $B(a, \varepsilon)$ see any point in $B(b, \varepsilon)$. So in this case there are two *neighborhoods* of a and b which do not see each other.

This observation allows us to begin the following inductive process: choose a point $t \in T$. By what we have seen, there are two points near t (call them t_0 and t_1) which do not see each other and belong to T. By the observation, there are two closed neighborhoods u_0, u_1 of t_0, t_1 respectively, which do not see each other. Continue as follow: choose t_{01}, t_{00} in $u_0 \cap T$ and two closed neighborhoods u_{01}, u_{00} of them which do not see each other and are contained in u_0; do the same in u_1. Continue like that, making sure that each closed neighborhood u_η, η being a sequence of length n of zeroes and ones, is of diameter smaller than $(\frac{1}{2})^n$. A sequence of points whose indices correspond to a branch in the tree of all finite sequences of zeroes and ones forms a Cauchy sequence, and therefore converges to a point of S. All those limit points form a perfect subset of S which is visually independent. This proves the theorem.

§2. Uncountable $\gamma(S)$

We have seen that for a closed set $S \subseteq R^d$ a non-countable $\beta(S)$ implies that $\alpha(S)$ (and therefore also $\beta(S)$ and $\gamma(S)$) equals the continuum in cardinality — because of the existence of a perfect, visually independent subset $P \subseteq S$. It remains to be seen whether the weaker hypothesis of a non-countable $\gamma(S)$ implies the same. We present now an example of a compact $S \subseteq R^2$ such that $\alpha(S) = \beta(S) = \aleph_0$ but $\gamma(S) = \aleph$. In fact, our set S will have a perfect seeing subset $B \subseteq S$ such that the convex closure of no three distinct points of B will be included in S.

2.1. Example. Let D be the unit disk. We first construct a set $A \subseteq \operatorname{int} D$ and a set $B \subseteq \operatorname{bd} D$. B is constructed in a manner similar to that of constructing the Cantor set: Let B_1 be the set consisting of two closed arcs, each of length $\frac{1}{2}$, one arc centered around $\langle -1, 0\rangle$, the other around $\langle 1, 0\rangle$. $B = \bigcap_{n<\infty} B_n$ where B_n is made of 2^n disjoint arcs and B_{n+1} is obtained from B_n by removing from each arc of B_n an open arc of length $(1-\varepsilon)$ times its own length which is centered around its middle.

$A = \bigcap_{n<\infty} A_n$ where $A_0 = \operatorname{int} D$ and A_{n+1} is obtained by removing 2^n disjoint open triangles from A_n. The removal of triangles is performed to fulfill the following demands:

(a) If $\langle x, y, z\rangle$ is a triple of points taken from three distinct arcs of B_{n+2}, then $\operatorname{con}(x, y, z) \not\subseteq A_{n+1}$.

(b) If $x, y \in B_n$, then the segment (x, y) is contained in A_n.

We describe now the induction step, in which we first define B_{n+2} and next A_{n+1}: If $\widehat{ab}$ is an arc in B_{n+1}, denote by p^+ the point of $B_{n+1} - \widehat{ab}$ such that the clockwise arc connecting p^+ with b is of minimal length and by p_- the point in $B_{n+1} - \widehat{ab}$ such that the clockwise arc connecting a with p_- is of minimal length. Let $a', b' \in \widehat{ab}$ be within ε of a, b respectively, ε being so small such that the intersection point p of (p^+, a') with (p_-, b') is on the same side of (a, b) as the origin. Obviously, there is an ε which is small enough to guarantee this (see Fig. 1). Now remove an arbitrary open triangle which is contained in the

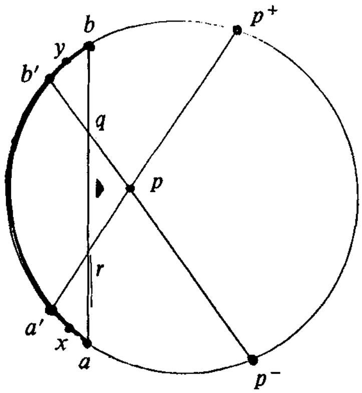

Fig. 1.

interior of $\Delta(p, q, r)$, where q and r are the intersection of (p_-, b') and of (p^+, a') with (a, b), respectively. Note that later removals of triangles will be made only in the areas bounded between the arc $\widehat{ab}$ and the chord (a, b) (or its analogs), thus assuring that each later triangle is contained in S_n. Also, a choice of a small ε ensures that the present 2^n removed triangles are mutually disjoint.

Let x, y, z be in B_{n+2}. If they are in three distinct arcs of B_{n+1}, they are taken care of by the induction hypothesis; so assume without loss of generality that $x \in \widehat{aa'}$, $y \in \widehat{bb'}$, with the same notation as in the figure. Clearly, the triangle we have removed intersects con(x, y, z). This takes care of the first demand. As for the second, choose an arbitrary pair of points $x, y \in B_{n+1}$. If both are taken neither from $\widehat{aa'}$ nor from $\widehat{bb'}$, then (x, y) is separated away from what we have removed by the chord (p^+, p_-); if both are in $\widehat{ab}$, then (x, y) is separated away from what is removed by the chord (a, b); and if (without loss of generality) $x \in \widehat{aa'}$ while $y \notin \widehat{ab}$, the choice of p^- guarantees what is required.

Note that we have removed the triangle in such a manner that for every ε there are only finitely many triangles which were removed from the open disk of radius $1 - \varepsilon$.

Now we define our set $S : S = B \cup (\text{con}(B) \cap A)$, namely the convex closure of B from which countably many open triangles are removed in the process described above. Clearly, S is compact, and its intersection with bd D is B. Furthermore

2.1.1. CLAIM. If $x, y \in B$ then $[x, y] \subseteq S$.

PROOF. This follows from demand (b). In the construction, suppose there was an n such that this segment was not in A_n. x and y are clearly in B_n — which contradicts demand (b).

2.1.2. CLAIM. $\alpha(S) = \beta(S) = \aleph_0$.

PROOF OF CLAIM. By Theorem 1.2 it is enough to prove that $\alpha(S)$ is countable. Assume to the contrary that there exists an uncountable visually independent subset V of S. As in any open disk of radius $(1 - 1/n)$ there are only finitely many triangular holes, it can contain only finitely many points of V; for the removal of k disjoint triangles from a convex set allows at most 3^k visually independent points. So there are only countably many members of V in int D. So without loss of generality all of its members are in B. But any two points of B see each other — contradiction!

Lastly, any three distinct points from B are such that their convex closure is not included in S. Assume $x, y, z \in B$ are three distinct points. Then there exists an n such that x, y, z belong to three distinct arcs of B_{n+1}. If so, then in A_n there is a triangular hole which is included in $\mathrm{con}(x, y, z)$. In particular, $\gamma(S) = \aleph$ (for B, being a perfect set, has cardinality $\aleph$).

This example shows that with a closed subset of the plane one can have a countable β and a non-countable γ. A natural question to be asked at this stage is whether a non-countable $\gamma(S)$ always equals the continuum in cardinality — or whether other values are possible. The question is not trivial — for if we consider more general sets (rather than only closed sets) γ can take non-countable values other than $\aleph$. Assume that the non-countable cardinal $\aleph_1$ is smaller than $\aleph$ (a situation which is known to be possible, or consistent with the axioms of set theory) and consider any subset of R^d of size $\aleph_1$. This set cannot contain any line segment, and therefore not any nontrivial convex subset. Its γ then equals $\aleph_1$ — which is not $\aleph$.

The following theorem will show that for a closed $S \subseteq R^2$, $\gamma(S)$ respects the continuum hypothesis, or in other words, for such an S if $\gamma(S)$ is non-countable, then it equals the continuum in cardinality. Amazingly enough, this cannot be proved (nor refuted) for R^3, using the usual axioms of set theory.

But let us consider the two dimensional case now. In the example we just saw we had really more than just $\gamma(S) = \aleph$: we had a perfect subset of S such that no three distinct points in it can be in one convex subset, but such that every two of its points see each other in S. We will prove that this is the typical case for closed $S \subseteq R^2$ with $\gamma(S) > \aleph_0$ and $\alpha(S) = \aleph_0$.

2.2. **Theorem.** *If $S \subseteq R^2$ is closed, and $\gamma(S) > \aleph_0$, then there is a perfect subset $P \subseteq S$, such that for all triples $p, q, r \in P$ of distinct points,* $\mathrm{con}(p, q, r) \not\subseteq S$. *Furthermore, if $\alpha(S) = \aleph_0$ then P can be chosen to be a seeing-subset of S.*

Proof of 2.2. Suppose that S is closed in the plane. Let U be the union of all open disks u of rational radius and rational center coordinates such that $u \cap S$ is a union of countably many convex sets. Clearly, $U \cap S$ is open in S and is a union of countably many convex sets. Denote by B the set $S - U$ — which will be referred to later as "the set of bad points". So B is closed, and if $x \in B$, then for every neighborhood u of x, $u \cap S$ is not the union of countably many convex sets.

