



ELSEVIER

Topology and its Applications 99 (1999) 135–235

---



---

**TOPOLOGY  
AND ITS  
APPLICATIONS**


---



---

www.elsevier.com/locate/topol

## Special subsets of ${}^{\text{cf}(\mu)}\mu$ , Boolean algebras and Maharam measure algebras<sup>☆</sup>

Saharon Shelah<sup>a,b,c,1</sup><sup>a</sup> *Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel*<sup>b</sup> *Department of Mathematics, Rutgers University, Busch Campus, Piscataway, NJ 08854, USA*<sup>c</sup> *Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA*

Received 12 June 1996; received in revised form 5 June 1999

---

### Abstract

The original theme of the paper is the existence proof of “there is  $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$  which is a  $(\lambda, J)$ -sequence for  $\bar{I} = \langle I_i : i < \delta \rangle$ , a sequence of ideals”. This can be thought of as a generalization to Luzin sets and Sierpinski sets, but for the product  $\prod_{i < \delta} \text{dom}(I_i)$ , the existence proofs are related to pcf.

The second theme is when does a Boolean algebra  $\mathbf{B}$  have a free caliber  $\lambda$  (i.e., if  $X \subseteq \mathbf{B}$  and  $|X| = \lambda$ , then for some  $Y \subseteq X$  with  $|Y| = \lambda$  and  $Y$  is independent). We consider it for  $\mathbf{B}$  being a Maharam measure algebra, or  $\mathbf{B}$  a (small) product of free Boolean algebras, and  $\kappa$ -cc Boolean algebras. A central case is  $\lambda = (\square_\omega)^+$ , or more generally,  $\lambda = \mu^+$  for  $\mu$  strong limit singular of “small” cofinality. A second one is  $\mu = \mu^{<\kappa} < \lambda < 2^\mu$ ; the main case is  $\lambda$  regular but we also have things to say on the singular case. Lastly, we deal with ultraproducts of Boolean algebras in relation to  $\text{irr}(-)$  and  $s(-)$  etc. © 1999 Published by Elsevier Science B.V. All rights reserved.

**Keywords:** Set theory; Boolean algebra; Maharam algebra; pcf; Caliber

**AMS classification:** Primary 03E05; 06E05; 03E35, Secondary 28A60; 04A20; 03E75

---

## 0. Introduction

### 1. The framework and an illustration

We define when “ $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$  is a  $(\lambda, I, J)$ -sequence for  $\bar{I} = \langle I_i : i < \delta \rangle$ ”, which means ( $I = J_\lambda^{\text{bd}}$  for simplicity) that each  $\eta_\alpha \in \prod_{i < \delta} \text{Dom}(I_i)$  and that for  $\bar{A} = \langle A_i : i < \delta \rangle \in \prod_{i < \delta} I_i$  for all large enough  $\alpha$ ,  $\eta_\alpha$  “run away” from  $\bar{A}$ , i.e., for the  $J$ -majority of

---

<sup>☆</sup> Publication no. 620. The research partially supported by NSF under grant #144-EF67 and by the Israel Science Foundation.

<sup>1</sup> E-mail: shelah@sunrise.huji.ac.il.

$i < \delta$ ,  $\eta_\alpha(i) \notin A_i$ . We give the easy existence if  $I_i$  is  $\kappa_i$ -complete and  $\langle \kappa_i: i < \delta \rangle$  are strictly increasing converging to a strong limit (singular)  $\mu$ ,  $\mu^+ = 2^\mu = \lambda$  (1.8). We define normality, explain how by the existence of such  $\bar{\eta}$ , coloring properties can be lifted (1.6). As an illustration we prove that (the well known result that), e.g., if  $\lambda = 2^{\aleph_\omega} = \beth_\omega^+$ , then  $\beth_\omega^+$  is not a free caliber of the Maharam measure algebra (i.e., some set  $X$  of  $\lambda$  elements, is non-independent, in fact in a more specific way). For this we use ideals related to the Erdős–Rado Theorem.

## 2. There are large free subsets

Why does the application in §1 involve  $\lambda$  “near” a strong limit singular  $\mu$  of cofinality  $\aleph_0$ ? We show that this was necessary: if  $\mu^{\aleph_0} < \lambda \leq 2^\lambda$  and  $\text{cf}(\lambda)$  is large enough ( $> \beth_2$  is OK,  $> 2^{\aleph_0}$  is almost OK, but involves more pcf considerations), then  $\lambda$  is a free caliber of the Maharam measure algebra.

## 3. Strong independence in Maharam measure algebras

We define when “ $\bar{\eta}$  is a super  $(\lambda, I, J)$ -sequence for  $\bar{I}$ ”. The strengthening is that we now can deal with  $n$ -tuples (any  $n < \omega$ ) and prove the easy existence (see 3.1, 3.2). We define for a set of  $\lambda$  intervals in a Boolean algebra variants of independence and strong negation of it (3.4) and apply it to prove existence of a strongly  $\lambda$ -anti-independent set in a Maharam measure algebra (3.6), which (by 3.7) suffices for having a subalgebra of dimension  $\lambda$  with no independent set of cardinality  $\lambda$ . This completes the consistency part of the solution of a problem, which was to characterize all cardinals  $\lambda$  which can have this property. The question was asked for  $\lambda = \aleph_1$  by Haydon and appeared in Fremlin’s book [5]. Haydon [8, 9] and Kunen [10] independently proved it to be consistent for  $\lambda = \aleph_1$  assuming CH. The question from [8] and [5] was what happens with  $\aleph_1$  under MA. Recently, Plebanek [13, 14] proved that under MA all regular cardinals  $\geq \aleph_2$  fail the property, and finally Fremlin [5] gave the negative answer to the original question of Haydon by showing that under MA the property fails for  $\aleph_1$ . Džamonja and Kunen [2,3] considered the general case (any  $\lambda$ ) and topological variants.

We prove here, e.g., if  $\lambda = \beth_{\omega+1} = \beth_\omega^+$ , then there is a Hausdorff compact zero-dimensional topological space with measure on the family of the Borel subsets such that it has dimension  $\lambda$ , so as a measure space is isomorphic to the Maharam measure space  $\mathcal{B}(\lambda)$ , but there is no homomorphism from  $X$  onto  ${}^\mu 2$  (see 3.8). We finish by some easy examples.

## 4. The interesting ideals and the direct pcf application

We return to our original aim: existence of  $\lambda$ -sequences for  $\bar{I}$ . In 4.1 we consider some ideals  $(J_A^{\text{bd}}, \prod_{\ell < n} J_\ell, J_{\langle \lambda_\ell: \ell < n \rangle}^{\text{bd}}) = \prod_\ell J_{\lambda_\ell}^{\text{bd}}$ , each  $\lambda_\ell$  regular, in the cases  $\lambda_\ell < \lambda_{\ell+1}$ ,  $\lambda_\ell >$

$\lambda_{\ell+1}, \lambda_\ell > 2^{\lambda_{\ell+1}}$ ). We point out (4.8) that for  $\bar{I} = \langle J_{\lambda_i}^{\text{bd}} : i < \delta \rangle$ , if  $\lambda = \text{tcf}(\prod_{i < \delta} \lambda_i / J_\delta^{\text{bd}})$  we get existence directly from the pcf theory. We then turn to the case  $I_i = \prod_{\ell < n_i} J_{\lambda_{i,\ell}}^{\text{bd}}$ , give a sufficient pcf condition for the existence when  $\langle \lambda_{i,\ell} : \ell < n \rangle$  is increasing (4.10) and then prove that this condition occurs not rarely (in 4.12), so if

$$\lambda = \prod_{i < \delta} \lambda_i / J_\delta^{\text{bd}},$$

$\lambda_i$  increasing, we can “group together” intervals of  $\lambda_i$ ; and the existence of such  $\langle \lambda_i : i < \delta \rangle$  is an important theme of pcf theory.

### 5. $\lambda$ -sequences for decreasing $\bar{\lambda}^i$ by pcf

We consider cases with  $I_i = J_{\langle \lambda_{i,\ell} : \ell < n_i \rangle}^{\text{bd}}$ ,  $\langle \lambda_{i,\ell} : \ell < n_i \rangle$  a decreasing sequence of regulars. We prove the existence by using twice cases of true cofinalities, and show that if the pcf structure is not so simple then there are such cases (e.g.,  $\beth_{\omega_i+1} > \beth_{\omega_i}^{+\omega}$ ). We concentrate on the case  $i < \delta \Rightarrow n_i = n$ , and then indicate the changes needed in the general case.

### 6. Products of Boolean algebras

Monk asks about the free caliber of products of  $B_i = \text{FBA}(\chi_i) =$  the free algebra with  $\chi_i$  generators, for  $i < \delta$ . In fact he asks whether  $\lambda = \beth_\omega^+$  is a free caliber of the product of the  $\text{FBA}(\beth_n)$  for  $n < \omega$ . But we think that the intention was to ask if  $\lambda = \text{cf}(\lambda) > 2^{|\delta|}$  is a free caliber of  $\prod_{i < \delta} B_i$ . Note that this product satisfies the  $(2^{|\delta|})^+$ -c.c. In fact it has cellularity  $2^{|\delta|}$ , so “tends to have free calibers”. We show that if there is a normal super  $(\lambda, J)$ -sequence  $\bar{\eta}$  for appropriate  $\bar{I} = \langle I_n : n < \omega \rangle$ , then  $\lambda$  is not a free caliber of  $\prod_{n < \omega} \text{FBA}(|\text{Dom } I_n|)$  (see 6.3, 6.3A), so a negative answer is possible. Now being “near a strong limit singular of cofinality  $\aleph_0$ ” is necessary as a result parallel to that of §2 holds (see 6.4).

Though the choice of  $\beth_\omega$  was probably just natural as the first case to consider, actually the product of uncountably many  $\text{FBA}(\chi_i)$ 's behave differently, e.g.,  $\prod_{i < \omega_1} \text{FBA}(\beth_i)$  has free caliber  $(\beth_{\omega_1})^+$ ! (see 6.5). The proof involves pcf considerations dealt with in §7. We turn to another problem of Monk [12, Problem 34], this time giving unambivalent solution. If  $\kappa$  is weakly inaccessible with  $\langle 2^\mu : \mu < \kappa \rangle$  not eventually constant, then there is a  $\kappa$ -c.c. Boolean algebra of cardinality  $2^{<\kappa}$  and no independent subsets of cardinality  $\kappa^+$  (see 6.8, using the existence of suitable trees). We note that results similar to countable products hold for the completion of  $\text{FBA}(\chi)$ .

We end by deducing from Gitik and Shelah [6] complementary consistency results (so, e.g., the first question is not answerable in ZFC) and phrasing the principles involved, so slightly sharpening the previous results. (See 6.11–6.14.) So together with the earlier part of the section we have answered [12, Problems 35, 36] and [12, Problems 32, 33] in the case we are near a strong limit singular cardinal.

## 7. A nice subfamily of function exists

For completeness we deal with the following:  $f_\alpha \in {}^\theta \text{Ord}$  for  $\alpha < \lambda$  are given,  $2^\theta < \lambda = \text{cf}(\lambda)$  and we would like to get approximation to “for some  $X \subseteq \lambda$ ,  $|X| = \lambda$ ,  $\langle f_\alpha : \alpha \in X \rangle$  is a  $\Delta$ -system”, continuing [26, Claim 6.6D]. We phrase a special case (7.1) and deal with some variants.

## 8. Consistency of “ $\mathcal{P}(\omega_1)$ has a free caliber” and discussion of pcf

We deal with another of Monk’s problems, [12, Problem 37], proving the consistency of “there is no complete Boolean algebra  $\mathbf{B}$  of cardinality  $2^{\aleph_1}$  with empty free caliber” (in fact  $\aleph_{\omega_1+1} = 2^{\aleph_1}$  is always a free caliber of  $\mathbf{B}$ ). The universe is obtained by adding  $\aleph_{\omega_1+1}$  Cohens to a model of ZFC + GCH, and the proof uses §7. We finish by discussing some pcf problems.

## 9. Having a $\lambda$ -sequence for a sequence of non-stationary ideals

We return to the original theme, for a more restricted case. We assume  $\lambda = \text{cf}(2^\mu)$  where  $\mu$  is strong limit singular, and in this section  $\lambda = 2^\mu$ , i.e.,  $2^\mu$  is regular (for the singular case see §10). We get quite strong results: (fix  $n(*) < \omega$  for simplicity) for some ideal  $J$  on  $\text{cf}(\mu)$  (usually  $J_{\text{cf}(\mu)}^{\text{bd}}$ , always close to it) we can find  $\langle \bar{\lambda}^i : i < \text{cf}(\mu) \rangle$ ,  $i < j \Rightarrow \max(\bar{\lambda}^i) < \min(\bar{\lambda}^j)$ ,  $\bar{\lambda}^i = \langle \lambda_{i,\ell} : \ell < n(*) \rangle$ ,  $\lambda_{i,\ell+1} > 2^{\lambda_{i,\ell}}$  ( $\lambda_{i,\ell}$  regular of course,  $\mu = \sup_{i < \text{cf}(\mu)} \lambda_{i,0}$ ), such that there is a  $(\lambda, J)$ -sequence for  $\bar{I} = \langle J_{\bar{\lambda}^i}^{\text{bd}} : i < \text{cf}(\mu) \rangle$ . This is nice (compare with §5) but we get much more:  $\bar{I}$  is a sequence of nonstationary ideals and even  $\langle \prod_{\ell < n(*)} J_{\lambda_{i,\ell}}^{\text{nst},\sigma} : i < \text{cf}(\mu) \rangle$  where  $J_\chi^{\text{nst},\sigma} = \{A : A \cap \{\delta < \chi : \text{cf}(\delta) = \sigma\}$  is not stationary} and  $\sigma = \text{cf}(\sigma) \in (\text{cf}(\mu), \mu)$ .

We then work more and get versions with club guessing ideals. We deal further with the version we get for the case  $\text{cf}(\mu) = \aleph_0$ . (So it is less clear which ideals  $J$  can be used.)

## 10. The power of a strong limit singular is itself singular: Existence

We do the parallel of the first theorem of §9 in the case  $2^\mu$  is singular.

## 11. Preliminaries to the construction of ccc Boolean algebras with no large independent sets

Here the problem is whether every  $\kappa$ -c.c. Boolean algebra has free caliber  $\lambda$ ; the case of being “near a strong limit singular  $\mu$  of cofinality  $< \kappa$ ” was considered in [18], we deal with the case  $\mu = \mu^{<\kappa} < \lambda < 2^\mu$ . Here we make the set theoretic preparation for a proof of the consistency of a negative answer with strong violation of GCH. We use

Boolean algebras generated by  $x_\alpha$ 's freely except for  $x_\alpha \cap x_\beta \cap x_\gamma = 0$  for  $\{\alpha, \beta, \gamma\} \in W$  for some set  $W$  of triples with intersection of any two having at most one element. The point is that the properties of “ $\bar{\eta}$  is a  $\lambda$ -sequence for  $\bar{I}$ ” with such ideals  $I$  (unlike the ones associated with the Erdős–Rado theorem) are preserved by adding many Cohens to  $\mu$  (where  $\mu \ll |\text{Dom}(I_i)|$  etc.).

## 12. Constructing ccc Boolean algebras with no large independent sets

We complete the consistency results for which the ground was prepared in §11. We construct the relevant Boolean algebra using a  $(\lambda, J)$ -sequence for  $\bar{I}, \bar{I}$  as there, using, as building blocks, Boolean algebras generated, e.g., from the triple system. So we will give sufficient conditions for the  $\kappa$ -c.c. and other properties of the Boolean algebra.

## 13. The singular case

We continue §11 and §12 by dealing here with the case  $\lambda$  is singular but  $(\forall \alpha < \lambda) (|\alpha|^{<\kappa} < \lambda)$ , note that the forcing from §12 essentially creates only such cases.

## 14. Getting free caliber for regular cardinals

We continue dealing with  $\kappa$ -c.c. Boolean algebras, giving a sufficient condition for  $\lambda$  being a free caliber, hence a consistency follows (complementing §11 and §12; together this solves [12, Problems 32, 33] in the case we are not near a strong limit singular cardinal; thus together with §6 this gives a solution).

## 15. On irr: The invariant of the ultraproduct, greater than the ultraproduct of invariants

We prove the consistency of

$$\text{irr}\left(\prod_{n < \omega} \mathbf{B}_n / \mathcal{D}\right) > \prod_{n < \omega} \text{irr}(\mathbf{B}_n) / \mathcal{D}$$

where  $\mathcal{D}$  is a nonprincipal ultrafilter on  $\omega$  and  $\text{irr}(\mathbf{B}) = \text{irr}_\omega(\mathbf{B})$  and  $\text{irr}_n(\mathbf{B}) = \sup\{|X|: X \subset \mathbf{B} \text{ and if } x_0, x_1, \dots, x_m \text{ are distinct members of } X, m < n \text{ then } x_0 \notin \langle x_1, \dots, x_m \rangle_{\mathbf{B}}\}$ . The way is to build  $\mathbf{B}_n$  with  $\text{irr}_n(\mathbf{B}_n) = \lambda^+$ ,  $\text{irr}_{2n+1}(\mathbf{B}_n) = \lambda$ ,  $\lambda = \lambda^{\aleph_0}$ . Our earlier tries as the approximation to  $\mathbf{B}_n$  did not work. So the point is a version of  $n$ -graded independence phrased as  $\langle F_\ell: \ell < n \rangle$ , then solve [12, Problem 26]. We then deal with  $s(\cdot)$ ,  $\text{hL}(\cdot)$ ,  $\text{hd}(\cdot)$  and  $\text{Length}(\cdot)$ , using the construction of §12 in ZFC, and solving [12, Problems 22, 46, 51, 55].

## 0. Introduction

Our original aim was to construct special subsets of  $\prod_{i < \delta} \lambda_i$ , concentrating particularly on the case when  $\lambda_i$  converge to a strong limit singular.

This continues [18] (so [25,28], Rosłanowski and Shelah [15]), but as these are essentially notes from the author's lectures in Madison, they are self-contained. (§1 and §4 just represent old material, adding an illustration for Maharam algebras.)

Some sections improve the general existence theorems. The main new point is the case when we use

$$I_i = \prod_{\ell < n_i} J_{\lambda_{\ell,i}}^{\text{bd}} \quad \text{with the } \lambda_{\ell,i} \text{'s regular decreasing}$$

(as well as the case of the nonstationary ideal). We shall discuss this below and give the definition after we first fix some notation.

### Notation.

(1)  $I$  denotes an ideal on a set  $\text{Dom}(I)$ , which means that  $I$  is a subset of  $\mathcal{P}(\text{Dom}(I))$  closed under (finite) unions and subsets,  $\text{Dom}(I) \notin I$ , and usually for simplicity, all singletons are assumed to belong to  $I$ .

$I$  is  $\kappa$ -complete if it is closed under unions of  $< \kappa$  elements.

(2)  $I, J$  denote ideals.

(3)  $I^+ = \{A \subseteq \text{Dom}(I): A \notin I\}$ .

(4) If  $A$  is a set of ordinals with no last member we let

$$J_A^{\text{bd}} = \{B \subseteq A: B \text{ a bounded subset of } A\}.$$

(5) The completeness of the ideal  $I$ ,  $\text{comp}(I)$  is the maximal  $\kappa$  such that  $I$  is  $\kappa$ -complete (it is necessarily a well-defined regular cardinal).

(6)  $[A]^\kappa = \{a \subseteq A: |a| = \kappa\}$ ,  $[A]^{<\kappa} = \{a \subseteq A: |a| < \kappa\}$ , etc.

(7)  $\text{cov}(\lambda, \mu, \theta, \sigma) = \text{Min}\{\mathcal{P}: \mathcal{P} \subseteq [\lambda]^{<\mu}, \text{ and for every } a \in [\lambda]^{<\theta} \text{ there are } \alpha < \sigma \text{ and } a_i \in \mathcal{P} \text{ for } i < \alpha \text{ such that } a \subseteq \bigcup_{i < \alpha} a_i\}$ .

**Definition.** We say  $\bar{\eta} = \langle \eta_\alpha: \alpha < \lambda \rangle$  is a  $(\lambda, I, J)$ -sequence for  $\bar{I} = \langle I_i: i < \delta \rangle$  if

(a)  $I$  is an ideal on  $\lambda$  (if not mentioned, we assume  $I = J_\lambda^{\text{bd}}$ ),  $I_i$  is an ideal on  $\text{Dom}(I_i)$ ,

(b)  $J$  is an ideal on  $\delta$  (if not mentioned, we assume  $J = J_\delta^{\text{bd}}$ ),

(c)  $\eta_\alpha \in \prod_{i < \delta} \text{Dom}(I_i)$ ,

(d) If  $X \in I^+$  then

$$\{i < \delta: \{\eta_\alpha(i): \alpha \in X\} \in I_i\} \in J.$$

By [18], if  $I_i$  is  $\kappa_i$ -complete,  $\kappa_i > \sum_{j < i} \kappa_j$ ,  $\mu = \sum_{i < \delta} \kappa_i$  strong limit,  $|\text{Dom}(I_i)| < \mu$  and  $2^\mu = \mu^+ = \lambda$ , then there is such a sequence. We recall this in §1.

As an example of the application of such  $\bar{\eta}$ , we presented the following (presented in 1.13): Suppose that  $\mathcal{B}$  is a Maharam measure algebra of dimension  $\geq \mu$ ,  $\text{cf}(\mu) = \aleph_0$ . Then we can find  $a_\alpha \in \mathcal{B}$  for  $\alpha < \lambda$  such that  $\text{Leb}(a_\alpha) > 0$  and

$$(\forall X \in [\lambda]^\lambda)(\exists n)(\forall \alpha_0 < \dots < \alpha_n \in X) \bigcap_{i \leq n} a_{\alpha_i} = 0.$$

A “neighborhood” of  $\mu$  being strong limit of cofinality  $\aleph_0$  is necessary.

Our usual case, which we call normal is:  $\kappa_i > \prod_{j < i} |\text{Dom}(I_j)|$  (this was not used in the measure algebra application, but it is still good to have).

**Main point.** The main new point of this paper is to build a  $(\lambda, I, J)$ -sequence  $\bar{\eta}$  for certain  $\bar{I}$  without using  $2^\mu = \mu^+$ . We describe the cases of  $\bar{I}$  which we can handle.

*Case 1:* The easiest case of  $I_i: I_i = J_{\lambda_i}^{\text{bd}}, \lambda = \text{cf}(\prod_{i < \delta} \lambda_i / J)$ . We only need to translate from the known pcf results.

*Case 2:*

$$I_i = \prod_{\ell < n_i} J_{\lambda_{\ell,i}}^{\text{bd}},$$

where  $\lambda_{\ell,i}$  are regular increasing with  $\ell$  and  $i$ , and  $J$  is an ideal on  $\{(i, \ell): i < \delta, \ell < n_i\}$  such that

$$(\forall X \in J)(\exists (J_\delta^{\text{bd}})^+ i) \left( \bigwedge_{\ell < n_i} (i, \ell) \notin X \right),$$

and where for ideals  $J_m$  ( $m < n$ )

$$\prod_{m < n} J_m^{\text{def}} \equiv \{X \subseteq \times_{m < n} \text{Dom}(J^m): \neg \exists^{J_0^+} x_0 \exists^{J_1^+} x_1 \dots \exists^{J_{n-1}^+} x_{n-1} ((x_0, \dots, x_{n-1}) \in X)\}.$$

Starting from reasonable pcf assumptions and working a little, we can handle this case as well.

*Main Case 3:*

$$I_i = \prod_{\ell < n_i} J_{\lambda_{\ell,i}}^{\text{bd}},$$

$\lambda_{\ell,i}$  regular decreasing with  $\ell$ .

We prove: If  $\bigwedge_i n_i = n$ , and  $\lambda_\ell = \text{tcf}(\prod_{i < \delta} \lambda_{\ell,i} / J')$  for  $\ell < n$ , then we can find  $\langle \eta_{\bar{\alpha}}: \bar{\alpha} \in \prod_{\ell < n} \lambda_\ell \rangle$  ( $\eta_{\bar{\alpha}}(i) \in \prod_{\ell < n} \lambda_{\ell,i}, i < \delta$ ) such that

$$\text{if } X \in \left( \prod_{\ell < n} J_{\lambda_\ell}^{\text{bd}} \right)^+ \stackrel{\text{def}}{=} (J_{(\lambda_\ell: \ell < n)}^{\text{bd}})^+$$

$$\text{then } \{i < \delta: \{\eta_{\bar{\alpha}}(i): \bar{\alpha} \in X\} \in (J_{(\lambda_{\ell,i}: \ell < n)}^{\text{bd}})\} \in J'.$$

*Interesting instances:*  $\lambda_{\ell,i}$  decreasing with  $\ell$  and  $i < j \Rightarrow \lambda_{\ell,i} < \lambda_{n,j}$ .

*Case 4:* Like Case 3, but using the nonstationary ideal, or nonstationary ideal restricted so some “large subset” of  $\lambda_{\ell,i}$  instead of  $J_{\lambda_{\ell,i}}^{\text{bd}}$ .

*Case 5:* Like Case 3 but using a suitable club guessing ideal ( $\text{id}^a(\bar{C}^{\ell,i})$ ).

On history, background etc. and on Boolean algebras, see Monk [11,12]. This works continues [18] and it evolved as follows. Getting the thesis of Carrieres, which was based on [22], we started thinking again on “free calibers”, this time on measure algebras. We noted that [18] gives the answer if, e.g.,  $\lambda = (\beth_\omega)^+ = \beth_{\omega+1}$ , and started to think of what is called here “there is a  $(\lambda, J)$ -sequence for  $\bar{I}$ ”. We started to lecture on it (§1, §4, then §5, §9, §10). Meanwhile Mirna Džamonja asked me if this doesn’t solve a problem from her thesis. This was not actually the case, but it became so in §3. Then she similarly brought me p. 256 of Monk [12] and this influenced most of the rest of the paper, while later I also looked at pages 255, 257 of [12], but not so carefully. Lastly, §15 is looking back at the problems from [15]. Some of the sections are (revisions of) notes from my lectures. So I would like to thank Christian Carrieres, Donald Monk and the participants of the seminar in Madison for their influence, and mainly Mirna Džamonja for god-mothering this paper in many ways, and last but not the least Diane Reppert for typing the paper, and even more for correcting and correcting and to David Fremlin who lately informed me that 1.13 was well known and 3.7, 3.12 have already appeared in Plebanek [13,14].

## 1. The framework and an illustration

We are considering a sequence  $\langle I_i : i < \delta \rangle$  of ideals, and we would like to find a sequence  $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$  of members of  $\prod_{i < \delta} \text{Dom}(I_i)$  which “runs away” from  $\bar{A} = \langle A_i : i < \delta \rangle$  when  $A_i \in I_i$  (see Definition 1.1 below).

When  $I_i$  is  $\kappa_i$ -complete,  $\kappa_i > \prod_{j < i} |\text{Dom}(I_j)|$ ,  $\mu = \sum_{i < \delta} \kappa_i$  strong limit singular,  $\lambda = \mu^+ = 2^\mu$ , this is easy. We present this (all from [18]) and, for illustration, an example.

### Definition 1.1.

- (1) We say that  $\bar{\eta}$  is a  $(\lambda, I, J)$ -sequence for  $\bar{I}$  if:
- $J$  is an ideal on  $\delta$  and  $I$  is an ideal on  $\lambda$ ,
  - $\bar{I} = \langle I_i : i < \delta \rangle$ , where  $I_i$  is an ideal on  $\text{Dom}(I_i)$ ,
  - $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$  where  $\eta_\alpha \in \prod_{i < \delta} \text{Dom}(I_i)$ ,
  - if  $X \in I^+$  then

$$\{i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i\} \in J.$$

- (2) We say  $\bar{\eta}$  is a weakly  $(\lambda, I, J)$ -sequence for  $\bar{I}$  if we weaken clause (d) to
- (d<sup>-</sup>) if  $X \in I^+$  then

$$\{i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i^+\} \in J^+.$$

- (3) We may omit  $J$  if  $J = J_\delta^{\text{bd}}$ , we may omit  $I$  if  $I = J_\lambda^{\text{bd}}$ , and then we may say “ $\bar{\eta}$  is a  $\lambda$ -sequence for  $\bar{I}$ ”.

We can replace  $\lambda$  by another index set.



**Definition 1.2.**

(1) We say  $\bar{\eta}$  is normally a  $(\lambda, I, J)$ -sequence for  $\bar{I}$  (or in short, “ $\bar{\eta}$  is normal”, when  $\bar{I}, I, J$  are clear) if:

(\*) for every  $i < \delta$ ,

$$\text{comp}(I_i) > |\{\eta_\alpha \upharpoonright i : \alpha < \lambda\}|.$$

(2) We say  $\bar{I} = \langle I_i : i < \lambda \rangle$  is normal if

$$\text{comp}(I_i) > \prod_{j < i} |\text{Dom}(I_j)|.$$

**Claim 1.3.** *If  $\bar{I} = \langle I_i : i < \delta \rangle$  is normal and  $\bar{\eta}$  is a  $(\lambda, I, J)$ -sequence for  $\bar{I}$  then  $\bar{\eta}$  is normal (i.e., normally a  $(\lambda, I, J)$ -sequence for  $\bar{I}$ ).*

**Proof.** As for each  $i < \delta$

$$|\{\eta_\alpha \upharpoonright i : \alpha < \lambda\}| \leq \left| \prod_{j < i} \text{Dom}(I_j) \right| = \prod_{j < i} |\text{Dom}(I_j)| < \text{comp}(I_i).$$

**Discussion 1.4.** Why is normality (and  $(\lambda, J)$ -sequences in general) of interest? Think for example of having, for each  $i < \delta$ , a coloring  $c_i$ , say a function with domain  $[\text{Dom}(I_i)]^2$  (or even  $[\text{Dom}(I_i)]^{<\aleph_0}$ ), call its range the set of colors. These colorings are assumed to satisfy “for every  $X \in I_i^+$  we can find some  $Y \subseteq X$  with  $Y \in I^+$ , such that  $c_i \upharpoonright [Y]^2$  (or  $[Y]^{<\aleph_0}$ ) is of some constant pattern”. Now using  $\bar{\eta}$  we can define a coloring  $c$  on  $[\lambda]^2$  (or  $[\lambda]^{<\aleph_0}$ ) “induced by the  $\langle c_i : i < \delta \rangle$ ”, e.g.,

$$c(\{\alpha, \beta\}) = c_{i(\alpha, \beta)}(\{\eta_\alpha(i(\alpha, \beta)), \eta_\beta(i(\alpha, \beta))\}),$$

where  $i(\alpha, \beta) = \text{Min}\{i : \eta_\alpha(i) \neq \eta_\beta(i)\}$ .

Now, normality (or weak normality) is a natural assumption, because of the following:

**Claim 1.5.** *If  $\bar{\eta}$  is normally a  $(\lambda, I, J)$ -sequence for  $\bar{I}$  (or weakly so) and  $X \in I^+$ , then the following set is  $= \delta \text{ mod } J$  (or  $\neq \emptyset \text{ mod } J$ ):*

$$Y = \left\{ i < \delta : \text{for some } v \in \prod_{j < i} \text{Dom}(I_j) \text{ and } X_i \in I_i^+ \text{ we have:} \right. \\ \left. (\forall x \in X_i)(\exists \alpha \in X)[v = \eta_\alpha \upharpoonright i \ \& \ x = \eta_\alpha(i)] \right\}.$$

**Proof.** Let  $X_i = \{\eta_\alpha(i) : \alpha \in X\}$ , by the definitions it is enough to prove

(\*) if  $X_i \in I_i^+$  then  $i \in Y$ .

Let  $Z_i = \{\eta_\alpha \upharpoonright i : \alpha < \lambda\}$ , so  $Z_i \subseteq \prod_{j < i} \text{Dom}(I_j)$  and  $|Z_i| < \text{comp}(I_i)$  by the normality of  $\bar{\eta}$ . Now for each  $v \in Z_i$  let us define

$$X_v^i = \{\eta_\alpha(i) : \alpha \in X \text{ and } \eta_\alpha \upharpoonright i = v\}.$$

Clearly  $X_i = \bigcup \{X_\nu^i : \nu \in Z_i\}$ , and  $I_i$  is  $|Z_i|^+$ -complete (as  $|Z_i| < \text{comp}(I_i)$ ). As  $X_i \in I_i^+$ , necessarily for some  $\nu \in Z$  we have  $X_\nu^i \in I_i^+$ . This exemplifies that  $i \in Y$ , as required.  $\square$

**Conclusion 1.6.** Assume

- (a)  $\bar{\eta}$  is a normal weak  $(\lambda, I, J)$ -sequence for  $\bar{I}$ .
- (b)  $\mathbf{c}_i$  is a function from  ${}^{\omega>}(\text{Dom}(I_i))$  to a set  $\mathbf{C}$  of colors (or from  $[\text{Dom}(I_i)]^{<\aleph_0}$ ).
- (c)  $\mathbf{d}$  is a function from  ${}^{\omega>}\varepsilon(*)$  (or from  $[\varepsilon(*)]^{<\aleph_0}$ ) to  $\mathbf{C}$ .
- (d)  $\mathbf{c}_i$  exemplifies  $I_i \not\rightarrow (\mathbf{d})$  which means
  - (\*) for every  $X \in I_i^+$  we can find distinct  $x_\zeta \in X$  for  $\zeta < \varepsilon(*)$  such that:
    - if  $n < \omega$  and  $\zeta_0 < \dots < \zeta_{n-1} < \varepsilon(*)$  then

$$\mathbf{c}_i(\langle x_{\zeta_0}, \dots, x_{\zeta_{n-1}} \rangle) = \mathbf{d}(\langle \zeta_0, \dots, \zeta_{n-1} \rangle)$$

$$\text{(or } \mathbf{c}_i(\{x_{\zeta_0}, \dots, x_{\zeta_{n-1}}\}) = \mathbf{d}(\{\zeta_0, \dots, \zeta_{n-1}\})\text{)}.$$

- (e) We define the coloring  $\mathbf{c}$  such that for all  $n < \omega$

$$\mathbf{c}(\langle \alpha_0, \dots, \alpha_{n-1} \rangle) = \mathbf{c}_i(\langle \eta_{\alpha_0}(i), \dots, \eta_{\alpha_{n-1}}(i) \rangle)$$

$$\text{(or } \mathbf{c}(\{\alpha_0, \dots, \alpha_{n-1}\}) = \mathbf{c}_i(\{\eta_{\alpha_0}(i), \dots, \eta_{\alpha_{n-1}}(i)\})\text{)}, \text{ when}$$

$$\ell < m < n \Rightarrow i = \text{Min}\{j < \delta : \eta_{\alpha_\ell}(j) \neq \eta_{\alpha_m}(j)\}.$$

Then  $\mathbf{c}$  exemplifies  $I \not\rightarrow (\mathbf{d})$ .