Moreover, if we had a neighborhood u such that $u \cap B \neq \emptyset$ and such that every $u' \subseteq u$, satisfying $u' \cap B \neq \emptyset$ satisfied that in it there were two bad points which do not see each other in S, then we could repeat the proof of Theorem 1.2 and get a perfect, visually independent subset of S — which is more than required. We therefore assume that in every neighborhood u we can by shrinking it, if necessary, have that all bad points see each other in S. In fact we assume from now on, without loss of generality, that the set B of bad points is a seeing subset of S. In particular, in no neighborhood u can all bad points be contained in a countable number of lines, for then the bad points would be a countable union of convex subsets of S.

The course the proof will take is the following: We define a perfect tree of closed subsets of S ordered by reverse inclusion, such that the diameters of the neighborhoods along a branch tend to zero and such that any three points taken from three different subsets are such that their convex closure is not contained in S. (A more precise definition will be given below.) If such a tree exists, then exists also a perfect subset of S — namely one point for each branch in the tree — with the property that the convex closure of no three of them is contained in S. So we assume that no such tree exists. We, nonetheless, try to construct one several times, when each time the failure is utilized to improve the starting conditions for another try. We eventually reach a situation in which we can guarantee success, and thus obtain a contradiction.

2.2.1. NOTATION. ${}^{<\omega}2$ is the tree of all finite sequences of zeroes and ones, where a finite sequence η is smaller than a finite sequence ζ if η is an initial segment of ζ. $\lg(\eta)$ is the length of η. A sequence η of length n has domain $0, \ldots, n-1$. By $\langle\ \rangle$ we denote the empty sequence. $\eta{}^\wedge\langle 0\rangle$ means the concatenation of η with zero, namely the sequence ζ such that $\lg(\zeta) = \lg(\eta) + 1$, $\eta < \zeta$ and $\zeta(\lg(\eta)) = 0$.

2.2.2. THE CONSTRUCTION. We wish now to describe a construction of a perfect tree of closed neighborhoods $T' = \{u_\eta \mid \eta \in T\}$ where:

(1) $T \subseteq {}^{<\omega}2$ is perfect (i.e. for every $\eta \in T$ there is an extension ζ of η such that both $\zeta{}^\wedge\langle 0\rangle$ and $\zeta{}^\wedge\langle 1\rangle$ are in T).

(2) $\eta < \zeta \Rightarrow u_\zeta \subseteq u_\eta$.

(3) For every $n > 0$ all sequences η in T of length n, except one sequence, η_n, have only one extension of length $n+1$ in T, while η_n has both extensions in T.

(4) $u_\eta \cap B$ is not empty.

(5) $\lg(\eta) = n \Rightarrow \operatorname{diam}(u_\eta) < 1/2^n$.

(6) If $\eta \neq \zeta$ then $u_\eta \cap u_\zeta = \varnothing$.

(7) If η_l, $1 \leqq l \leqq 3$ are three distinct nodes in T, then $x_1 \in u_{\eta_l}$ implies $\mathrm{con}(x_1, x_2, x_3) \not\subseteq S$.

The construction is carried out by induction as follows: at each stage we have a finite list of disjoint neighborhoods, $\langle u_\nu \mid \nu \in {}^n 2 \rangle$ satisfying the property

(∗) If ν_1, ν_2, ν_3 are three distinct sequences of length n, $x_1 \in u_{\eta_l}$, $1 \leqq l \leqq 3$, then $\mathrm{con}(x_1, x_2, x_3) \not\subseteq S$.

Note that (∗) implies that no three bad points taken from three distinct neighborhoods are collinear, because we assume that all bad points see each other.

From this list, all neighborhoods but one are either untouched or shrunk a little, and one is replaced by two disjoint subneighborhoods, retaining the property (∗).

A success in the construction yields the theorem, since to each branch in T corresponds one point of B, the collection of these points being a perfect set, and the convex closure of no three of them is contained in S; in case of failure we have

2.2.3. Claim. We can find a finite sequence $\langle x_1, \ldots, x_n \rangle$, $n \geqq 2$, of points from B and a neighborhood u not containing them such that

(1) $u \cap B \neq \varnothing$.

(2) u is disjoint to any of the lines determined by a pair of points x_l, x_m for $1 \leqq l < m \leqq n$,

(3) for every point $y \in u$ there exists $1 \leqq l \leqq n-1$ such that $\mathrm{con}(y, x_l, x_n) \subseteq S$.

Proof of Claim. Suppose the construction of T' fails at height h while trying to split u_η into two neighborhoods. Fix a bad point $x_\nu \in u_\nu \cap B$ for every neighborhood $\nu \neq \eta$ which is already defined. Our failure means in particular that for any pair of points $x, y \in u_\eta \cap B$ there exists ν such that $\mathrm{con}(x, y, x_\nu) \subseteq S$; for otherwise we could shrink each u_ν around x_ν and pick two small enough neighborhoods around x and y to meet the requirements (remember that S is closed). Label now these x_νs as $\langle x_1, \ldots, x_{n-1} \rangle$. Pick $x_n \in u_\eta \cap B$ and find a $y \in u \cap B$ which is not on any of the lines determined by two x_ls. This is possible by the assumption that not all bad points in y are contained in a line. Pick a small enough neighborhood $y \in u'$ which avoids all these lines. So for every bad point $y \in u$ there exists a $1 \leqq l \leqq n-1$ such that $\mathrm{con}(x_n, x_l, y) \subseteq S$.

2.2.4. Claim. We can find an open neighborhood u' such that $u' \cap B \neq \varnothing$

and two points x_1, x_2 outside of u' such that for every $y \in u' \cap B$, $\text{con}(x_1, x_2, y) \subseteq S$.

PROOF. Start with what we have by 2.2.3. Define the following subsets B_l, $1 \leqq l \leqq n-1$ of $B \cap u$: B_l is the set of all $y \in B \cap u$ such that $\text{con}(x_n, x_l, y) \subseteq S$. Now by 2.2.3 we know that these sets cover $u \cap B$. By the closedness of S, each of them is clearly closed. Define by induction a descending sequence of neighborhoods u_l for $0 \leqq l \leqq n-1$ such that:

(1) $u_l \cap B \neq \varnothing$,
(2) $u_l \subseteq u_{l+1}$,
(3) $u_{l+1} \cap B_l = \varnothing$,
(4) $u = u_0$.

In the induction step avoid — if possible — B_l by shrinking the present neighborhood. As the B_ns cover $u \cap B$ there must be some $l < n-1$ such that B_l is dense in $u_{l+1} \cap B$. But the latter is closed — therefore it covers all of $B \cap u_{l-1}$. Denote u_{l-1} by u' (shrinking it, if necessary, to avoid (x_l, x_m)) and denote x_n, x_l by x_1, x_2 to obtain the claim.

Draw now the x-axis from x_1 to x_2 with zero being their average. By the claim, u' is in one side of the axis — let us call it the positive side. By applying some affine transformation we may assume without loss of generality that the y-axis passes through u'.

2.2.5. CLAIM. No two different points $b_1, b_2 \in u \cap B$ are above each other, namely $x(b_1) = x(b_2) \Rightarrow b_1 = b_2$.

PROOF. Suppose b_1 is exactly above b_2. This implies that $b_2 \in \text{int}\,\Delta(x_1, b_1, x_2)$ which is impossible, for this interior is included in S, and therefore b_2 would have an open neighborhood wholly in S, contrary to its badness.

2.2.6. COROLLARY. *Denote by C the projection of $B \cap u$ on the x-axis. Then there is some function f whose domain is C and whose graph is $B \cap u$. Furthermore, f is a continuous function, even satisfying a Lipschitz condition.*

PROOF. Obvious by Fig. 2; the slope of f at b cannot excede the slopes of (x_1, b) or (x_2, b) (or again a bad point would be inside a open neighborhood contained in S).