[Why? If  $X \in I^+$ , let  $Y$  be the set as in Claim 1.5, hence  $Y \in J^+$ . Pick an  $i \in Y$ , so there is  $X_i \in I_i^+$  and  $\nu$  exemplifying that  $i \in Y$ . Let  $\{x_\zeta : \zeta < \varepsilon(*)\}$  exemplify that  $I_i \not\rightarrow (\mathbf{d})$ . For  $\zeta < \varepsilon(*)$ , let  $\alpha_\zeta \in X$  be such that  $\eta_{\alpha_\zeta} \upharpoonright i = \nu$  and  $\eta_{\alpha_\zeta}(i) = x_\zeta$ . Hence for all  $n < \omega$  and  $\zeta_0 < \dots < \zeta_{n-1} < \varepsilon(*)$  we have

$$\mathbf{c}(\langle \alpha_{\zeta_0}, \dots, \alpha_{\zeta_{n-1}} \rangle) = \mathbf{c}_i(\langle x_{\zeta_0}, \dots, x_{\zeta_{n-1}} \rangle) = \mathbf{d}(\langle \zeta_0, \dots, \zeta_{n-1} \rangle).]$$

**Comments 1.6A.**

- (1) Of course in 1.6 we can restrict ourselves to coloring of pairs. Note that the conclusion works for all  $\mathbf{d}$ 's simultaneously. Also, additional properties of the  $\mathbf{c}_i$ 's are automatically inherited by  $\mathbf{c}$ , see 1.7 below.
- (2) We can also be interested in colors of  $n$ -tuples,  $n \geq 3$ , where  $i < \delta$  as in clause (e) of 1.6 does not exist.
- (3) What is the gain in the conclusion?

A reasonable gain is “catching” more cardinals, i.e., if  $I_i = J_{\lambda_i}^{\text{bd}}$ ,  $I = J_\lambda^{\text{bd}}$ , then in addition to having an example for  $\lambda_i$  we have one for  $\lambda$ . A better gain is when  $I$  is simpler than the  $I_i$ 's. The best situation is when we essentially can get  $I = J_\lambda^{\text{bd}}$ ,  $J = J_\delta^{\text{bd}}$  for all normal  $\bar{I}$  with  $\langle |\text{Dom}(I_i)| : i < \delta \rangle$  increasing with limit  $\mu$ . Assuming a case of GCH this is trivially true.

Normally we can find many tuples for which there is  $i < \delta$  as in clause (e) of 1.6.

**Fact 1.7.** In 1.6 if  $\theta = (2^{|\delta|})^+$ , or at least  $\theta = \text{cf}(\theta)$  &  $(\forall \alpha < \theta)(|\alpha|^{|\delta|} < \theta)$  then:

(\*) for every  $X \in [\lambda]^\theta$ , we can find  $Y \in [X]^\theta$  and  $i < \delta$  and a 1-to-1 function  $h$  from  $Y$  into  $\text{Dom}(I_i)$  such that

$$c(\langle \alpha_0, \dots, \alpha_{n-1} \rangle) = c_i(\langle h(\alpha_0), \dots, h(\alpha_{n-1}) \rangle)$$

for  $\alpha_0, \dots, \alpha_{n-1} \in Y$  (actually  $h(\alpha) = \eta_\alpha(i)$ , where for all  $\alpha$  we have  $\eta_\alpha \upharpoonright i = \nu$  for some  $\nu \in \prod_{j < i} \text{Dom}(I_j)$ ).

**Proof.** By the  $\Delta$ -system lemma applied to  $\{\{\eta_\alpha \upharpoonright i : i < \delta\} : \alpha \in X\}$ . More elaborately, let  $\chi$  be large enough, and let  $M \prec (\mathcal{H}(\chi), \in, <_\chi^*)$  be such that  $\{\theta, X, I, J, \bar{I}, \bar{\eta}\} \subseteq M$  and  $M^\delta \subseteq M$ , while  $\|M\| < \theta$  and  $M \cap \theta$  is an ordinal  $< \theta$ . If we choose  $\alpha \in X \setminus M$ , then we can choose  $i < \delta$  such that  $\eta_\alpha \upharpoonright i \in M$ ,  $\eta_\alpha \upharpoonright (i+1) \notin M$  (exists as  $M^\delta \subseteq M$ ). Now notice that for some such  $\alpha$  and  $i$  the set  $Z \stackrel{\text{def}}{=} \{\eta_\beta(i) : \beta \in X, \eta_\beta \upharpoonright i = \eta_\alpha \upharpoonright i\}$  has cardinality  $\theta$ . Let  $h : Z \rightarrow X$  be such that  $\gamma \in Z \Rightarrow \eta_{h(\gamma)} \upharpoonright i = \eta_\alpha \upharpoonright i$  and  $\eta_{h(\gamma)}(i) = \gamma$ . Lastly let  $Y = \text{Rang}(h)$ .  $\square$

**Lemma 1.8.** Assume

- (a)  $I_i$  is a  $\kappa_i$ -complete ideal on  $\lambda_i$  for  $i < \delta$ , and  $\delta$  is a limit ordinal,
- (b)  $\kappa_i = \text{cf}(\kappa_i) > \sum_{j < i} \kappa_j$ ,
- (c)  $\mu = \sup_{i < \delta} \kappa_i = \sup_{i < \delta} \lambda_i$ ,
- (d)  $\text{cf}(I_i, \subseteq) \leq \mu^+$  (usually in applications it is  $< \mu$  as usually  $2^{\lambda_i} < \mu$ ; the cofinality is that of a partially ordered set),
- (e)  $\lambda = \mu^+ = \mu^{|\delta|}$  (so  $\lambda = \lambda^{|\delta|}$ ; note that  $\mu^{|\delta|} \geq \mu^{\text{cf}(\mu)} \geq \mu^+$  always).

Then some  $\bar{\eta}$  is a  $\mu^+$ -sequence for  $\langle I_i : i < \delta \rangle$ .

**Remark 1.9.**

- (1) We shall focus on the case  $\mu$  as strong limit singular,  $\delta = \text{cf}(\mu)$ .
- (2) We can weaken the requirement  $\lambda = \mu^+$ , but not now and here.

**Proof of 1.8.** Let  $\mathcal{Y}_i \subseteq I_i$  be cofinal,

$$|\mathcal{Y}_i| \leq \lambda.$$

So  $|\prod_{i < \delta} \mathcal{Y}_i| \leq \lambda^{|\delta|} = \lambda$ , and we can list  $\prod_{i < \delta} \mathcal{Y}_i$  as  $\langle \langle A_i^\zeta : i < \delta \rangle : \zeta < \lambda \rangle$ , where  $A_i^\zeta \in \mathcal{Y}_i$ .

For  $\zeta < \lambda$ , let  $\langle \beta(\zeta, \varepsilon) : \varepsilon < \mu \rangle$  list  $\{\beta : \beta < \max\{\mu, \zeta\}\}$  (or  $\{\beta : \beta \leq \zeta\}$ ).

Now we choose for  $\zeta < \lambda$ , a function  $\eta_\zeta \in \prod_{i < \delta} \lambda_i$ . Let  $\eta_\zeta(i)$  be any member of

$$\lambda_i \setminus \bigcup \left\{ A_i^{\beta(\zeta, \varepsilon)} : \varepsilon < \sum_{j < i} \kappa_j \right\}.$$

[Why can we choose such  $\eta_\zeta(i)$ ? Because  $A_i^{\beta(\zeta, \varepsilon)} \in I_i$  and  $I_i$  is  $\kappa_i$ -complete and  $\kappa_i > \sum_{j < i} \kappa_j$ .]

We claim that  $\bar{\eta} \stackrel{\text{def}}{=} \langle \eta_\zeta : \zeta < \lambda \rangle$  is as required. Let  $X$  be unbounded  $\subseteq \lambda$ , we need to show  $Y$  is co-bounded in  $\delta$ , where

$$Y \stackrel{\text{def}}{=} \{i : \{\eta_\alpha(i) : \alpha \in X\} \in I_i^+\}.$$

Let  $A_i^* = \{\eta_\alpha(i) : \alpha \in X\}$  for every  $i \notin Y$ . Let  $A_i^* \stackrel{\text{def}}{=} \emptyset$  for  $i \in Y$ . Let  $A_i \in \mathcal{Y}_i$ ,  $A_i \supseteq A_i^*$ . Let  $\zeta < \lambda$  be such that  $\langle A_i : i < \delta \rangle = \langle A_i^\zeta : i < \delta \rangle$ . So for every  $\alpha \in X \setminus (\zeta + 1)$ , for every  $i < \delta$  large enough  $\eta_\alpha(i) \notin A_i$ .

[Large enough means: Just that letting  $\varepsilon = \varepsilon_{\alpha, \zeta} < \mu$  be such that  $\zeta = \beta(\alpha, \varepsilon)$  and letting  $i^* = i_{\alpha, \zeta}^*$  be such that  $\sum_{j < i^*} \kappa_j > \varepsilon$ , then  $i \in [i^*, \delta) \Rightarrow \eta_\alpha(i) \notin A_i$ .]  $\square$

**Example 1.10.**  $\lambda = \mu^+ = 2^\mu$ ,  $\mu$  strong limit of cofinality  $\aleph_0$ . Let  $\mu = \sum_{n < \omega} \mu_n$ . Without loss of generality  $\mu_{n+1} > \beth_{n+7}(\mu_n)$ . Let  $D_n = [\beth_{n+3}(\mu_n)^+]^n$ .

$$I_n \stackrel{\text{def}}{=} \{X \subseteq D_n : \text{there is } h : X \rightarrow 2^{\mu_n} \text{ such that for no infinite } A \text{ is } h \upharpoonright [A]^n \text{ constant}\}.$$

**Fact 1.11.**  $I_n$  is an ideal.

**Fact 1.11A.** The ideal  $I_n$  is not trivial (so  $D_n \notin I_n$ ).

[Why? By the Erdős–Rado Theorem, see 1.14–1.15 for a detailed explanation.]

**Fact 1.12.**  $I_n$  is  $\mu_n^+$ -complete.

[Why? If  $h_i : D_n \rightarrow 2^{\mu_n}$  ( $i < \mu_n$ ), then there is  $h : D_n \rightarrow 2^{\mu_n}$  such that  $h(x) = h(y) \Rightarrow \bigwedge_i h_i(x) = h_i(y)$ .]

**Conclusion 1.12A.** So, by Lemma 1.8, there is  $\bar{\eta} = \langle \eta_i : i < \mu^+ = \lambda \rangle$  which is a  $\lambda$ -sequence for  $\langle I_n : n < \omega \rangle$ .

We apply Conclusion 1.12A to measure algebras getting a well known result:

**Application 1.13.** Assume  $\lambda = \mu^+$  and  $\mu$  is a strong limit singular of cofinality  $\aleph_0$  (i.e., as in 1.10). If  $\mathcal{B}$  is a measure algebra (Maharam) of dimension  $\geq \mu$ , we can find  $a_\alpha \in \mathcal{B}$  for  $\alpha < \lambda$  with  $\text{Leb}(a_\alpha) > 0$  for each  $\alpha$ , and such that for every  $X \in [\lambda]^\lambda$  we can find  $n^* < \omega$ ,  $\alpha_1, \dots, \alpha_{n^*} \in X$  such that

$$\mathcal{B} \models \bigcap_{\ell=1}^{n^*} a_{\alpha_\ell} = 0.$$

**Proof.** Let  $\bar{\eta}$  and  $I_n$  be as in Conclusion 1.12A (all in the content of Example 1.10). Let  $\langle x_{n, \alpha} : n < \omega, \alpha < \beth_{n+3}(\mu_n)^+ \rangle$  be independent in the sense of measure, all elements of  $\mathcal{B}$  and of measure  $1/2$ .

For any  $\eta \in \prod_{n < \omega} D_n$ , let

$$y_{\eta,n} = y_{\eta(n)} = 1 - \bigcap_{\beta \in \eta(n)} x_{n,\beta} - \bigcap_{\beta \in \eta(n)} (1 - x_{n,\beta}).$$

Note that  $\bigcap_{\beta \in \eta(n)} x_{n,\beta}$  has measure  $2^{-n}$ . So  $\text{Leb}(y_{\eta,n}) = 1 - 2 \cdot 2^{-n}$  (hence  $\text{Leb}(y_{\eta(n)}) > 0$  if  $n \geq 2$ ). Let

$$y_\eta = \bigcap_{n \geq 5} y_{\eta,n} \in \mathcal{B}.$$

So  $\text{Leb}(y_\eta) \geq 1 - 2 \cdot \sum_{n \geq 5} 2^{-n} = 1 - 2 \cdot 2^{-4} = 1 - 2^{-3} > 1/2$ . We let  $a_\alpha = y_{\eta_\alpha}$  for  $\alpha < \lambda$ . We check that  $\langle a_\alpha : \alpha < \lambda \rangle$  is as required. Suppose  $X \in [\lambda]^\lambda$ . So, as  $\bar{I}$  is normal, for some  $n > 5$  and  $v \in \prod_{\ell < n} D_\ell$  we have

$$Y_v =: \{ \eta_\alpha(n) : \alpha \in X, \eta_\alpha \upharpoonright n = v \} \in I_n^+.$$

(Note that  $v$  is not really needed for the rest of the proof.)

So there is  $\{ \gamma_\ell : \ell < \omega \} \subseteq \beth_{n+3}(\mu_n)^+$  increasing such that

$$[\{ \gamma_\ell : \ell < \omega \}]^n \subseteq Y_v.$$

We use just  $\langle \gamma_\ell : \ell < 2n - 1 \rangle$ .

For  $u \in [\{ \gamma_\ell : \ell < 2n - 1 \}]^n$  let  $\alpha(u) \in X$  be such that

$$\eta_{\alpha(u)}(n) = u.$$

It is enough to show that in  $\mathcal{B}$

$$\bigcap_u y_{\eta_{\alpha(u)}} = \bigcap_u a_{\alpha(u)} = 0.$$

So suppose that there is  $z \in \mathcal{B}$  with  $\text{Leb}(z) > 0$  and such that  $z \leq \bigcap_u y_{\eta_{\alpha(u)}}$ . Then without loss of generality

$$\ell < 2n - 1 \Rightarrow z \leq x_{n,\gamma_\ell} \vee z \leq 1 - x_{n,\gamma_\ell}.$$

Case 1.  $|\{ \ell : z \leq x_{n,\gamma_\ell} \}| \geq n$ . Let  $u \in [\{ \gamma_\ell : \ell < 2n - 1 \}]^n$  be such that

$$\bigwedge_{\gamma_\ell \in u} (z \leq x_{n,\gamma_\ell}).$$

So  $z \leq \bigcap_{\gamma_\ell \in u} x_{n,\gamma_\ell}$ . But  $z \leq y_{\eta_{\alpha(u)}} \leq 1 - \bigcap_{\gamma \in u} x_{n,\gamma}$ , a contradiction.

Case 2. Not Case 1. So  $|\{ \ell : z \leq 1 - x_{n,\gamma_\ell} \}| \geq n$  and continue as above using  $1 - x_{n,\gamma_\ell}$ .  $\square$

Let us elaborate on the ideals used above.

**Definition 1.14.** For  $n, \lambda, \varepsilon$  let

$$\text{ERJ}_\lambda^{n,\varepsilon} = J_\lambda^{n,\varepsilon} = \{ A \subseteq [\lambda]^n : \text{there is no } w \subseteq \lambda, \text{otp}(w) = \varepsilon \text{ and } [w]^n \subseteq A \},$$

$$\text{ERI}_{\lambda,\mu}^{n,\varepsilon} = I_{\lambda,\mu}^{n,\varepsilon} = \left\{ A \subseteq [\lambda]^n : \text{there are } A_i \in J_\lambda^{n,\varepsilon} \text{ for } i < i(*) < \mu \right. \\ \left. \text{such that } A = \bigcup_{i < i(*)} A_i \right\}.$$

**Fact 1.15.**

(1)  $I_{\lambda, \mu}^{n, \varepsilon}$  is a  $\text{cf}(\mu)$ -complete ideal on  $[\lambda]^n$ , not necessarily proper (see (2)).  $J_{\lambda}^{n, \varepsilon}$  is not necessarily an ideal.

(2)  $I_{\lambda, \mu}^{n, \varepsilon}$  is a proper ideal, i.e.,  $[\lambda]^n \notin I_{\lambda, \mu}^{n, \varepsilon}$  iff

$$\chi < \mu \Rightarrow \lambda \rightarrow (\varepsilon)_{\chi}^n.$$

(3)  $I_n = I_{\beth_{n+7}(\mu_n)^+, (2^{\mu_n})^+}^{n, \omega}$  (where  $I_n$  and  $\langle \mu_n : n < \omega \rangle$  are from 1.10).

(4) In the proof of 1.13 we could have used less, for example,

$$I_n = I_{\beth_{n+1}(\mu_n)^+, \mu_n^+}^{n, 2n+1}$$

as  $\beth_{n+1}(\mu_n)^+ \rightarrow (\mu_n^+)_{\mu_n}^n$  for  $n \geq 1$ .

**Proof.** (3) *First direction:* Let  $A \in I_n$ , so there is  $h: A \rightarrow 2^{\mu_n}$  witnessing it. Let  $A_i = h^{-1}(i)$  for  $i < 2^{\mu_n}$ . Now  $X \subseteq \lambda$ ,  $|X| \geq \aleph_0 \Rightarrow [X]^n \not\subseteq A_i$ , by the choice of  $A$ . Hence

$$A_i \in J_{\beth_{n+7}(\mu_n)^+}^{n, \omega}.$$

Hence

$$A \in I_{\beth_{n+7}(\mu_n)^+, (2^{\mu_n})^+}^{n, \omega}.$$

*Second direction:* Let  $A \in I_{\beth_{n+7}(\mu_n)^+, (2^{\mu_n})^+}^{n, \omega}$ , so there are  $A_i$  (for  $i < i(*) < (2^{\mu_n})^+$ ) such that  $A_i \in J_{\beth_{n+7}(\mu_n)^+}^{n, \omega}$  and  $A = \bigcup_{i < i(*)} A_i$ .

Renaming, without loss of generality  $i(*) \leq 2^{\mu_n}$ , and let

$$A'_i = \begin{cases} A_i \setminus \bigcup_{j < i} A_j & \text{if } i < i(*), \\ \emptyset & \text{otherwise, i.e., if } i \in [i(*), 2^{\mu_n}). \end{cases}$$

So  $\langle A'_i : i < i(*) \rangle$  is a partition of  $A$ . As  $A_i \in J_{\beth_{n+7}(\mu_n)^+}^{n, \omega}$ , we know that  $\neg(\exists X \subseteq \beth_{n+7}(\mu_n)^+$  infinite  $([X]^n \subseteq A_i)$ ). Hence, letting  $\kappa = \beth_{n+7}(\mu_n)^+$

$$\neg(\exists X \subseteq \kappa \text{ infinite } ([X]^n \subseteq A_i)).$$

Define  $h: A \rightarrow 2^{\mu_n}$  by

$$h(\bar{\alpha}) = i \quad \text{iff } \bar{\alpha} \in A'_i,$$

so  $h$  witnesses  $A \in I_n$ .  $\square$

**Definition 1.16.**

(1) A set  $W \subseteq [\lambda]^{<\aleph_0}$  is called a *ccc base* if

$$(*) \quad \text{for } u \neq v \text{ in } W, |u \cap v| < |u|/2.$$

(2) For  $W \subseteq [\lambda]^{<\aleph_0}$  let

$$I_{\lambda}[W] = \{A \subseteq \lambda : W \cap [A]^{<\aleph_0} = \emptyset\},$$

$$I_{\lambda, \kappa}[W] = \{A \subseteq \lambda : A \text{ is the union of } < \kappa \text{ members of } I_{\lambda}[W]\}.$$

- (3) For a Boolean algebra  $\mathbf{B}$  we define  $I_{\mathbf{B},\kappa}$  by letting:  $X \in I_{\mathbf{B},\kappa}$  iff  $X \subseteq \mathbf{B} \setminus \{1\}$  is the union of  $< \kappa$  ideals of  $\mathbf{B}$ .

**Claim 1.17.**

(1) Assume

- (a)  $\bar{\eta}$  is a  $(\lambda, J)$ -sequence for  $\bar{I} = \langle I_i : i < \delta \rangle$ , and  $\text{cf}(\lambda) > \delta$ .  
 (b) For  $i < \delta$ , the function  $h_i : \text{Dom}(I_i) \rightarrow \lambda_i$  satisfies

$$\alpha < \lambda_i \Rightarrow \{x \in \text{Dom}(I_i) : h_i(x) < \alpha\} \in I_i.$$

Let  $\bar{h} = \langle h_i : i < \delta \rangle$  and let  $f_\alpha = \bar{h} \circ \eta_\alpha \stackrel{\text{def}}{=} \langle h_i(\eta_\alpha(i)) : i < \delta \rangle \in \prod_{i < \delta} \lambda_i$ .

Then

- (c)  $(\forall f \in \prod \lambda_i) (\forall J_\lambda^{\text{bd}} \gamma < \lambda) (f <_J f_\gamma)$ .  
 (d) for some club  $E$  of  $\lambda$ , we have  
 (d)<sub>E</sub> if  $\alpha < \varepsilon \leq \beta < \lambda$  and  $\varepsilon \in E$  then  $f_\alpha <_J f_\beta$ .  
 (So if  $X \in [\lambda]^\lambda$ ,  $(\forall \delta \in E) |X \cap (\delta, \min(E \setminus (\delta + 1))]| \leq 1$  then  $\langle f_\alpha : \alpha \in X \rangle$  is  $<_J$ -increasing cofinal in  $\prod_{i < \delta} \lambda_i$ .)  
 (2) If  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ ,  $E$  satisfies (d)<sub>E</sub> (and of course  $\sup_{i < \delta} \lambda_i < \lambda$ ) and  $\mu < \lambda$  then without loss of generality for  $X$  as in (d)<sub>E</sub> the sequence  $\bar{f} \upharpoonright X$  is  $\mu$ -free (see below). Moreover  $\bar{f}$  is  $(\mu, E)$ -free (see below clause (1) of 1.18), if  $(*)$  or just the weaker  $(*)'$  or just  $(*)''$  below holds where:

**Definition 1.18.**

- (1)  $\bar{f}$  is  $\mu$ -free if for  $X \in [\lambda]^{<\mu}$  we can find  $\bar{s} = \langle s_\alpha : \alpha \in X \rangle$ ,  $s_\alpha \in J$  such that

$$[\alpha < \beta \ \& \ \alpha \in X \ \& \ \beta \in X \ \& \ i \in J \setminus s_\alpha \setminus s_\beta] \Rightarrow f_\alpha(i) < f_\beta(i).$$

- (2)  $\bar{f}$  is  $(\mu, E)$ -free if for  $X \in [\lambda]^{<\mu}$  we can find  $\bar{s} = \langle s_\alpha : \alpha \in X \rangle$ ,  $s_\alpha \in J$  such that

$$[\alpha \leq \delta < \beta \ \& \ \alpha \in X \ \& \ \delta \in E \ \& \ \beta \in X \ \& \ i \in \delta \setminus s_\alpha \setminus s_\beta] \Rightarrow f_\alpha(i) < f_\beta(i).$$

- (3) For  $\lambda > \mu$  and  $\bar{\lambda} = \langle \lambda_i : i < \delta \rangle$  we consider the conditions

- $(*)$   $\lambda = \chi^+$ ,  $\chi = \text{cf}(\chi) \geq \mu$  for some  $\chi$ ,  
 $(*)'$   $\mu = \lim_J \lambda_i$  and  $\{\delta < \lambda : \text{cf}(f(\delta)) < \mu\} \in I[\lambda]$ ,  
 $(*)''$  there is  $\bar{f}' = \langle f'_\alpha : \alpha < \lambda \rangle$  which is  $<_J$ -increasing cofinal in  $(\prod_{i < \delta} \lambda_i, <_J)$  and is  $\mu$ -free.

**Remark.** This applies to the construction in §4, §5, etc. (e.g., construction from  $\lambda = \prod_{i < \delta} \lambda_i / J_\delta^{\text{bd}}$ ).

## 2. There are large free subsets

The reader may wonder if really something like  $\lambda = \text{cf}(\lambda) \in (\mu, 2^\mu]$  for  $\mu$  strong limit singular, is necessary for 1.13. As in [18], the answer is yes, though not for the same reason.

Of course, in what follows, Maharam measure algebra can be replaced by any measure algebra. The interesting case is  $(\exists \chi)(\chi < \lambda \leq \chi^{\aleph_0})$ .

**Fact 2.1.** *Let  $\mathcal{B}$  be a Maharam measure algebra. If  $\beth_2 \leq \mu = \mu^{\aleph_0} < \text{cf}(\lambda) \leq \lambda \leq 2^\mu$  and  $a_\alpha \in \mathcal{B}^+$  (so  $\text{Leb}(a_\alpha) > 0$ ) for  $\alpha < \lambda$  are pairwise distinct, then for some  $X \in [\lambda]^\lambda$  we have:*

(\*) *any nontrivial Boolean combination of finitely many members of  $\{a_\alpha: \alpha \in X\}$  has positive measure.*

**Proof.** Let  $\{x_i: i < i(*)\}$  be a basis of the Maharam measure algebra (so each  $x_i$  has measure  $1/2$  and  $x_i$ 's are measure-theoretically independent). So for each  $\alpha < \lambda$  we can find ordinals  $i(\alpha, n) < i(*)$  for  $n < \omega$ , and a Boolean term  $\tau_\alpha$  such that  $a_\alpha = \tau_\alpha(x_{i(\alpha,0)}, x_{i(\alpha,1)}, \dots)$ . Note that this equality is only modulo the ideal of null sets.

Without loss of generality, each  $\tau_\alpha$  is a countable intersection of a countable union of finite Boolean combinations of the  $x_i$ 's. Again without loss of generality,  $\langle i(\alpha, n): n < \omega \rangle$  is with no repetition. Note that without loss of generality

$$i(*) = \{i(\alpha, n): \alpha < \lambda \text{ and } n < \omega\}.$$

Hence without loss of generality  $i(*) \leq \lambda$ , hence without loss of generality  $i(*) = \lambda$ . By the Engelking–Karłowicz Theorem [4], we can divide  $\lambda$  to  $\mu$  sets  $\langle X_\zeta: \zeta < \mu \rangle$  such that

(\*)<sub>1</sub> the sets  $A_{\zeta,n} \stackrel{\text{def}}{=} \{i(\alpha, n): \alpha \in X_\zeta\}$  for each  $\zeta$  satisfy:  $\langle A_{\zeta,n}: n < \omega \rangle$  are pairwise disjoint.

As the number of possible terms  $\tau_\alpha \leq 2^{\aleph_0} \leq \mu$ , without loss of generality

(\*)<sub>2</sub> if  $\alpha, \beta \in X_\zeta$  then  $\tau_\alpha = \tau_\beta$ , call it  $\tau^\zeta$ .

Note also

(\*)<sub>3</sub> if  $Y \subseteq X_\zeta$  then

$$\text{ind}(Y) =: \{\alpha \in Y: \text{for no } m < \omega \text{ and } \beta_0, \dots, \beta_{m-1} \in Y \cap \alpha \text{ do we have:}$$

$$a_\alpha \in \text{the complete subalgebra generated by}$$

$$\{x_{i(\beta_\ell, n)}: \ell < m, n < \omega\}\},$$

$$\text{satisfies } |\text{ind}(Y)| + 2^{\aleph_0} \geq |Y|.$$

[Why? We can prove by induction on  $\alpha \notin \text{ind}(Y)$  that for some  $m < \omega$  and  $\beta_0, \beta_1, \dots, \beta_{m-1} \in \text{ind}(Y) \cap \alpha$  we have  $a_\alpha \in$  the complete subalgebra of  $\mathcal{B}$  “generated” by  $\{x_{i(\beta_\ell, n)}: \ell < m, n < \omega\}$ , when  $x_{i(\gamma, m)}$  are considered generated by  $a_0$ . Now for each  $m < \omega$  and  $\beta_0, \dots, \beta_{m-1} \in \text{ind}(Y)$ , the number of  $a_\alpha$  such that  $a_\alpha \in$  the subalgebra generated by  $\{x_{i(\beta_\ell, n)}: \ell < m, n < \omega\}$  is at most continuum.]

As  $\text{cf}(\lambda) > \mu$ , for at least one  $\zeta < \mu$ ,  $|X_\zeta| = \lambda$ , hence by (\*)<sub>3</sub> we have  $|\text{ind}(X_\zeta)| = \lambda$ .

So, without loss of generality

(\*)<sub>4</sub> (a) the sets  $A_n = \{i(\alpha, n): \alpha < \lambda\}$  are pairwise disjoint,

(b)  $\tau_\alpha = \tau$  for  $\alpha < \lambda$ ,

(c) for no  $m < \omega$  and  $\beta_0 < \dots < \beta_m < \lambda$  do we have  $a_{\beta_m} \in$  the complete subalgebra generated by  $\{x_{i(\beta_\ell, n)}: \ell < m, n < \omega\}$ .

Now for each  $\alpha < \lambda$  we define an ideal  $I_\alpha$  on  $\omega$ : it is the ideal generated by the sets

$$Z_{\alpha, \beta} =: \{n < \omega: i(\beta, n) = i(\alpha, n)\} \quad \text{for } \beta < \alpha$$



and let (where  $\text{ch}_A(n)$  is 1 if  $n \in A$  and 0 if  $n \notin A$ )

$$J = \left\{ A \subseteq \omega: \tau(z_0, z_2, \dots, z_{2n}, \dots) \right. \\ \left. = \tau(z_{0+\text{ch}_A(0)}, z_{2+\text{ch}_A(1)}, \dots, z_{2n+\text{ch}_A(n)}, \dots) \right\}.$$

As  $\{x_i: i < i(*)\}$  is free (in the measure theoretic sense), for  $A \in J$ , and  $\{\alpha_m: n < \omega\}$  and  $\{\beta_n: n < \omega\}$  such that  $\alpha_n < i(*)$  are with no repetition and  $\beta_n < i(*)$  with no repetition, we have the following:

If  $(\forall m, n < \omega)[\alpha_n = \beta_m \Leftrightarrow n = m \ \& \ n \notin A]$  then  $\tau(x_{\alpha_0}, \dots) = \tau(x_{\beta_0}, \dots)$  (just apply the definition of  $J$  to  $\langle x_{\alpha_0}, x_{\beta_0}, x_{\alpha_1}, \dots \rangle$ ). By transitivity of equality we get

$$(\forall n < \omega)[n \notin A \Rightarrow \alpha_n = \beta_n] \Rightarrow \tau(x_{\alpha_0}, \dots) = \tau(x_{\beta_0}, \dots).$$

Hence  $J$  is closed under subsets and (finite) unions. By clause (c) of  $(*)_4$  we know that  $\omega \notin I_\alpha$ ; so  $I_\alpha$  is an ideal on  $\omega$  though it is possible that singletons are not in  $I_\alpha$  (a violation of a convention in §0). [In fact we could have eliminated this violation, but there is no reason to put extra work for it.] Also  $I_\alpha \subseteq J$ .

Now, the number of possible ideals on  $\omega$  is at most  $\beth_2 \leq \mu < \text{cf}(\lambda)$ , so it suffices to prove

$(*)_5$  if  $Y \subseteq \lambda$ ,  $\alpha \in Y \Rightarrow I_\alpha \subseteq I$ , where  $I$  is an ideal on  $\omega$  (so  $\omega \notin I$  but singletons may or may not belong to  $I$ ) extending  $J$ , then any finite Boolean combination of  $\{a_\alpha: \alpha \in Y\}$  has positive measure.  $\square$

**Proof of  $(*)_5$ .** Let  $\beta_0 < \dots < \beta_{m-1}$  be from  $Y$ . Let

$$A = \{n < \omega: \text{for some } \ell < k < m \text{ we have } i(\beta_\ell, n) = i(\beta_k, n)\}.$$

By the definition of  $Z_{\alpha, \beta}$ , clearly  $A \in I$ . For  $Z \subseteq i(*)$  let  $\mathcal{B}^*[Z]$  be the complete subalgebra of  $\mathcal{B}$  generated by  $\{x_\beta: \beta \in Z\}$ . We let  $\mathcal{B}^* \stackrel{\text{def}}{=} \mathcal{B}^*[Z]$  if  $Z = \{i(\beta_\ell, n): \ell < m, n \in A\}$ . Let  $\mathcal{B}_\ell^* \stackrel{\text{def}}{=} \mathcal{B}^*[\{i(\beta_\ell, n): n \in A\}]$ .

As  $\mathcal{B}_\ell^*$  is complete, for each  $\ell < m$  we can find  $b_\ell^-, b_\ell^+ \in \mathcal{B}_\ell^*$  such that

- (i)  $b_\ell^- \leq a_{\beta_\ell} \leq b_\ell^+$ ,
- (ii) if  $c \in \mathcal{B}_\ell^*$  then  $c \leq a_{\beta_\ell} \Rightarrow c \leq b_\ell^-$  and  $c \geq a_{\beta_\ell} \Rightarrow c \geq b_\ell^+$ .

By the definition of  $\mathcal{B}^*$  and assumptions on  $\langle x_i: i < i(*) \rangle$  and  $\langle a_\alpha: \alpha < \lambda \rangle$  clearly

- $(*)_6$  if  $\{i(\beta_\ell, n): n \in A\} \subseteq Z$  and  $\{i(\beta_\ell, n): n \in \omega \setminus A\} \cap Z = \emptyset$  and  $Z \subseteq i(*)$  then
- (ii) $_Z$  if  $c \in \mathcal{B}^*[Z]$ , then  $c \leq a_{\beta_\ell} \Rightarrow c \leq b_\ell^-$  and  $c \geq a_{\beta_\ell} \Rightarrow c \geq b_\ell^+$ .