We denote by θ or ϕ real numbers in $[-\pi, \pi)$, and call such numbers *directions*. For a pair of points a, b in the plane we denote by $\text{dir}(a, b)$ the

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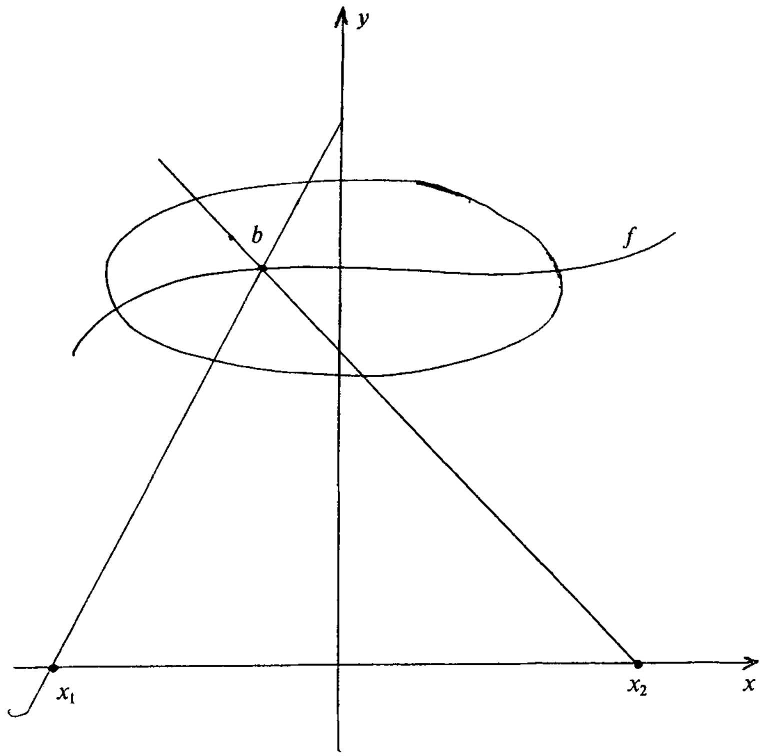

Fig. 2.

direction from a to b, if it is in $[-\pi, \pi)$, and its minus otherwise. We call a neighborhood u *relevant* if $u \cap B \neq \emptyset$. We call a direction which is obtained by a pair of bad points a "relevant direction", and we call the relevant direction obtained by pairs of points in u "the relevant directions of u".

We assume now that we cannot construct a tree of neighborhoods as described above, and utilize this assumption for more claims.

2.2.7. LEMMA. *For every relevant neighborhood u and $\varepsilon > 0$ there is a direction θ and a relevant sub-neighborhood $u' \subseteq u$ such that for every relevant direction ϕ of u', $|\phi - \theta| < \varepsilon$.*

PROOF. Suppose this is not the case for a given u and ε, and we shall eventually construct the tree of neighborhoods. Without loss of generality, $\varepsilon = 2\pi/n$ for some n, by shrinking ε if necessary. We look at the the set of intervals

$$\left\{\left[-\pi + \frac{i\varepsilon}{2}, -\pi + \frac{(i+1)\varepsilon}{2}\right) \middle| 0 \leqq i \leqq n-1\right\}.$$

By our assumption, in any sub-neighborhood there is a counterexample to the claim, namely there appear two relevant directions whose difference is greater than ε; such a pair of directions cannot lie either in the same interval or in two adjacent intervals of our list. (The first and last interval are also considered to be adjacent.). As the intervals cover $[-\pi, \pi)$, every counterexample determines a pair of non-adjacent intervals of the list. There is a finite number of possibilities for such pairs — let us order them in a list. Now try, by shrinking the neighborhood to a relevant sub-neighborhood, to avoid all counterexamples which determine the first possibility, and so on by induction. This process must end before the last possibility, by the assumption that such pairs always exist. Namely, there is a fixed pair of non-adjacent intervals and a relevant subneighborhood u'' such that in every relevant subneighborhood $v \subseteq u''$ there are two relevant directions, one in each interval. To put it differently, there is a relevant subneighborhood $u'' \subseteq u$ and a fixed pair of intervals such that counterexamples that determine this pair appear densely in $B \cap u''$.

Call the fixed pair of intervals I_1 and I_2. Without loss of generality $\inf I_2 > \sup I_1$. We work in u'' and construct our tree of neighborhoods. Here is how: we add two other induction hypotheses, namely that for every $\eta \neq \nu$ all directions between points of u_η and points of u_ν are in I_1. Let u_0 and u_1 be two relevant (disjoint) sub-neighborhoods such that for every pair of bad points $x_1 \in u_1$, $x_2 \in u_2$, $\mathrm{dir}(x_1, x_2) \in I_1$: Take two such *points* whose direction is in int I_1, noting that not all pairs of bad points have $\inf I_1$ as their direction, and shrink the neighborhoods around them until this property is obtained. Suppose that at some stage of the induction we have to split the neighborhood u_η. As usual, fix witnesses, namely fix one bad point in every other neighborhood such that all these points are in general position. Pick a pair of bad points whose direction is in I_1, and which is nonetheless different from any of the directions determined by the witnesses and such that all the points picked are in general position. Let z be any witness. So $\Delta(z, x_1, x_2)$ is a triangle. Note that the directions of all the edges in this triangle are in I_1 — so it has an obtuse angle (for without loss of generality $\varepsilon < \pi/2$). In a close enough neighborhood of the vertex of this angle find a pair of points whose direction is in I_2. By replacing the vertex with one of them, if necessary, the other will be seen to reside in the interior of $\Delta(z, x_1, x_2)$. This shows that the triangle is bad, and completes the induction step.

But since we assume that the construction is impossible, we must accept the claim.

2.2.8. LEMMA. *For every relevant neighborhood u, two bad points $b_1, b_2 \in u$ and $\varepsilon > 0$, there are two disjoint relevant sub-neighborhoods u_1, u_2, $u_1 \subseteq B(b_1, \varepsilon)$, $u_2 \subseteq B(b_2, \varepsilon)$ such that the union of the convex closure of relevant directions of u_1 with the convex closure of relevant directions of u_2 is disjoint from*

$$\mathrm{con}(\{\mathrm{dir}(x_1, x_2) \mid \langle x_1, x_2 \rangle \in u_1 \times u_2\}).$$

PROOF. Pick three bad points c_l, $0 < l < 4$ in $B(b_1, \varepsilon)$ and three bad points c_{l+3} in $B(b_2, \varepsilon)$ with all six points in general position, and moreover such that there are no two parallel line segments in the configuration. This is possible, by the fact that in no relevant neighborhood are the bad points contained in a finite number of lines. Fix disjoint neighborhoods u_l, one around each point, which are so small such that if $\langle k, l \rangle \neq \langle m, n \rangle$ then

$$\mathrm{con}(\{\mathrm{dir}(x, y) \mid \langle x, y \rangle \in u_k \times u_l\}) \cap \mathrm{con}(\{\mathrm{dir}(x, y) \mid \langle x, y \rangle \in u_m \times u_n\}) = \varnothing .$$

So these convex closures are a set of 15 disjoint sub-intervals of $[-\pi, \pi)$. Let δ be the minimal distance obtained between two of these intervals. Using 2.2.7 shrink every neighborhood further, if necessary, to determine in it the possible relevant directions up to $\delta/2$. So for each u_l $\mathrm{con}(\{\mathrm{dir}(x, y) \mid x, y \in u_l\})$ can have non-empty intersection with at most one of the disjoint intervals

$$\mathrm{con}(\{\mathrm{dir}(x, y) \mid x \in u_l, y \in u_m\})$$

where $0 < l < 4$, $3 < m < T$. Since this defines a choice of at most six elements out of a set of nine elements, at least three elements are left unpicked, namely there are pairs $0 < l < 4$, $3 < m < 7$ such that the set of directions obtained between u_l and u_m is disjoint to *any* of the sets $\mathrm{con}(\mathrm{dir}(x, y) \mid x, y \in u_j \cap B)$, in particular for $j = l, m$. So this proves the claim.

If u_1, u_2 are two neighborhoods, we call $\mathrm{con}(\{\mathrm{dir}(x, y) \mid \langle x, y \rangle \in u_1 \times u_2\})$ "the directions between u_1 and u_2".

We recall that we have restricted attention to some relevant neighborhood in which all bad points are arranged on the graph of some continuous function f, and that in particular every pair of bad points has a rightmost one and a leftmost one. Suppose we pick a pair of bad points b_1, b_2 and blow them up to two small open neighborhoods u_1, u_2 such that u_2, u_2 are disjoint and their projections on the x-axis are also disjoint. The last claim assures us that there are two subneighborhoods $v_1 \subseteq u_1$, $v_2 \subseteq u_2$ such that the convex closure of the relevant directions of each v_i is disjoint from the convex closure of the directions between v_1 and v_2. Up to reflection in the x-axis there are exactly two possibilities: (A) the relevant directions in both v_1 and v_2 are greater than the directions between v_1 and v_2; (B) the relevant directions in v_1 are smaller than the directions between v_1 and v_2 while those in v_2 are greater (see Fig. 3).

Fig. 3.