Obviously, for some Boolean terms  $\tau_\ell^-, \tau_\ell^+$  we have

$$b_\ell^- = \tau_\ell^-(\dots, x_{i(\beta_\ell, n)}, \dots)_{n \in A},$$

$$b_\ell^+ = \tau_\ell^+(\dots, x_{i(\beta_\ell, n)}, \dots)_{n \in A}.$$

Now, clearly  $\tau_\ell^- = \tau^-$  and  $\tau_\ell^+ = \tau^+$  for some fixed  $\tau^-$  and  $\tau^+$ . Also  $b_\ell^- < b_\ell^+$  as otherwise  $\omega \setminus A \in J$ . Let  $b_\ell = b_\ell^+ - b_\ell^-$  so  $\text{Leb}(b_\ell) > 0$ , and for some term  $\tau^*$ ,  $b_\ell = \tau^*(\dots, x_{i(\beta_\ell, n)}, \dots)_{n \in A}$ , and let  $b = \bigcap_{\ell < m} b_\ell \in \mathcal{B}^*$ .

Clearly

$(*)_7$   $\text{Leb}(b) > 0 \Rightarrow$  any Boolean combination of the  $a_{\beta_\ell}$  ( $\ell < m$ ) has positive measure.

[Why? Prove it on  $\{a_{\beta_\ell}: \ell < m'\}$  by induction on  $m' \leq m$  using  $(*)_6$ .]

For proving  $\text{Leb}(\bigcap_{\ell < m} b_\ell) > 0$ , we define an equivalence relation  $E$  on  $\omega$ :

$$n_1 E n_2 \quad \text{iff} \quad \text{for every } \ell < k < m \text{ we have} \\ i(\beta_\ell, n_1) = i(\beta_k, n_1) \Leftrightarrow i(\beta_\ell, n_2) = i(\beta_k, n_2).$$

Clearly  $E$  has finitely many equivalence classes, say  $A_0, A_1, \dots, A_{k(*)-1}$ . For  $k_1 \leq k(*)$  and  $\bar{\ell} = \langle \ell_k: k_1 \leq k < k(*) \rangle$  let

$$Z_{k_1, \bar{\ell}} = \{ \tau^*(\dots, x_{i(\gamma_n, n)}, \dots): \text{for every } k < k(*), \text{ for some } \ell < m \text{ we have} \\ \langle \gamma_n: n \in A_k \rangle = \langle i(\beta_n, \ell): n \in A_k \rangle, \text{ but} \\ \text{if } k \geq k_1 \text{ then } \ell = \ell_k \}.$$

We prove by induction on  $k < k(*)$  that for any  $\bar{\ell}$

$$c_{\bar{\ell}} \stackrel{\text{def}}{=} \text{Leb}\left(\bigcap \{b: b \in Z_{k, \bar{\ell}}\}\right) > 0.$$

(In fact the measure does not depend on  $\bar{\ell}$ .)

For  $k = k(*)$  we have  $\{b_\ell: \ell < m\} \subseteq Z_{k, \langle \rangle}$  so we are done.

*The case  $k = 0$ :* It is trivial:  $Z_{0, \bar{\ell}}$  is a singleton  $\{\tau^*(\dots, x_{i(\gamma_n, n)}, \dots)\}$ , where  $\gamma_n \in A_n$  so obviously is not zero.

*The case  $k_1 + 1$ :* So let  $\bar{\ell} = \langle \ell_k: k_1 + 1 \leq k < k(*) \rangle$ , and we know that for each  $n < k(*)$  the element  $d_\ell = c_{\langle n \rangle \bar{\ell}}$  is  $> 0$ . For  $\zeta < m$  let  $f_\zeta$  be a function from  $Y = \{i(\beta_n, \ell): \ell < \omega\}$  such that if  $\ell \in A_k$  then  $k \in [k_1 + 1, k(*)] \Rightarrow n = \ell_k$  and  $k = k_1 \Rightarrow n = 0$  and  $k < k_1 \Rightarrow n < m$  into  $\lambda$ ,  $f_\zeta$  is one to one,  $f_\zeta$  is the identity on  $Y^* = \{i(\beta_n, \ell) \in Y: \ell \notin A_{k_1}\}$  and  $\langle \text{Rang}(f_\zeta \upharpoonright (Y \setminus Y_0^*)): \zeta < m \rangle$  are pairwise disjoint and

$$\ell \in A_{k_1} \Rightarrow f_\zeta(i(\beta_0, \ell)) = i(\beta_\zeta, \ell).$$

Now we can imitate the proof of  $(*)_7$  and get  $\bigcap_{n < m} d_n > 0$ . Let  $Y_\zeta = \text{Rang}(f_\zeta)$ , and note that  $f_0$  is the identity and  $Y_0 = Y$ . Clearly  $f_\zeta$  induces an isomorphism from  $\mathcal{B}[Y_0]$  onto  $\mathcal{B}[Y_\zeta]$ . Call it  $\hat{f}_\zeta$  and easily  $d_\zeta \stackrel{\text{def}}{=} \hat{f}_\zeta(d_0)$ . So we can imitate the proof of  $(*)_7$  and get  $\bigcap_{n < m} d_n > 0$ . But

$$c_{\bar{\ell}} = \bigcap_{n < m} c_{\langle n \rangle \bar{\ell}} = \bigcap_{n < m} d_n > 0$$

as required.  $\square$

## Discussion 2.2.

- (1) The proof of 2.1 gives more, almost a division to  $\leq \mu$  subfamilies of independent elements (in the Boolean algebra sense), see 2.16 below.
- (2) We may wonder if " $\mu \geq \beth_2$ " is necessary. Actually it almost is not (see 2.5 below) but cf  $\lambda > 2^{\aleph_0}$  is essential (see 3.10 below).

We shall see below (in 2.5) what we can get from the proof of 2.1.

**Definition 2.3.** For a Boolean algebra  $\mathbf{B}$  we say  $\langle \langle a_\alpha, b_\alpha \rangle : \alpha < \alpha^* \rangle$  is an explicitly independent sequence of intervals in  $\mathbf{B}$  if

- (a)  $\mathbf{B} \models a_\alpha < b_\alpha$ ,  
 (b) if  $u_0, u_1 \subseteq \alpha^*$  are finite disjoint then

$$\mathbf{B} \models \bigcap_{\alpha \in u_0} b_\alpha \cap \bigcap_{\alpha \in u_1} (-a_\alpha) > 0.$$

**Claim 2.4.** Assume

(\*)<sub>Y</sub>[X] (a)  $|X| = \chi$  and  $\mathcal{B}(X)$  is a Maharam measure algebra with free basis  $\{x_i : i \in X\}$ . For  $Z \subseteq X$  we let  $\mathcal{B}(Z)$  be the complete subalgebra of  $\mathcal{B}(X)$  generated by  $\{x_i : i \in Z\}$ .

(b)<sub>Y</sub>  $a_\alpha \in \mathcal{B}^+$  (i.e.,  $\text{Leb}(a_\alpha) > 0$ ) for  $\alpha \in Y$  and  $\beta < \alpha \Rightarrow a_\beta \neq a_\alpha$ , while  $|Y| = \lambda$ .

(1) If  $\lambda = \text{cf}(\lambda) > \aleph_1$  then for some  $Y' \in [Y]^\lambda$ ,  $Z \in [X]^{<\lambda}$  and  $a_\alpha^- \leq a_\alpha^+$  from  $\mathcal{B}(Z)$  we have:

- (i) for  $c \in \mathcal{B}(Z)$  we have  $c \leq a_\alpha \Rightarrow c \leq a_\alpha^-$  and  $a_\alpha \leq c \Rightarrow a_\alpha^+ \leq c$ ,  
 (ii) if  $u \in [Y']^{<\aleph_0}$ ,  $\eta \in {}^u 2$  and

$$\bigcap \{a_\alpha^+ : \alpha \in u, \eta(\alpha) = 1\} \cap \bigcap \{1 - a_\alpha^- : \alpha \in u, \eta(\alpha) = 0\} \neq 0,$$

then  $\bigcap_{\alpha \in u} a_\alpha^{[\eta(\alpha)]} \neq 0$ , where  $c^{[0]} = -c$ ,  $c^{[1]} = c$ .

(2) Assume  $\inf\{\text{Leb}(a_\alpha \Delta b) : b \in \langle a_\beta : \beta < \alpha \rangle\}_{\mathcal{B}} > 0$  for  $\alpha \in Y$ . Then in part (1) we can demand  $a_\alpha^- < a_\alpha^+$ . Hence

(\*) there is  $Y'' \in [Y']^\lambda$  such that  $\langle a_\alpha : \alpha \in Y'' \rangle$  is independent iff there is  $Y'' \in [Y']^\lambda$  such that  $\langle (a_\alpha^-, a_\alpha^+) : \alpha \in Y'' \rangle$  is explicitly independent. (See Definition 2.3 above.)

(3) If  $|Y| = \lambda > |X| = \chi$  and  $\chi_1 < \chi$ ,  $\sigma = \text{cov}(\chi, \chi_1^+, \aleph_1, 2) < \lambda$  then  $Y$  can be represented as the union of  $\leq \sigma$  subsets  $Y'$  such that for each there is  $Z \in [\chi]^{<\chi_1}$  satisfying  $\{a_\alpha : \alpha \in Y'\} \subseteq \mathcal{B}(Z)$ .

(4) If the clause (a) below holds then we can represent  $Y$  as the union of  $\leq \mu$  subsets  $Y'$  each satisfying (c) below (and (b)<sub>Y'</sub>),

(c)<sub>Y'</sub>  $a_\alpha = \tau(\dots, x_{i(\alpha, n)}, \dots)_{n < \omega}$ ,  $n \neq m \Rightarrow i(\alpha, n) \neq i(\alpha, m)$  and the sets  $A_n(Y') = \{i(\alpha, n) : n < \omega\}$  are pairwise disjoint, where

- (a) (i)  $2^{\aleph_0} \leq \mu = \mu^{\aleph_0}$  and  $2^\mu \geq \lambda$  or at least  
 (ii)  $2^{\aleph_0} \leq \mu$  and the density of the  $(< \aleph_1)$ -base product  ${}^\omega \chi$  is  $\leq \mu$ .

(5) If  $Y'$  is as in (4), i.e., satisfies clause (c), then any finite intersection of  $a_\alpha$ 's (for  $\alpha \in Y'$ ) is not zero.

(6) If  $Y'$  is as in (4), i.e., satisfies clause (c) then  $Y'$  is the union of  $\leq \beth_2$  subsets  $Y''$ , such that

(\*)<sub>Y''</sub> there is an algebra  $M$  with universe  $Y''$  and  $\leq \beth_1$  functions (with finite arity, of course) such that:

if  $[u \subseteq Y'', \alpha \in u \Rightarrow \alpha \notin \text{cl}_M\{u \cap \alpha\}]$ , then  $\langle a_\alpha : \alpha \in u \rangle$  is independent.

**Proof.** Straight and/or included in the proof of 2.1.  $\square$

**Claim 2.5.** In 2.1 we can weaken “ $\mu \geq \beth_2$ ” to “ $\mu \geq 2^{\aleph_0}$ ” or even “ $\text{cf}(\lambda) > 2^{\aleph_0}$ ” except possibly when  $\lambda$  is singular but  $\boxtimes$  below fails:

$\boxtimes$  for any countable set  $\mathfrak{a}$  of regulars,  $|\text{pcf}(\mathfrak{a})| \leq \aleph_0$  or  $(*)$  from 2.6.

**Proof.** Without loss of generality we assume  $(*)_4$  from the proof of 2.1 (as the proof of 2.1 up to that point works here too). Let  $J$  be as there, so  $J$  is an ideal on  $\omega$ , so

(+)  $J$  is an ideal on  $\omega$  and  $\langle i(\alpha, n): n < \omega \rangle / J$  for  $\alpha < \lambda$  are pairwise distinct; by the following observation 2.6 for some ideal  $I$  on  $\omega$  extending  $J$  and  $X \in [\lambda]^\lambda$ , we have

$$\alpha \in X \ \& \ \beta \in X \ \& \ \alpha \neq \beta \Rightarrow \{n: i(\alpha, n) = i(\beta, n)\} \in I.$$

This is enough for continuing with the old proof of 2.1.  $\square$

**Fact 2.6.**

(1) If  $J$  is an ideal on  $\kappa$ ,  $\langle f_\alpha / J: \alpha < \lambda \rangle$  are pairwise distinct functions in  ${}^\kappa \text{Ord}$  and  $\theta = \text{cf}(\lambda) > 2^\kappa$  then for some ideal  $I$  on  $\kappa$  extending  $J$  and  $X \in [\lambda]^\lambda$  we have:

$$\alpha \in X \ \& \ \beta \in X \ \& \ \alpha \neq \beta \Rightarrow f_\alpha \neq_I f_\beta$$

except possibly when

$(*)$   $\lambda$  is singular and  $\neg \boxtimes_\kappa$ , where

$\boxtimes_\kappa$  for any set  $\mathfrak{a}$  of regular cardinals  $> \kappa$  we have  $|\mathfrak{a}| \leq \kappa \Rightarrow |\text{pcf}(\mathfrak{a})| \leq \kappa$ ,

(2) We can replace  $(*)$  by

$(*)'$   $\lambda$  is singular and  $\neg \boxtimes_{\lambda, \kappa}^+$  or  $\neg \boxtimes_{\lambda, \kappa}^-$ , where

$\boxtimes_{\lambda, \kappa}^+$  for no set  $\mathfrak{a}$  of regular cardinals  $> \kappa$ , do we have  $|\mathfrak{a}| \leq \kappa$  and  $\lambda = \sup(\lambda \cap \text{pcf}(\mathfrak{a}))$ ,

$\boxtimes_{\lambda, \kappa}^-$  there are no  $\chi$ ,  $\text{cf}(\lambda) = \theta < \chi < \lambda$  and increasing sequences  $\bar{\lambda}^\zeta = \langle \lambda_i^\zeta: i < \kappa \rangle$  of regular cardinals  $\in (2^\kappa, \chi)$  such that  $\langle \max \text{pcf}\{\lambda_i^\zeta: i < \kappa\}: \zeta < \theta \rangle$  is increasing with limit  $\lambda$  but for every ultrafilter  $\mathcal{D}$  on  $\kappa$  we have

$$\sup \left\{ \text{tcf} \left( \prod_{i < \kappa} \lambda_i^\zeta / \mathcal{D} \right) : \zeta < \theta \right\} < \lambda.$$

**Proof.** (1) Follows by (2).

(2) The proof is split to cases.

Case 1:  $\lambda$  is regular. By [20, 6.2].

Case 2:  $\lambda$  singular. First note

**Fact 2.6A.**  $\neg \boxtimes_{\lambda, \kappa}^+ \Rightarrow \boxtimes_{\lambda, \kappa}^-$ .

[Why? Let  $\mathfrak{a}$  exemplify  $\neg \boxtimes_{\lambda, \kappa}^+$ , let  $\theta_\varepsilon \in \text{pcf}(\mathfrak{a}) \setminus \{\lambda\}$  be increasing for  $\varepsilon < \theta$  with limit  $\lambda$ . Let  $\mathfrak{b}_\varepsilon \subseteq \mathfrak{a}$  be such that  $\theta_\varepsilon = \max \text{pcf}(\mathfrak{b}_\varepsilon)$  and let  $\langle \lambda_\zeta: \zeta < \kappa \rangle$  list  $\mathfrak{a}$  and let  $\lambda_\zeta^\varepsilon$  be:  $\lambda_\zeta$  if  $\lambda_\zeta \in \mathfrak{b}_\varepsilon$  and  $(2^\kappa)^+$  if  $\lambda_\zeta \notin \mathfrak{b}_\varepsilon$ . Now  $\bar{\lambda}^\varepsilon = \langle \lambda_\zeta^\varepsilon: \zeta < \kappa \rangle$  exemplifies  $\neg \boxtimes_{\lambda, \kappa}$ . First  $\max \text{pcf}\{\lambda_\zeta^\varepsilon: \zeta < \kappa\} = \theta_\varepsilon < \lambda$  and  $\theta_\varepsilon$  is increasing with limit  $\sup(\lambda \cap \text{pcf}(\mathfrak{a}))$ . Secondly, for every ultrafilter  $\mathcal{D}$  on  $\kappa$  for each  $\varepsilon$  we have  $\text{tcf}(\prod_{\zeta < \kappa} \lambda_\zeta^\varepsilon / \mathcal{D})$  is  $(2^\kappa)^+$  or

is  $\text{tcf}(\prod_{\zeta < \kappa} \lambda_\zeta / \mathcal{D})$ . (Simplify the first case if  $\{\zeta < \kappa: \lambda_\zeta \notin \mathbf{b}_\varepsilon\} \in \mathcal{D}$  and the second case if  $\{\zeta < \kappa: \lambda_\zeta \in \mathbf{b}_\varepsilon\} \in \mathcal{D}$ .) So now if  $\text{tcf}(\prod_{\zeta < \kappa} \lambda_\zeta / \mathcal{D}) \geq \lambda$  implies  $\text{tcf}(\prod_{\zeta < \kappa} \lambda_\zeta^\varepsilon / \mathcal{D}) = (2^\kappa)^+$  as the latter is  $\leq \theta_\varepsilon < \lambda$ , so really there is no ultrafilter  $\mathcal{D}$  on  $\kappa$  for which

$$\sup \left\{ \text{tcf} \left( \prod_{\zeta < \kappa} \lambda_\zeta^\varepsilon / \mathcal{D} \right) : \varepsilon < \theta \right\} < \lambda,$$

so the second demand in  $\boxtimes_{\lambda, \kappa}^-$  holds.]  $\square$

Now we assume  $\boxtimes_{\lambda, \kappa}^-$ . For every regular  $\sigma \in (2^\kappa, \lambda)$  we apply 7.3 to  $\langle f_\alpha: \alpha < \sigma \rangle$ , so we can find  $A_\sigma \subseteq \kappa$  and  $\langle \gamma_{\sigma, i}: i < \kappa \rangle$  such that

(\*)<sub>0</sub> for every sequence  $\langle \beta_i: i \in A_\sigma \rangle$  satisfying  $\beta_i < \gamma_{\sigma, i}$  there are  $\sigma$  ordinals  $\alpha < \sigma$  for which

$$i \in A_\sigma \Rightarrow \beta_i < f_\alpha(i) < \gamma_{\sigma, i},$$

$$i \in \kappa \setminus A_\sigma \Rightarrow f_\alpha(i) = \gamma_{\sigma, i},$$

(\*)<sub>1</sub>  $B \in J \Rightarrow \sigma \in \text{pcf}\{\text{cf}(\gamma_{\sigma, i}): i < \kappa, i \in A_\sigma, i \notin B\}$ .

Let  $J_\sigma = \{B \subseteq \kappa: \max \text{pcf}\{\text{cf}(\gamma_{\sigma, i}): i \in \kappa \setminus A_\sigma \text{ and } i \in B\} < \sigma\}$ , so clearly  $\sigma = \text{tcf}(\prod_{i < \kappa} \text{cf}(\gamma_{\sigma, i}) / J_\sigma)$  and  $J \subseteq J_\sigma$ . Let  $A'_\sigma$  be such that  $A'_\sigma \subseteq A_\sigma$ , and  $\sigma = \max \text{pcf}\{\text{cf}(\gamma_{\sigma, i}): i \in A'_\sigma\}$ . Also, as  $\theta = \text{cf}(\lambda) > 2^\kappa$ , for some  $A' \subseteq \kappa$  (infinite) the set  $\Theta = \{\sigma: 2^\kappa < \sigma = \text{cf}(\sigma) < \theta \text{ and } A'_\sigma = A'\}$  is unbounded in  $\lambda$ . Let  $\langle \sigma_\varepsilon: \varepsilon < \theta \rangle$  be an increasing unbounded sequence of members of  $\Theta$ , such that its limit is  $\lambda$ . Apply 7.3 (see case 1) to  $\langle g_\varepsilon \upharpoonright A': \varepsilon < \theta \rangle$ , where  $g_\varepsilon(i) = \gamma_{\sigma_\varepsilon, i}$ , and get  $\langle \beta_i^*: i \in A' \rangle$  and  $B' \subseteq A'$  such that

(\*) if  $\langle \beta_i: i \in A' \rangle$  satisfies  $i \in A' \Rightarrow \beta_i < \beta_i^*$  then for unboundedly many ordinals  $\varepsilon < \theta$

$$i \in B' \Rightarrow \beta_i < \gamma_{\sigma_\varepsilon, i} < \beta_i^*,$$

$$i \in A' \setminus B' \Rightarrow \gamma_{\sigma_\varepsilon, i} = \beta_i^*.$$

Can  $B' = \emptyset$ ? This would mean that for some unbounded  $X \subseteq \theta$  we have

$$\varepsilon \in X \Rightarrow (\forall i \in A') [\gamma_{\sigma_\varepsilon, i} = \beta_i^*],$$

hence  $\{\sigma_\varepsilon: \varepsilon \in X\} \subseteq \text{pcf}\{\text{cf}(\beta_i^*): i \in A'\}$ , so  $\{\text{cf}(\beta_i^*): i \in A'\}$  has pcf of cardinality  $\geq \theta > 2^\kappa$  whereas  $|A'| \leq \kappa$ , contradiction, so really  $B' \neq \emptyset$ .

As we are assuming  $\neg \boxtimes_{\lambda, \kappa}^-$ , there is an ultrafilter  $\mathcal{D}$  on  $A'$  such that

$$\lambda \leq \sup \left\{ \text{tcf} \left( \prod_{i \in A'} \gamma_{\sigma_\varepsilon, i} / \mathcal{D} \right) : \varepsilon < \theta \right\}.$$

Clearly

$$\text{tcf} \left( \prod_{i \in A'} \gamma_{\sigma_\varepsilon, i} / \mathcal{D} \right) \leq \sigma_\varepsilon < \lambda$$

(by the choice of  $A'_{\sigma_\varepsilon} = A'$ ). Without loss of generality  $\sigma_\varepsilon > \theta$  for each  $\varepsilon < \theta$ . So we can choose, for each  $\varepsilon$ , a function  $h_\varepsilon \in \prod_{i \in A'} \gamma_{\sigma_\varepsilon, i}$  such that

(\*) if  $\zeta < \theta$  and  $\zeta \neq \varepsilon$ , while  $\langle \gamma_{\sigma_\zeta, i}: i \in A' \rangle \leq_{\mathcal{D}} \langle \gamma_{\sigma_\varepsilon, i}: i \in A' \rangle$  then

$$\langle \gamma_{\sigma_\zeta, i}: i \in A' \rangle <_{\mathcal{D}} h_\varepsilon.$$

(Note that  $\langle \gamma_{\sigma_\zeta, i}: i \in A' \rangle \neq_{\mathcal{D}} \langle \gamma_{\sigma_\varepsilon, i}: i \in A' \rangle$  because of the cofinalities of the respective ultraproducts.) So, considering  $\mathcal{D}$  as an ultrafilter on  $\kappa$ :

$$X_\varepsilon = \left\{ \alpha < \sigma_\varepsilon: h_\varepsilon <_{\mathcal{D}} f_\alpha <_{\mathcal{D}} \langle \gamma_{\sigma_\varepsilon, i}: i < \kappa \rangle, \text{ but} \right. \\ \left. \beta < \alpha \Rightarrow \neg(f_\alpha \leq_{\mathcal{D}} f_\beta <_{\mathcal{D}} \langle \gamma_{\sigma_\varepsilon, i}: i < \kappa \rangle) \right\}$$

has cardinality  $\sigma_\varepsilon$ . So  $X = \bigcup_{\varepsilon < \sigma} X_\varepsilon$  is as required.

We may wonder whether we can remove or at least weaken the assumption (\*); the answer is:

**Claim 2.7.** *If  $\kappa \leq \lambda$  and  $\theta = \text{cf}(\lambda) < \lambda$ , and  $\boxtimes_{\lambda, \kappa}^-$  (from 2.6) then for some  $f_\alpha \in {}^\kappa \lambda$  (for  $\alpha < \lambda$ ) the conclusion of 2.6(1) fails.*

**Proof.** Let  $\chi, \lambda_i^\zeta$  ( $i < \kappa, \zeta < \theta$ ) be as in  $\boxtimes_{\lambda, \kappa}^-$ .

Let  $\mathfrak{a}_\zeta =: \{\lambda_i^\zeta: i < \kappa\}$ , and  $\sigma_\zeta = \max \text{pcf}(\mathfrak{a}_\zeta)$ . Without loss of generality  $\langle \sigma_\zeta: \zeta < \theta \rangle$  is increasing with limit  $\lambda$ . By [25, II, §3] for each  $\zeta < \theta$  we can find  $\langle f_\alpha^\zeta: \alpha < \sigma_\zeta \rangle$  be such that:

$$\mathfrak{b} \subseteq \mathfrak{a}_\zeta \Rightarrow |\{f_\alpha^\zeta \upharpoonright \mathfrak{b}: \alpha < \max \text{pcf}(\mathfrak{a}_\zeta)\}| = \max \text{pcf}(\mathfrak{b}).$$

Define  $\langle f_\alpha: \alpha < \lambda \rangle$  by:  $f_\alpha(i) = f_\alpha^{\zeta(\alpha)}(\lambda_i^\zeta)$  where  $\zeta(\alpha) = \min\{\zeta: \sigma_\zeta > \alpha\}$ . Now check.  $\square$

### Discussion 2.8.

- (1) So if  $2^\kappa < \lambda$ ,  $\theta = \text{cf}(\lambda)$  then 2.7 shows that 2.6 is the best possible. (Of course we still do not know if  $\boxtimes_{\lambda, \kappa}^-$  is possible.) See more in 3.11.
- (2) Note: If  $\text{cf}(\lambda) > 2^\kappa$ , and

$$(\forall \mathfrak{a})(\mathfrak{a} \subseteq \text{Reg} \ \& \ |\mathfrak{a}| \leq \kappa < \min(\mathfrak{a}) \Rightarrow |\text{pcf}(\mathfrak{a})| \leq |\mathfrak{a}|),$$

then  $\square_{\lambda, \kappa}^-$  cannot occur as without loss of generality

$$J_\zeta = \{A \subseteq \kappa: \max \text{pcf}\{\lambda_i^\zeta: i \in A\} < \max \text{pcf}\{\lambda_i^\zeta: i < \kappa\}\}$$

does not depend on  $\zeta$ .

## 3. Strong independence in Maharam measure algebras

**Claim 3.1.** *Assume*

- (a)  $I_i$  is a  $\kappa_i$ -complete ideal on  $\lambda_i$  for  $i < \delta$ ,
- (b)  $\kappa_i > \sum_{j < i} \kappa_j$ ,
- (c)  $\mu = \sup_{i < \delta} \kappa_i$  is strong limit singular,
- (d)  $\lambda_i < \mu$ ,
- (e)  $\lambda = \mu^+ = 2^\mu$ .

Then there is  $\bar{\eta}$  a super  $\lambda$ -sequence for  $\langle I_i: i < \delta \rangle$ , where

**Definition 3.2.** We say  $\bar{\eta}$  is a super  $(\lambda, J)$ -sequence for  $\langle I_i: i < \delta \rangle$  if, in addition (to the demands in 1.8)

(\*) for every  $n < \omega$  and  $\beta_{\alpha,\ell} < \lambda$  (for  $\alpha < \lambda$ ,  $\ell < n$ ) increasing with  $\ell$ , pairwise distinct we have

$$\left\{ i < \delta: \{ \langle \eta_{\beta_{\alpha,\ell}}(i): \ell < n \rangle: \alpha < \lambda \} \in \prod_{\ell < n} I_i \right\} \in J.$$

Moreover

(\*) if  $n < \omega$ ,  $\beta_{\alpha,\ell} < \lambda$  (for  $\alpha < \lambda$ ,  $\ell < n$ ),  $\beta_{\alpha,\ell} < \beta_{\alpha,\ell+1}$ , and the  $\beta_{\alpha,\ell}$  are pairwise distinct then for some  $A \in J$  we have:

if  $m < \omega$ ,  $i_0 < i_1 < \dots < i_{m-1}$  belong to  $\delta \setminus A$ , then

$$\{ \langle \langle \eta_{\beta_{\alpha,\ell}}(i_t): \ell < n \rangle: t < m \rangle: \alpha < \lambda \} \in \left( \prod_{t < m} \left( \prod_{\ell < n} I_{i_t} \right) \right)^+.$$

**Proof.** Like the proof of 1.8.  $\square$

**Example 3.3.**  $\lambda = \mu^+ = 2^\mu$ ,  $\mu = \sum_{i < \kappa} \lambda_i$ ,  $i < j \Rightarrow \delta = \kappa < \lambda_i < \lambda_j < \mu$  and each  $\lambda_i$  is measurable with a  $(\aleph_0 + \sum_{j < i} \lambda_j)^+$ -complete normal (or just Ramsey for  $n_i$ ) ultrafilter  $\mathcal{D}_i$  on  $\lambda_i$ .

Let  $\bar{n} = \langle n_i: i < \kappa \rangle$ ,  $i < n_i < \omega$  (if  $\kappa = \aleph_0$ ,  $n_i = i$  we may omit it),

$$I_i = \{ A \subseteq [\lambda_i]^{n_i}: \text{for some } B \in \mathcal{D}_i \text{ we have } [B]^{n_i} \cap A = \emptyset \}.$$

Then

(\*)<sub>1</sub> Claim 3.1 applies,

(\*)<sub>2</sub> for every  $m < \omega$  and  $X \in \prod_{\ell < m} I_i$  we can find  $A \in \mathcal{D}_i$  such that:

$$\{ \bar{s}: \bar{s} = \langle s_\ell: \ell < m \rangle, s_\ell \in [A]^{n_i}, s_\ell < s_{\ell+1} \} \cap X = \emptyset.$$

**Definition 3.4.**

(1) For a Boolean algebra  $\mathbf{B}$  we say  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$  is a strongly independent sequence of intervals if

(a)  $\mathbf{B} \models a_\alpha < b_\alpha$ ,

(b) if  $\mathbf{B}'$  is a Boolean algebra extending  $\mathbf{B}$  and  $n < \omega$ ,  $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \alpha^*$  and  $\mathbf{B}' \models "a_{\alpha_\ell} \leq x_\ell \leq b_{\alpha_\ell}"$  for  $\ell < n$ , then any nontrivial Boolean combination of  $\langle x_\ell: \ell < n \rangle$  is nonzero (in  $\mathbf{B}'$ ).

(2) We say, for a Boolean algebra  $\mathbf{B}$  that  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$  is a  $\lambda$ -anti-independent sequence of intervals if

(a)  $\mathbf{B} \models a_\alpha \leq b_\alpha$ ,

(b) if  $\mathbf{B}'$  is a Boolean algebra extending  $\mathbf{B}$  and  $X \in [\alpha^*]^\lambda$  and  $\mathbf{B}' \models "a_\alpha \leq x_\alpha \leq b_\alpha"$  for  $\alpha \in X$ , then there are  $n < \omega$  and  $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$  from  $X$  such that some nontrivial Boolean combination of  $\langle x_{\alpha_\ell}: \ell < n \rangle$  is zero.

(3) We say  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$  is an independent sequence of intervals in the Boolean algebra  $\mathbf{B}$  if letting  $\mathbf{B}'$ ,  $x_\alpha$  be as in 3.5 below, we have  $\langle x_\alpha: \alpha < \alpha^* \rangle$  is independent (in  $\mathbf{B}'$ ).

- (4) We say  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$  is a strongly  $\lambda$ -anti-independent sequence of intervals for the Boolean algebra  $\mathbf{B}$  if:
- $\mathbf{B} \models a_\alpha \leq b_\alpha$ ,
  - if  $\mathbf{B}'$ ,  $X$ ,  $x_\alpha$  ( $\alpha \in X$ ) are as in 3.4(2)(b) above, then the Boolean subalgebra of  $\mathbf{B}'$  generated by  $\{x_\alpha: \alpha \in X\}$  contains no free subset of cardinality  $\lambda$ .
- (5) We say  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$  is mediumly  $\lambda$ -anti-independent (sequence of intervals of the Boolean algebra  $\mathbf{B}$ ) if
- $\mathbf{B} \models a_\alpha \leq b_\alpha$ ,
  - if  $\mathbf{B}'$  is the free extension of  $\mathbf{B}$  for  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$  (see 3.5), then the Boolean subalgebra of  $\mathbf{B}'$  generated by  $\{x_\alpha: \alpha < \alpha^*\}$  contains no free subalgebra of cardinality  $\lambda$ .

**Definition 3.5.** We say that  $\mathbf{B}' = \mathbf{B}'(\mathbf{B}, \langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle)$ , or  $\mathbf{B}'$  is the free extension of  $\mathbf{B}$  for  $\langle (a_\alpha, b_\alpha): \alpha < \alpha^* \rangle$ , if

- (\*)  $\mathbf{B}'$  is the algebra freely generated by  $\mathbf{B} \cup \{x_\alpha: \alpha < \alpha^*\}$  except for the equations:
- the equations which  $\mathbf{B}$  satisfies,
  - $a_\alpha \leq x_\alpha \leq b_\alpha$ , for  $\alpha < \alpha^*$ .

**Observation 3.5A.**

- In 3.4(3), if  $\mathbf{B} \subseteq \mathcal{B}(\alpha_0)$ ,  $\alpha_0 + \omega + \alpha^* \leq \alpha_1$  then we can embed  $\mathbf{B}'$  into  $\mathcal{B}(\alpha_1)$  over  $\mathbf{B}$ .
- There are obvious implications among the notion from Definition 3.4 and some equivalences: independent (3.4(3)) with explicitly independent; and strongly independent with “(a) of 3.4(1) and if  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m < \alpha^*$  with no repetition,

$$\mathbf{B} \models \left( \bigcap_{\ell=1}^n a_{\alpha_\ell} \cap \bigcap_{\ell=1}^m (-b_{\beta_\ell}) > 0 \right).$$

**Lemma 3.6.** Assume  $\mu$  is strong limit singular of countable cofinality and  $\lambda = \mu^+ = 2^\mu$ . Then in  $\mathcal{B}(\mu)$ , (the Maharam measure algebra of dimension  $\mu$ ) we can find a sequence  $\langle (a_\alpha, b_\alpha): \alpha < \mu \rangle$  such that:

- $\mathcal{B}(\mu) \models a_\alpha < b_\alpha$ ,
- $\langle (a_\alpha, b_\alpha): \alpha < \lambda \rangle$  is strongly  $\lambda$ -anti-independent.

**Remark.** What is the difference with 1.13? Note that 3.4(ii)(b) speaks of “no free subset of the Boolean algebra”, not just of the set.

**Proof.** (1) Let  $\mu = \sum_{n < \omega} \lambda_n^0$  (we may demand  $\beth_{n+8}(\lambda_n^0) < \lambda_{n+1}^0 < \mu$ ) and let  $I_n$  be  $\text{ERI}_{\beth_{n-1}(\lambda_n^0)^+, (\lambda_n^0)^+}^{n,h}$  (see Definition 1.14, they were used in the proof of 1.13). Let  $\bar{\eta} = \langle \eta_\alpha: \alpha < \lambda \rangle$  be as guaranteed by 3.1 (so  $\text{lg}(\eta_\alpha) = \omega$ ,  $\eta_\alpha(n) \in [\lambda_n]^n$ , where  $\lambda_n = \beth_{n-1}(\lambda_n^0)^+$ ). So  $I_{n+1}$  is  $|\text{Dom}(I_n)|^+$ -complete (we could also have  $\langle I_n: n < \omega \rangle$  is normal). Renaming, let  $x_\alpha^n$  (for  $n < \omega$ ,  $\alpha < \lambda_n$ ) be the free generators of the Maharam algebra.

Define for  $\alpha < \lambda$  and  $m < \omega$

$$a_{\alpha,m}^* = \bigcap \{x_\beta^m: \beta \text{ appears in } \eta_\alpha(m)\},$$



$$b_{\alpha,m}^* = \bigcup \{(1 - x_\beta^m : \beta \text{ appears in } \eta_\alpha(m))\}.$$

We define by induction on  $n$ , the elements  $a_{\alpha,n}$ ,  $b_{\alpha,n}$  as follows: for  $n < 5$  let  $a_{\alpha,n} = 0$ ,  $b_{\alpha,n} = 1$ . For  $n \geq 5$  we let  $a_{\alpha,n} = a_{\alpha,n-1} \cup (a_{\alpha,n}^* \cap b_{\alpha,n})$  and  $b_{\alpha,n} = b_{\alpha,n-1} \cap (b_{\alpha,n}^* \cup a_{\alpha,n})$ . We can prove by induction on  $n < \omega$  that  $a_{\alpha,n-1} \leq a_{\alpha,n} \leq b_{\alpha,n} \leq b_{\alpha,n-1}$ . We can compute the measure, e.g., let  $(b_{\alpha,n} - a_{\alpha,n}) = \prod \{1 - 2^{-(\ell-1)} : 5 \leq \ell \leq n\}$ .

Let  $a_\alpha = \bigcup_{n < \omega} a_{\alpha,n} \in \mathcal{B}(\mu)$ ,  $b_\alpha = \bigcap_{n < \omega} b_{\alpha,n} \in \mathcal{B}(\mu)$ .

So clearly  $\mathcal{B}(\mu) \models a_\alpha \leq b_\alpha$ , and by the measure computations above,  $\mathcal{B}(\mu) \models a_\alpha < b_\alpha$ . So  $\langle (a_\alpha, b_\alpha) : \alpha < \lambda \rangle$  is a sequence of intervals. Suppose  $\mathbf{B}$ ,  $c_\alpha$  (for  $\alpha < \lambda$ ), is a counterexample to the conclusion so there is an independent subset  $\{d_\alpha : \alpha < \lambda\}$  of  $\langle c_\alpha : \alpha < \lambda \rangle_{\mathbf{B}} \subseteq \mathbf{B}$ . Thus, for each  $\alpha < \lambda$  for some  $k_\alpha < \omega$  and a Boolean term  $\tau = \tau_\alpha(x_0, \dots, x_{k_\alpha-1})$  and some  $\beta_{\alpha,0} < \beta_{\alpha,1} < \dots < \beta_{\alpha,k_\alpha-1}$  we have  $d_\alpha = \tau_\alpha(c_{\beta_{\alpha,0}}, c_{\beta_{\alpha,1}}, \dots, c_{\beta_{\alpha,k_\alpha-1}})$ .

As we can replace  $\{d_\alpha : \alpha < \lambda\}$  by any subset of the same cardinality without loss of generality  $\tau_\alpha = \tau$ , so let  $k_\alpha = k(*)$ .

Similarly, by the  $\Delta$ -system argument without loss of generality for some  $k < k(*)$  we have

$$\ell < k \Rightarrow \beta_{\alpha,\ell} = \beta_\ell \quad \text{and} \quad \alpha(1) < \alpha(2) \Rightarrow \beta_{\alpha(1),k(*)-1} < \beta_{\alpha(2),k}.$$

Let  $X_n = \{\langle \eta_{\beta_{\alpha,\ell}}(n) : k \leq \ell < k(*) \rangle : \alpha < \lambda\} \subseteq {}^{(k(*)-k)}([\lambda_n]^n)$ . So we know that  $B = \{n < \omega : n \geq k(*) - k \text{ and } X_n \in (\prod_{\ell=k}^{k(*)-1} I_n)^+\} \in J^+$ . Let  $n \in B$ . We can find a function  $h : X_n \rightarrow \lambda$  such that

$$\bar{t} \in X_n \text{ \& } h(\bar{t}) = \alpha \Rightarrow \bar{t} = \langle \eta_{\beta_{\alpha,\ell}}(n) : k \leq \ell < k(*) \rangle.$$

Let  $m(*) < \omega$  be large enough, a power of 2 for simplicity.

As  $X_n \in (\prod_{\ell=k}^{k(*)-1} I_n)^+$ , we can find  $\langle S_\ell : \ell \in [k, k(*)] \rangle$  and  $\langle u_{\bar{s}} : \bar{s} \in S_\ell \rangle$  for  $\ell \in [k, k(*)]$  such that

- $S_k = \{\langle \rangle\}$ ,
- $u_{\bar{s}} \in [\lambda_n]^{m(*)}$ ,
- the  $u_{\bar{s}}$ 's are pairwise disjoint,
- $S_{\ell+1} = \{\bar{s} \cap \langle w \rangle : \bar{s} \in S_\ell, w \in [u_{\bar{s}}]^n\}$ ,
- $S_{k(*)} \subseteq X_n$ .

(We just do it by induction on  $\ell$  using the definition of  $\prod_{\ell=k}^{k(*)-1} I_n$  and the definition of  $I_\ell$ .)

So it suffices to show that  $\langle d_{h(\bar{t})} : \bar{t} \in S_{k(*)} \rangle$  is not independent. For this just note:

- ( $\otimes$ ) for every  $\varepsilon \in \mathbb{R}^{>0}$  if  $n$  is large enough compared to  $k(*)$ ,  $1/\varepsilon$ , and  $m(*)$  is large enough compared to  $n$  then for every ultrafilter  $\mathcal{D}$  on  $\mathcal{B}(\mu)$  we can by downward induction on  $\ell = k, \dots, k(*) - 1$  find  $u_{\bar{s}}^- \in [u_{\bar{s}}]^{m(*)/2^{k(*)-\ell}}$  and  $\eta_{\bar{s}} \in {}^{\{k, \dots, k(*)-1\}}2$  for  $\bar{s} \in S_\ell$  such that:  $\bar{s} \sqsubseteq \bar{t} \in S_{\ell+1}$  and  $\ell \leq \ell_1 < k(*)$  and  $\alpha \in u_{\bar{t}}^- \Rightarrow [x_\alpha^n \in \mathcal{D} \equiv \eta_{\bar{s}}(\ell_1) = 1]$ .

Now let  $\eta^* = \eta_{\langle \rangle}$  (i.e.,  $\eta_{\bar{s}}$  for the unique  $\bar{s} \in S_0$ ) and for  $m < k(*)$  letting  $S'_m = \{\bar{s} \in S_m : \text{if } \ell < m \text{ then } \bar{s}(\ell) \in [u_{\bar{s}}^-]_\ell^n\}$ , we have  $\bar{s} \in S'_{k(*)} \Rightarrow d_{h(\bar{s})} \in \mathcal{D}$  or  $\bar{s} \in S'_{k(*)} \Rightarrow d_{h(\bar{s})} \notin \mathcal{D}$ .

So to prove that  $\langle d_\alpha : \alpha < \lambda \rangle$  is not independent it suffices to find  $S \subseteq S_{k(*)}$  such that

$$\otimes_S \bigcap_{\alpha \in S} d_\alpha \cap \bigcap_{\alpha \in S_{k(*)} \setminus S} \neg d_\alpha = 0,$$

or equivalently

$\otimes'_S$  for no ultrafilter  $\mathcal{D}$  on  $\mathcal{B}(\mu)$  do we have

$$\alpha \in S_{k(*)} \Rightarrow [d_\alpha \in \mathcal{D} \equiv \alpha \in S].$$

By the argument above it will suffice to have

$\otimes''_S$  if  $\langle u_{\bar{s}}^- : \bar{s} \in \bigcup \{S'_\ell : \ell < k(*)\}$  satisfies:  $S'_0 = S_0, S'_\ell \subseteq S_\ell,$

$$\bar{s} \in S'_\ell \Rightarrow u_{\bar{s}}^- \in [u_{\bar{s}}^-]^{m(*)/2^{2k(*)-\ell}}$$

and  $S'_{\ell+1} = \{\bar{s} \frown \langle w \rangle : \bar{s} \in S'_\ell \text{ and } w \in [u_{\bar{s}}^-]^n\}$  then  $S \cap S'_{k(*)} \notin \{\emptyset, S\}.$

Now, not only that this is trivial by the probabilistic existence proof á la Erdős but the proof gives much more than enough.  $\square$

**Claim 3.7.** Assume

(\*)  $\lambda$  is regular  $> \aleph_0$  and  $\langle (a_\alpha, b_\alpha) : \alpha < \lambda \rangle$  is a strongly (or just mediumly)  $\lambda$ -anti-independent sequence of pairs from  $\mathcal{B}(\lambda)$  satisfying  $a_\alpha < b_\alpha.$

Then:

(a) There is  $\mathbf{B}'$ , such that:

( $\alpha$ )  $\mathbf{B}'$  is a subalgebra of  $\mathcal{B}(\lambda),$

( $\beta$ )  $\mathbf{B}'$  has cardinality  $\lambda$  and even dimension  $\lambda,$

( $\gamma$ ) there is no subset of  $\mathbf{B}'$  of cardinality  $\lambda$  which is independent.

(b) Let  $\mathbf{B}'$ ,  $x_\alpha (\alpha < \lambda)$  be as in 3.5, then the Boolean algebra in clause (a) can be chosen isomorphic to  $\langle x_\alpha : \alpha < \lambda \rangle_{\mathbf{B}'}$ .

**Proof.** Straight. Clause (a) follows from clause (b). For clause (b) apply Definition 3.4(5) and 3.5A. (Note: we can use  $\mathcal{B}' \subseteq \mathcal{B}(\lambda + \lambda).$ ) It has already been done by Plebanek [13].  $\square$

**Conclusion 3.8.** For  $\lambda$  as in 3.6 (i.e.,  $\lambda = \mu^+ = 2^\mu, \mu$  strong limit  $> \text{cf}(\mu) = \aleph_0$ ) or just as in (\*) of 3.7, we have

(\*) there is a topological space  $X$  which is Hausdorff, compact zero-dimensional, with a measure Leb on the Borel sets such that it has dimension  $\lambda,$  so as a measure space is isomorphic to  $\mathcal{B}(\lambda)$  but there is no homomorphism from  $X$  onto  ${}^\lambda 2.$

**Proof.** By 3.6(a), (\*) of 3.7 holds so we can restrict ourselves to this case. So by 3.7 we know that clause (a) of 3.7 holds. Now it follows that (\*) holds, more specifically, that the Čech–Stone compactification of  $\mathbf{B}'$  (i.e., the set of ultrafilters of  $\mathbf{B}'$  with the natural topology) and the measure of  $\mathbf{B}'$  (which is just the restriction of the one on  $\mathcal{B}(\lambda)$ ) satisfies (\*) of 3.8.  $\square$

**Example 3.9.** Assume  $\mathcal{B}$  is a Maharam measure algebra of dimension  $\mu$  and free basis  $\langle x_\alpha : \alpha < \mu \rangle, \mu \geq \lambda > \text{cf}(\lambda) = \aleph_0.$  Then (\*) $_{2,\lambda}$  below holds, where

(\*) $_{2,\lambda}$  there are positive pairwise distinct members  $a_\alpha$  of  $\mathcal{B}(\mu)$  for  $\alpha < \mu,$  such that for every  $X \in [\lambda]^\lambda$  for some  $\alpha \neq \beta$  from  $X, a_\alpha \cap a_\beta = 0.$

**Proof.** Trivial: let  $\lambda = \sum_{n < \omega} \lambda_n$ ,  $\lambda_n < \lambda_{n+1}$  and for  $\alpha \in (\bigcup_{\ell < n} \lambda_\ell, \lambda_n)$  we let  $a_\alpha = x_{\omega+\alpha} \cap (x_n - \bigcup_{m < n} x_m)$ .  $\square$

**Fact 3.10.** Suppose  $\aleph_0 < \text{cf}(\lambda) < \lambda$  and there are positive  $b_\alpha \in \mathcal{B}(\text{cf}(\lambda))$  for  $\alpha < \text{cf}(\lambda)$  such that for every  $X \in [\text{cf}(\lambda)]^{\text{cf}(\lambda)}$  for some  $m < \omega$  and  $\beta_0, \dots, \beta_m \in X$  we have  $\text{Leb}(\bigcap_{\ell \leq m} b_{\beta_\ell}) = 0$  and  $\mu \geq \lambda$ . Then we can find pairwise distinct  $a_\alpha \in \mathcal{B}(\lambda)$  for  $\alpha < \lambda$  such that for every  $X \in [\lambda]^\lambda$  for some  $m < \omega$ ,  $\beta_0, \dots, \beta_m \in X$  we have  $\text{Leb}(\bigcap_{\ell \leq m} a_{\beta_\ell}) = 0$ , i.e.,  $\mathcal{B}(\lambda) \models \bigcap_{\ell \leq m} a_{\beta_\ell} = 0$ .

**Proof.** Like the proof of 3.9 replacing  $x_n - \bigcup_{m < n} x_m$  (for  $n < \omega$ ) by  $b_\alpha$  (for  $\alpha < \text{cf}(\lambda)$ ). (Just say that if  $\text{cf}(\lambda)$  is a precaliber of  $\mathcal{B}$  then so is  $\lambda$ .)  $\square$

**Remark 3.10A.**

- (1) By 2.1 we have in 3.10 that necessarily  $\text{cf}(\lambda) \leq \beth_2$  is normally  $\text{cf}(\lambda) \leq \beth_1$ .
- (2) Note that 3.11 elaborates 2.7 above and 3.12 is complementary to §2.

**Example 3.11.** Assume  $\aleph_0 \leq \sigma \leq \theta = \text{cf}(\lambda) \leq 2^\sigma \leq \mu < \lambda$ ,

$$\lambda = \sup\{\max \text{pcf}(\mathfrak{a}): \mathfrak{a} \subseteq \text{Reg} \cap \mu \setminus 2^\sigma, |\mathfrak{a}| = \sigma, [\mathfrak{a}]^{<\sigma} \subseteq J_{<\max \text{pcf}(\mathfrak{a})}[\mathfrak{a}], \text{ and} \\ \sup(\text{pcf}(\mathfrak{a}) \setminus \{\max \text{pcf}(\mathfrak{a})\}) \leq \mu\}$$

and there is  $\mathcal{A} \subseteq [\sigma]^\sigma$  such that  $|\mathcal{A}| \geq \theta$  and

$$A \neq B \ \& \ A \in \mathcal{A} \ \& \ B \in \mathcal{A} \Rightarrow |A \cap B| < \sigma.$$

Or just for no uniform ultrafilter  $\mathcal{D}$  on  $\sigma$  do we have  $|\mathcal{D} \cap \mathcal{A}| \geq \sigma$ .

Then we can find ordinals  $i(\alpha, \varepsilon)$  for  $\alpha < \lambda$ ,  $\varepsilon < \sigma$  such that

- (a) for  $\alpha \neq \beta$ ,  $\{\varepsilon: i(\alpha, \varepsilon) \neq i(\beta, \varepsilon)\}$  is infinite. Moreover
- (a)<sup>+</sup> for any  $\lambda' < \lambda$  for some ultrafilter  $\mathcal{D}$  on  $\sigma$ ,  $\{\{i(\alpha, \varepsilon): \varepsilon < \sigma\}/\mathcal{D}: \alpha < \lambda\}$  has cardinality  $\geq \lambda'$ ,
- (b) for no ultrafilter  $\mathcal{D}$  on  $\sigma$  do we have  $\{\{i(\alpha, \varepsilon): \varepsilon < \sigma\}/\mathcal{D}: \alpha < \lambda\}$  have cardinality  $\lambda$ .

[Why? Let

$$\lambda = \sum_{\zeta < \theta} \lambda_\zeta, \quad \lambda_\zeta < \lambda, \quad \lambda_\zeta = \max \text{pcf}(\mathfrak{a}_\zeta),$$

$$|\mathfrak{a}_\zeta| = \sigma, \quad [\mathfrak{a}_\zeta]^{<\sigma} \subseteq J_{<\lambda_\zeta}[\mathfrak{a}_\zeta], \quad \mu \geq \sup(\text{pcf}(\mathfrak{a}_\zeta) \setminus \{\lambda_\zeta\}).$$

Let  $f_\alpha^\zeta \in \prod \mathfrak{a}_\zeta$  for  $\zeta < \theta$ ,  $\alpha < \lambda_\zeta$  be such that  $\langle f_\alpha^\zeta: \alpha < \lambda_\zeta \rangle$  is  $< J_{<\lambda_\zeta}[\mathfrak{a}_\zeta]$ -increasing cofinal and  $\mathfrak{b} \in J_{<\lambda_\zeta}(\mathfrak{a}_\zeta) \Rightarrow \mu \geq |\{f_\alpha^\zeta \mid \mathfrak{b}: \alpha < \lambda_\zeta\}|$ . Let  $\mathcal{A} = \{A_\zeta: \zeta < \theta\}$ , let  $\mathfrak{a}_\zeta = \{\tau_\varepsilon^\zeta: \varepsilon \in A_\zeta\}$ . Lastly  $i(\alpha, \varepsilon)$  is

$$f_\alpha^\zeta(\varepsilon) \quad \text{if } \bigcup_{\xi \leq \zeta} \lambda_\xi \leq \alpha < \lambda_\zeta \ \& \ \varepsilon \in A_\zeta, \\ \zeta \quad \text{if } \bigcup_{\xi < \zeta} \lambda_\xi \leq \alpha < \lambda_\zeta \ \& \ \varepsilon \notin A_\zeta.$$

Now check.]

**Remark 3.11A.** There are easy sufficient conditions: if  $2^\sigma < \mu^1 \leq \mu$ ,  $\text{cf}(\mu^1) = \sigma$ ,  $\text{pp}(\mu^1) \geq \lambda$ ,  $(\forall \chi < \mu^1)(\text{cf}(\chi) \leq \sigma \rightarrow \text{pp}(\chi) < \mu^1)$  and  $\lambda < \mu^{+\omega}$  or at least  $\lambda = \sup\{\chi : \mu < \chi = \text{cf}(\chi) < \lambda \text{ and } \neg(\exists \mathfrak{a})(\mathfrak{a} \subset \text{Reg} \cap \chi \setminus \mu \ \& \ |\mathfrak{a}| \leq \sigma \ \& \ \chi \in \text{pcf}(\mathfrak{a}))\}$ .

**Example 3.12.** Assume

- (a)  $\aleph_0 < \theta = \text{cf}(\lambda) \leq 2^{\aleph_0} < \mu < \lambda$ ,
- (b) there is a  $\theta$ -Luzin subset of  ${}^\omega 2$ .

Then

- ( $\alpha$ ) there are pairwise disjoint  $a_\alpha \in \mathcal{B}(\mu)$  for  $\alpha < \lambda$  such that for no  $X \in [\lambda]^\lambda$  is  $\langle a_\alpha : \alpha \in X \rangle$  free,
- ( $\beta$ ) moreover for  $X \in [\lambda]^\lambda$  for some  $n < \omega$  and  $\beta_0 < \beta_1 < \dots < \beta_n$  from  $X$  we have  $\mathcal{B}(\lambda) \models \bigcap_{\ell \leq n} a_{\beta_\ell} = 0$ .

**Proof.** (Has already appeared in Plebanek [13].) By 3.10 it suffices to prove its assumption. Let for  $n < \omega$ ,  $\langle c_{n,\ell} : \ell < (n+1)^2 \rangle$  be a sequence of pairwise disjoint members of  $\mathcal{B}(\omega)$  with union 1, each with each with measure  $1/(n+1)^2$ . For  $\eta \in \prod_{n < \omega} (n+1)^2$  let  $b_\eta = \bigcap_{n < \omega} (1 - c_{\eta, \eta(n)})$ . Now suppose

- (\*)  $X \subseteq {}^\omega 2$ ,  $|X| = \theta$ , and if  $Y \in [X]^\theta$  then for some  $n < \omega$  and  $v \in \prod_{\ell < n} (\ell+1)^2$  we have

$$\{\ell : \ell < (n+1)^2\} = \{\eta(n) : \eta \upharpoonright n = v, \eta \in Y\}.$$

So  $\{b_\eta : \eta \in X\}$  is as required. Lastly from clause (b) of the assumption there is  $X$  as required in (\*) so, we are done.  $\square$

**Remark 3.13.**

- (1) So we can weaken clause (b) of the assumption to (\*) from the proof, or variants of it.
- (2) Note that strong negation of (b) of 3.12 which is consistent, implies the inverse situation.

#### 4. The interesting ideals and the direct pcf application

Our problem, the existence of  $(\lambda, I, J)$ -sequences for  $\bar{I}$ , depends much on the ideals  $I_i$  we use. Under strong set theoretic assumptions, there are  $\lambda$ -sequences  $\bar{\eta}$  by 1.8 (and 3.1); but we would like to prove their existence (i.e., in ZFC). For some ideals, by [25] we will have many cases of existence, e.g., when  $I_i$  is  $J_{\lambda_i}^{\text{bd}}$ ,  $\lambda_i$  regular. But we are more interested in the existence for more complicated ideals. The first step up are  $J_{\bar{\lambda}}^{\text{bd}}$  with  $\bar{\lambda}$  a (finite) strictly increasing sequence of cardinals. The proof for them is not much harder than with the  $J_{\lambda}^{\text{bd}}$ 's. We then consider the central ideal here:  $J_{\bar{\lambda}}^{\text{bd}}$  for  $\bar{\lambda}$  a (strictly) decreasing sequence of regular cardinals, and explain why the existence of  $\bar{\eta}$  for these ideals is more useful. We also consider their strong relative which comes from the nonstationary ideal. We would of course love to have even stronger ideals but there are indications that for those which we considered and failed, the failure is not completely due to incompetence, i.e., there

are related independence results (see later). We commence this section by reviewing some general definitions, some of them used earlier in the paper.

**Definition 4.1.**

- (1) For a set  $A$  of ordinals with no last element (mainly  $A = \lambda = \text{cf}(\lambda)$ )

$$J_A^{\text{bd}} = \{B: B \subseteq A \text{ is bounded}\}.$$

- (2) If  $A \subseteq \text{Ord}$  is such that  $\text{cf}(\text{otp}(A)) > \aleph_0$  and  $A$  stationary in  $\text{sup}(A)$ , we let

$$J_A^{\text{nst}} = \{B \subseteq A: B \text{ is not a stationary subset of } \text{sup}(A)\}.$$

- (3) If  $A \subseteq \text{Ord}$ ,  $\theta = \text{cf}(\theta) < \text{cf}(\text{otp}(A))$  and

$$\{\delta < \text{sup}(A): \delta \in A, \text{cf}(\delta) = \theta\}$$

is a stationary subset of  $\text{sup}(A)$ , then let

$$J_A^{\text{nst},\theta} = \{B \subseteq A: \{\delta \in B: \text{cf}(\delta) = \theta\} \text{ is a nonstationary subset of } \text{sup}(A)\}.$$

**Definition 4.2.**

- (1) For an ideal  $J$  let  $(\exists^{J^+} x)\varphi(x)$  mean that

$$\{x \in \text{Dom}(J): \varphi(x)\} \in J^+.$$

- (2) For an ideal  $J$  let  $(\forall^J x)\varphi(x)$  mean

$$\{x \in \text{Dom}(J): \neg\varphi(x)\} \in J.$$

**Definition 4.3.**

- (1)  $J = \prod_{\ell < n} J_\ell$  is the following ideal on  $\prod_{\ell < n} \text{Dom}(J_\ell)$ : for  $X \subseteq \prod_{\ell < n} \text{Dom}(J_\ell)$  we have

$$X \in J^+ \text{ iff } (\exists^{J_0^+} x_0)(\exists^{J_1^+} x_1) \cdots (\exists^{J_{n-1}^+} x_{n-1})[\langle x_0, \dots, x_{n-1} \rangle \in X].$$

- (2) If  $\bar{\lambda} = \langle \lambda_\ell: \ell < n \rangle$  we let:

(a)  $J_{\bar{\lambda}}^{\text{bd}} = \prod_{\ell < n} J_{\lambda_\ell}^{\text{bd}}.$

- (b) If  $\text{cf}(\lambda_\ell) > \aleph_0$  for  $\ell < n$  then we let

$$J_{\bar{\lambda}}^{\text{nst}} = \prod_{\ell < n} J_{\lambda_\ell}^{\text{nst}}.$$

- (c) If  $\text{cf}(\lambda_\ell) > \theta = \text{cf}(\theta)$  for  $\ell < n$  then we let

$$J_{\bar{\lambda}}^{\text{nst},\theta} = \prod_{\ell < n} J_{\lambda_\ell}^{\text{nst},\theta}.$$

- (d) If  $\bar{\theta} = \langle \theta_\ell: \ell < n \rangle$  and  $\text{cf}(\lambda_\ell) > \theta_\ell = \text{cf}(\theta_\ell)$  for  $\ell < n$  then we let

$$J_{\bar{\lambda}}^{\text{nst},\bar{\theta}} = \prod_{\ell < n} J_{\lambda_\ell}^{\text{nst},\theta_\ell}.$$

**Claim 4.4.** If  $\bar{\lambda} = \langle \lambda_\ell: \ell < n \rangle$  is a strictly increasing sequence of regular cardinals then the following conditions (a)–(d) on  $X \subseteq \prod_{\ell < n} \lambda_\ell = \text{Dom}(J_{\bar{\lambda}}^{\text{bd}})$  are equivalent:

- (a)  $X \in (J_{\bar{\lambda}}^{\text{bd}})^+;$

(b) for no  $\bar{\alpha} \in \prod_{\ell < n} \lambda_\ell$  do we have

$$(\forall \bar{\beta} \in X)(\neg(\bar{\alpha} < \bar{\beta})), \quad \text{where } \bar{\beta} < \bar{\alpha} =: \bigwedge_{\ell < n} \beta_\ell < \alpha_\ell;$$

(c) we can find  $\langle \alpha_\eta : \eta \in \bigcup_{m \leq n} \prod_{\ell < m} \lambda_\ell \rangle$  such that:

- (i)  $\alpha_\eta < \lambda_{\text{lg}(\eta)}$ ,
- (ii)  $\alpha_{\eta \frown \langle i \rangle} < \alpha_{\eta \frown \langle j \rangle}$  for  $i < j < \lambda_{\text{lg}(\eta)+1}$ ,
- (iii)  $\eta \in \prod_{\ell < n} \lambda_\ell \Rightarrow \langle \alpha_{\eta \upharpoonright \ell} : \ell \leq n \rangle \in X$ ;
- (d) like (c), adding
  - (iv)  $\alpha_\eta = \alpha_\nu \Rightarrow \eta = \nu$ .

**Proof.** Straight. For (b)  $\Rightarrow$  (c) use induction on  $n = \text{lg}(\bar{\lambda})$ , see the proof at the end of the proof of 4.11, of (\*) there.  $\square$

**Discussion 4.4A.** From 4.4, we see that for  $X \in (J_{\bar{\lambda}}^{\text{bd}})^+$  there are patterns which necessarily occur as subsets of  $X$ . These are essentially like the branches (= maximal nodes) of a tree with  $n$  levels, with a linear order on each level and with no dependencies between the different levels. These patterns were explored in [28,15,18]. The patterns considered there can be represented as a set  $\Delta \subseteq \prod_{\ell < n} B_\ell$ ,  $B_\ell \subseteq \text{Ord}$  such that  $\eta(i) = \nu(i) \Rightarrow \eta \upharpoonright i = \nu \upharpoonright i$  (i.e., treeness). Now look at  $J_{\bar{\lambda}}^{\text{bd}}$ , where the gain is that  $\Delta$  does not have a tree, that is, we have any  $\Delta \subseteq \prod_{\ell < n} B_\ell$ ,  $B_\ell \subseteq \text{Ord}$ , so that  $\eta, \nu \in \Delta$  can have  $\{\ell < n : \eta(\ell) = \nu(\ell)\}$  being arbitrary (rather than being an initial segment), of course this depends on the ideal.

**Claim 4.5.** Assume  $\bar{J} = \langle J_\ell : \ell < n \rangle$  and  $J_\ell$  is a  $\kappa_\ell$ -complete ideal on  $\lambda_\ell$ . We also demand  $\kappa_\ell > \lambda_k$  when  $\ell > k$ . Let  $J = \prod_{\ell < n} J_\ell$ .

(1) The following conditions on  $X \subseteq \prod_{\ell < n} \lambda_\ell$  are equivalent:

- (a)  $X \in J^+$ ;
- (b) for no  $\bar{A} = \langle A_\ell : \ell < n \rangle$ ,  $A_\ell \in J_\ell$  do we have

$$\bar{\beta} \in X \Rightarrow \bigvee_{\ell} \beta_\ell \in A_\ell;$$

- (c) we can find  $\langle \alpha_\eta : \eta \in \bigcup_{m \leq n} \prod_{\ell < m} \lambda_\ell \rangle$  such that  $\alpha_\eta < \lambda_{\text{lg}(\eta)}$  and
- (\*) for each  $\nu \in \prod_{\ell < n} \lambda_\ell$  we have

$$\langle \alpha_{\nu \upharpoonright (\ell+1)} : \ell < n \rangle \in X, \quad \{\alpha_{\nu \upharpoonright \ell \frown \langle \gamma \rangle} : \gamma < \lambda_{\text{lg}(\eta)+1}\} \in J_{\text{lg}(\eta)}^+.$$

(2) If  $[A \subseteq \lambda_\ell \ \& \ |A| < \lambda_\ell] \Rightarrow A \in J_\ell$  then we can add

- (d) like (c), but adding
- (iii)  $\alpha_{\nu \frown \langle i \rangle} < \alpha_{\nu \frown \langle j \rangle}$  if  $i < j < \lambda_{\text{lg}(\nu)+1}$ .

**Proof.** Similar to 4.4.  $\square$

**Claim 4.6.** Let  $\bar{\lambda} = \langle \lambda_\ell : \ell < n \rangle$  be a decreasing sequence of regular cardinals.

(1) If  $\lambda_\ell > 2^{\lambda_{\ell+1}}$  for  $\ell < n$ , then:

- (\*) for every  $A \in (J_{\bar{\lambda}}^{\text{bd}})^+$ , there are  $A_\ell \in (J_{\lambda_\ell}^{\text{bd}})^+$  such that  $\prod_{\ell < n} A_\ell \subseteq A$ .

- (2) If  $J = \prod_{\ell < n} J_\ell$  and  $J_\ell$  is a  $(2^{\lambda_{\ell+1}})^+$ -complete ideal on  $\lambda_\ell$ , then (\*) holds, with  $J$  in place of  $J_\lambda^{\text{bd}}$  and  $J_\ell$  in place of  $J_{\lambda_\ell}^{\text{bd}}$ .
- (3) For every  $A \in (J_\lambda^{\text{bd}})^+$  and  $k < \omega$  we can find  $B_\ell \in [\lambda_\ell]^k$  such that  $\prod_{\ell < n} B_\ell \subseteq A$ .
- (4) In (3), instead of  $k$  and  $J_{\lambda_\ell}^{\text{bd}}$  (for  $\ell < n$ ) we can use any  $\kappa$  and  $((\lambda_{\ell+1})^\kappa)^+$ -complete ideal  $J_\ell$  on  $\lambda_\ell$  for  $\ell < n$ .

**Proof.** E.g., (3). We prove it by induction on  $n$ .

$n = 1$ . Trivial, as singletons are in the ideal.

$n + 1$ . Let  $X_0 = \{\alpha < \lambda_0: \{\bar{\alpha} \in \prod_{\ell=1}^{n-1} \lambda_\ell: \langle \alpha \rangle \hat{\ } \bar{\alpha} \in A\} \in (\prod_{\ell=1}^{n-1} J_{\lambda_\ell}^{\text{bd}})^+\}$ .

Clearly,  $X_0 \in (J_{\lambda_0}^{\text{bd}})^+$ .

By the induction hypothesis, for each  $\alpha \in X_0$ , there is  $\langle B_\ell^\alpha: \ell = 1, \dots, n-1 \rangle$ , such that

$$B_\ell^\alpha \in [\lambda_\ell]^k \quad \text{and} \quad \prod_{\ell=1}^{n-1} B_\ell^\alpha \subseteq \left\{ \bar{\alpha} \in \prod_{\ell=1}^{n-1} \lambda_\ell: \langle \alpha \rangle \hat{\ } \bar{\alpha} \in A \right\} \stackrel{\text{def}}{=} \bar{B}^\alpha.$$

So  $X_0$  is the union of  $\prod_{\ell=1}^{n-1} \lambda_\ell^k = \lambda_1^k$  sets  $X_0[\bar{B}] = \{\alpha \in X_0: \bar{B}^\alpha = \bar{B}\}$ , so for some  $\bar{B}$ ,  $|X_0[\bar{B}]| \geq k$  and let  $B_0 =$  first  $k$  members of  $X_0, \bar{B}$ .  $\square$

**Definition 4.7.** For a partial order  $P$  let  $\text{tcf}(P) = \lambda$  iff there is an increasing cofinal sequence of length  $\lambda$  in  $P$  (tcf – stands for true cofinality); so, e.g.,  $(\omega, <) \times (\omega_1, <)$  has no true cofinality, but  $\text{tcf} \prod (\aleph_n, <)/\mathcal{D}$  is well defined if  $\mathcal{D}$  is an ultrafilter on  $\omega$ .