By an argument we have seen before, we know that there is a relevant neighborhood (without loss of generality, u itself) in which either possibility (A) or possibility (B) appears densely.

2.2.9. LEMMA. *If possibility (A) appears densely in u then we can carry out the construction.*

PROOF. Proceed by induction with the following additional induction hypotheses. First, if η precedes ν lexicographically then the projection of u_η on the x-axis is disjoint from the projection of u_ν on the x-axis and is to the left of it; second, if $\eta \neq \nu$, then the convex closure of the relevant direction of u_η as well as the convex closure of the relevant direction of u_ν is disjoint from and greater than the convex closure of the directions between u_η and u_ν.

To start the construction pick any two bad points, and near them small enough neighborhoods to get possibility (A). When splitting a neighborhood u_η at the induction step, pick in u_η two disjoint neighborhoods v_0, v_1 according to possibility (A). Fix witnesses in the other neighborhoods. Let $z \in u_\nu$ be a witness which lies to the right of v_1. Pick $b_0 \in v_0$ and shrink v_1 until it lies entirely above all the lines connecting points of v_0 with z (it has to be above, because by the induction hypotheses the relevant directions between v_0 and v_1 are greater than the directions between v_0 and u_ν). Now pick $c_1, c_2 \in v_1$ in such a way that b_0, c_1, c_2 and z are in general position and such that c_2 is the rightmost of the c_is. As $\mathrm{dir}(c_1, c_2)$ is greater than both $\mathrm{dir}(b_0, z)$ and $\mathrm{dir}(b_0, c_2)$, and as c_1 is on the same side of (b_0, z) as c_2, $c_1 \in \operatorname{int} \Delta(b_0, c_2, z)$. Therefore its convex closure is not contained in S. We call such triangles "bad triangles". Shrink now u_ν around z, v_0 around b_0 and v_1 around c_2 to get the property (*) for these neighborhoods. Proceed to the next witness to perform an analogous procedure until all witnesses are handled. This guarantees that property (*) is retained. The additional induction hypotheses are trivial.

So we assume that possibility (A) does not appear densely in u (for otherwise we are done). Therefore possibility (B) appears densely. In that case we can prove

2.2.10. **Lemma.** *Either we can carry out the construction or there is an interval C such that $C \cap \operatorname{Dom} f$ is a non-empty set on which f is (up to reflection of the y-axis) convex.*

Proof. We add the following induction hypotheses to the construction: if η precedes ν lexicographically, then u_η and u_ν are disjoint from each other, have disjoint projections on the y-axis, have disjoint projections on the x-axis and the projection of u_η on the x-axis is to the left of the projection of u_ν on the x-axis. Furthermore, the convex closure of the relevant direction of u_η is smaller than the convex closure of the directions between u_η and u_ν while the convex closure of the relevant direction of u_ν is greater than the convex closure of the directions between u_η and u_ν. Also, if J_η is the projection of u_η on the x-axis and $x \in [\inf J_\eta, \sup J_\eta] \cap \operatorname{Dom} f$, then $f(x) \in u_\eta$. This last additional hypothesis is possible by continuity of f.

Let us only demonstrate the induction step: coming to split u_η, we assume that f restricted to the projection of u_η is not convex — for otherwise we are done. So there are three xs, $x_1 < x_2 < x_3$ in J_η such that $\langle x_2, f(x_2)\rangle$ is above the line segment $(\langle x_1, f(x_1)\rangle, \langle x_3, f(x_3)\rangle$. So $\langle x_2, f(x_2)\rangle$ belongs to u_η. Now find two neighborhoods v_0, v_1 close enough to x_1 and x_3 respectively and small enough to satisfy both that any line segment passing from v_0 to v_1 is below $\langle x_2, f(x_2)\rangle$ and that v_0, v_2 satisfy possibility (B). By further shrinking, meet the other induction hypotheses. Now let z be any witness, say to the right of u_η, and let $b_0 \in v_0$, $b_1 \in v_1$ be two bad points that form a triangle with z. As

$$\operatorname{dir}(b_0, b_1) < \operatorname{dir}(b_0, \langle x_2, fx_2)\rangle) < \operatorname{dir}(b_0, z),$$

it must be that $\langle x_2, f(x_2)\rangle \in \Delta(b_0, b_1, z)$. This means that this is a bad triangle. In the same manner handle all other witnesses.

So if we reject the construction, we must suppose that f is convex somewhere.

We have reached a stage in which we can carry out a weaker construction, which we describe now. Let $<_{lx}$ be the lexicographic order on sequences. Suppose $\eta_1 <_{lx} \eta_2 <_{lx} \eta_3$ are three sequences. Then either there exists a natural number h such that $\eta_1 \restriction h = \eta_2 \restriction h \neq \eta_3 \restriction h$ or there exists an h such that $\eta_1 \restriction h \neq \eta_2 \restriction h = \eta_3 \restriction h$. Intuitively, either η_1 separates from η_2 before η_3 does, or η_1 separates from η_2 after η_3 does. This observation is valid also for infinite sequences of zeroes and ones. In the weaker construction we demand instead of (∗) the weaker demand

@ if $\eta_1 <_{lx} \eta_2 <_{lx} \eta_3$ and η_1 separates from η_2 before η_3 does, then for any three points taken one from each u_{η_i}, $1 \leqq i \leqq 3$ their convex closure is not contained in S (Fig. 4).

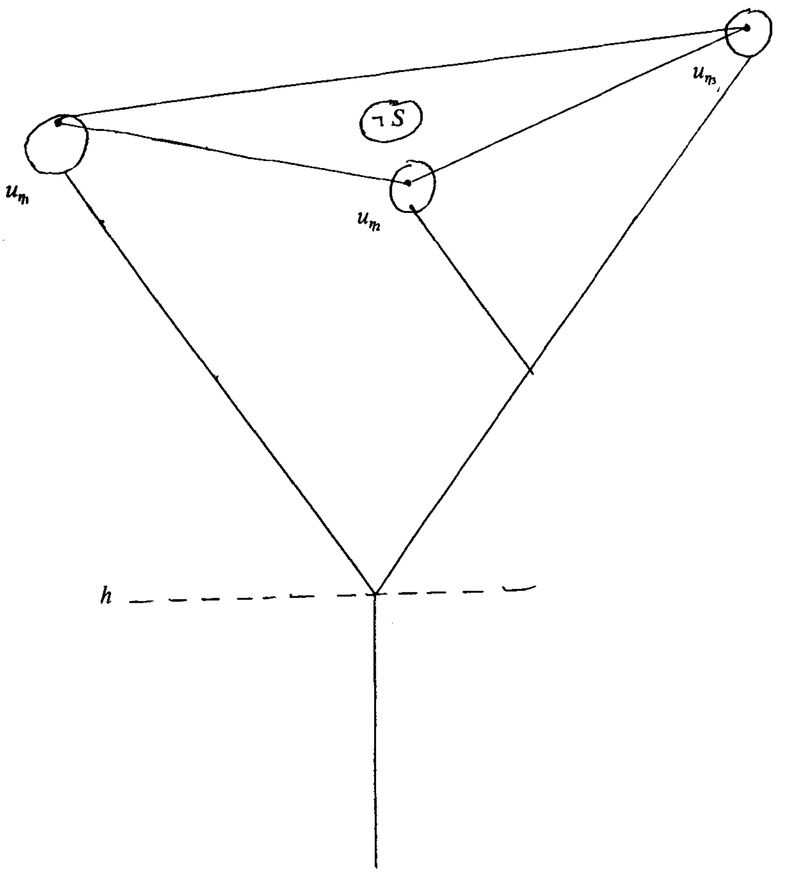

Fig. 4.

This is certainly weaker — for in the perfect set resulting from the construction we know that only "half" of the triangles are bad. But we can prove

CLAIM. If f is convex, then we can carry out the construction with the condition (*) replaced by (@).

PROOF. Suppose f is convex on Dom $f \cap J$ where J is some interval. We work in J and carry out the construction.

We define by induction a tree of disjoint intervals J_η, $\eta \in T \subseteq {}^{<\omega}2$, with u_η being any closed neighborhood containing $f''(J_\eta)$. We note the following simple

2.2.11. FACT. In any open neighborhood of a bad point x there are three points whose convex closure is not contained in S.

PROOF OF FACT. If the convex closure of any three bad points in u were in S, then con$(B \cap u)$ would be contained in S, by Carathéodory's theorem.

So we start with three bad points on the convex graph whose convex closure is not contained in S. As often before, we can assume that these points together with any finite number of bad points to be chosen in what follows are in general position. We find three disjoint open intervals, one around each x-coordinate of each point, and three open neighborhoods as required. The intervals are labeled as J, $J_{\langle 10\rangle}$ and $J_{\langle 11\rangle}$ from left to right.