**Fact 4.8.**

- (1) If  $J \supseteq J_\delta^{\text{bd}}$  is an ideal,  $\lambda_i = \text{cf}(\lambda_i) > \delta$ , for  $i < \delta$  and  $\lambda = \text{tcf}(\prod_{i < \delta} \lambda_i / J)$ , then there is a  $(\lambda, J)$ -sequence  $\bar{\eta} = \langle \eta_\alpha: \alpha < \lambda \rangle$  for  $\langle J_{\lambda_i}^{\text{bd}}: i < \delta \rangle$ .
- (2) If  $\lambda_i$  is increasing in  $i$  then  $\langle J_{\lambda_i}^{\text{bd}}: i < \delta \rangle$  is normal (hence  $\bar{\eta}$  is normal) provided that  $\delta = \omega$  or at least
- (\*)<sub>1</sub>  $\lambda > \prod_{j < i} \lambda_j$  for  $i < \delta$ .
- (3) If we just ask  $\bar{\eta}$  to be normal it suffices to demand
- (\*)<sub>2</sub>  $\lambda_i > \max \text{pcf}\{\lambda_j: j < i\}$  for  $i < \delta$ .

**Proof.** In  $\prod_{i < \delta} \lambda_i / J$ , there is a cofinal increasing sequence  $\langle f_\alpha: \alpha < \lambda \rangle$ . It is as required, as we now show. Let  $X \in [\lambda]^\lambda$ , let  $X_i = \{f_\alpha(i): \alpha \in X\}$  for  $i < \delta$ . Define  $f \in \prod_{i < \delta} \lambda_i$ :

$$f(i) = \begin{cases} \sup(X_i) + 1 & \text{if } \sup(X_i) < \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

But  $\langle f_\alpha: \alpha < \lambda \rangle$  is cofinal, so for some  $\alpha_0 < \lambda$ ,  $f <_J f_{\alpha_0}$ . Now  $X \in [\lambda]^\lambda$ , so for some  $\alpha_1$ , we have  $\alpha_0 < \alpha_1 \in X$ . As  $\langle f_\alpha: \alpha < \lambda \rangle$  is increasing,  $f_{\alpha_0} <_J f_{\alpha_1}$ , hence  $f <_J f_{\alpha_1}$ . So  $A = \{i: f(i) \geq f_{\alpha_1}(i)\} \in J$ . But  $f_{\alpha_1}(i) \in X_i$ , so  $i \in \delta \setminus A \Rightarrow \lambda_i = \sup(X_i)$ .

(2) Easy.

(3) By [25, II, 3.5].  $\square$

**Comment 4.9.**

- (1) This is good, e.g., to lift a coloring of the  $\lambda_i$ 's to one of  $\lambda$ . But we would like to have an upgrade as well.
- (2) The kind of assumptions of 4.8 is the central interest in [25].

**Claim 4.10.** Assume  $\bar{\lambda}^i = \langle \lambda_{i,\ell} : \ell < n_i \rangle$  is an increasing sequence of regulars  $> \delta$  for  $i < \delta$ . Also assume that  $J$  is an ideal on  $\{(i, \ell) : i < \delta, \ell < n_i\}$  and

$$\lambda = \text{tcf}\left(\prod_{i,\ell} \lambda_{i,\ell} / J\right),$$

and for some ideal  $J'$  on  $\delta$ , we have  $J' \supseteq J_\delta^{\text{bd}}$  and  $J$  is generated by

$$\{(i, n) : n < n_i, i \in A\} : A \in J'\}.$$

Then there is a  $(\lambda, J')$ -sequence  $\bar{\eta}$  for  $\langle J_{\bar{\lambda}^i}^{\text{bd}} : i < \delta \rangle$ .

- (2)  $\langle J_{\bar{\lambda}^i}^{\text{bd}} : i < \delta \rangle$  is normal (hence  $\bar{\eta}$  above is normal) if
  - (\*)<sub>1</sub>  $\delta = \omega$  and  $i < j < \delta \Rightarrow \lambda_{i,n_i-1} < \lambda_{j,0}$ , or
  - (\*)<sub>2</sub>  $\prod \{\lambda_{i,\ell} : i < j, \ell < n_j\} < \lambda_{j,0}$ .
- (3) If we ask just  $\bar{\eta}$  to be normal it suffices to demand
  - (\*)<sub>3</sub>  $\max \text{pcf}\{\lambda_{i,\ell} : i < j, \ell < n_j\} < \lambda_{j,0}$ .

**Proof.** Again, let  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  be  $<_J$ -increasing cofinal. Let  $\eta_\alpha(i) = \langle f_\alpha(i, \ell) : \ell < n_i \rangle \in \prod \bar{\lambda}^i$ . Let  $X \in [\lambda]^\lambda$ . Let  $X_i = \{\eta_\alpha(i) : \alpha \in X\}$ . If  $X_i \in J_{\prod \bar{\lambda}^i}^{\text{bd}}$ , then there is  $\bar{\alpha}^i \in \prod \bar{\lambda}^i = \prod_{\ell < n_i} \lambda_{i,\ell}$  such that

$$(*) \quad \bar{\beta} \in X_i \Rightarrow \bigvee_{\ell < n_i} \beta_\ell < \alpha_\ell^i.$$

(We return to this at the end of the proof.)

So let  $f \in \prod_{i,\ell} \lambda_{i,\ell}$  be given by  $f(i, \ell) = \alpha_\ell^i$ . So, as before, for some  $\alpha \in X$ ,  $f <_J f_\alpha$ . So

$$A = \left\{ i : \bigwedge_{\ell < n_i} f((i, \ell)) \geq f_\alpha((i, \ell)) \right\} \in J'.$$

Now for  $i \in \delta \setminus A$  we have  $X_i \notin J_{\prod \bar{\lambda}^i}^{\text{bd}}$ .

[Why (\*)? Prove the existence of  $\bar{\alpha}^i$ , for notational convenience denoted here by  $\bar{\beta}$ , by induction on  $n_i$ . Here we use “increasing  $\bar{\lambda}^i$ ”.

$n_i = 1$ . Clear.

$n_i = k + 1$ . For  $\alpha < \lambda_{i,0}$  define

$$X_{i,\alpha} = \{ \bar{\beta} \upharpoonright [1, n_i) : \bar{\beta} \in X_i \}.$$

So we know that for some  $\gamma_0 < \lambda_{i,0}$

$$\alpha \in [\gamma_0, \lambda_{i,0}] \Rightarrow X_{i,\alpha} \in J_{\prod_{\ell=1}^{n-1} \lambda_{i,\ell}}^{\text{bd}}.$$



So for each such  $\alpha$  we have  $\bar{\beta}^\alpha \in \prod_{\ell=1}^n \lambda_{i,\ell}$  as given by the induction hypothesis. Let

$$\beta_\ell = \begin{cases} \gamma_0 + 1 & \text{if } \ell = 1, \\ \bigcup \{ \beta_\ell^\alpha : \alpha \in [\gamma_0, \lambda_{i,0}] \} & \text{otherwise.} \end{cases}$$

Why is the latter  $< \lambda_{i,\ell}$ ? As  $\lambda_{i,0} < \text{cf}(\lambda_{i,\ell})$ .]  $\square$

**Question 4.11.** Are there many cases fitting the framework of 4.10?

**Answer 4.11A.** Not so few. E.g., for any  $\kappa$ , for many  $\lambda = \text{cf}(\lambda)$  we have that  $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i / J_\kappa^{\text{bd}})$  for some sequence  $\langle \lambda_i : i < \kappa \rangle$ . E.g., if  $\aleph_0 < \text{cf}(\delta) = \kappa$  and  $\kappa < \mu = \beth_\delta < \lambda = \text{cf}(\lambda) \leq \beth_{\delta+1}$  or just  $\aleph_0 < \kappa = \text{cf}(\mu) < \lambda = \text{cf}(\lambda) \leq \mu^\kappa$  and  $(\forall \chi < \mu)[\chi^\kappa < \mu]$  then there is an increasing sequence of regulars  $\langle \lambda_i : i < \kappa \rangle$  with limit  $\beth_\delta$  or  $\mu$ , respectively as above. [Why? see [25, VIII §1, 2.6].] Even if  $\kappa = \aleph_0$  this holds for many  $\lambda$ 's, e.g., if  $\mu < \lambda < \mu^{+\omega_1}$  or just  $|\{ \chi : \mu < \chi < \lambda \text{ and } \chi = \aleph_\chi \}| < \mu$  see [25, IX] and use 4.12 below.

Note that by the pcf theorem (see [25, VIII, 2.6]).

**Claim 4.12.** Assume  $I$  to be an ideal on  $\delta$ , and  $\lambda_{i,\ell} = \text{cf}(\lambda_{i,\ell}) > |\delta|$  for  $i < \delta$  and  $\ell < n_i$  and  $0 < n_i < \omega$ . Then the following are equivalent:

(a) For every  $\langle k_i : i < \delta \rangle \in \prod_{i < \delta} n_i$  we have

$$\lambda = \text{tcf}\left(\prod_{i < \delta} \lambda_{i,k_i} / I\right).$$

(b) Letting

$$I' = \left\{ A \subseteq \bigcup_{i < \delta} \{i\} \times n_i : \text{for some } B \in I \text{ we have } A \subseteq \bigcup_{i \in B} \{i\} \times n_i \right\},$$

we have  $\prod \lambda_{i,n} / I'$  has true cofinality  $\lambda$ .

**Proof.** Let  $A^*, B^*$  be a partition of  $\bigcup_{i < \delta} \{i\} \times n$  such that

$$\lambda = \max \text{pcf}\{\lambda_{i,n} : (i,n) \in A^*\} \quad \text{and} \quad \lambda \notin \text{pcf}\{\lambda_{i,n} : (i,n) \in B^*\}$$

(exists by the pcf theorem). Now:

(a)  $\Rightarrow$  (b) If  $\prod_{i,n} \lambda_{i,n} / I'$  does not have true cofinality  $\lambda$ , then for some  $A \in (I')^+$  we have that  $\prod_{(i,n) \in A} \lambda_{i,n} / I'$  has true cofinality  $\lambda' \neq \lambda$  (here we use the pcf theorem) and without loss of generality  $A \subseteq A^* \vee A \subseteq B^*$ , hence  $\lambda \notin \text{pcf}\{\lambda_{i,n} : (i,n) \in A\}$ . Let  $B = \{i < \delta : (\exists n < n_i)[(i,n) \in A]\}$ , so by the definition of  $I'$  we know  $B \in I^+$ . So, for  $i \in B$  we can choose  $k_i \in \{0, \dots, n_i - 1\}$  such that  $(i, k_i) \in A$ . So  $\{(i, k_i) : i \in B\} \subseteq A$  hence

$$\text{pcf}\{\lambda_{i,k_i} : i \in B\} \subseteq \text{pcf}\{\lambda_{i,k} : (i,k) \in A\},$$

but  $\lambda$  does not belong to the later, hence not to the former, contradicting (a).

$\neg$ (a)  $\Rightarrow$   $\neg$ (b) So there is  $\langle k_i : i < \delta \rangle \in \prod_{i < \delta} n_i$  such that  $\neg[\text{tcf}(\prod \lambda_{i,k_i} / I) = \lambda]$  hence by the pcf theorem, for some  $A \in (I)^+$ , let, e.g.,  $\max \text{pcf}\{\lambda_{i,k_i} : i \in A\} < \lambda$ . Let  $B =$

$\{(i, k_i) : i \in A\}$ , so clearly  $\max \text{pcf}\{\lambda_{i,k_i} : (i, k_i) \in B\} < \lambda$ . But by the definition of  $I'$ , we have  $B \in (I')^+$  so we get contradiction to (b).  $\square$

**Remark 4.13.** See more on related topics in [17].

## 5. $\lambda$ -sequences for decreasing $\bar{\lambda}^i$ by pcf

**Discussion 5.1.** Our aim here is to get “decreasing  $\bar{\lambda}$ ” from “increasing  $\bar{\lambda}$ ” (for  $J_{\bar{\lambda}}^{\text{bd}}$ ), in some sense, to “make gold from lead”. We do this by using pcf assumptions, then proving that these assumptions are very reasonable.

(Note: when we cannot materialize the pcf assumptions the situation is close to SCH, and then we have other avenues for construction of  $\lambda$ -sequences for some  $I$ , e.g., (1.8, 3.1).)

In the following claim the interesting case is when  $\lambda_\ell$  are increasing,  $\bar{\lambda}^i = \langle \lambda_{\ell,i} : \ell < n \rangle$  decreasing sequence of regular cardinals,  $\lambda_{\ell,i} > \prod_{j < i, m < n} \lambda_{m,j}$ , or at least  $\lambda_{\ell,i} > \max \text{pcf}\{\lambda_{m,j} : m < n, j < i\}$ .

**Claim 5.2.** Assume

- (a)  $\bar{\lambda} = \langle \lambda_\ell : \ell < n \rangle$ ,  $\bar{\lambda}^i = \langle \lambda_{\ell,i} : \ell < n \rangle$  for  $i < \delta$ ,
- (b)  $I$  is an ideal on  $\delta$ ,
- (c)  $\lambda_\ell = \text{tcf}(\prod_{i < \delta} \lambda_{\ell,i} / I)$  for  $\ell < n$ ,
- (d)  $f^{\bar{\lambda}} = \langle f_{\ell,\alpha} : \alpha < \lambda_\ell \rangle$  is  $<_I$ -increasing and cofinal in  $\prod_{i < \delta} \lambda_{\ell,i}$ ,
- (e)  $\delta < \lambda_{\ell,i} = \text{cf}(\lambda_{\ell,i})$ ,
- (f) for  $\bar{\alpha} \in \prod_{\ell < n} \lambda_\ell$  let  $f_{\bar{\alpha}}$  be defined by  $f_{\bar{\alpha}}(i) = \langle f_{\ell,\alpha_\ell}(i) : \ell < n \rangle \in \prod_{\ell < n} \lambda_{\ell,i}$ .

Then for any  $X \in (J_{\bar{\lambda}}^{\text{bd}})^+$  we have

$$\{i : \{f_{\bar{\alpha}}(i) : \bar{\alpha} \in X\} \in J_{\bar{\lambda}^i}^{\text{bd}}\} \in I.$$

**Proof.** Let  $X_i = \{f_{\bar{\alpha}}(i) : \bar{\alpha} \in X\}$  and let  $B = \{i < \delta : X_i \in J_{\bar{\lambda}^i}^{\text{bd}}\}$ .

Assume  $B \in I^+$  and we shall get a contradiction. For each  $i \in B$ ,  $m < n$  and  $\bar{\alpha} \in \prod_{\ell < m} \lambda_{\ell,i}$ , let

$$X_{\bar{\alpha}}^i = \left\{ \bar{\beta} \in \prod_{\ell=m}^{n-1} \lambda_{\ell,i} : \bar{\alpha} \hat{\ } \bar{\beta} \in X_i \right\}$$

and let

$$g_i(\bar{\alpha}) = \min \left\{ \gamma \leq \lambda_{m,i} : \text{if } \beta \in [\gamma, \lambda_{m,i}) \text{ then } X_{\bar{\alpha} \hat{\ } \langle \beta \rangle}^i \in J_{\prod_{\ell=m+1}^{n-1} \lambda_{\ell,i}}^{\text{bd}} \right\}.$$

This definition just unravels the definition of  $J_{\bar{\lambda}^i}^{\text{bd}}$ ; note

- (\*)  $X_{\langle \rangle}^i = X_i \in J_{\bar{\lambda}^i}^{\text{bd}}$ ,
- (\*)' if  $X_{\bar{\alpha}}^i \in J_{\prod_{\ell \geq \text{lg}(\bar{\alpha})} \lambda_{\ell,i}}^{\text{bd}}$  then  $g_i(\bar{\alpha}) < \lambda_{\text{lg}(\bar{\alpha})}$ .

Now we choose by induction on  $m < n$  ordinals  $\alpha_m < \lambda_m$  such that for  $m \leq n$  we have

$$(*)_m \quad B_m =: \left\{ i \in B: X_{\langle f_{\ell, \alpha_\ell}(i): \ell < m \rangle}^i \in J_{\prod_{\ell \geq m} \lambda_{\ell, i}}^{\text{bd}} \right\} = B \text{ mod } I.$$

So, stipulating  $J_{\prod_{\ell \geq n} \lambda_{\ell, i}}^{\text{bd}} = \{\emptyset\}$ , the ideal on  $\{\langle \rangle\}$ , we have that  $(*)_0$  holds with  $B = B_0$ .

If  $(*)_m$  is true, clearly

$$\langle g_i(\langle f_{\ell, \alpha_\ell}(i): \ell < m \rangle): i \in B_m \rangle$$

is in  $\prod_{i < \delta} \lambda_{m, i}$ . But  $B_m \in I^+$  and  $\langle f_{m, \alpha}: \alpha < \lambda_m \rangle$  is  $<_I$ -increasing cofinal in  $\prod_{i < \delta} \lambda_{m, i}$ .

So for some  $\alpha_m$

$$B'_m = \{ i \in B_m: g_i(\langle f_{\ell, \alpha_\ell}(i): \ell < m \rangle) \geq \alpha_m \} \in I.$$

Defining  $B_{m+1}$  using this  $\alpha_m$ , we easily obtain

$$B_{m+1} \supseteq B_m \setminus B'_m \quad \text{so we see that } (*)_{m+1} \text{ holds.}$$

So

$$\bar{\alpha} = \langle \alpha_\ell: \ell < n \rangle \in \prod_{\ell < n} \lambda_\ell$$

is well defined.

In the inductive definition of  $\alpha_m$ , any larger  $\alpha'_m$  would serve in place of  $\alpha_m$  (of course it would influence the future choices). So, in addition to  $(*)_m$ , we can demand

$$(**)_m \quad \left\{ \bar{\beta} \in \prod_{\ell=m}^{n-1} \lambda_\ell: \langle \alpha_\ell: \ell < m \rangle \frown \bar{\beta} \in X \right\} \in (J_{\lambda \upharpoonright \{m, n\}}^{\text{bd}})^+.$$

So from  $(**)_n$  we get  $\langle \alpha_\ell: \ell < n \rangle \in X$  hence for all  $i$  we have  $\langle \rangle \in X_{\langle f_{\ell, \alpha_\ell}(i): \ell < n \rangle}^i$ , by the definition. But

$$B_n = \left\{ i \in B: X_{\langle f_{\ell, \alpha_\ell}(i): \ell < n \rangle}^i \in J_{\prod_{\ell \geq n} \lambda_{\ell, i}}^{\text{bd}} \right\} = B \text{ mod } I,$$

so  $B_n \neq \emptyset$ , and if  $i \in B_n$  this means  $X_{\langle f_{\ell, \alpha_\ell}(i): \ell < n \rangle}^i \in J_{\prod_{\ell \geq n} \lambda_{\ell, i}}^{\text{bd}} = \{\emptyset\}$  so  $X_{\langle f_{\ell, \alpha_\ell}(i): \ell < n \rangle}^i = \emptyset$ , contradicting the previous sentence.  $\square$

In fact, more generally,

**Claim 5.3.** Assume

- (a)  $\bar{\eta}^\ell = \langle \eta_\alpha^\ell: \alpha < \lambda \rangle$  in an  $(I, J, \lambda)$ -sequence for  $\langle I_{i, \ell}: i < \delta \rangle$  for each  $\ell < n$ ,
- (b)  $I_i = \prod_{\ell < n} I_{i, \ell}$ ,
- (c)  $\bar{\eta} = \langle \eta_\alpha: \alpha < \lambda \rangle$ , where  $\eta_\alpha \in \prod_{i < \delta} \text{Dom}(I_i)$  and

$$\eta_\alpha(i) = \langle \eta_\alpha^\ell(i): \ell < n \rangle.$$

Then  $\bar{\eta}$  is an  $(I, J, \lambda)$ -sequence for  $\langle I_i: i < \delta \rangle$ .

**Proof.** Like the proof of 5.2.  $\square$

**Claim 5.4.** Assume

- (a)  $\lambda = \text{tcf}(\prod_{i < \delta} \theta_{\ell,i} / J)$  for  $\ell < n$  and  $\theta_{\ell,i}$  are increasing with  $\ell$ ;  
 (b)  $\theta_{\ell,i} = \text{tcf}(\prod_{\varepsilon < \varepsilon_i} \tau_{\ell,i,\varepsilon} / J_i)$  and  $\tau_{\ell,i,\varepsilon}$  are regular decreasing with  $\ell$ , i.e.,  $\tau_{\ell,i,\varepsilon} \geq \tau_{\ell+1,i,\varepsilon}$  (the interesting case is  $>$ ).

Let

$$J^* = \left\{ A: A \subseteq \{(\ell, i, \varepsilon): \ell < n, i < \delta, \varepsilon < \varepsilon_i\} \text{ and } \bigwedge_{\ell} (\forall^J i) (\forall^{J_i} \varepsilon) [(\ell, i, \varepsilon) \notin A] \right\},$$

and let

$$I_{i,\varepsilon} = \prod_{\ell < n} J_{\tau_{\ell,i,\varepsilon}}^{\text{bd}}.$$

Then

$$\lambda = \text{tcf} \left( \prod_{i,\varepsilon} \tau_{\ell,i,\varepsilon} / J^* \right)$$

and we can find  $\bar{\eta}_\alpha \in \prod_{i,\varepsilon} I_{i,\varepsilon}$  for  $\alpha < \lambda$  such  $\langle \eta_\alpha: \alpha < \lambda \rangle$  is a  $(\lambda, J^*)$ -sequence for  $\langle I_{i,\varepsilon}: i, \varepsilon \rangle$ .

**Proof.** Straight. (Using 5.3 and [25, I, 2.10].)  $\square$

**Example 5.5.** Assume

- (\*)  $\langle \lambda_i: i < \delta \rangle$  is a strictly increasing sequence of regulars,

$$\delta < \lambda_0, \quad \lambda = \text{tcf} \left( \prod_{i < \delta} \lambda_i / J_\delta^{\text{bd}} \right).$$

**Discussion 5.5A.** This may seem a strong assumption, but getting such representations is central in [25]. If  $\mu$  is strong limit singular

$$\otimes \quad \aleph_0 < \kappa = \text{cf}(\mu) < \mu < \lambda = \text{cf}(\lambda) \leq 2^\mu,$$

then there is such  $\langle \lambda_i: i < \text{cf}(\mu) \rangle$ ,  $\lambda_i < \mu = \sup(\lambda_i)$ . So without loss of generality  $2^{\lambda_i} < \lambda_{i+1}$  (see 4.11A).

Now fix  $n$  for simplicity. Let

$$\lambda_{\ell,i} = \lambda_{n \times i + n - \ell}.$$

So

$$\bar{\lambda}^i = \langle \lambda_{\ell,i}: \ell < n \rangle \text{ is strictly decreasing.}$$

In 4.12 an example is given for 5.2. For 5.4 we have, e.g.,

**Claim 5.6.** Assume

- (a)  $\mu$  is strong limit,

- (b)  $\aleph_0 = \text{cf}(\mu) < \mu$ ,  
 (c)  $2^\mu \geq \mu^{+\omega+1} = \lambda$  [also  $\mu^{+\omega_4+\omega+1} = \lambda$  is OK, or just  $\lambda = \mu^{+\delta+1} < \text{pp}^+(\mu)$  and  $\text{cf}(\mu^{+\delta}) < \mu$ ].

Then:

- (1) We can find  $\lambda_{\ell,i}, k_\ell$  such that ( $\ell \leq i < \omega$ ):  
 (A)  $\lambda_{i,\ell} < \mu = \sum_{m,j} \lambda_{m,j}$ ,  
 (B)  $2^{\lambda_{\ell+1,i}} < \lambda_{\ell,i}$  and  $2^{\lambda_{0,i}} < 2^{\lambda_{i+1,i+1}}$ ,  
 (C)  $\text{tcf}(\prod_{i < \omega} \lambda_{\ell,i} / J_\omega^{\text{bd}}) = \mu^{+k_\ell}$ ,  
 (D)  $0 < k_m < k_{m+1} < \omega$ ,  
 (E)  $\lambda = \text{tcf}(\prod_{m < \omega} \mu^{+k_m} / J_\omega^{\text{bd}})$ .  
 (2) For every  $n < \omega$ , we can find  $J, \lambda'_{\ell,i}$  ( $\ell < n, i < \omega$ ) such that:  
 (i) there is  $\bar{\eta}$  a  $\lambda$ -sequence for  $\langle J_{\langle \lambda'_{\ell,i} : \ell < n \rangle}^{\text{bd}} : i < \omega \rangle$ ,  
 (ii)  $2^{\lambda'_{\ell+1,i}} < \lambda'_{\ell,i}$ ,  
 (iii)  $2^{\lambda'_{0,i}} < \lambda'_{n-1,i+1}$ ,  
 (iv)  $(\forall A \in J)(\exists^\infty i) [n \times \{i\} \cap A = \emptyset]$ .

**Remark 5.7.**

- (1) This claim can be used with no further reference to pcf: just for any  $\mu$  as in (a)–(c), we have  $\bar{\eta}$  for which we can construct colorings, objects, etc.  
 (2) There are theorems with  $n$  increasing, they are somewhat cumbersome.  
 Of course, we can use

$$I'_m = \prod_{i=n_m}^{n_m+1} J_{\langle \lambda_{\ell,i} : \ell < n_m \rangle}^{\text{bd}}.$$

- (3) Note:  $2^\mu \geq \mu^{\omega+1}$  is a strong negation of  $2^\mu = \mu^+$  which was very useful here. (Our general theme is:  $\neg\text{SCH}$  is a good hypothesis) and we shall deal with closing the gap.  
 (4) Note: if  $2^\mu = \mu^{+n(*)}$ , we can prove nice things with  $I = J_{\langle \mu^{+n(*)-\ell}, \ell < n(*) \rangle}^{\text{bd}}$ .  
 (5) If  $\aleph_0 < \text{cf}(\mu) < \mu$  the parallel claim is even easier, and  $\mu$  being a strong limit is necessary only for (B).

**Proof of 5.6.** (1) We will just give a series of quotations.

First  $\text{cf}(\mu^{+\omega}) = \aleph_0$ , so by [25, II, 1.6], there is an increasing sequence  $\langle \theta_i : i < \omega \rangle$  of regulars with limit  $\mu^{+\omega}$  such that

$$\lambda = \mu^{+\omega+1} = \text{tcf}\left(\prod_{i < \omega} \theta_i / J_\omega^{\text{bd}}\right),$$

so for  $i$  large enough  $\theta_i > \mu$ . So without loss of generality  $\bigwedge_i \theta_i > \mu$ .

So let  $\theta_i = \mu^{+k_i}$ ,  $k_i \in (0, \omega)$  strictly increasing. By [25, IX, 5.9, p. 408], we have  $\text{pp}(\mu) > \mu^{+k_i}$ . (We would like to have  $\text{pp}(\mu) = 2^\mu$ , but only “almost proved”.) This means by the no hole theorem [25, II, 2.3] that for some countable set  $\mathfrak{a}_\ell$  of regulars

$< \mu$ ,  $\mu = \sup(\mathfrak{a}_\ell)$  and  $\mu^{+k_\ell} \in \text{pcf}(\mathfrak{a}_\ell)$ . So by the pcf theorem, without loss of generality  $\mu^{+k_\ell} = \max \text{pcf}(\mathfrak{a}_\ell)$  and  $\mu^+, \dots, \mu^{+(k_\ell-1)} \notin \text{pcf}(\mathfrak{a}_\ell)$  (alternatively use [25, VIII, §1]).

So necessarily

$$\mu^{+k_\ell} = \text{tcf}\left(\prod \mathfrak{a}_\ell / J_{\mathfrak{a}_\ell}^{\text{bd}}\right).$$

Let  $\mu = \sum_{n < \omega} \mu_n$ ,  $\mu_n < \mu_{n+1} < \mu$ . We start choosing  $\lambda_{\ell,i}$  by induction on  $i$ , for all  $i$  by downward induction on  $\ell$ , so that

$$\lambda_{\ell,i} > \mu_i, \quad \lambda_{\ell,i} \in \mathfrak{a}_i,$$

and (B) holds. So, as  $\lambda_{\ell,i} \in \mathfrak{a}_i$  and  $\lambda_{\ell,i}$  is increasing with  $i$ , with limit  $\mu$ , we have

$$\text{tcf}\left(\prod_i \lambda_{\ell,i} / J_\omega^{\text{bd}}\right) = \mu^{+k_\ell}.$$

(2) Let  $h: \omega \rightarrow \omega$  be such that  $(\forall m)(\exists^{\aleph_0} i)(h(i) = m)$ . Choose by induction on  $i$ ,  $\lambda'_{\ell,i} \in \{\lambda_{h(i),m}: m < \omega\}$  such that (b) + (c) of (2) hold.

For each  $i$  we do this by downward induction on  $\ell$ . Then apply the last theorem.  $\square$

We may deal with all  $n$ 's at once, at some price. The simplest case is:

**Claim 5.8.** Assume

(a)  $\langle \mathfrak{A}_\ell: \ell < \omega \rangle$  is a sequence of pairwise disjoint sets,

(b)  $\lambda = \text{tcf}(\prod_{n < \omega} \theta_n / J_\omega^{\text{bd}})$ ,

(c)  $\theta_n = \text{tcf}(\prod_{\ell < \omega} \tau_{n,\ell} / J_\omega^{\text{bd}})$ ,  $\tau_{n,\ell}$  regular  $> \aleph_0$ ,

(d)  $h: \omega \rightarrow \omega$  is such that  $|h^{-1}(\{n\})| = \aleph_0$ ,  $J = \{A \subseteq \omega \times \omega: (\forall J_\omega^{\text{bd}} n) (\forall J_\omega^{\text{bd}} m) (h(n) = \emptyset = A \cap \{m\} \times [h(n), 2h(n)])\}$ .

Then there is a  $(\lambda, J)$ -sequence for  $\langle J_{\tau_{n,\ell}}^{\text{bd}}: (n, \ell) \in \omega \times \omega \rangle$ .

**Proof.** Straight.  $\square$

**Remark 5.9.**

(1) We can replace  $\langle \theta_n: n < \omega \rangle$  by  $\langle \theta_i: i < \delta \rangle$ .

(2) Another way to get an example for 5.4 is to have  $\langle \mu_i: i < \kappa \rangle$  increasing continuous,  $\kappa = \text{cf}(\kappa) > \aleph_0$ ,  $\kappa < \mu_0$ ,  $\mu = \mu_\kappa = \sum_{i < \kappa} \mu_i$ ,  $\text{cf}(\mu_i) \leq |\delta|$ ,  $\text{pp}_{|\delta|}(\mu_i) < \mu_{i+1}$ ,  $\chi_i = |\text{Reg} \cap [\mu_i, \text{pp}_{|\delta|}^+(\mu_i))|$ ,  $S \subseteq \kappa$  stationary such that for every  $S' \subseteq S$  stationary we have  $\prod_{i \in S'} \chi_i > \chi_\kappa$ .

(3) In all the cases here we can get normality as in §4.

(4) See 1.16, 1.17.

## 6. Products of Boolean algebras

Monk asks [12, Problem 35, p. 15]:

**Monk's Problem 6.1.** Does  $\prod_{n < \omega} \text{FBA}(\beth_n)$  have free caliber  $\beth_\omega^+$ ? Here:

**Notation 6.1A.**  $\mathbf{FBA}(\beta)$  is the Boolean algebra freely generated by  $\langle x_\alpha: \alpha < \beta \rangle$ .

**Definition 6.2.**

- (1) We say that the cardinal  $\lambda$  is a free caliber of the Boolean algebra  $\mathbf{B}$  if for every  $X \in [\mathbf{B}]^\lambda$  there is  $Y \in [X]^\lambda$  such that  $Y$  is independent in  $\mathbf{B}$ .
- (2)  $\text{FreeCal}(\mathbf{B}) = \{\lambda \leq |\mathbf{B}|: \lambda \text{ is a free caliber of } \mathbf{B}\}$ .

We show that, e.g., if  $\beth_\omega^+ = 2^{\beth_\omega}$  then the answer is NO.

**Claim 6.3.** Assume:

- (a) there is a normal<sup>2</sup> super  $(\lambda, J)$ -sequence  $\bar{\eta}$  for  $\bar{I} = \langle I_i: i < \delta \rangle$ ,
- (b)  $I_i = \text{ERI}_{\lambda_i, \kappa_i}^2 =: \{X \subseteq [\lambda_i]^2: \text{for some } h: \chi \rightarrow \kappa_i, |\text{Rang } h| < \kappa_i, \text{ and for no } u \in [\lambda_i]^{\aleph_0} \text{ do we have } (h \upharpoonright [u]^2 \text{ constant}) \& [u]^2 \subseteq X\}$ ,
- (c)  $\delta < \omega_1$ .

Then  $\lambda$  is not a free caliber of  $\prod_{i < \delta} \mathbf{FBA}(\lambda_i)$ .

**Remark 6.3A.** By 3.1, if  $\lambda = \mu^+ = 2^\mu$ ,  $\mu$  strong limit  $> \aleph_0 = \text{cf}(\mu)$ , then we can find such  $\kappa_i, \lambda_i < \mu$  and  $\bar{\eta}$  for  $\delta = \omega$ .

**Proof.** By renaming without loss of generality

$$(*)_1 \quad \eta_\alpha(i) \geq \sum_{j < i} \lambda_j.$$

Let  $f_\alpha(i) = \{f_\alpha^0(i), f_\alpha^1(i)\}$ ,  $f_\alpha^0(i) < f_\alpha^1(i) (< \lambda)$ . First we deal with the case  $\delta = \omega$ , as its notation is simpler. Let  $\mathbf{B}_n = \mathbf{FBA}(\lambda_n)$  be freely generated by  $\{x_\alpha^n: \alpha < \lambda_n\}$ . We define  $g_\alpha^* \in \prod_{n < \omega} \mathbf{B}_n$  for  $\alpha < \lambda$  by

$$g_\alpha^*(\ell) = \bigcap_{k < \ell} (x_{f_\alpha^0(k)}^\ell - x_{f_\alpha^1(k)}^\ell).$$

Note:

- $\otimes_1$  for  $\alpha < \beta < \lambda$ , we have  $g_\alpha^*, g_\beta^*$  are distinct elements of  $\prod_{n < \omega} \mathbf{B}_n$ ,
  - $\otimes_2$  if  $f_n^0(\beta) = f_n^1(\alpha)$  and  $m > n$  then  $\mathbf{B}_m \models g_\alpha^*(m) \cap g_\beta^*(m) = 0$ .
- [Why? As  $x_{f_n^0(\alpha)}^m - x_{f_n^1(\alpha)}^m$  is disjoint to  $x_{f_n^0(\beta)}^m - x_{f_n^1(\beta)}^m$ .]
- $\otimes_3$  if  $n < \omega$  and for  $i = 1, 2$  we have  $\alpha_i, \beta_i < \lambda$  and  $f_n^0(\beta_i) = f_n^1(\alpha_i)$  and

$$\bigwedge_{k < n} f_k^0(\alpha_1) = f_k^0(\alpha_2) \quad \text{and} \quad \bigwedge_{k < n} f_k^1(\beta_1) = f_k^1(\beta_2),$$

then

$$\prod_{n < \omega} \mathbf{B}_n \models g_{\alpha_1}^* \cap g_{\beta_1}^* = g_{\alpha_2}^* \cap g_{\beta_2}^*.$$

[Why? Check each coordinate in the product, for  $m > n$  use  $\otimes_2$  to show that both sides are zero, and if  $m \leq n$  use the last two assumptions.]

<sup>2</sup> If  $\bar{I}$  is normal, i.e.,  $\kappa_{i+1} > \lambda_i$ , the normality of  $\bar{\eta}$  follows.

Now if  $X \in [\lambda]^\lambda$  then there are such  $\alpha_1, \alpha_2, \beta_1, \beta_2$  (using the choice of  $\bar{\eta}$  and its normality).  $\square$

**Claim 6.4.** Assume:

- (\*) (a)  $\mu = \mu^\theta < \lambda = \text{cf}(\lambda) \leq 2^\mu$ , and  $\langle \chi_i : i < \theta \rangle$  a sequence of cardinals, or
- (b)  $2^\theta < \lambda = \text{cf}(\lambda)$  and in the  $(< \theta^+)$ -base product topology on  ${}^{\text{sup}(\chi_i)} 2$  the density is  $< \lambda$ , or at least in the box product topology on  $\prod_{i < \theta} (\chi_i 2)$  (where each  $\chi_i 2$  has Tychonoff topology) has density  $< \lambda$ .

Then  $\prod_{i < \theta} \text{FBA}(\chi_i)$  has free caliber  $\lambda$ .

**Proof.** As in §2.  $\square$

Probably the choice of the product of  $\langle \text{FBA}(\bar{\eta}_n) : n < \omega \rangle$  in the original question was chosen just as the simplest case, as is often done. But in this case the products of uncountably many free Boolean algebras behave differently.

**Claim 6.5.** Assume  $\lambda = \text{cf}(\lambda) > 2^\theta$ ,  $\text{cf}(\theta) > \aleph_0$  and  $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$ . Then  $\prod_{i < \theta} \text{FBA}(\chi_i)$  has free caliber  $\lambda$ .

**Proof.** First assume a stronger assumption

- (\*)  $\lambda = \mu^+$ ,  $\text{cf}(\mu) = \theta > \aleph_0$  and  $(\forall \alpha < \mu)(|\alpha|^\theta < \mu)$ ,

or alternatively

- (\*)<sup>-</sup>  $\lambda = \text{cf}(\lambda)$  and  $\mu > 2^\theta$  are as in 7.3 below and we assume  $i < \theta \Rightarrow \chi_i \leq \mu$ .

(This was our first proof. It possibly covers all cases under some reasonable pcf hypothesis, and illuminates the method.)

Let  $g_\alpha^* \in \prod_{i < \theta} \text{FBA}(\chi_i)$  for  $\alpha < \lambda$  be pairwise distinct, and we should find  $X \in [\lambda]^\lambda$  such that  $\langle g_\alpha^* : \alpha \in X \rangle$  is independent. Let

$$g_\alpha^*(i) = \tau_{\alpha,i}(x_{\beta_{\alpha,i,0}}, x_{\beta_{\alpha,i,1}}, \dots, x_{\beta_{\alpha,i,m(\alpha,i)-1}}),$$

where  $\tau_{\alpha,i}$  is a Boolean term. Without loss of generality no  $x_{\beta_{\alpha,i,\ell}}$  is redundant,  $\beta_{\alpha,i,m}$  increasing with  $m$ . As  $2^\theta < \lambda = \text{cf}(\lambda)$  without loss of generality  $\tau_{\alpha,i} = \tau_i$  and so  $m(\alpha,i) = m(i)$ . Let  $f_\alpha$  be the function with domain  $\theta$ ,  $f_\alpha(i) = \langle \beta_{\alpha,i,\ell} : \ell < m(i) \rangle$ . Let  $f_\alpha^{[\ell]}(i) = \beta_{\alpha,i,\ell}$ , so  $\text{Dom}(f_\alpha^{[\ell]}) = \{i < \theta : \ell < m(i)\}$ .

If (\*) holds then by 7.1 and 7.2 (see below) we have

- ⊙ there are  $u^*, m^*, v, \bar{\beta}^*, X$  such that
  - (a)  $u^* \in [\theta]^\theta$  and  $X \in [\lambda]^\lambda$ ,  $\mathcal{D}$  an ultrafilter on  $u^*$  disjoint to  $J_\theta^{\text{bd}}$ ,
  - (b)  $i \in u^* \Rightarrow m(i) = m^*$ ,
  - (c)  $v \subseteq m^*$  but  $v \neq m^*$ , and  $\lambda = \text{tcf} \prod \beta_{\ell,i}^* / \mathcal{D}$  for  $\ell \in m^* \setminus v$ ,
  - (d)  $\bar{\beta}^* = \langle \beta_{\ell,i}^* : \ell < m^*, i \in u^* \rangle$ ,
  - (e)  $\ell \in v \Rightarrow \langle f_\alpha^{[\ell]} \upharpoonright u^* : \alpha \in X \rangle$  is  $<_{J_{u^*}^{\text{bd}}}$ -increasing and cofinal in  $\prod_{i \in u^*} \beta_{\ell,i}^*$ ,
  - (f)  $\ell \in m^* \setminus v \Rightarrow f_\alpha^{[\ell]} \upharpoonright u^* = \langle \beta_{\ell,i}^* : i \in u^* \rangle$ ,
  - (g) for every  $\bar{\gamma} \in \prod_{\ell \in v, i \in u^*} \beta_{\ell,i}^*$  for  $\lambda$  ordinals  $\alpha \in X$  we have,

$$i \in u^* \ \& \ \ell \in v \Rightarrow \gamma_{\ell,i} < f_\alpha^{[\ell]}(i) < \beta_{\ell,i}^*,$$



- (h) if  $\ell \in v$ ,  $\alpha \in X$ ,  $i \in u^*$  then  $f_\alpha^{[\ell]}(i) > \sup\{\beta_{\ell_1, i_1}^* : \beta_{\ell_1, i_1}^* < \beta_{\ell, i}^* \text{ where } \ell_1 < m^* \text{ and } i_1 < \theta\}$  and  $\alpha < \beta \in X$  implies: for every  $i \in u^*$  large enough we have  $f_\beta^{[\ell]}(i) > \max\{f_\alpha^{[\ell_1]}(i_1) : \beta_{\ell_1, i_1}^* = \beta_{\ell, i}^* \text{ and } \ell_1 < m^* \text{ and } i_1 < \theta\}$  (the interesting case is  $i_1 = i$ ).

Now for any  $n < \omega$ , and  $\alpha_0 < \dots < \alpha_{n-1}$  from  $X$ , we have

⊗ for arbitrarily large  $i \in u^*$

$$\langle f_{\alpha_0}(i), f_{\alpha_1}(i), \dots, f_{\alpha_{n-1}}(i) \rangle \\ = \langle \langle \beta_{\alpha_0, i, \ell} : \ell < m^* \rangle, \langle \beta_{\alpha_1, i, \ell} : \ell < m^* \rangle, \dots, \langle \beta_{\alpha_{n-1}, i, \ell} : \ell < m^* \rangle \rangle$$

is as in a  $\Delta$ -system, in fact

$$\beta_{\alpha_{k(1)}, i, \ell(1)} = \beta_{\alpha_{k(2)}, i, \ell(2)} \Rightarrow (k(1), \ell(1)) = (k(2), \ell(2)) \vee (\ell(1) = \ell(2) \in v).$$

As  $v \neq \{0, 1, \dots, m^* - 1\}$  and in  $\tau$  no variable is redundant clearly

⊗' for every  $i \in u^*$  large enough,  $\langle \tau(x_{\beta_{\alpha_0, i, 0}}, \dots), \tau(x_{\beta_{\alpha_1, i, 0}}, \dots), \dots \rangle$  is independent.

This implies that  $\langle g_{\alpha_\ell}^* : \ell < n \rangle$  is independent (in  $\prod_{i < \theta} \text{FBA}(\chi_i)$ ) as required.

If we do not have  $(*)$  or  $(*)^-$ , by  $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$  and  $2^\theta < \lambda = \text{cf}(\lambda)$  without loss of generality for some  $\tau = \tau(x_1, \dots, x_{n-1})$  and infinite  $u \subseteq \theta$ , and some  $X \in [\lambda]^\lambda$  we have:  $\langle f_\alpha \upharpoonright u : \alpha \in X \rangle$  is with no repetition,  $\tau_{\alpha, i} = \tau$  for  $\alpha \in X$ ,  $i \in u$ . So without loss of generality  $u = \theta$ . Then we can find an ultrafilter  $\mathcal{D}$  on  $\theta$  as in 7.4 below and then the proof above works.  $\square$

**Comment 6.6.** Before we use 7.4, we wonder if “ $\chi_i \leq \mu$ ” is necessary in  $(*)^-$  of 6.5. This is quite straight. We can omit it if

$$a \subseteq \text{Reg} \cap \lambda \setminus \mu, |a| \leq \theta \Rightarrow \max \text{pcf}(a) < \lambda.$$

**Problem 6.7.**

- (1) Which of the following statements is consistent with ZFC:
  - (a)  $\mu$  is strong limit,  $\text{cf}(\mu) = \aleph_0$ , and for every  $\lambda \in \text{Reg} \cap (\mu, 2^\mu]$  and cardinals  $\chi_n$  such that  $\mu = \sum_{n < \omega} \chi_n$ ,  $\lambda$  is a free caliber of  $\prod_{n < \omega} \text{FBA}(\chi_n)$ , (What about “some such  $\lambda$ ”? See 6.11 below.)
  - (b) The same for all such  $\mu$ .
- (2) Can you prove in ZFC that for some strong limit  $\mu$ ,  $\theta = \text{cf}(\mu) < \mu$  and for some set  $\langle a_i : i < \sigma \rangle$  where  $\sigma = \theta^+$  or  $\sigma = (2^\theta)^+$ , pairwise disjoint there is  $\lambda \in (\mu, 2^\mu] \cap \bigcap_{i < \sigma} \text{pcf}(a_i)$ .

Now we turn to another of Monk’s problems.

**Claim 6.8.** Assume

$(*) \kappa > \aleph_0$  is weakly inaccessible and  $\langle 2^\mu : \mu < \kappa \rangle$  is not eventually constant.

Then

- (a) there is a  $\kappa$ -c.c. Boolean algebra of cardinality  $2^{< \kappa}$ , with no independent subset of cardinality  $\kappa^+$ .

**Proof.** There are sequences  $\langle (\mathcal{I}_i, \mathcal{J}_i) : i < \kappa \rangle$ ,  $\langle (\kappa_i, \lambda_i) : i < \kappa \rangle$  such that  $\mathcal{J}_i$  is a dense linear order of cardinality  $\lambda_i$  and  $\mathcal{I}_i \subseteq \mathcal{J}_i$  a dense subset of  $\mathcal{J}_i$  of cardinality  $\kappa_i$ ,  $\langle \kappa_i : i < \kappa \rangle$  increasing with limit  $\kappa$ , and  $\lambda_j > \sum_{i < j} 2^{\kappa_i}$  ( $\geq \sum_{i < j} \lambda_i$ ), by [26, 3.4].

Let  $\mathbf{B}_i$  be  $\text{Intalg}(\mathcal{J}_i)$ , the Boolean algebra of closed-open intervals of  $\mathcal{J}_i$ . Let  $\mathbf{B}$  be the free product of  $\{\mathbf{B}_i : i < \kappa\}$ , so  $\mathbf{B}$  extends each  $\mathbf{B}_i$  and each element of  $\mathbf{B}$  is a Boolean combination of finitely many elements of  $\bigcup_{i < \kappa} \mathbf{B}_i$ . It is straight to check  $\mathbf{B}$  is as required:

$$(*)_1 \quad |\mathbf{B}| = \sum_{i < \kappa} |\mathbf{B}_i| + \aleph_0 = \sum_{i < \kappa} \lambda_i = \sum_{i < \kappa} 2^{\kappa_i} = 2^{< \kappa},$$

(\*)<sub>2</sub>  $\mathbf{B}$  satisfies the  $\kappa$ -c.c.

[Why? Let  $a_i \in \mathbf{B} \setminus \{0\}$  for  $i < \kappa$ , so let  $a_i = \tau_i(b_{i,0}, \dots, b_{i,n_i-1})$  for  $i < \kappa$ ,  $b_{i,\ell} \in \mathbf{B}_{\alpha_{i,\ell}}$ . As we can replace  $a_i$  by any  $a'_i$ ,  $0 < a'_i \leq a_i$  without loss of generality  $a_i = \bigcap_{\ell < n_i} b_{i,\ell}$ ,  $b_{i,\ell} \in \mathbf{B}_{\alpha_{i,\ell}} \setminus \{0\}$ . So without loss of generality  $\alpha_{i,0} < \alpha_{i,1} < \dots < \alpha_{i,n_i}$ . As  $\kappa > \aleph_0$  is regular and as we can replace  $\langle a_i : i < \kappa \rangle$  by  $\langle a_i : i \in X \rangle$  whenever  $X \in [\kappa]^\kappa$ , without loss of generality for some  $m$ ,  $\bigwedge_{\ell < m} \alpha_{i,\ell} = \alpha_\ell$  and  $i < j$  &  $\{\ell, k\} \subseteq [m, n] \Rightarrow \alpha_{i,\ell} < \alpha_{j,k}$ . Let  $a'_i = \bigcap_{\ell < m} b_{i,\ell}$ , so clearly

$$a'_i \wedge a'_j \neq 0 \Leftrightarrow a_i \wedge a_j \neq 0 \Leftrightarrow \bigwedge_{\ell < m} b_{i,\ell} \cap b_{j,\ell} \neq 0.$$

But  $\mathbf{B}_i$  satisfies the  $\kappa$ -Knaster condition (as  $\kappa = \text{cf}(\kappa) > \text{density}(\mathcal{J}_i)$ ), so can we finish.]

(\*)<sub>3</sub>  $\mathbf{B}$  has no independent subset of cardinality  $\kappa^+$ .

[Why? Let  $a_i \in \mathbf{B}$  for  $i < \kappa^+$ , let  $a_i = \tau_i(b_{i,0}, \dots, b_{i,n_i-1})$  and let  $b_{i,\ell} \in \mathbf{B}_{\alpha_{i,\ell}} \setminus \{0, 1\}$ . We can replace  $\langle a_i : i < \kappa^+ \rangle$  by  $\langle a_i : i \in X \rangle$  for  $X \in [\kappa^+]^{\kappa^+}$ , so without loss of generality  $\tau_i = \tau$ ,  $n_i = n$  and  $\alpha_{i,\ell} = \alpha_\ell$ . Let  $b_{i,\ell} = \bigcup_{k \in u_{i,\ell}} [x_{i,\ell,k}, x_{i,\ell,k+1})$  where  $\bar{x}^{i,\ell} = \langle x_{i,\ell,k} : k \leq k_{i,\ell} \rangle$  is an increasing sequence of elements of  $\{-\infty\} \cup \mathcal{J}_i \cup \{\infty\}$ ,  $x_{i,\ell,0} = -\infty$ ,  $x_{i,\ell,k_{i,\ell}} = \infty$ ,  $u_{i,\ell} \subseteq k_{i,\ell}$ . We can find  $y_{i,\ell,k} \in \mathcal{I}_i$  such that  $x_{i,\ell,k} < y_{i,\ell,k} < x_{i,\ell,k+1}$ . Without loss of generality  $k_{i,\ell} = k_\ell$ ,  $y_{i,\ell,k} = y_{\ell,k}$ ,  $u_{i,\ell} = u_\ell$ .

Without loss of generality  $y_{i,\ell,k} = y_{\ell,k}$ . For a finite  $A \subseteq \mathbf{B}$  let  $\text{at}(A) = \text{at}(A, \mathbf{B})$  be the number of atoms in the Boolean subalgebra of  $\mathbf{B}$  which  $A$  generates (all this was mainly for clarity). Now for any finite  $u \subseteq \kappa^+$

$$\begin{aligned} \text{at}(\{a_i : i \in u\}, \mathbf{B}) &\leq \text{at}(\{b_{i,\ell} : i \in u, \ell < n\}, \mathbf{B}) \leq \prod_{\ell < n} \text{at}(\{b_{i,\ell} : i \in u\}, \mathbf{B}_{\alpha_{i,\ell}}) \\ &\leq \prod_{\ell < n} \text{at}(\{x_{i,\ell,k} : i \in u, k < k_\ell\}, \mathbf{B}_{\alpha_{i,\ell}}) \leq \prod_{\ell < n} \left( \sum_{k < k_\ell} (|u| + 1) \right) \leq k^* \times |u|^n \end{aligned}$$

for  $k^* = \max\{k_\ell + 1 : \ell < n\}$ . So if  $u$  is large enough this is  $< 2^{|u|}$ , showing non-independence.]  $\square$

**Claim 6.9.** Let  $\mathbf{B}$  be the completion of  $\text{FBA}(\chi)$

(1)  $\lambda$  is not a free caliber of  $\mathbf{B}$  if

$$(*) \quad \lambda = \mu^+ = 2^\mu, \mu \leq \chi, \mu \text{ strong limit singular of cofinality } \aleph_0,$$

(2)  $\lambda$  is a free caliber of  $\mathbf{B}$  if

$$(*) \quad \mu = \mu^{\aleph_0} < \lambda = \text{cf}(\lambda) \leq 2^\mu, \chi \geq \lambda, \text{ or at least}$$

(\*)'  $\chi \geq \mu$ ,  $\mu < \lambda = \text{cf}(\lambda) \leq 2^\lambda$ ,  $\mu$  strong limit singular of cofinality  $\aleph_0$  and the  $(< \aleph_1)$ -box product topology on  ${}^X \omega$  has density  $< \lambda$ .

**Proof.** (1) By 6.3, 6.3A's proofs.

(2) If (\*) use 6.5, if (\*)' the proof is similar.  $\square$

**Remark 6.9A.** We can deal with singular cardinals similarly as in the earlier proofs.

**Claim 6.10.** In the earlier claims if

(\*)<sub>1</sub>  $\lambda = \mu^{++}$ , or at least if

(\*)<sub>2</sub>  $\mu < \lambda$ , and  $[\alpha < \lambda \Rightarrow \text{cf}([\alpha]^\theta, \subseteq) < \lambda]$ ,  $\chi = \sup_{i < \theta} \chi_i$

then “in the  $(\leq \theta^+)$ -box product topology,  ${}^X \theta$  has density  $< \lambda$ ” can be replaced by “in the  $(< \theta^+)$ -box product topology,  ${}^\mu \theta$  has density  $< \lambda$ ”.

**Conclusion 6.11.**

(1) Let  $\ell \in \{1, 2\}$  for simplicity. The following questions cannot be answered in ZFC (assuming the consistency of large cardinals). Assume  $\beth_\omega^{+\ell} \leq \beth_{\omega+1}$

(a) <sub>$\ell$</sub>  Does  $\prod_{n < \omega} \text{FBA}(\beth_n)$  have free caliber  $\beth_\omega^{+\ell}$ ?

(b) <sub>$\ell$</sub>  Does the completion of  $\text{FBA}(\beth_\omega)$  have free caliber  $\beth_\omega^{+\ell}$ ?

(c) <sub>$\ell$</sub>  Does the completion of  $\text{FBA}(\beth_\omega^{+\ell})$  have free caliber  $\beth_\omega^{+\ell}$ ?

(2) Moreover we can add

for  $x \in \{a, b, c\}$  even  $(*)_1 + (*)_2$ , and  $\neg(*)_1 + \neg(*)_2$ .

**Proof.** (1) Let  $\ell = 2$ . By Gitik and Shelah [6] it is consistent with ZFC that with the  $(< \aleph_1)$ -box product topology,  $(\beth_\omega^+)^{\omega}$  has density  $\leq \beth_\omega^+$ , so we can use 6.4 (using 6.11 of course). For the other direction by Gitik and Shelah [6] the necessary assumptions for 6.3, 6.9(2) are consistent.

For  $\ell = 1$ , if  $\beth_\omega^+ = 2^{\beth_\omega}$  then the answer is NO by 6.2, 6.9.

To get consistency for  $\lambda = \beth_\omega^+$  we need dual: in  ${}^\mu \omega$ , for every  $\mu^+$  open sets there is a point belonging to  $\mu^+$  of them (this is phrased in 6.12 below). This too is proved consistent in [6].

(2) Similarly.  $\square$

**Definition 6.12.**  $\text{Pr}_{\theta, \sigma}(\lambda, \mu)$  means:

if  $f_\alpha$  is a partial function from  $\mu$  to  $\theta$  such that  $|\text{Dom}(f_\alpha)| < \sigma$  for  $\alpha < \lambda$ ,  
then some  $f \in {}^\mu \theta$  extends  $\lambda$  of the functions  $f_\alpha$ .

If  $\sigma = \theta$  we may omit it.

**Claim 6.13.** In Claim 6.11 the assumption on the density of box products can be replaced by cases of Definition 6.12:

(a) [2.1] Assume  $\mathbf{B} = \mathcal{B}(\chi)$  is a Maharam measure algebra of dimension  $\chi$ ,  $\text{cf}(\lambda) > 2^{\aleph_0}$  and  $\text{cf}(\lambda) > \beth_2 \vee \lambda = \text{cf}(\lambda) \vee \boxtimes_{\lambda, \aleph_0}^-$ . If  $\text{Pr}_{\aleph_0}(\lambda, \chi)$  then  $\mathbf{B}$  has  $\lambda$  as a free caliber.

- (b) [6.4] Assume  $2^\theta < \lambda = \text{cf}(\lambda)$ ,  $\chi = \sup_{i < \theta} \chi_i$ . If  $\text{Pr}_\theta(\lambda, \chi)$  then  $\prod_{i < \theta} \text{FBA}(\chi_i)$  has free caliber  $\lambda$ .

**Proof.** Straight.  $\square$

In fact cases of Pr are essentially necessary and sufficient conditions.

**Claim 6.14.** Assume  $\lambda = \text{cf}(\lambda) > 2^{\aleph_0}$ , and  $\chi_n$  are cardinal. The following conditions are equivalent

- (a)  $\prod_{n < \omega} \text{FBA}(\chi_n)$  has free caliber  $\lambda$ ;  
 (b) if for  $\alpha < \lambda$ ,  $i < \omega$ ,  $(u_i^\alpha, v_i^\alpha)$  is a pair of disjoint finite subsets of  $\chi_i$  then for some  $X \in [\lambda]^\lambda$  we have

$$i < \omega \Rightarrow \bigcup_{\alpha \in X} u_i^\alpha \cap \bigcup_{\alpha \in X} v_i^\alpha = \emptyset,$$

i.e., if  $f_i^\alpha$  is a finite function from  $\chi_i$  to  $\{0, 1\}$  for  $i < \omega$ ,  $\alpha < \lambda$ , then for some  $\langle f_i : i < \omega \rangle$

$$(\exists^\lambda \alpha < \lambda)(\forall i < \omega) f_i^\alpha \subseteq f_i.$$

**Proof.** Straight.  $\square$

**Discussion 6.15.** For measure, the parallel seems cumbersome. We still may like to be more concrete on the dependencies appearing. Note

$\otimes_1$  in 3.6, we can have  $\bar{x} = \langle x_\alpha : \alpha < \lambda \rangle$  satisfies

- (\*) $_{\mathbf{B}, \bar{x}}$  for every  $X \in [\lambda]^\lambda$ ,  $m < \omega$ , and  $\beta(\alpha, k) < \lambda$  for  $\alpha < \lambda$ ,  $k < 2m$  pairwise distinct, for every  $n$  large enough there are pairwise distinct  $\alpha_0, \dots, \alpha_{2n-1} \in X$  such that

$$0 = \bigcap_{\ell < n} \left( \bigcup_{k < m} (x_{\beta(\alpha_{2\ell}, k)} \Delta x_{\beta(\alpha_{2\ell+1}, k)}) \right),$$

$\otimes_2$  if (\*) $_{\mathbf{B}, \bar{x}}$  holds then the Boolean algebra  $\mathbf{B}' = \langle x_\alpha : \alpha < \lambda \rangle_{\mathbf{B}}$  has no independent subset of cardinality  $\lambda$ . Moreover, if  $x'_\alpha \in \mathbf{B}'$  for  $\alpha < \lambda$  are distinct, then

- (\*) $_{\mathbf{B}', \langle x'_\alpha : \alpha < \lambda \rangle}$ .

## 7. A nice subfamily of functions exists

We expand and continue on [26, 6.6D], [20, 6.1].

**Claim 7.0.** Assume

- (A)  $\lambda = \text{cf}(\lambda) \geq \mu > 2^\kappa$ ,  
 (B)  $\mathcal{D}$  is a  $\mu$ -complete<sup>3</sup> filter on  $\lambda$  containing the co-bounded subsets of  $\lambda$ ,

<sup>3</sup>In parts (0), (1),  $\mu = (2^\kappa)^+$  is OK.

- (C)  $f_\alpha : \kappa \rightarrow \text{Ord}$  for  $\alpha < \lambda$ ,  
 (D)  $\alpha < \mu \Rightarrow \mathcal{D}$  is  $|\alpha|^{\leq \kappa}$ -complete.

Then

- (0) We can find  $w \subseteq \kappa$  and  $\bar{\beta}^* = \langle \beta_i^* : i < \kappa \rangle$  such that:  $i \in \kappa \setminus w \Rightarrow \text{cf}(\beta_i^*) > 2^\kappa$  and for every  $\bar{\beta} \in \prod_{i \in \kappa \setminus w} \beta_i^*$  for  $\lambda$  ordinals  $\alpha < \lambda$  (even a set in  $\mathcal{D}^+$ ) we have  $\bar{\beta} < f_\alpha \upharpoonright (\kappa \setminus w) < \bar{\beta}^* \upharpoonright (\kappa \setminus w)$ ,  $f_\alpha \upharpoonright w = \bar{\beta}^* \upharpoonright w$ , and  $\sup\{\beta_j^* : \beta_j^* < \beta_i^*\} < f_\alpha(i) < \beta_i^*$ .
- (1) We can find a partition  $\langle w_\ell^* : \ell < 2 \rangle$  of  $\kappa$ ,  $X \in \mathcal{D}^+$  and  $\langle A_i : i < \kappa \rangle$ ,  $\langle \bar{\lambda}_i : i < \kappa \rangle$ ,  $\langle h_i : i < \kappa \rangle$ ,  $\langle n_i : i < \kappa \rangle$  such that:
- $A_i \subseteq \text{Ord}$ ,
  - $\bar{\lambda}_i = \langle \lambda_{i,\ell} : \ell < n_i \rangle$  and  $2^\kappa < \lambda_{i,\ell} \leq \lambda_{i,\ell+1} \leq \lambda$ ,
  - $h_i$  is an order preserving function from  $\prod_{\ell < n_i} \lambda_{i,\ell}$  onto  $A_i$  so  $n_i = 0 \Leftrightarrow |A_i| = 1$  (the order on  $\prod_{\ell < n_i} \lambda_{i,\ell}$  being lexicographic  $<_{\ell x}$ ),
  - $i < \kappa$  &  $\alpha \in X \Rightarrow f_\alpha(i) \in A_i$ , and we let  $f_\alpha^*(i, \ell) = [h_i^{-1}(f_\alpha(i))](\ell)$ , so  $f_\alpha^* \in \prod_{i < \kappa, \ell < n_i} \lambda_{i,\ell}$ ,
  - $i \in w_0^* \Rightarrow n_i = 0$  (so  $|A_i| = 1$ ),
  - if  $i \in w_1^*$  then  $|A_i| \leq \lambda$ , hence  $|\bigcup_{i \in w_1^*} A_i| \leq \lambda$ ,
  - if  $g \in \prod_{\ell < n_i} \lambda_{i,\ell}$  then  $\{\alpha \in X : g < f_\alpha^*\} \in \mathcal{D}^+$ ,
  - $\mu \leq \max \text{pcf}\{\lambda_{i,\ell} : i \in w_1^* \text{ and } \ell < n_i\} \leq \lambda$  when  $w_1^* \neq \emptyset$  (so, e.g., under GCH  $\max \text{pcf}\{\text{cf}(\lambda_{i,\ell}) : i \in w_1^* \text{ and } \ell < n_i\} = \lambda$ ).
- (2) In part (1) we can add  $(*)_1$  to the conclusion if (E) below holds,  
 $(*)_1$  if  $\lambda_{i,\ell} \in [\mu, \lambda)$  then  $\lambda_{i,\ell}$  is regular.  
 (E) For any set  $\mathfrak{a}$  of  $\leq \kappa$  singular cardinals from the interval  $(\mu, \lambda)$ , we have  $\max \text{pcf}\{\text{cf}(\chi) : \chi \in \mathfrak{a}\} < \lambda$ .
- (3) Assume in part (1) that (F) below holds. Then we can demand  $(*)_2$ .  
 $(*)_2$   $\lambda_\ell^i \geq \mu$  for  $i \in w_2$ ,  $\ell < n_i$ .  
 (F)  $\text{cf}(\mu) > \kappa$  and  $\alpha < \mu \Rightarrow \mathcal{D}$  is  $[|\alpha|^{\leq \kappa}]^+$ -complete.
- (4) If in part (1) in addition (G) below holds, then we can add  
 $(*)_3$   $\lambda \in \text{pcf}_{\sigma\text{-complete}}\{\lambda_\ell^i : i \in w_1^* \text{ and } \ell < n_i\}$  if  $w_1^* \neq \emptyset$ ,  
 moreover  
 $(*)_4$  if  $\ell_i < n_i$  for  $i \in w_1^*$  then  $\lambda \in \text{pcf}_{\sigma\text{-complete}}\{\text{cf}(\lambda_{\ell_i}^i) : i \in w_1^*\}$ .  
 (G) (i)  $(\forall \alpha < \lambda)(|\alpha|^{< \sigma} < \lambda)$  and  $\sigma = \text{cf}(\sigma) > \aleph_0$ ,  
 (ii)  $\mathcal{D}$  is  $\lambda$ -complete.
- (5) If in part (1) in addition (H) below holds then we can add  
 $(*)_5$  if  $m < m^*$ ,  $A \in J_m$  and  $\ell_i < n_i$  for  $i \in \kappa \setminus A$  (so  $w_0^* \subseteq A$ ) then  $\lambda \in \text{pcf}\{\lambda_{\ell_i}^i : i \in \kappa \setminus A\}$ .  
 (H) (i)  $m^* < \omega$  and  $J_m$  an  $\aleph_1$ -complete ideal on  $\kappa$  for  $m < m^*$ ,  
 (ii)  $\mathcal{D}$  is  $\lambda$ -complete.

### Remark.

- (1) If  $\lambda_{i,\ell}$  is singular we can replace it with a sequence  $\langle \gamma_{i,\ell_1} : \zeta < \text{cf}(\lambda_{i,\ell}) \rangle$ , and the index set  $\langle \langle \alpha : \alpha < \lambda_{i,\ell} \rangle \rangle$  by  $\langle \langle \zeta, \gamma \rangle : \zeta < \text{cf}(\lambda_{i,\ell}) \text{ and } \gamma < \gamma_{i,\ell_2} \rangle \rangle$ , and  $\gamma_{i,\ell_1}$ , are replaced by sequences of regular cardinals. Not clear if all this helps.
- (2) The reader may concentrate on the case (F) + (G)(ii) holds.

**Proof.** (0) By part (1).

(1) Let  $\chi$  be regular large enough. Choose  $N$  such that

- (i)  $N < (\mathcal{H}(\chi), \in)$ ,
- (ii)  $2^\kappa + 1 \subseteq N$  and  $\|N\| = 2^\kappa$ ,
- (iii)  $\kappa, \mu, \lambda, \mathcal{D}$  and  $\langle f_\alpha: \alpha < \lambda \rangle$  belong to  $N$ ,
- (iv)  $N^\kappa \subseteq N$ .

Next choose  $\delta(*) < \lambda$  which belongs to  $B^* = \bigcap \{B \in \mathcal{D}: B \in N\}$ , which is the intersection of  $\leq 2^\kappa < \mu$  members of  $\mathcal{D}$ . Necessarily  $B^* \in \mathcal{D}$  so  $\delta(*)$  exists. For each  $i < \kappa$  let

$$Y_i =: \{A \in N: A \text{ a set of ordinals and } f_{\delta(*)}(i) \in A\},$$

clearly  $Y_i \neq \emptyset$  as  $\bigcup_{\gamma < \lambda} (f_\gamma(i) + 1) \in N$ , hence there is a set  $A_i \in Y_i$  of minimal order type. As  $N^\kappa \subseteq N$  clearly  $\bar{A} =: \langle A_i: i \in \kappa \rangle$  belongs to  $N$ .

Let us define:

$$w_0^* =: \{i < \kappa: |A_i| = 1\},$$

$$w_1^* =: \{i < \kappa: 2^\kappa < |A_i| \leq \lambda\}.$$

Now note

$$(*)_1 \quad A_i \neq \emptyset.$$

[Why? As  $A_i \in Y_i$  hence  $f_{\delta(*)}(i) \in A_i$ .]

$$(*)_2 \quad |A_i| = 1 \text{ iff } A_i = \{f_{\delta(*)}(i)\} \text{ iff } f_{\delta(*)}(i) \in N \text{ (iff } i \in w_0^*).$$

[Why? Think.]

$$(*)_3 \quad \text{Without loss of generality } A_i \subseteq \{f_\alpha(i): \alpha < \lambda\}.$$

[Why? As  $\{f_\alpha(i): \alpha < \lambda\} \in Y_i$  and  $A_i \cap \{f_\alpha(i): \alpha < \lambda\} \in Y_i$ .]