Coming to split a neighborhood u_η, we fix witnesses ony in those u_νs for which $\nu <_{\mathrm{lx}} \eta$ — because we have to take care only of condition (@) instead of (*). For the induction step we need some auxiliary definitions.

Let $\Delta(t, q, r)$ be such that its convex closure is not contained in S, where t, q, r are on the graph of f, and $x(t) < x(q) < x(r)$. We note that for every point z on the graph of f such that $x(z) < x(t)$, the segment (z, q) divides int $\Delta(t, q, r)$ into two open triangles by the convexity of f, and that since S is closed, at least one of these two open triangles contains points of $\neg S$, namely is bad.

Since $\Delta(t, q, r)$ is bad, we can choose a point $s \in \neg S \cap \operatorname{int} \Delta(t, q, r)$. Observe the ray R from q through s. All points z on the graph of f which are to the left of t — except, maybe, one point, in case this ray meets the graph — lie either above or below R. Respectively, $\Delta(z, t, q)$ or $\Delta(z, q, r)$ are bad (maybe both), having to contain s in their interior. We choose a function F defined for each bad triangle as above and satisfying $F(t, q, r) < x(t)$ with the property that for every point z such that $x(z) < F(t, q, r)$, $\Delta(z, t, q)$ is bad and every point z such that $x(z) > F(t, q, r)$, $\Delta(z, q, r)$ is bad (Fig. 5).

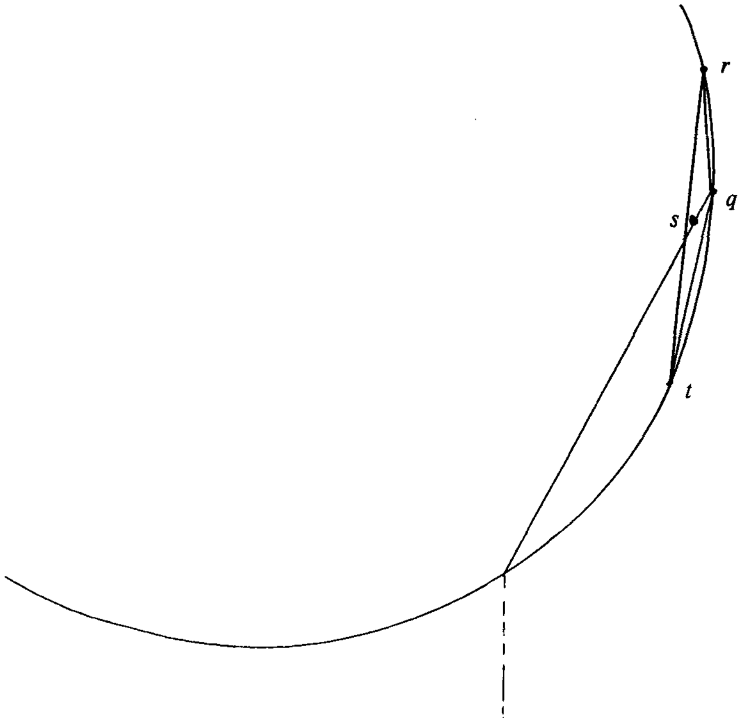

Fig. 5.

In the induction step, trying to split u_η, witnesses fixed in all neighborhoods which are to the left of u_η, observe that one of the following possibilities must occur: (1) there exists a bad triangle $\Delta(t, q, r)$, $t, q, r \in u_\eta \cap B$ with $F(t, q, r) \in J'$; (2) for every bad triangle as in (1), $F(t, q, r) < \inf J_\eta$. If the first possibility occurs, then for every witness e, satisfying $x(e) < F(t, q, r)$, $\Delta(e, t, q)$ is bad, and we are through with the induction step by shrinking other neighborhoods and choosing the new neighborhoods, one around t, the other around q. If it is the other possibility that occurs, we give up temporarily, and start the whole construction again inside u_η. Now at the induction step we can always choose a bad triangle $\Delta(t, q, r)$ as above in the neighborhood to be split, and choose the two sub-neighborhoods around q and r. This proves the claim.

So we have at hand now a perfect set $P = \{b_\alpha \mid \alpha \in Br(T)\} \subseteq B$ where $T \subseteq {}^{<\omega}2$ is a perfect tree and $\mathrm{Br}(T)$ is the set of infinite branches of T, such that the points of the set lie on the graph of a convex function f, and such that $\alpha_1 <_{\mathrm{lx}} \alpha_2$ implies that $x(b_{\alpha_1}) < x(b_{\alpha_2})$. Furthermore, if $\alpha_1 <_{\mathrm{lx}} \alpha_2 <_{\mathrm{lx}} \alpha_3$ are infinite branches of T and α_1 separates from α_2 before α_3 does then $\mathrm{con}(b_{\alpha_1}, b_{\alpha_2}, b_{\alpha_3}) \not\subseteq S$.

CLAIM. There is a perfect subset $P' \subseteq P$ such that for all triples $p, q, r \in P'$, $\mathrm{con}(p, q, r) \not\subseteq S$.

PROOF. We go over the construction of P again, and "thin out" the tree T to a sub-tree T' which is also perfect. Namely, when coming to split a neighborhood u_η, $\eta \in T$, we do not necessarily take $u_{\eta\hat{}\langle 0\rangle}$ and $u_{\eta\hat{}\langle 1\rangle}$ as the next two disjoint sub-neighborhoods, but take u_ν, u_ζ where ν, $\zeta \in T$ are two extensions of η picked as described below.

We define analogously to the definition of F before, a function G on bad triangles of P, with its values on the x-axis on the right side of each triangle. At the induction step we fix witnesses (from P) only in the neighborhoods which are to the right of u_η, because we already have the property (@). Now if there exists a bad triangle $\Delta(p, q, r)$ with vertices p, q, r, $x(p) < x(q) < x(r)$ in $P \cap u_\eta$ such that $G(\Delta(p, q, r)) \in J_\eta$, then pick two neighborhoods around q and r. Otherwise, start again in J_η, and in what follows always pick neighborhoods around p, q in such a situation. This guarantees that all triangles in P' are bad, and thus completes the proof of the theorem.

Combining Theorems 1.2 and 2.2 we see that there are only two types of closed planar sets S with uncountable $\gamma(S)$: either there is a perfect, visually independent subset $V \subseteq S$, and in this case $\alpha(S) = \aleph$, or (as in Example 2.1) $\alpha(S) = \beta(S) = \aleph_0$ and there is a perfect seeing subset $P \subseteq S$ such that for all distinct triples $p, q, r \in P^*$, $\mathrm{con}(p, q, r) \not\subseteq S$, and in this case clearly $\gamma(S) = \aleph$.

To complete the classification of closed planar sets, we recall that if $\alpha(S) = m$ then by [PS] $\gamma(S) \leqq m^6$. The remaining case is when there are visually independent subsets of arbitrary finite size, but not infinite such subsets. In this case $\alpha(S) \geqq \aleph_0$ trivially, on the one hand, and on the other hand

2.3. FACT. A closed set S in a Euclidean space is locally starlike with relation to every one of its points iff there is no infinite visually independent subset of the form of a convergent sequence together with its limit.

PROOF. Suppose S is not locally star-like with relation to $x \in S$. Find x_0, then, that does not see x. As S is closed, x_0 does not see a neighborhood u_0

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of x. Continue by induction to find $x_{n+1} \in u_n$. The set $\{\langle x_n \mid n \in N\rangle \cup \{x\}\}$ is a visually independent set as required. Conversely, suppose such a visually independent subset of S exists; then clearly S is not locally star-like in relation to x.

The fact assures us that if there is no infinite visually independent subset of S, then S is locally star-like with relation to every point. So the union of all convex kernels of $S \cap u_n$, where $\langle u_n \mid n \in N\rangle$ enumerates all rational neighborhoods in which S is starlike, is a countable cover of S with convex subsets. So we have established that if there are arbitrarily large, but not infinite, visually independent subsets of S, then $\gamma(S) = \aleph_0$.

3. More on uncountable γ

In this section we phrase a necessary and sufficient condition for closed sets in $\mathbf{R}^d$ to have uncountable γ. Then we sketch the proof mentioned before, of the independence of the size of an uncountable $\gamma(S)$ for a closed $S \subseteq \mathbf{R}^3$.