Hence

$$(*)_4 \quad \text{If } i \in \kappa \setminus w_0^* \text{ then } |A_i| \leq \lambda.$$

Let

$$K_i = \{(\bar{\lambda}, \bar{\beta}) \in N: \text{for some } n, \bar{\lambda} = \langle \lambda_\ell: \ell < n \rangle \in N, \text{ and}$$

$$\bar{\beta} = \langle \beta_\eta: \eta \in \prod_{\ell < n} \lambda_\ell \rangle \in N \text{ and } \beta_\eta \in \{f_\alpha(i): \alpha < \lambda\} \text{ and}$$

$$f_{\delta(*)}(i) \in \{\beta_\eta: \eta \in \prod_{\ell < n} \lambda_\ell\} \text{ and: if } \eta <_{\ell x} \nu \text{ are from } \prod_{\ell < n} \lambda_\ell$$

$$\text{then } \beta_\eta \leq \beta_\nu\}.$$

We define a partial order  $<^*$  on  $K_i$

$$(\bar{\lambda}^1, \bar{\beta}^1) <^* (\bar{\lambda}^2, \bar{\beta}^2) \quad \text{iff} \quad \left\{ \beta_\eta^1: \eta \in \prod_{\ell} \lambda_\ell^1 \right\} \subseteq \left\{ \beta_\eta^2: \eta \in \prod_{\ell} \lambda_\ell^2 \right\}$$

and:

$$\text{otp}\left(\prod_{\ell} \lambda_\ell^1, \leq_{\ell x}\right) < \text{otp}\left(\prod_{\ell} \lambda_\ell^2, \leq_{\ell x}\right) \quad \text{or}$$

$$\text{otp}\left(\prod_{\ell} \lambda_\ell^1, \leq_{\ell x}\right) = \text{otp}\left(\prod_{\ell} \lambda_\ell^2, \leq_{\ell x}\right) \quad \text{and} \quad \text{lg}(\bar{\lambda}^1) < \text{lg}(\bar{\lambda}^2) \quad \text{or}$$

$$\text{otp}\left(\prod_{\ell} \lambda_\ell^1, \leq_{\ell x}\right) = \text{otp}\left(\prod_{\ell} \lambda_\ell^2, \leq_{\ell x}\right), \quad \text{lg}(\bar{\lambda}^1) = \text{lg}(\bar{\lambda}^2) \quad \text{and}$$

$$\bigvee_{k < \text{lg}(\bar{\lambda}^1)} \left[ \lambda_{\text{lg}(\bar{\lambda}^1)-1-k}^1 < \lambda_{\text{lg}(\bar{\lambda}^2)-1-k}^2 \text{ and } \bigwedge_{\ell < k} \lambda_{\text{lg}(\bar{\lambda}^1)-1-\ell}^1 = \lambda_{\text{lg}(\bar{\lambda}^2)-1-\ell}^2 \right].$$

(\*)<sub>5</sub>  $(K_i, \leq^*) \subseteq N$  is a partial order which is a well quasi order (i.e., no strictly decreasing  $\omega$ -chains).

[Why? Reflect.]

(\*)<sub>6</sub> There is  $(\bar{\lambda}, \bar{\beta}) \in K_i$  such that  $\bigwedge_{\ell < \text{lg}(\bar{\lambda})} \lambda_\ell \leq |A_i|$ .

[Why? By (\*)<sub>7</sub> below.]

(\*)<sub>7</sub>  $\text{otp}(A_i) \leq |A_i|^n$  for some  $n < \omega$ .

[Why? By Dushnik and Milner [1], we can find  $A_{i,n} \subseteq A_i$  for  $n < \omega$  such that  $A_i = \bigcup_{n < \omega} A_{i,n}$  and  $\text{otp}(A_{i,n}) \leq |A_i|^n$ . So as  $A_i \in N$  there is such sequence  $\langle A_{i,n} : n < \omega \rangle$  in  $N$  so  $A_{i,n} \in N$  hence for some  $n$  we have  $f_{\delta(*)}(i) \in A_{i,n} \in N$ , so by the choice of  $A_i$  clearly  $\text{otp}(A_i) \leq |A_i|^n$ .]

So we can find a  $<^*$ -minimal  $(\bar{\lambda}^i, \bar{\beta}^i) \in K_i$  and let  $n_i = \text{lg}(\bar{\lambda}^i)$ . Note:

(\*)<sub>8</sub> we can above in the choice of  $A_i$  demand  $A_i = \{\beta_\eta^i : \eta \in \prod_{\ell < n_i} \lambda_\ell^i\}$ , where  $(\bar{\lambda}^i, \bar{\beta}^i)$  is a  $<^*$ -minimal in  $K_i$ ,

(\*)<sub>9</sub>  $\lambda_\ell^i \leq \lambda_{\ell+1}^i \leq \lambda$  for  $\ell < n_i$ .

[Why? The second inequality by (\*)<sub>4</sub> and (\*)<sub>6</sub>, the first inequality as otherwise by renaming we can omit  $\lambda_{\ell+1}^i$  and contradict the  $<^*$ -minimality of  $(\bar{\lambda}^i, \bar{\beta}^i)$ .]

Let  $\langle \eta_i^* : i < \kappa \rangle$  be such that  $\beta_{\eta_i^*}^i = f_{\delta(*)}(i)$  and  $\eta_i^* \in \prod_{\ell < n_i} \lambda_\ell^i$ .

(\*)<sub>10</sub>  $\lambda_{i,\ell} > 2^\kappa$  [trivial].

Let  $Y = \{\alpha < \lambda : \text{for every } i < \kappa \text{ we have } f_\alpha(i) \in A_i\}$ , as  $\bar{f} \in N$  and  $\langle A_i : i < \kappa \rangle \in N$  necessarily  $Y \in N$ . Also  $Y \in \mathcal{D}^+$  because  $\delta(*) \in Y$  and the choice of  $\delta(*)$ . So for  $\alpha \in Y$  we can choose  $\langle \eta_i^\alpha : i < \kappa \rangle$  such that  $\eta_i^\alpha \in \prod_{\ell < n_i} \lambda_\ell^i$  and  $f_\alpha(i) = \beta_{\eta_i^\alpha}^i$ . We now define  $f_\alpha^* \in \prod_{\ell < n_i} \lambda_\ell^i$  for  $\alpha < \lambda$ :  $f_\alpha^*(i, \ell) = \eta_i^\alpha(\ell)$ .

Note:

(\*)<sub>11</sub>  $\langle \bar{\lambda}^i : i < \kappa \rangle, \langle \bar{\beta}^i : i < \kappa \rangle$  and  $\bar{f}$ , hence  $\bar{f}^* = \langle f_\alpha^* : \alpha \in Y \rangle$  belong to  $N$ .

(\*)<sub>12</sub>  $\eta_i^\alpha(\ell) = f_\alpha^*(i, \ell) \in [\sup(N \cap \lambda_\ell^i), \lambda_\ell^i]$  and  $\alpha \in Y \Rightarrow f_\alpha^*(i, \ell) < \lambda_\ell^i$ .

[Why? Reflect.]

(\*)<sub>13</sub> for every  $g \in \prod_{\ell < n_i} \lambda_\ell^i$  and  $X \in [Y]^\lambda \cap N$  such that  $\delta(*) \in X$  there is  $\alpha \in X$  such that

$$g < f_\alpha^* \quad \text{i.e.,} \quad i < \kappa \text{ and } \ell < n_i \Rightarrow g(i, \ell) < f_\alpha^*(i, \ell).$$

[Why? If not, there is such  $g$ , so as  $\langle (\bar{\lambda}^i, \bar{\beta}^i) : i < \kappa \rangle, \bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  and  $X, Y$  belong to  $N$  also  $\bar{f}^* = \langle f_\alpha^* : \alpha \in X \rangle$  belongs to  $N$ , so all the requirements on  $g$  are first order with parameters from  $N$ , so without loss of generality  $g \in N$ . Now  $\delta(*) \in X$  cannot satisfy the requirement hence there are  $i < \kappa, \ell < n_i$  such that  $g(i, \ell) > f_{\delta(*)}^*(i, \ell)$  contradicting (\*)<sub>12</sub>.]

Let

$$Z_i = \left\{ \eta \in \prod_{i < n_i} \lambda_\ell^i : \text{if } v \in \prod_{\ell < n_i} \lambda_\ell^i \text{ and } v <_{\ell x} \eta \text{ then } \beta_v^i < \beta_\eta^i \right\},$$

$$Z_i^+ = \{\eta \upharpoonright k : \eta \in Z_i \text{ and } k \leq n_i\}.$$

As  $(\bar{\lambda}^i, \bar{\beta}^i) \in N$  clearly also  $Z_i, Z_i^+ \in N$ .

(\*)<sub>14</sub> If  $i < \kappa, k < n_i$  then  $\lambda_k^i = \text{otp}\{\eta(k): (\eta_i^* \upharpoonright k) \triangleleft \eta \in Z_i\}$ .

[Why? Let  $Z'_i = \{\eta \in Z_i: \lambda_k^i > \text{otp}\{v \in Z_i: \eta \upharpoonright k \triangleleft v \in Z_i\}\}$ . So  $\eta_i^* \in Z'_i \in N$ , by renaming

$$\eta \in Z'_i \Rightarrow \lambda_k^i > \sup\{v \in Z_i: \eta \upharpoonright k \triangleleft v \in Z_i\},$$

and if  $\lambda_k^i$  is regular we get a contradiction to (\*)<sub>8</sub> as in the proof of (\*)<sub>9</sub>. If  $\lambda_k^i$  is singular, we by renaming get the desired equality.]

Hence

(\*)<sub>15</sub> without loss of generality  $\langle \beta_\eta^i: \eta \in \prod_{\ell < n_i} \lambda_\ell^i \rangle$  is increasing (with  $<_{\ell x}$ ),

(\*)<sub>16</sub>  $\mu \leq \sup\{\lambda_\ell^i: i \in w_1^* \text{ and } \ell < n_i\}$ .

[Why? Otherwise let  $\mu > \mu_0 = \sup\{\lambda_\ell^i: i \in w_1^* \text{ and } \ell < n_i\}$ , and so  $B^* \stackrel{\text{def}}{=} \{\beta_\eta^i: i < \kappa, \eta \in \prod_{\ell < n_i} \lambda_\ell^i\}$  has cardinality  $\mu_0$  so there is  $\mathcal{P} \in N, |\mathcal{P}| < \lambda, \mathcal{P} \subseteq [\mu_0]^{\leq \kappa}$  and  $\mathcal{P}$  is cofinal in  $([\mu_0]^{\leq \kappa}, \subseteq)$ . (Why? By assumption (D).) Note that if for some  $X \in (\mathcal{D} + Y)^+, \bar{f} \upharpoonright X$  is constant we are done. Otherwise

$$a \in \mathcal{P} \Rightarrow \{\alpha < \lambda: \text{Rang}(f_\alpha) \subseteq a\} = \emptyset \text{ mod } \mathcal{D}$$

but  $\mathcal{D}$  is  $\mu$ -complete hence

$$X^* =: \{\alpha \in Y: (\exists a \in \mathcal{P})[\text{Rang}(f_\alpha) \subseteq a]\} = \emptyset \text{ mod } \mathcal{D}$$

and  $X^* \in N$  and  $\delta(*) \in X^*$ , contradicting the choice of  $X^*$ .]

(\*)<sub>17</sub>  $\max \text{pcf}\{\lambda_\ell^i: i \in w_1^* \text{ and } \ell < n_i\} \leq \lambda$ .

[Why? By (\*)<sub>13</sub>.]

(\*)<sub>18</sub>  $\lambda_\ell^i$  has cofinality  $> 2^\kappa$ .

[Why? Otherwise we can decrease it, getting a contradiction to the  $<^*$ -minimality of  $(\bar{\lambda}^i, \bar{\beta}^i)$ .]

The conclusion can be checked easily

(2) Let  $\mathfrak{a} = \{\text{cf}(\lambda_\ell^i): \lambda_\ell^i \text{ is singular and } \mu \leq \lambda_\ell^i < \lambda\}$  and use (E).

(3) Easy.

(4) Assume that the desired conclusion fails. For this we choose not just one model  $N$  but an  $(\omega + 1)$ -tree of models. More precisely, we choose by induction on  $i \leq \omega$  a sequence  $\langle N_\eta: \eta \in T_i \rangle$  such that

(a)  $T_i \subseteq {}^i \lambda$ ,

(b)  $j < i \text{ \& } \eta \in T_i \Rightarrow \eta \upharpoonright j \in T_j$ ,

(c)  $|T_i| < \lambda$ ,

(d)  $N_\eta \prec (\mathcal{H}(\chi), \in)$  satisfies (i)–(iv) from the proof of part (1),

(e) for  $\eta \in T_i, \eta \in N_\eta$  and  $\langle N_\nu: \nu \in \bigcup_{j < i} T_j \rangle \in N_\eta$  and

$$\nu \triangleleft \eta \Rightarrow N_\nu \prec N_\eta \text{ \& } N_\nu \in N_\eta,$$

(f) if  $i = 0$ , then  $T_i = \{\langle \rangle\}$ ,

(g) if  $i$  is  $\omega$ , then  $T_i = \{\eta \in {}^i \lambda: (\forall j < i)(\eta \upharpoonright j \in T_j)\}$ ,

(h) if  $i = j + 1, \eta \in T_j$  and  $\langle a_{\eta, \varepsilon}: \varepsilon < \varepsilon_\eta < \lambda \rangle$  list  $[\sup(N_\eta \cap \lambda)]^{< \sigma}$ , then

$$\{v \in T_i: \eta \triangleleft v\} = \{\eta \frown \langle \alpha \rangle: \alpha < \varepsilon_\eta\},$$

and  $a_{\eta, \varepsilon} \in N_{\eta \frown \langle \varepsilon \rangle}$ ,



$$(i) T = \bigcup_{i \leq \omega} T_i.$$

There is no problem to carry out the definition (note that  $\varepsilon_\eta < \lambda$  by assumption (G)(i) and  $|T_{m+1}| < \lambda$  as in addition  $\lambda$  is regular, and  $|T_\omega| < \lambda$  by assumption (G)(i) as  $\sigma > \aleph_0$ ). Now

$$B^* = \bigcap \{B \in \mathcal{D}: \text{for some } \eta \in T \text{ we have } B \in N_\eta\}$$

being the intersection of  $\leq |T| + 2^\kappa < \lambda$  sets in  $\mathcal{D}$ , belongs to  $\mathcal{D}$  (using assumption (G)(ii)), so choose  $\delta(*) \in B^*$ . Now we choose by induction on  $k < \omega$ ,  $\eta_k \in T_k$  and  $w_0^k, w_1^k, \langle \bar{\lambda}^{i,k}, \bar{\beta}^{i,k} \rangle: i < \kappa \in N_{\eta_k}$  as in the proof of (1) for  $N_{\eta_k}$ , such that  $w_0^k \subseteq w_0^{k+1}$ ,  $\eta_k \triangleleft \eta_{k+1}$  and  $(\forall i \in w_1^k)[\langle \bar{\lambda}^{i,k+1}, \bar{\beta}^{i,k+1} \rangle <^* \langle \bar{\lambda}^{i,k}, \bar{\beta}^{i,k} \rangle]$ . The last assertion holds by the assumption toward contradiction and basic pcf.

If  $\bigcup_{k < \omega} w_0^k = \kappa$ , then  $f_{\delta(*)} \in N_{\bigcup_k \eta_k}$ , hence  $\delta(*) \in N_{\bigcup_k \eta_k}$ , contradiction. If  $i \in \kappa \setminus \bigcup_{k < \omega} w_0^k$ , then  $\langle \bar{\lambda}^{i,k}, \bar{\beta}^{i,k} \rangle: k < \omega$  is strictly decreasing in  $K_i$  (more exactly in  $\bigcup_{k < \omega} K_i[N_{\eta_k}]$ ), contradicting a parallel of  $(*)_{11}$ .

(5) We choose by induction on  $t \in \omega$  the objects  $N_t, \delta_t, \bar{A}^t = \langle A_i^t: i < \kappa \rangle, \langle \bar{\lambda}_i^t, \bar{\beta}_i^t \rangle: i < \kappa \rangle, \langle h_i^t: i < \kappa \rangle, K_i^t$  such that

- (a) for each  $t$ , they are as required in the proof of part (1),
- (b)  $N_t \in N_{t+1}$ ,  $K_i^t \subseteq K_i^{t+1}$  and  $\langle \bar{\lambda}_i^{t+1}, \bar{\beta}_i^{t+1} \rangle \leq^* \langle \bar{\lambda}_i^t, \bar{\beta}_i^t \rangle$  in  $K_i^{t+1}$ ,
- (c) for each  $t$  for some  $m_t < m^*$  we have

$$\{i < \kappa: \langle \bar{\lambda}_i^{t+1}, \bar{\beta}_i^{t+1} \rangle <^* \langle \bar{\lambda}_i^t, \bar{\beta}_i^t \rangle\} = \kappa \text{ mod } J_{m_t}.$$

No problem to carry it out by assumption toward contradiction. So for some  $m$ ,  $\{t: m_t = m\}$  is infinite, contradicting “ $J_m$  is  $\aleph_1$ -complete, and for each  $i < \kappa$ ,  $\bigcup_t K_i^t$  well ordered by  $<^{**}$ ”.  $\square$

See Section 9 for actually some consequences.

**Notation.** If  $f$  is a function from, say,  $\theta$  to the ordinals, and  $\bar{g}$  is a sequence of length  $\theta$  of functions from the ordinals to the ordinals, then  $f^* = f^{\bar{g}}$  is a function from the ordinals to the ordinals defined by  $f^*(i) = g_i(f(i))$ .

We spell out a special case.

**Fact 7.1.** Assume

$$(*) \quad 2^\theta < \mu, \text{cf}(\mu) = \theta \quad \text{and} \quad (\forall \alpha < \mu)(|\alpha|^\theta < \mu),$$

and  $\lambda = \mu^+$ .

Then:

(1) For every sequence  $\bar{f} = \langle f_\alpha: \alpha < \lambda \rangle$  of functions from  $\theta$  to the ordinals, we can find  $u^* \in [\theta]^\theta$  and  $\bar{\beta}^* = \langle \beta_i^*: i \in u^* \rangle$  such that one of the following cases occurs:

$(*)_1$  for some  $X \in [\lambda]^\lambda$ ,  $f_\alpha \upharpoonright u^* = \bar{\beta}^*$  for  $\alpha \in X$ ,

$(*)_2$  if  $\theta > \aleph_0$  then  $\beta_i^*$  is a limit ordinal (for every  $i \in u^*$ ), and  $\langle \text{cf}(\beta_i^*): i \in u^* \rangle$  is strictly increasing with limit  $\mu$  and  $\lambda = \text{tcf}(\prod_{i \in u^*} \text{cf}(\beta_i^*) / J_{u^*}^{\text{bd}})$  and for every  $\bar{\gamma} \in \prod_{i \in u^*} \beta_i^*$  for  $\lambda$  ordinals  $\alpha < \lambda$  we have

$$(\forall i \in u^*)(\gamma_i < f_\alpha(i) < \beta_i^*),$$

if  $\theta = \aleph_0$  then for some strictly increasing sequence  $\bar{\lambda} = \langle \lambda_i : i \in u^* \rangle$  of regular cardinals with limit  $\mu$ ,  $\lambda = \text{tcf}(\prod_{i \in u^*} \lambda_i / J_{u^*}^{\text{bd}})$  and for some  $\bar{g} = \langle g_i : i < \theta \rangle$ ,  $g_i : \text{Ord} \rightarrow \lambda_i$ , we have: for every  $\bar{\gamma} \in \prod_{i \in u^*} \lambda_i$  for  $\lambda$  ordinals  $\alpha < \lambda$  we have

$$i \in u^* \Rightarrow \gamma_i < f_\alpha^{\bar{g}}(i) < \lambda_i,$$

(\*)<sub>3</sub>  $\beta_i^*$  is a limit ordinal of cofinality  $\lambda$  for  $i \in u^*$  and for some  $X \in [\lambda]^\lambda$  we have:  $i \in u^* \Rightarrow \langle f_\alpha(i) : \alpha \in X \rangle$  is strictly increasing with limit  $\beta_i^*$  and for  $\alpha \in X$ , the interval  $[f_\alpha(i), \beta_i^*)$  is disjoint to

$$\{f_\beta(j) : \beta \in X \text{ and } j \in u^* \setminus \{i\} \text{ \& } \beta_j \neq \beta_i \text{ or } \beta < \alpha \text{ and } j \in u^*\}.$$

(2) Assume  $\theta > \aleph_0$ . For every sequence  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  of pairwise distinct functions from  $\theta$  to  ${}^{\omega} \text{Ord}$  such that  $|\{f_\alpha(i) : \alpha < \lambda\}| < \lambda$  for  $i < \theta$ , we can find  $u^* \in [\theta]^\theta$  and  $n^* \in [1, \omega)$  and  $v \subseteq n^*$  nonempty and  $\bar{\beta}^* = \langle \beta_{\ell,i}^* : \ell < n^*, i \in u^* \rangle$  such that for each  $i$

(a) for  $\ell \in v$  we have that  $\beta_{\ell,i}^*$  is a limit ordinal,  $\langle \text{cf}(\beta_{\ell,i}^*) : i \in u^* \rangle$  is strictly increasing with limit  $\mu$  and  $\lambda = \text{tcf}(\prod_{i \in u^*} \text{cf}(\beta_{\ell,i}^*) / J_{u^*}^{\text{bd}})$ , and also for  $i < j$  in  $u^*$ , and  $\ell, k \in v$  we have  $\text{cf}(\beta_{\ell,i}^*) < \text{cf}(\beta_{k,j}^*)$ ,

(b) for every  $\bar{\gamma} \in \prod_{\ell,i} \beta_{\ell,i}^*$  for  $\lambda$  ordinals  $\alpha < \lambda$  we have

$$(\forall i < u^*)(\forall \ell \in v)[\gamma_{\ell,i} < (f_\alpha(i))(\ell) < \beta_{\ell,i}^*] \quad \text{and}$$

$$(\forall i \in u^*)(\forall \ell \in n^* \setminus v)[f_\alpha(i)(\ell) = \beta_{\ell,i}^*].$$

(3) In part (2), we can replace  $u^* \in [\theta]^\theta$  by  $u \in J^+$  for any normal ideal  $J$  on  $\theta$ . Moreover if  $\{\delta < \theta : (\forall \alpha < \text{cf}(\delta))(|\alpha|^{<\sigma} < \text{cf}(|\delta|))\}$  is stationary then  $\text{Rang}(f_\alpha) \subseteq {}^{\sigma} \text{Ord}$  is fine. If we omit the assumption  $|\{f_\alpha(i) : \alpha < \lambda\}| < \lambda$ , instead of  $v$  we have a partition  $(v_1, v_2, v_3)$  of  $\{\ell : \ell < n^*\}$  such that clause (a) holds for  $\ell \in v_2$ , clause (b) holds with  $\ell \in v_2 \cup v_3$ ,  $\ell \in v_1$  instead of  $\ell \in v$ ,  $\ell \in n^* \setminus v$ , and the parallel of (\*)<sub>3</sub> holds for  $\ell \in v_3$ .

**Proof.** (1) By 7.0(0)–(2) we know that

⊗ there is  $\langle \beta_i^* : i < \theta \rangle$  and  $w^* \subseteq \theta$  such that letting  $u^* = \theta \setminus w^*$  we have:

(a) for every  $\bar{\gamma} \in \prod_{i \in u^*} \beta_i^*$  for  $\lambda$  ordinals  $\alpha < \lambda$  we have

$$i \in w^* \Rightarrow f_\alpha(i) = \beta_i^*,$$

$$i \in u^* \Rightarrow \gamma_i < f_\alpha(i) < \beta_i^*,$$

and moreover  $(w_0^*, w_1^*)$ ,  $\langle \lambda_{i,\ell} : i \in w_1^*, \ell < n_i \rangle$ ,  $X$ ,  $\bar{g} = \langle g_i : i < \theta \rangle$  as there (so  $w_1^* = u^*$ ); clearly  $\lambda_{i,\ell} \leq \lambda$  and without loss of generality  $\lambda_{i,m_i} = \lambda$ .

Case 1.  $|u^*| < \theta$ . So for some  $X \in [\lambda]^\lambda$  we have  $\langle f_\alpha \upharpoonright w^* : \alpha \in X \rangle$  is constant. Easily (\*)<sub>1</sub> holds.

Case 2. For some unbounded subset  $u'$  of  $\theta$  and  $\langle m_i : i \in u' \rangle$ ,  $m_i < n_i$  and  $\sup\{\lambda_{i,\ell} : i \in u', \ell < m_i\} \leq \mu^* < \mu$  and  $\lambda_{i,m_i} = \lambda$ .

Clearly (\*)<sub>3</sub> holds and we get  $X$  by “thinning”: choose by induction on  $\gamma < \lambda$  the  $\gamma$ th member  $\alpha_\gamma < \lambda$  of  $X$ , fixing  $\langle g_i^{-1}(f_\alpha(i)) \upharpoonright m_i : i \in u' \rangle$ .

Case 3. Neither Case 1 nor Case 2.

Let  $\mu = \sum_{i < \theta} \mu_i$ ,  $\mu_i < \mu$  increasing with  $i$ . Choose  $j_i \in u^*$  such that  $j_i$  is the minimal  $j > \bigcup_{\zeta < i} j_\zeta$ ,  $\lambda_{j_i, n_i-1} > \mu_i + \sum_{\zeta < i} \lambda_{j_\zeta}$ , and let  $m_{j_i} < n_{j_i}$  be the minimal  $m$  such that  $\lambda_{j_i, m} > \mu_i + \sum_{\zeta < i} \lambda_{j_\zeta}$  and the  $j_i$  is such that  $\lambda_{j_i, m_i} < \mu$ .

If  $\theta > \aleph_0$ , by Fodor's lemma, replacing  $\langle j_i: i < \theta \rangle$  by a subsequence, without loss of generality  $\mu^* =: \sup\{\lambda_{j_i, m}: i < \theta, m < m_{j_i}\} < \mu$ , and without loss of generality  $\langle h_{j_i}^{-1}(f_\alpha(j_i)) \upharpoonright m_{j_i}: i < \theta \rangle = x$  is the same for all  $\alpha \in X$ .

Choose  $u^* = \{j_i: i < \theta\}$ ,  $\lambda_i = \lambda_{j_i, m_i}$ , which is regular by 7.0(2). Now,  $\langle \lambda_j: j \in u^* \rangle$  is a strictly increasing sequence of regular cardinals with limit  $\mu$ , and hence  $\prod_{j \in u^*} \lambda_j / J_{u^*}^{\text{bd}}$  is  $\mu$ -directed and hence  $\lambda$ -directed. But, by 7.0,  $\{\langle h_{j_i}^{-1}(f_\alpha(i))(m_i): i \in u^* \rangle: \alpha \in X\}$  is unbounded in it (or use "max pcf $\{\lambda_{i, \ell}: i < \theta, \ell < n_i\} \leq \lambda$ "). So  $\lambda = \text{tcf}(\prod_{j \in u^*} \lambda_j / J_{u^*}^{\text{bd}})$ . Let  $g_j$  be defined by  $g_j(\gamma) = (h_j^{-1}(\gamma))(m_j)$ , and we are done.

(2) First without loss of generality  $\text{lg}(f_\alpha(i)) = n^*$ , i.e., does not depend on  $\alpha$ , secondly, e.g., by successive applications of part (1).

(3) Similar.  $\square$

### Conclusion 7.2. For

(1) In 7.1(1),  $(*)_2$  and  $(*)_3$  implies

$(*)'_2$  there are  $u^*$ ,  $\beta^* = \langle \beta_i^*: i \in u^* \rangle$  and  $X$  such that

(a)  $u^* \in [\theta]^\theta$ ,

(b)  $X \in [\lambda]^\lambda$ ,

(c)  $\langle f_\alpha \upharpoonright u^*: \alpha \in X \rangle$  is  $<_{J_{u^*}^{\text{bd}}}$ -increasing if  $\theta > \aleph_0$ , and  $\langle f_\alpha^{\bar{g}} \upharpoonright u^*: \alpha \in X \rangle$  is  $<_{J_{u^*}^{\text{bd}}}$ -increasing if  $\theta = \aleph_0$  (for appropriate  $\bar{g}$ ),

(d) if  $\theta > \aleph_0$  then for every  $\bar{\gamma} \in \prod_{i \in u^*} \beta_i^*$  there are  $\lambda$  ordinals  $\alpha \in X$  such that

$$i \in u^* \Rightarrow \gamma_i < f_\alpha(i) < \beta_i^*,$$

if  $\theta = \aleph_0$ ,  $\lambda_i = \text{Rang}(g_i)$  then for every  $\bar{\gamma} \in \prod_{i \in u^*} \lambda_i$  there are  $\lambda$  ordinals  $\alpha \in X$  such that

$$i \in u^* \Rightarrow \gamma_i < f_\alpha^{\bar{g}}(i) < \lambda_i,$$

(e) if  $(*)_3$  then:

(i)  $\alpha < \beta$  from  $X \Rightarrow f_\alpha \upharpoonright u^* < f_\beta \upharpoonright u^*$ ,

(ii) if  $i \neq j$  are in  $u^*$  and  $\beta_i^* < \beta_j^*$  then  $\alpha \in X \Rightarrow f_\alpha(j) > \beta_i^*$ ,

(iii) if  $i, j \in u^*$ ,  $\beta_i^* = \beta_j^*$  and  $\alpha < \beta$  are from  $X$  then  $f_\alpha(i) < f_\beta(j)$ .

(2) Similarly for 7.1(2), getting  $\odot$  from the proof of 6.5.

**Proof.** Straight. Choose the  $\gamma$ th member of  $X$  for  $\gamma < \lambda$ , by induction on  $\gamma$ .  $\square$

Similarly we can prove

### Claim 7.3. Assume

(A)  $\lambda = \text{cf}(\lambda) > 2^\theta$ ,

(B)  $\mu = \min\{\mu: \mu^\theta \geq \lambda\}$ ,  $\text{cf}(\mu) = \theta > \aleph_0$ ,

(C) if  $\mathfrak{a} \subseteq \text{Reg} \cap \mu \setminus 2^\theta$ ,  $|\mathfrak{a}| \leq \theta$ ,  $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ , then for some  $\mathfrak{b} \subseteq \mathfrak{a}$ ,  $\lambda = \text{tcf}(\prod \mathfrak{b}/[\mathfrak{b}]^{<\theta})$ . (Note: this holds if

$$\mathfrak{d} \subseteq \text{Reg} \setminus 2^\theta \text{ \& } |\mathfrak{d}| \leq \theta \Rightarrow |\text{pcf}(\mathfrak{d})| \leq \theta.$$

Why? Now  $\langle \mathfrak{b}_\theta[\mathfrak{a}]: \theta \in \text{pcf}(\mathfrak{a}) \rangle$  is well defined and  $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$  so letting  $\text{pcf}(\mathfrak{a}) \cap \lambda$  be  $\langle \theta_\zeta: \zeta < \theta \rangle$ , choose  $\mu_\zeta \in \mathfrak{b}_\lambda[\mathfrak{a}] \setminus \bigcup_{\xi < \zeta} \mathfrak{b}_{\theta_\xi}[\mathfrak{a}]$ , and let  $\mathfrak{b} = \{\mu_\zeta: \zeta < \theta\}$ .)

Then the conclusions of 7.1, 7.2 hold.

**Proof.** Similar.  $\square$

#### Fact 7.4.

(1) Assume

- (A)  $\lambda = \text{cf}(\lambda) > 2^\theta$  and  $n < \omega$ ,
- (B)  $f_\alpha^\ell \in {}^\theta \text{Ord}$  for  $\ell < n$ ,  $\alpha < \lambda$ ,
- (C)  $\alpha \neq \beta \Rightarrow \langle f_\alpha^\ell: \ell < n \rangle \neq \langle f_\beta^\ell: \ell < n \rangle$ ,
- (D)  $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$ .

Then we can find an ultrafilter  $\mathcal{D}$  on  $\theta$  (possibly a principal one) and  $X \in [\lambda]^\lambda$ ,  $v \subseteq n$  and  $f_\ell \in {}^\theta \text{Ord}$  for  $\ell < n$  such that  $v = \emptyset$ ,

- (a) for  $\ell \in n \setminus v$  and  $\alpha \in X$  we have  $f_\alpha^\ell/\mathcal{D} = f_\ell/\mathcal{D}$ ,
- (b) for  $\alpha < \beta$  from  $X$  and  $\ell, m \in v$  such that  $f_\ell/\mathcal{D} = f_m/\mathcal{D}$  (e.g.,  $\ell = m$ ) we have  $f_\alpha^\ell/\mathcal{D} < f_\beta^m/\mathcal{D}$ ,
- (c) if  $\ell, m < n$  and  $f_\ell/\mathcal{D} < f_m/\mathcal{D}$  and  $\alpha, \beta$  are from  $X$  then  $f_\alpha^\ell/\mathcal{D} < f_\beta^m/\mathcal{D}$ .

(2) Assume

- (a)  $\lambda = \text{cf}(\lambda) > 2^\theta$  and  $(\forall \alpha < \lambda)(|\alpha|^{<\sigma} < \lambda)$  and  $\aleph_1 + |\varepsilon(*)|^+ \leq \sigma = \text{cf}(\sigma)$  and
- (b)  $f_\alpha^\varepsilon \in {}^\theta \text{Ord}$  for  $\varepsilon < \varepsilon(*)$  and  $\alpha < \lambda$ ,
- (c)  $I$  is a  $\sigma$ -complete filter on  $\theta$ ,
- (d)  $\mathcal{D}$  is a  $\lambda$ -complete filter on  $\lambda$  to which all co-bounded subsets of  $\lambda$  belong.

Then we can find  $X, v, f_\varepsilon$  (for  $\varepsilon < \varepsilon(*)$ ) and  $\bar{w}, J$  such that

- ( $\alpha$ )  $X \in [\lambda]^\lambda$ ,
- ( $\beta$ )  $f_\varepsilon \in {}^\theta \text{Ord}$  for  $\varepsilon < \varepsilon(*)$ ,
- ( $\gamma$ )  $J$  is a  $\sigma$ -complete ideal on  $\theta$  extending  $I$ ,
- ( $\delta$ )  $\bar{w} = \langle w_\varepsilon: \varepsilon < \varepsilon(*) \rangle$ ,  $w_\varepsilon \subseteq \theta$ ,
- ( $\varepsilon$ ) if  $\alpha \in X$  and  $\varepsilon < \varepsilon(*)$  then  $f_\alpha^\varepsilon \upharpoonright w_\varepsilon = f_\varepsilon \upharpoonright w_\varepsilon$ ,
- ( $\zeta$ ) if  $\alpha < \beta$  are from  $X$  then  $\varepsilon < \varepsilon(*) \Rightarrow f_\alpha^\varepsilon < f_\beta^\varepsilon \pmod{(J + w_\varepsilon)}$ , moreover

$\{i < \theta: \text{for some } \zeta, \xi < \varepsilon(*) \text{ we have } i \notin w_\zeta, i \notin w_\xi \text{ and}$

$$f_\zeta(i) \leq f_\xi(i) \text{ but } f_\alpha^\zeta(i) \geq f_\beta^\xi(i)\} \in J,$$

( $\eta$ ) if  $\alpha \in X$  and  $i < \theta$ ,  $\zeta, \xi < \varepsilon(*)$ ,  $f_\zeta(i) < f_\xi(i)$  then  $f_\zeta(i) < f_\alpha^\xi(i)$ ,

( $\theta$ ) if  $2^{|\varepsilon(*)|} < \sigma$  then  $\varepsilon < \varepsilon(*) \Rightarrow w_\varepsilon \in J \vee \theta \setminus w_\varepsilon \in J$ .