3.1. THEOREM. *For a closed set $S \subseteq \mathbf{R}^d$ the following conditions are equivalent*:

(1) *$\gamma(S)$ is uncountable*;

(2) *there is a continuous*, 1–1 *function $f: {}^{\omega}(d+1) \to S$ such that if $\eta_0, \eta_1 \cdots \eta_{d-1} \in {}^{\omega}(d+1)$ are $d+1$ distinct sequences satisfying that there is an integer h such that for $0 \leqq i < j < d$, $\eta_i \restriction h = \eta_j \restriction h$ but $\eta_i \restriction (h+1) \neq \eta_j \restriction (h+1)$, then* $\mathrm{con}(f(\eta_0), \ldots, f(\eta_d)) \not\subseteq S$.

PROOF. Suppose $\gamma(S)$ is uncountable. Let B denote the subset of S consisting of all points X such that for every open neighborhood u of x, $u \cap S$ is not a countable union of convex subsets. Set B is closed and uncountable. We define f by approximations, namely for each finite sequence η we define $F(\eta) := u_\eta$ a closed, relevant neighborhood, with $u_\eta \cap u_\nu = \emptyset$ whenever $\eta \neq \nu$, and with $u_{\eta_1} \subseteq u_{\eta_2}$ whenever $\eta_1 > \eta_2$. To get uniform continuity we may demand that $\lg(\eta) \geqq n$ implies that $\mathrm{diam}(u_\eta) < 1/n$. The crucial condition we wish to satisfy is that if $\eta_0, \ldots, \eta_d$ are all $d+1$ successors of some ν, then for any $(d+1)$-tuple $x_0, \ldots, x_d$ such that $x_u \in u_{\eta_i}$, $\mathrm{con}(x_0, \ldots, x_d) \not\subseteq S$. All conditions are easily satisfied in defining F. In particular, Carathéodory's theorem assures us that in any relevant u_η there are $d+1$ points of B whose convex closure is not contained in S. Blow up around them the required neighborhoods.

For the other direction, suppose that f is given. If, contrary to 1, S were a countable union of convex subsets, $S = \bigcup C_n$, then by the Baire category theorem and the fact that ${}^{\omega}(d+1)$ is a complete metric space, there would be an n such that $f^{-1}(C_n)$ was dense above some finite η. So we could pick $(d+1)$ infinite sequences that pass through all immediate successors of η. By the demand on f the convex closure of their images under f would not be contained in S — contrary to the assumption that these images lie in a convex subset of S.

3.2. Example. We define a set $S \subseteq R^3$ which has an uncountable γ, but which cannot have a perfect subset all of whose 4-tuples form bad 4-gons, and which consistently is the union of $\aleph_1$ convex subsets, for $\aleph_1$ strictly smaller than $\aleph$.

Let us construct on the three dimensional sphere D a perfect subset B, and exclude from the interior of the ball open 4-gons to get a continuous function $f: {}^{\omega}3 \rightarrow D$ with the property that con$(f(\nu), f(\eta_1), f(\eta_2), f(\eta_3))$ is defected (i.e. has non-empty intersection with one of the 4-gons we exclude) if and only if there is a finite sequence η such that three of the four sequences, say η_1, η_2, η_3, pass through the immediate successors of η, while η and ν are incomparable (see Fig. 6).

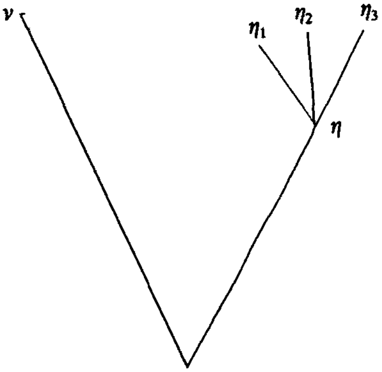

Fig. 6.

The construction is, as usual, by approximations, and we describe only the induction step, in which we split a spherical cap to three subcaps. Choose three arbitrary vertices, $a, b, c \in u_\eta$, of an equilateral triangle and draw around it a circle C, on the sphere, which separates it from all other neighborhoods, but does not touch it (see Fig. 7).

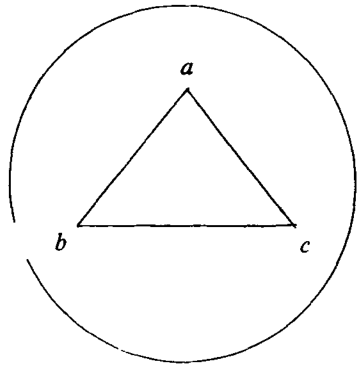

Fig. 7.

Remove now an open 4-gon from $\bigcap_{x\in C}\operatorname{con}(x, a, b, c)$. Evidently, this removal defects all 4-gons of the form $\operatorname{con}(a, b, c, d)$ where d is taken from another neighborhood. Also, if we look at a 4-gon of the form $\operatorname{con}(a, b, d, e)$ where d and e are in other neighborhoods, we see that this 4-gon is separated away from what we have removed, by the choice of C. Now blow up a, b and c to sufficiently small closed neighborhoods.

As in Example 2.1, we take the convex closure of B and subtract from it all the 4-gons removed in the construction. This is a compact set S. An argument identical to that in 2.1 shows that $\alpha(S)=\aleph_0$. An easy application of 3.1 shows that $\gamma(S)>\aleph_0$ (simply use the fact that above every finite sequence there is a 4-tuple of the type described in Fig. 8). So B is the set of bad points of S. But is $\gamma(S)=\aleph$? First, let us observe that there cannot be a perfect set $P\subseteq B$ such that all its 4-tuples form bad 4-gons; for in any perfect subset of B must appear the type of four points shown in Fig. 8, and by the construction such four points are the vertices of a 4-gon whose convex closure is contained in S. This means that the analog of 3.2 for the 3-space is not true.

But we know even more: note that if T is a perfect subtree of ${}^{\omega}3$ such that through every finite sequence η there are at most two immediate successors through which sequences of T pass, then by our construction of B, the convex closure of all points of B indexed by the branches of T is a convex subset of S, because for every four such points the convex closure is contained in S.

Now it is possible, using the technique of iterated forcing, to construct a model of set theory in which the continuum equals $\aleph_2$, and ${}^{\omega}3$ is the union of

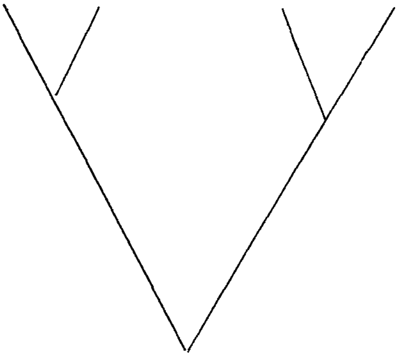

Fig. 8.

$\aleph_1$ such trees. So in this model of set theory $\gamma(S) = \aleph_1$. This means that we cannot prove even the corollary of 3.2 — namely that $\gamma(S)$ respects the continuum hypothesis — for closed sets in R^3.

For the interested reader we sketch here the forcing construction. Start with a model of CH. Then force with a finite support iteration $P = \langle P_i, Q_i \mid i \leqq \omega_1 \rangle$ where a condition of Q_i is a finite function from ${}^{\omega}3^{[V^{P_i}]}$ to ω which satisfies that no three branches which separate simultaneously are mapped to the same member of ω. So a generic set of Q_i gives a decomposition of the old branches into $\aleph_0$ desirable sets. A simple Δ-system argument shows that each Q_i, and therefore also each P_i, satisfy the countable chain condition. So in $V^{[P]}$ we have a decomposition of ${}^{\omega}3$ into $\aleph_1$ desirable sets, while during the iteration $\aleph_2$ reals were added. In a similar way we can get a model in which the continuum equals some $\lambda = \lambda^{\aleph_0}$, while our set S is the union of $\kappa = \mathrm{cf}(\kappa) < \lambda$ convex subsets.

4. Other restrictions on $\alpha(S)$

We know that for a closed $S \subseteq R^d$ a countable $\alpha(S)$ does not imply a countable $\gamma(S)$. But maybe stronger restrictions than merely countability of visually independent subsets imply this.

4.1. Definition. Let us denote by $C_n(S)$ the condition "every visually independent subset $V \subseteq S$ has at most n accumulation points" for $n \in N$ and by $C_{<\omega}(S)$ the condition "every visually independent subset of S has only finitely many accumulation points". Denote by $C_\omega(S)$ the conditions "every visually independent subset of S has a countable closure".

We mention here that in case $\gamma(S)$ is not countable, S cannot be locally star-like with respect to every point (for then the convex kernels of the rational neighborhoods in which S is starlike would produce a countable cover of convex subsets), therefore there always exists a convergent sequence of points of S which together with their limit constitute a visually independent set (see 2.3).

This means that $C_0(S)$ (which really makes sense only for non-bounded sets) does imply that $\gamma(S)$ is countable, for all R^d.

The following example will show that in dimension greater then or equal to three, we cannot hope for more than that.

4.2. THEOREM. *There exists a set $S \subseteq R^3$ such that $\gamma(S)$ is uncountable, but such that every visually independent subset of S has at most one accumulation point.*

PROOF. This is a variation on the previous example. The difference is that now, instead of removing open 4-gons from the interior of the 3-ball, we remove open pyramidal indentations. We again wish to construct a perfect set B on the sphere in countably many approximations, with the property that the type described in Fig. 6 is a bad type. This will again assure the uncountability of γ. However, we add the following demand: if b_1 and b_2 are two different points of B, then there are open neighborhoods u_1 and u_2 of b_1 and b_2 respectively, such that every point in $S \cap u_1$ sees in S every point in $u_2 \cap S$. We describe now the induction step. Coming to split the spherical cap u_η, indent into the ball a pyramidal indentation through u_η which is so flat, that it can hide from no point of what is left of u_η no point from any of the other u_νs. Next pick three points on u_η such that the plane they span meets the indentation. We leave the details to the reader.

By the same argument which was used in the previous examples, we see that every set of visually independent points can have only a finite intersection with any ball of smaller radius; so a set of visually independent points may have accumulation points only on the sphere. By the additional demand, such an accumulation point is unique.

So it is left to see whether for closed sets in the plane, C_n implies a countable γ. We prove here a conjecture of G. Kalai:

4.3. THEOREM. *If $S \subseteq R^2$ is closed, then $C_1(S)$ implies that $\gamma(S) \leqq \aleph_0$. In other words, if every visually independent subset has at most one accumulation point, then S is a countable union of convex sets.*

PROOF. Suppose that $S \subseteq R^2$ is closed, and that $\gamma(S) > \aleph_0$; we intend to construct a visually independent subset of S with two accumulation points. Define as usual $B \subseteq S$, the set of all (bad) points, which have no neighborhood in which S is a countable union of convex subsets. By the assumption of $\gamma(S)$, B is a non-empty perfect subset. We also may assume, without loss of generality, that B is a seeing subset, for otherwise we would have as in the proof of 1.2 an uncountable visually independent set (which has uncountably many accumulation points).

We construct now by induction on n a visually independent set $\langle x_i, y_i \mid i < n \rangle$ and two neighborhoods u_n, v_n such that $u_n \cap B \neq \emptyset$ and $v_n \cap B \neq \emptyset$ with the following properties: (1) for all $i < n$, x_i, y_i do not see any point of $u_n \cup v_n$; (2) $\operatorname{diam}(u_n)$, $\operatorname{diam}(v_n) < 1/n$ and $u_0 \cap v_0 = \emptyset$; (3) if $m > n$ then $u_m \subseteq u_n$ and $v_m \subseteq v_n$; (4) $x_{n+1} \in u_n$ and $y_{n+1} \in v_n$.

Clearly, if the construction is carried out successfully for all n, the set $\langle x_n, y_n \mid n \in N \rangle$ is a visually independent set which converges to two points of B, $\bigcap_n u_n$ and $\bigcap_n v_n$.

At the stage $n = 0$ pick two points of b and let u_0 and v_0 be two sufficiently small disjoint neighborhoods of those points. We break the stage $n + 1$ into two parts. We first pick x_n, then y_n. First we pick a triangle $\Delta(b_1, b_2, b_3)$ whose vertices belong to $u_n \cap B$ and such that $\operatorname{con}(b_1, b_2, b_3) \not\subseteq S$. We know that such a triangle exists by the fact that $\operatorname{con}(B \cap u_n) \not\subseteq S$ and Carathéodory's theorem. Pick a hole inside $\Delta(b_1, b_2, b_3)$, namely a connected component of $\mathbf{R}^2 - S$. By the assumption that B is a seeing subset, all the edges of the triangle are in S, and therefore the hole is contained in the interior of the triangle.

4.3.1. CLAIM. There is an ellipse of maximal area that is contained in the hole and its boundary touches the hole's boundary at three points such that the ellipse's center is contained in their convex closure.

PROOF OF CLAIM. Pick a maximal ellipse which is contained in the hole. By applying an affine transformation we may assume it is the unit circle. We assume to the contrary that the intersection of the ellipse's boundary with the hole's boundary does not contain the origin in its convex closure. Without loss of generality, this intersection lies in the lower half plane. Pick a constant $0 < c < 1/2$ and define the ellipse

$$E_\varepsilon = \{\langle x, y \rangle \mid x^2 + y^2 - \varepsilon(y - c)(y + 1) \leqq 1\}.$$

This is a stretching of the circle upwards which fixes its lowest point, with center

$$\left\langle 0, \frac{\varepsilon(1-c)}{2(1-\varepsilon)} \right\rangle .$$

As there is a minimum to the distance between the hole's boundary and the arc $\{\langle x, y\rangle \mid x^2+y^2=1, y \geqq c\}$, a choice of a sufficiently small ε guarantees that E_ε is contained in the hole. A direct calculation shows that the area of E_ε is greater than that of the unit disk, thus contradicting its maximality. This shows that the origin is in the convex closure of the intersection of the boundaries, therefore it is in the convex closure of some three intersection points.

So we pick such an ellipse and three such points. Each of these three points does not see (at least) an open half-plane determined by the tangent to the ellipse at that point. These three half planes cover the whole plane (because the ellipse's center is contained in the convex closure of the points). So at least one of these half planes contains points of $B \cap v_n$. Let x_n be one of those three points which does not see a point $b \in v_n \cap B$. By the definition of x_n, there must be a vertex, say b_1, that x_n does not see. Let u' be a neighborhood of b_1 not seen by x_n and let v' be a neighborhood of b not seen by x_n.

Now change the roles, and let v' act as u_n and let u' act as v_n in the choice of x_n to choose y_n and two neighborhoods $v_{n+1} \subseteq v_n$ and $u_{n+1} \subseteq u_n$. Lastly, shrink the chosen neighborhoods further, if necessary, to have sufficiently small diameter as required by the construction. It is easy to verify that the choice fulfills the induction hypotheses.

4.4. Problem. Calculate for which n does C_n imply a countable γ for closed $S \subseteq \mathbf{R}^2$.

5. Non-closed sets with finite α and $\gamma = \aleph$

5.1. Example. We define $A \subseteq \mathbf{R}^4$ with $\alpha(A) = 2$ and $\gamma(A) = \aleph$.

Define the curve $M = \{(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta) \mid 0 \leqq \theta \leqq 2\pi\}$ and let $K = \mathrm{con}(M)$. $\mathrm{ext}(K)$, the set of extremal points of K, lies on a 4-sphere, so we note that $\mathrm{bd}(K) = \bigcap_{q \neq p \in M}(p, q)$.

Claim. Every edge (p, q) is an edge of K.

Proof. Let $p = M(\phi_1)$, $q = M(\phi_2)$. Observe the function

$$u(\theta) = (1 - \cos(\theta - \phi_1))(1 - \cos(\theta - \phi_2)).$$

Clearly, $u(\theta) \geqq 0$ and equality holds only when $\theta = \phi_1$ or $\theta = \phi_2$. Using trigonometric identities, we can express $u(\theta) = a_0 + a_1 \cos\theta + a_2 \sin\theta +$

$a_3 \cos 2\theta + a_4 \sin 2\theta$ where the a_is are constants that depend only on ϕ_1, ϕ_2. Now define the function $F(x_1, x_2, x_3, x_4) = a_0 + a_i x_i$, $1 \leqq i \leqq 4$. The non-negtivity of u assures us that $F^{-1}(0)$ is a supporting plane of K — so we have proven the claim.

An easy corollary of the claim is that for any two open segments (p, q) and (r, t) the intersection is empty.

We make use now of [ER], which assures the existence of a triangle free graph of size $\aleph$ whose chromatic number is also $\aleph$, to pick such a graph whose vertices are the points on M. To define our set S, remove from K all open segments (p, q) such that p, q are connected by an edge of the graph we picked.

Now as the chromatic number of the graph is $\aleph$, it is clear that we cannot cover S by less than $\aleph$ seeing subsets. Note that if p, q do not see each other in S it must be that $p, q \in M$ and that (p, q) was removed, by the claim and its corollary. So we cannot have a visually independent set of size 3 because the graph has no triangles.

5.2. EXAMPLE. We define a Borel set $S \subseteq \mathbf{R}^4$ with $\alpha(S) = 3$ and $\gamma(S) = \aleph$.

Enumerate in a sequence $\langle u_n \mid n \in N \rangle$ all rational open neighborhoods. By induction pick x_n such that $x_n \in u_n$ and such that $x_1, \ldots, x_n$ are in general position. There is no difficulty in carrying out this process. Now let $S = \mathbf{R}^4 - \{x_n\}_{n \in N}$. If $v_1, v_2, v_3, v_4 \in S$ is a visually independent set, then either there are three x_ns on a line — in case all the v_is are collinear — or there are at least 5 x_ns in a 3-space, contrary to the definition of $\{x_n\}$. So we establish $\alpha(S) \leqq 3$. The reverse inequality is easy. As for the γ, the set of points (t, t^2, t^3, t^4), where t is a real number, is such that every 4-tuple of them forms a simplex from which a point was removed — therefore $\gamma(S) = \aleph$.

5.3. EXAMPLE. We define a Borel set $S \subseteq \mathbf{R}^3$ with $\alpha(S) = 4$ and $\gamma(S) = \aleph$.

Enumerate all rational neighborhoods. At the nth stage remove $x_n \in u_n - \mathrm{Cl}(Q(x_1, \ldots, x_{n-1}))$, where $\mathrm{Cl}(Q(x_1, \ldots, x_{n-1}))$ is the algebraic closure of the field extension obtained from the rationals by adding the $3n - 3$ coordinates of the xs. So the set $\{x_n\}$ is dense, and the set of the coordinates of its members is an algebraically independent set. Let S be $R^3 - \{x_n\}$. As before, $\gamma(S)$ is clearly the continuum. Let us see why $\alpha(S) \geqq 4$. Pick any non-degenerate 4-gon T. Pick a small enough neighborhood around the middle of each of the 6 edges of T. The neighborhoods are so small, such that any choice of six points, one point from each neighborhood, determines a non-degenerate 4-gon whose edges pass through these points. Use the density of the x_n to pick one x_n inside each neighborhood. So each 3-tuple of xs determines a side of a 4-gon. The algebraic independence of the xs assures that the vertices of this 4-gon are not xs. As in each edge there is a missing point, the vertices are a visually independent set.

For the converse inequality, note that if $\{p_1, \ldots, p_5\}$ is a visually independent set, no four ps are co-planar, for then we would have at least five xs, which are required to hide the ps from each other, in one plane. So the p_is are in general position. This means we need exactly 10 x_n to interrupt seeing. Now add to the field of rationals the 15 coordinates of the p_is. An addition of the coordinates of each x of the 10 points on the edges increases the transcendental degree of the field by at most 1, for it is on an edge whose endpoints are in the field. So we can have all ten xs in a field of transcendental degree of no more than 25, while by the construction 30 is required. Contradiction.

5.4. Example. We define a set $S \subseteq \mathbf{R}^2$ with $\alpha(S) = 5$ and $\gamma(S) = \aleph$.

Enumerate all rational neighborhoods. At the nth stage remove $x_n \in u_n - \mathrm{Cl}(Q(x_1, \ldots, x_{n-1}))$, where $\mathrm{Cl}(Q(x_1, \ldots, x_{n-1}))$ is the algebraic closure of the field extension obtained from the rationals by adding the $2n-2$ coordinates of the xs. So the set $\{x_n\}$ is dense, and the set of the coordinates of its members is an algebraically independent set. Let S be $R^2 - \{x_n\}$. As in the previous example, obviously $\gamma(S) = \aleph$. Now let us see why $\alpha(S) \geqq 5$.

In Fig. 9 we first choose four x_ns, without loss of generality x_1, x_2, x_3, x_4 which are near $\langle 0, 1\rangle$, $\langle -1, 0\rangle$, $\langle 0, -1\rangle$ and $\langle 1, 0\rangle$ respectively. Assume, for simplicity, that $x_1 = \langle 0, 1\rangle$, $x_2 = \langle -1, 0\rangle$ and so on. Let $p_0 = \langle 0, 0\rangle$. Let $p_1 = \langle 0, 2\rangle$. Let $p \in (p_1, \langle -2, 0\rangle)$ be such that $d(p_1, p) = 1/2$. Within ε of p pick x_5 and let p_2 be the intersection point between the x-axis and (p_1, x_5). So p_2 is near $\langle -2, 0\rangle$. Do the same to find p_3 and p_4. Pick p_5 on the y-axis, an epsilon below p_1, and let $x_8 \in (p_4, p_5)$ be such that $|d(x_8, p_4) - 1/2| < \varepsilon$. So p_0 does not see any of the other p_is, and p_i does not see neither p_{i+1} nor p_{i+2}. But maybe p_1 sees p_4. In this case imagine that the x_ns are hinges, and that (p_i, p_{i+1}) are rods, and pull p_1 upwards. Follow the arrows to see that p_5 will move upwards faster than p_1, so in a short way p_1 and p_5 merge. As two xs are already removed from the y-axis, p_1 and p_5 do not merge at a point which was removed. This yields a visually independent set, $\{p_0, \ldots, p_4\}$, of size five.

It remains to be seen why $\alpha(S) \leqq 5$. So assume to the contrary that $a_1, \ldots, a_6$

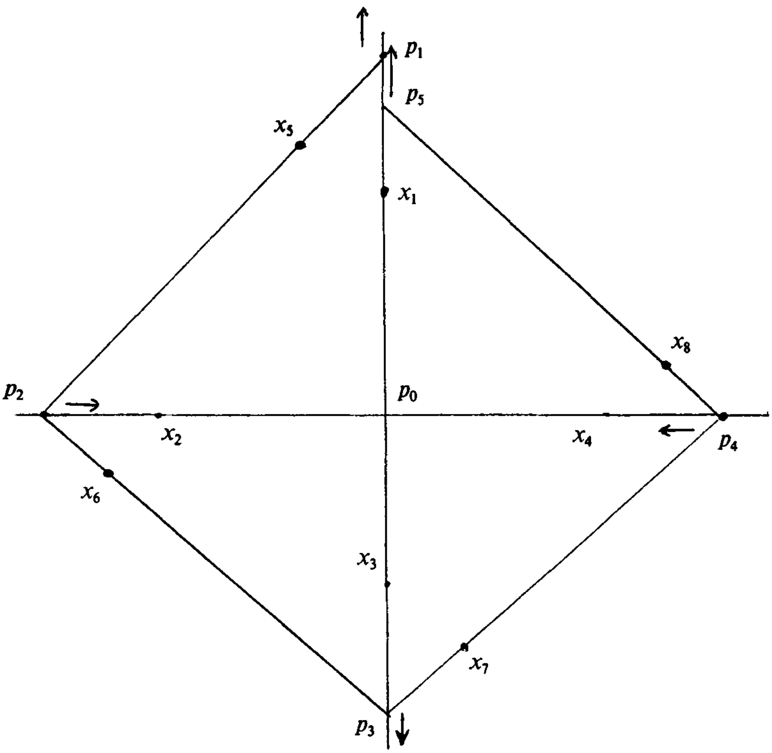

Fig. 9.

are six points which constitute a visually independent subset of S. This means that in every interval (c_i, c_j) there is a point $c \in \{x_n\}$. We pick a set of such cs of minimal size and order those points in a list $c_1, \ldots, c_m$ where $m \leqq 15$. Define ν_k as the number of intervals (c_i, c_j) such that $c_k \in (a_i, c_j)$, but for no $l < k$, $c_l \in (a_i, a_j)$. We compute now a bound λ_k of the transcendental degree of $Q(a_1, \ldots, a_6, c_1, \ldots, c_k)$ by recursion. $\lambda_0 = 12$ — for we have twelve coordinates of the a_is. If $\nu_k = 1$, then λ_k is $\lambda_{k_1} + 1$, for once one of the coordinates of c_k is added to a field which contains the c_is, the other is a solution of a linear equation. In case $\nu_k \geqq 2$, the coordinates of c_k are already in the field containing the c_is, being the solution of a linear system of equations. In this case $\lambda_k = \lambda_{k-1}$. Note that $\nu_k \leqq 3$ (for there are only six c_is), and that the value 3 can be assumed at most once by ν_k. So, in the first case assume that $\nu_k \leqq 2$ for all $k \leqq m$. Then we can write $\lambda_0 = 12$ and for $k > 0$, $\lambda_k = \lambda_{k-1} + 2 - \nu_k$. This gives that $\lambda_m = 12 + 2m - \Sigma\nu_k = 12 + 2m - 15 < 2m$ — a contradiction to the algebraic independence of the $2m$ coordinates of $c_1, \ldots, c_m$. In the second case we assume without loss of generality that $\nu_1 = 3$ and write $\lambda_1 = 12$, $\lambda_k = \lambda_{k-1} + 2 - \nu_k$. Now the recursion has only $m - 1$ steps, so we have $\lambda_m = 12 + 2(m-1) - \Sigma\nu_k = 12 + 2m - 2 - 12 < 2m$, again a contradiction.

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