(3) We can combine 7.0(1) with part (2) (having  $\langle \lambda_{i,\ell}^\varepsilon: \ell < \ell_i^\varepsilon \rangle$ ).

**Remark.** We can prove also the parallel of 7.0(5).

**Proof.** (1) Like the proof of 7.0(4) or by part (2) for  $\sigma = \aleph_1$ .

(2) We repeat the proof of 7.0(4) except that  $T \subseteq \bigcup_{i < \sigma} i\lambda$ . After defining  $B^* \in \mathcal{D}$  and choosing  $\delta(*)$ , for  $\eta \in T$ ,  $\varepsilon < \varepsilon(*)$  and  $i < \theta$  we let  $\beta_{\varepsilon,i,\eta} = \min[N_\eta \cap \text{Ord} \setminus f_{\delta(*)}^\varepsilon(i)]$  and  $w_{\varepsilon,\eta} = \{i < \theta: f_{\delta(*)}^\varepsilon(i) \in N_\eta\}$ .

So clearly

$$(*)_1 \quad \eta \triangleleft \nu \in T \Rightarrow (\forall \varepsilon < \varepsilon(*))(\forall i < \theta)(\beta_{\varepsilon,i,\eta} \geq \beta_{\varepsilon,i,\nu} \ \& \ w_{\varepsilon,\eta} \subseteq w_{\varepsilon,\nu})$$

and

$$(*)_2 \quad i \notin w_{\varepsilon,\eta} \Rightarrow \text{cf}(\beta_{\varepsilon,i,\eta}) > 2^\theta.$$

Let  $J_\eta$  is the  $\sigma$ -ideal on  $\theta$  generated by

$$I \cup \{w \subseteq \theta: \text{for some } \varepsilon < \varepsilon(*) \text{ we have } w \subseteq \theta \setminus w_{\varepsilon,\eta} \text{ and} \\ \lambda > \max \text{pcf}\{\text{cf}(\beta_{\varepsilon,i,\eta}): i < w\}\}.$$

If for some  $\eta$ ,  $\theta \notin J_\eta$  then we are done (choosing the  $\alpha$ th member of  $X$  by induction on  $\alpha$ ). So assume that  $\eta \in T \Rightarrow \theta \in J_\eta$ . We now choose by induction on  $\zeta < \sigma$ , a sequence  $\eta_\zeta \in T_i$  such that  $\xi < \zeta \Rightarrow \eta_\xi = \eta_\zeta \upharpoonright j$  and

$$\zeta = \xi + 1 \Rightarrow \{i < \theta: (\exists \varepsilon < \varepsilon(*))(\beta_{\varepsilon,i,\eta_\zeta} > \beta_{\varepsilon,i,\eta_\xi})\} = \theta \text{ mod } I.$$

For some  $\varepsilon < \varepsilon(*)$  and infinite  $Y \subseteq \theta$  we have:

$$\xi \in Y \Rightarrow Z_\xi = \{i < \theta: \beta_{\varepsilon,i,\xi} > \beta_{\varepsilon,i,\xi+1}\} = \theta \text{ mod } I.$$

But for  $\xi < \zeta$  we have  $\beta_{\varepsilon,i,\xi+1} \geq \beta_{\varepsilon,i,\zeta}$  by  $(*)_1$ . Without loss of generality  $\text{otp}(Y) = \omega$ . As  $I$  is  $\sigma$ -complete and  $\sigma \geq \aleph_0$ , there is an  $i \in \bigcap \{Z_\xi: \xi \in Y\}$ , and  $\langle \beta_{\varepsilon,i,\xi}: \xi \in Z \rangle$  is strictly decreasing, a contradiction.

Now for  $\zeta = 0$ ,  $\zeta$  limit there are no “serious” demands and for  $\zeta$  successor ordinal we use  $\theta \in J_\eta$ .

(3) Left to the reader (and not used).  $\square$

**Fact 7.4A.** Assume

- (A)  $\lambda = \mu^+$ ,  $\mu > 2^\theta$ ,  $\theta = \text{cf}(\mu) > \aleph_0$ ,
- (B)  $|\varepsilon(*)|^+ + \aleph_0 < \theta$ ,
- (C)  $f_\alpha^\varepsilon \in {}^\theta \text{Ord}$  for  $\varepsilon < \varepsilon(*)$ ,  $\alpha < \lambda$ ,
- (D)  $(\forall \alpha < \mu)(|\alpha|^\theta < \mu)$ .

Then we can find a stationary  $S \subseteq \{\delta < \theta: \text{cf}(\delta) \geq |\varepsilon(*)|^+ + \aleph_0\}$  and unbounded subset  $X'$  of  $\lambda$  and  $S_\varepsilon \subseteq S$  and  $f_\varepsilon \in {}^S \text{Ord}$  for  $\varepsilon < \varepsilon(*)$

- (a) for  $\varepsilon < \varepsilon(*)$  we have  $\alpha \in X \Rightarrow f_\alpha^\varepsilon \upharpoonright S_\varepsilon = f_\varepsilon \upharpoonright S_\varepsilon$ ,
- (b) for  $\varepsilon_1 < \varepsilon(*)$  and  $\alpha < \beta$  from  $X$  if  $S_{\varepsilon_1,\zeta} = \{i \in S: f_{\varepsilon_1}(i) \leq f_\zeta(i)\} \setminus S_{\varepsilon_1} \setminus S_\zeta$  is unbounded in  $\theta$  then  $f_\alpha^{\varepsilon_1} \upharpoonright S_{\varepsilon_1,\zeta} < f_\beta^\zeta \text{ mod } J_{S_{\varepsilon_1,\zeta}}^{\text{bd}}$ ,
- (c) if  $\zeta$ ,  $\varepsilon < \varepsilon(*)$ ,  $f_\zeta(i) < f_\varepsilon(i)$ , and  $\alpha \in X$  then  $f_\zeta(i) < f_\alpha^\varepsilon(i)$ ,
- (d) if  $2^{|\varepsilon(*)|} < \theta$  then  $\varepsilon < \varepsilon(*) \Rightarrow S_\varepsilon \in \{\emptyset, S\}$ .

**Proof.** Let  $\bar{f}^\varepsilon = \langle f_\alpha^\varepsilon: \alpha < \lambda \rangle$ , let  $\chi$  be large enough and  $\langle \lambda_\varepsilon: \varepsilon < \theta \rangle$  be increasing continuous with limit  $\mu$ , and choose by induction on  $\zeta < \theta$ , an elementary submodel  $N_\zeta$  of  $(\mathcal{H}(\chi), \in, <_\zeta^*)$  of cardinality  $(\lambda_\zeta)^\theta$  such that  $(\lambda_\zeta)^\theta \subseteq N_\zeta$ ,  ${}^\theta(N_\zeta) \subseteq N_\zeta$ ,  $\{\bar{f}^\varepsilon: \varepsilon < \varepsilon(*)\} \in N_\zeta$ , and  $\langle N_\xi: \xi < \zeta \rangle \in N_\zeta$ .

Choose  $\delta(*) \in \lambda \setminus \bigcup_{\zeta < \theta} N_\zeta$ , possible as  $|\bigcup_{\zeta < \theta} N_\zeta| = |\sum_{\varepsilon < \theta} (\lambda_\varepsilon)^\theta| = \mu < \lambda$ . For each  $\zeta < \theta$ ,  $\varepsilon < \varepsilon(*)$  and  $i < \theta$  let  $\beta_{\varepsilon,i,\zeta}^* = \min(N_\zeta \cap \text{Ord} \setminus f_{\delta(*)}^\varepsilon(i))$ .

For each limit  $i < \theta$  of cofinality  $> |\varepsilon(*)|$  look at  $\langle \beta_{\varepsilon,i,\zeta}^* : \zeta < i \rangle$ , it is a non-increasing sequence of ordinals, hence it is constant on some end segment, i.e., for some  $j_{\varepsilon,i} < i$  we have

$$j_{\varepsilon,i} \leq \zeta < i \Rightarrow \beta_{\varepsilon,i,\zeta}^* = \beta_{\varepsilon,i,j_{\varepsilon,i}}^*.$$

As  $\text{cf}(i) > |\varepsilon(*)|$ , necessarily  $j_i = \sup\{j_{\varepsilon,i} : \varepsilon < \varepsilon(*)\}$  is  $< i$ , hence for some  $j(*) < \theta$  the set

$$S = \{i < \theta : \text{cf}(i) > |\varepsilon(*)|, i \text{ a limit ordinal}\}$$

is stationary. The rest should be clear.  $\square$

**Remark.** We can demand  $S \subseteq S^*$  in 7.4 if  $S^* \subseteq \{\delta < \theta : \text{cf}(\delta) \geq |\varepsilon(*)|^+ + \aleph_0\}$  is stationary.

**Discussion 7.5.** We may wonder what occurs for ultraproducts of free Boolean algebras  $\prod_{i < \theta} \text{FBA}(\chi_i)/\mathcal{D}$  (or even reduced products, recall  $\text{FBA}(\chi_i)$  is the free Boolean algebra generated, say, by  $\{x_\alpha : \alpha < \chi_i\}$  freely). Now

- (\*)<sub>1</sub> if  $\mathcal{D}$  is  $\aleph_1$ -complete, the situation is as in the  $\theta > \aleph_0$  case for products;
- (\*)<sub>2</sub> if

$$(\exists A_0, A_1, \dots) \left( \bigwedge_{n < \omega} A_n \in \mathcal{D} \ \& \ \bigcap_{n < \omega} A_n = \emptyset \right),$$

the situation is as in the  $\theta = \aleph_0$  case.

**Claim 7.6.** Assume

- (A)  $\lambda = \mu^{++}$ ,  $\mu > 2^\theta$ ,
- (B)  $f_\alpha : \theta \rightarrow \text{Ord}$  for  $\alpha < \lambda$ .

Then we can find  $\bar{u}^* = \langle u_0^*, u_1^*, u_2^* \rangle$ ,  $\bar{\beta}^*$ ,  $X$  such that

- (a)  $\langle u_0^*, u_1^*, u_2^* \rangle$  is a partition of  $\theta$ ,
- (b)  $\bar{\beta} = \langle \beta_i^* : i < \theta \rangle$ ,
- (c)  $X \in [\lambda]^\lambda$  (we can use an appropriate ideal  $J$  on  $\lambda$  and demand  $X \in J^+$ ),
- (d)  $\alpha \in X \Rightarrow f_\alpha \upharpoonright u_0^* = \langle \beta_i^* : i \in u_0^* \rangle$ ,
- (e) if  $i \in u_1^*$  then  $\langle f_\alpha(i) : \alpha \in X \rangle$  is strictly increasing with limit  $\beta_i^*$  (so  $\text{cf}(\beta_i^*) = \lambda$ ),
- (f)  $i \in u_2^* \Rightarrow 2^\theta < \text{cf}(\beta_i^*) < \mu$ ,
- (g) for every  $\bar{\gamma} \in \prod_{i \in u_2^*} \beta_i^*$  for  $\lambda$  ordinals  $\alpha \in X$  we have

$$i \in u_2^* \Rightarrow \gamma_i < f_\alpha(i) < \beta_i^*,$$

- (h) if there are  $\langle \lambda_{i,\ell} : i < \theta, \ell < n_i \rangle$ ,  $\langle A_i : i < \theta \rangle$ ,  $\langle h_i : i < \theta \rangle$  as in 7.0,  $i \in u_0^* \Leftrightarrow n_i = 0$ ,

$$i \in u_1^* \Leftrightarrow n_i > 0 \ \& \ \lambda_{i,0} = \lambda \Leftrightarrow n_i > 0 \ \& \ (\forall \ell)(\lambda_{i,\ell} = \lambda),$$

and if  $u_0^* \neq \theta$ , then  $\lambda = \max \text{pcf}\{\lambda_{i,\ell} : i < \theta, \ell < n_i\}$  and  $\mu^+ \notin \max \text{pcf}\{\lambda_{i,\ell} : i < \theta, \ell < n_i\}$ .

**Proof.** Let  $\bar{C} = \langle C_\alpha: \alpha < \mu^+ \rangle$  be such that  $\text{otp}(C_\alpha) \leq \theta^+$ ,  $[\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$ ,  $C_\alpha$  a set of successor ordinals and the set

$$S^* = \{\delta < \lambda^+: \text{cf}(\delta) = \theta^+ \text{ and } \alpha = \text{sup}(C_\alpha)\}$$

is stationary (exists by [24, §1].)

Let  $\bar{f} = \langle f_\alpha: \alpha < \lambda \rangle$  be given. Let  $\chi$  be strong limit such that  $\bar{f} \in \mathcal{H}(\chi)$ . We choose  $M_\alpha$  by induction on  $\alpha < \mu^+$  such that

- ( $\alpha$ )  $M_\alpha < (\mathcal{H}(\chi), \in, <_\chi^*)$ ,
- ( $\beta$ )  $\|M_\alpha\| = 2^\theta$  and  $2^\theta + 1 \subseteq M_\alpha$  and  ${}^{\theta \geq} (M_\alpha) \subseteq M_\alpha$ ,
- ( $\gamma$ )  $\lambda, \bar{f}, \bar{C}$  and  $\alpha$  belong to  $M_\alpha$ ,
- ( $\delta$ )  $\langle M_\beta: \beta < \alpha \rangle$  belongs to  $M_\alpha$  and  $\beta \in C_\alpha \Rightarrow M_\beta < M_\alpha$ .

Now for every  $\beta \in \lambda \setminus \bigcup_{\alpha < \mu^+} M_\alpha$  we define a function  $g_\alpha^\beta \in {}^\theta(M_\alpha \cap \text{Ord})$  and a function  $F_\beta$  from  $\mu^+$  to  $\mu^+$ , as follows

$$(*)_2 \quad g_\alpha^\beta(i) = \min(M_\alpha \cap \chi \setminus f_\beta(i)).$$

[Why is it well defined? As  $\bar{f} \in M_\alpha$  also  $\bigcup\{f_\gamma(i) + 1: \gamma < \lambda\} \in M_\alpha$  and  $f_\beta(i)$  is smaller than that ordinal.]

We let

$$\begin{aligned} u_{\alpha,0}^\beta &= \{i < \theta: f_\beta(i) \in M_\alpha\}, \\ u_{\alpha,1}^\beta &= \{i < \theta: f_\beta(i) \notin M_\alpha \text{ and } \text{cf}(g_\alpha^\beta(i)) = \lambda\}, \\ u_{\alpha,2}^\beta &= \{i < \theta: \text{cf}(g_\alpha^\beta(i)) \leq \mu^+ \text{ and } f_\beta(i) \notin M_\alpha\}. \end{aligned}$$

Note.  $f_\beta(i) \notin M_\alpha \Rightarrow \lambda \geq \text{cf}(g_\alpha^\beta(i)) > 2^\theta$ .

[Why? If  $i \in \theta \setminus u_{\alpha,0}^\beta$  and  $\lambda < \text{cf}(g_\alpha^\beta(i))$ , then

$$\bigcup\{f_\gamma(i): \gamma < \lambda \text{ and } f_\gamma(i) < g_\alpha^\beta(i)\}$$

belongs to  $M_\alpha$  and contradicts the choice of  $g_\alpha^\beta(i)$ . If  $i \in \theta \setminus u_{\alpha,0}^\beta$  and  $\text{cf}(g_\alpha^\beta(i)) \leq 2^\theta$  then  $g_\alpha^\beta(i) = \text{sup}(M_\alpha \cap g_\alpha^\beta(i))$ .]

Similarly choose  $\langle A_{\alpha,i}^\beta: i < \theta \rangle$ ,  $\langle h_{\alpha,i}^\beta: i < \theta \rangle$ ,  $\langle \lambda_{\alpha,i,\ell}^\beta: i < \theta, \ell < n_{\alpha,i}^\beta \rangle$ ,  $\langle f_{\alpha,\gamma}^{*,\beta}: \gamma \in X_\alpha^\beta \rangle$  as in 7.0(1). Let  $U_\alpha^\beta = \{(i, \ell): i < \theta, \ell < n_i\}$ ; this is  $\text{Dom}(f_{\alpha,\gamma}^{*,\beta})$  for  $\gamma \in X_\alpha^\beta$ . Let

$$J = J_{\beta,\alpha} = \{u \subseteq U_\alpha^\beta: \mu^+ > \max \text{pcf}\{\text{cf}(\lambda_{\alpha,i,\ell}^\beta): (i, \ell) \in u\}\}.$$

By the pcf theorem [25, VIII, 2.6] there is  $W_\alpha^\beta \subseteq U_\alpha^\beta$  such that:

$$\begin{aligned} \mu^+ &\notin \text{pcf}\{\text{cf}(\lambda_{\alpha,i,\ell}^\beta): (i, \ell) \in U_\alpha^\beta \setminus W_\alpha^\beta\}, \\ \mu^+ &\geq \max \text{pcf}\{\text{cf}(\lambda_{\alpha,i,\ell}^\beta): (i, \ell) \in W_\alpha^\beta\}. \end{aligned}$$

If  $W_\alpha^\beta \notin J$  let  $\bar{h}_{\beta,\alpha} = \langle h_{\alpha,\beta,\gamma}: \gamma < \mu^+ \rangle \in M_\alpha$  be  $<_{J \upharpoonright W_\alpha^\beta}$ -increasing and cofinal in  $\prod_{(i,\ell) \in W_\alpha^\beta} \lambda_{\alpha,i,\ell}^\beta$ . Then for some  $\gamma = \gamma(\alpha, \beta) < \mu^+$ ,

$$f_\beta^* < h_{\alpha,\beta,\gamma(\alpha,\beta)} \pmod J.$$

In fact any  $\gamma' \in [\gamma(\alpha, \beta), \mu^+)$  will do, and now we let  $F_\beta(\alpha) = \gamma(\alpha, \beta)$ . If  $W_\alpha^\beta \in J_{\beta, \alpha}$  we let  $F_\beta(\alpha) = \alpha + 1$ .

So the set  $E_\beta = \{\delta < \mu^+ : \delta \text{ a limit ordinal such that } (\forall \alpha < \delta) F_\beta(\alpha) < \delta\}$  is a club of  $\mu^+$ . Hence there is  $\delta = \delta_\beta \in S^* \cap \text{acc}(E_\beta)$  (i.e.,  $\delta = \sup(E_\beta \cap \delta)$  and  $\delta \in S^*$ ). Now for each  $(i, \ell) \in U_\alpha^\beta$  the sequence  $\langle g_\alpha^\beta(i, \ell) : \alpha \in C_{\delta_\beta} \rangle$  is non-increasing as  $\langle M_\alpha : \alpha \in C_{\delta_\beta} \rangle$  is increasing. Hence it is eventually constant, and similarly  $(\bar{\lambda}_{\alpha, i}^\beta, h_{\alpha, i}^\beta), A_{\alpha, i}^\beta$  as in 7.0(2) (any freedom left—choose the  $<^*_\chi$ -first), so easily  $(\bar{\lambda}_{\alpha, i}^\beta, h_{\alpha, i}^\beta) : \alpha \in C_{\delta_\beta}$  is eventually constant; say for  $\alpha \in C_{\delta_\beta} \setminus \alpha^*(\beta, i)$ . But  $\text{otp}(C_{\delta_\beta}) = \theta^+$  so  $\alpha^*(\beta) = \sup\{\alpha^*(\beta, i) : i < \theta\}$  is  $< \delta_\beta$ , and reflection shows that

$$\alpha \in C_{\delta_\beta} \setminus (\alpha^*(\beta) + 1) \Rightarrow W_\alpha^\beta \in J_{\beta, \alpha}.$$

Choose such  $\alpha_\beta^\otimes$ . So for some  $\alpha^\otimes, \delta^\otimes, \langle (\bar{\lambda}_i, h_i) : i < \theta \rangle$  we have

$$X = \left\{ \beta < \lambda : \beta \notin \bigcup_{\alpha < \mu^+} M_\alpha \text{ and } \alpha_\beta^\otimes = \alpha^\otimes, \delta_\beta = \delta^\otimes, \right. \\ \left. \bar{\lambda}_{\alpha_\beta^\otimes, i}^\beta = \bar{\lambda}_i, h_{\alpha_\beta^\otimes, i}^\beta = h_i \text{ for } i < \theta \right\}$$

belongs to  $[\lambda]^\lambda$ . Now we continue as in 7.0.  $\square$

### Claim 7.7.

- (1) In 7.6 we can replace  $\lambda = \mu^{++}$ , by  $\lambda = \tau^+$ ,  $\tau = \text{cf}([\tau]^{\leq \mu}, \subseteq)$  using [24, §2].
- (2) Also if  $\lambda$  is weakly inaccessible  $> \beth_\omega$ ,  $(\forall \alpha < \lambda) [\lambda > \text{cf}([\alpha]^{\leq \mu}, \subseteq)]$  we can get 7.6.

## 8. Consistency of “ $\mathcal{P}(\omega_1)$ has a free caliber” and discussion of pcf

This solves [12, Problem 37].

**Claim 8.1.** Assume for simplicity GCH and  $P$  is adding  $\aleph_{\omega_1}$  Cohen reals. In  $V^P$  we have  $2^{\aleph_0} = \aleph_{\omega_1}$ ,  $2^{\aleph_1} = \aleph_{\omega_1+1}$  and

- (\*) there is no complete Boolean algebra  $\mathbf{B}$  of cardinality  $2^{\aleph_1}$  such that  $\text{FreeCal}(\mathbf{B}) = \emptyset$ . In fact for any complete Boolean algebra  $\mathbf{B}$  of cardinality  $2^{\aleph_1}$  we have  $\aleph_{\omega_1+1} \in \text{FreeCal}(\mathbf{B})$ .

**Proof.** Clearly (as if the Boolean algebra  $\mathbf{B}$  has cardinality  $2^{\aleph_1} = \aleph_{\omega_1+1}$  and satisfies the ccc then (\*) holds, i.e.,  $\aleph_{\omega_1+1} \in \text{FreeCal}(\mathbf{B})$ , because  $V^P \models “(\aleph_{\omega_1})^{\aleph_0} = \aleph_{\omega_1}”$ , [22], otherwise we can reduce to the case  $\mathbf{B} = \mathcal{P}(\omega_1)$ ) it is enough to show

- (\*)<sub>1</sub>  $V^P \models \aleph_{\omega_1+1} \in \text{FreeCal}(\mathcal{P}(\omega_1))$ .

So let  $p^* \in P$

$$p^* \Vdash_P “\langle q_\alpha : \alpha < \aleph_{\omega_1+1} \rangle \text{ is a sequence of distinct elements of } \mathcal{P}(\omega_1)”.$$

Note.  $P = \{f : f \text{ is finite function from } \aleph_{\omega_1} \text{ to } \{0, 1\}\}$  so without loss of generality  $p^* = \emptyset$ . So  $P_A = \{f \in P : \text{Dom}(f) \subseteq A\} \triangleleft P$  for any  $A \subseteq \aleph_{\omega_1}$ .



For each  $\alpha < \aleph_{\omega_1+1}$  and  $i < \omega_1$  there is a maximal antichain  $\langle f_{\alpha,i,n} : n < \omega \rangle$  of  $P$  and sequence of truth values  $\langle t_{\alpha,i,n} : n < \omega \rangle$  such that

$$f_{\alpha,i,n} \Vdash_P "i \in a_\alpha \text{ iff } t_{\alpha,i,n}."$$

Let  $A_\alpha = \bigcup_{i < \omega_1, n < \omega} \text{Dom}(f_{\alpha,i,n}) \cup \text{Dom}(p^*)$ , so  $A_\alpha \in [\aleph_{\omega_1}]^{\leq \aleph_1}$ . Let  $A_\alpha = \{\gamma_{\alpha,j} : j < j_\alpha\}$ ,  $\gamma_{\alpha,j}$  strictly increasing with  $j$ .

As  $V \models 2^{\aleph_1} < \aleph_{\omega_1+1}$ , without loss of generality

- (\*)<sub>2</sub> (a)  $j_\alpha = j^*$ ,  
 (b) the truth value of " $\gamma_{\alpha,j} \in \text{Dom}(f_{\alpha,i,n})$ " and the value of  $f_{\alpha,i,n}(\gamma_{\alpha,j})$  do not depend on  $\alpha$ .

Let  $\underline{a}$  be the Mostowski collapse of the name, i.e.,  $\underline{a} = \text{OP}_{j^*, A_\alpha}(\underline{a}_\alpha)$  for each  $\alpha$  (without loss of generality it does not depend on  $\alpha$ ). [Remember  $\text{OP}_{A,B}(\beta) = \alpha$  iff  $\alpha \in A, \beta \in B$ ,  $\text{opt}(\beta \cap B) = \text{otp}(\alpha \cap A)$ .] We apply 7.0(1) to  $f_\alpha : j^* \rightarrow \aleph_{\omega_1}$ ,  $f_\alpha(j) = \gamma_{\alpha,j}$  and get  $\langle w_\ell^* : \ell < 2 \rangle$ ,  $X \in [\aleph_{\omega_1+1}]^{\aleph_{\omega_1+1}}$  and  $\langle \lambda_{j,\ell} : j < j^*, \ell < n_j \rangle$ , and  $\bar{h} = \langle h_j : j < j^* \rangle$ . For  $i < \omega_1$  let  $w_i = \{j < j^* : j \in w_0^* \text{ or } \lambda_{j,n_j-1} \leq \aleph_i\}$ .

We call  $\langle (g_i^0, g_i^1, \xi_i) : i < \omega_1 \rangle$  a witness above  $f^*$  if:

- (i)  $f^*, g_i^0, g_i^1 \in P$  and  $p^* \leq f^*$ ,  
 (ii)  $f^* \leq g_i^0$ ,  
 (iii)  $f^* \leq g_i^1$ ,  
 (iv)  $\text{Dom}(g_i^\ell) \subseteq j^*$ ,  
 (v)  $\langle \text{Dom}(g_i^0) \cup \text{Dom}(g_i^1) \setminus \text{Dom}(f^*) : i < \omega_1 \rangle$  are pairwise disjoint,  
 (vi)  $g_i^0 \Vdash " \xi_i \in \underline{a} "$ ,  
 (vii)  $g_i^1 \Vdash " \xi_i \notin \underline{a} "$ ,  
 (viii)  $\xi_i < \omega_1$  and  $\xi_i \neq \xi_j$  for  $i \neq j$ .

Shrinking  $X$  (still unbounded in  $\aleph_{\omega_1+1}$ ) we get:

- ⊠ if  $\alpha < \beta$  are from  $X$  then there is  $i < \omega_1$  such that

$$j \in j^* \setminus w_i \wedge \lambda_{j,n_j-1} > \aleph_i \Rightarrow (h_j^{-1}(\gamma_{\alpha,j}))(m) < (h_j^{-1}(\gamma_{\beta,j}))(m), \quad \text{and} \\ j \in j^* \setminus w_i \wedge j_1 < j^* \Rightarrow \gamma_{\alpha,j_1} \neq \gamma_{\beta,j_1}.$$

**Fact.** There are  $f^*$  and a witness  $\langle (g_i^0, g_i^1, \xi_i) : i < \omega_1 \rangle$  above  $f^*$  and  $X \subseteq \aleph_{\omega_1+1}$  unbounded and an ideal  $J \supseteq J_{\omega_1}^{\text{bd}}$  on  $\omega_1$  such that: letting

$$u_{\alpha,i} = \text{OP}_{A_\alpha, j^*}(\text{Dom}(g_i^0) \cup \text{Dom}(g_i^1) \setminus \text{Dom}(f^*))$$

⊕ if  $\alpha < \beta$  are in  $X$  then

$$\{i : u_{\alpha,i} \cap u_{\beta,i} \neq \emptyset\} \in J.$$

We show how to finish the proof assuming the fact, and then prove the fact. For some unbounded  $X \subseteq \aleph_{\omega_1+1}$  we have  $\alpha \in X \Rightarrow f^{**} = \text{OP}_{A_\alpha, j^*}(f^*)$ , i.e., does not depend on  $\alpha \in X$ . (As there are  $\leq |P| = \aleph_{\omega_1} < \aleph_{\omega_1+1}$  possibilities.)

We shall prove

$$f^{**} \Vdash_P " \langle a_\alpha : \alpha \in X \rangle \text{ is independent (as a family of subsets of } \omega_1), \\ \text{even modulo } J_{\omega_1}^{\text{bd}}."$$

This is more than enough.

If not then for some  $n < \omega$  and pairwise distinct  $\alpha_1, \dots, \alpha_{2n} \in X$ , we have:

$$\neg \left( f^{**} \Vdash \left\langle \bigcap_{\ell=1}^n \underline{a}_{\alpha_\ell} \cap \bigcap_{\ell=n+1}^{2n} (\omega_1 \setminus \underline{a}_{\alpha_\ell}) \text{ is unbounded in } \omega_1 \right\rangle \right).$$

So for some  $f^1, f^{**} \leq f^1 \in P$ , and  $\zeta < \omega_1$  we have

$$\boxtimes \quad f^1 \Vdash_P \left\langle \bigcap_{\ell=1}^n \underline{a}_{\alpha_i} \cap \bigcap_{\ell=n+1}^n (\omega_1 \setminus \underline{a}_{\alpha_i}) \subseteq \zeta \right\rangle.$$

Now letting

$$\begin{aligned} g_{\alpha,i}^0 &= g_i^0 \circ \text{OP}_{j^*, A_\alpha} & \text{and} \\ g_{\alpha,i}^1 &= g_i^1 \circ \text{OP}_{j^*, A_\alpha} \end{aligned}$$

we have

$$\text{Dom}(g_{\alpha,i}^0) \cup \text{Dom}(g_{\alpha,i}^1) \subseteq \{\gamma_{\alpha,j} : j < j^*\}.$$

Let

$$\begin{aligned} B &= \{i < \omega_1 : \xi_i < \zeta\} \in J, \\ B_{\ell,m} &= \{i : u_{\alpha_\ell,i} \cap u_{\alpha_m,i} \neq \emptyset\} \in J \quad \text{for } \ell \neq m, \\ B_\ell &= \{i : \text{Dom}(f^1) \cap (\text{Dom}(g_{\alpha,i}^0) \cup \text{Dom}(g_{\alpha,i}^1)) \neq \text{Dom}(f^{**})\} \in J \\ &\quad (\text{in fact is finite}). \end{aligned}$$

So we can find  $i \in \omega_1 \setminus \bigcup_{\ell \neq m} B_{\ell,m} \setminus \bigcup_{\ell} B_\ell \setminus B$  (because the set of inappropriate  $i$ 's is in  $J$ ).

So  $f^2 = f^1 \cup \bigcup_{\ell=1}^n g_{\alpha_\ell,i}^0 \cup \bigcup_{\ell=n+1}^{2n} g_{\alpha_\ell,i}^1 \in P$  forces that the intersection from  $\boxtimes$  is not  $\subseteq \zeta$ , contradicting the choice of  $f^1$ .  $\square$

**Proof of the Fact.** We divide the proof into two cases, depending on the answer to:

**Question.** Is there  $\zeta < \omega_1$  such that: for no  $g^0, g^1 \in P_{j^*}$  above  $f^*$  and  $\xi \in [\zeta, \omega_1)$  do we have

$$g^0 \upharpoonright w_\zeta = g^1 \upharpoonright w_\zeta, \quad g^0 \Vdash \langle \xi \in \underline{a} \rangle, \quad g^1 \Vdash \langle \xi \notin \underline{a} \rangle?$$

*Case A:* The answer is YES. For some unbounded  $X \subseteq \aleph_{\omega_1+1}$  and  $\langle \gamma_j^{**} : j \in w_\zeta \rangle$  we have

$$j \in w_\zeta \ \& \ \alpha \in X \Rightarrow \gamma_{\alpha,j} = \gamma_j^{**}.$$

So  $\underline{a}$  is actually a  $P_{\{\gamma_j^{**} : j \in w_\zeta\}}$ -name. So for  $\alpha \in X$ ,  $\underline{a}_\alpha$  depends only on  $\{f \in \mathcal{G}_P : \text{Dom}(f) \subseteq \{\gamma_{\alpha,i} : i \in w_\zeta\}\}$ .

Hence there are  $\leq \aleph_1 < \aleph_{\omega_1}$  such names, a contradiction.

*Case B:* The answer is NO. So for every  $\zeta < \omega_1$ , we have  $\langle \xi_\zeta^*, g_\zeta^0, g_\zeta^1 \rangle$  giving the counterexample for  $\zeta$ , without loss of generality  $\text{Dom}(g_\zeta^0) = \text{Dom}(g_\zeta^1)$ . As  $\langle j(\zeta) : \zeta < \omega_1 \rangle$

is increasing continuous, by Fodor's lemma we can find  $S \subseteq \omega_1$  stationary and  $\zeta^* < \omega_1$  and  $n^*$  such that

$$\zeta \in S \Rightarrow (\text{Dom}(g_\zeta^0) \cap w_\zeta) \cup (\text{Dom}(g_\zeta^1) \cap w_\zeta) \subseteq w_{\zeta^*},$$

and  $\langle \text{Dom}(g_\zeta^0) \cup \text{Dom}(g_\zeta^1) : \zeta \in S \rangle$  forms a  $\Delta$ -system with heart  $v$ , and  $g_\zeta^0 \upharpoonright v = g_\zeta^1 \upharpoonright v$  does not depend on  $\zeta$ , and we call it  $f^*$ . Also  $\text{Dom}(g_\zeta^\ell \setminus w_{\zeta^*})$  has  $n^*$  elements and  $\zeta_1 < \zeta \in S \Rightarrow \xi_{\zeta_1}^* < \zeta$ .

Let  $\langle \varepsilon(i) : i < \omega_1 \rangle$  be a (strictly) increasing sequence listing  $S$ , and  $\xi_i = \xi_{\varepsilon(i)}^*$ . For  $\ell < n^*$ ,  $\alpha \in X'$  and  $i < \omega_1$  we let  $f_\alpha^\ell(i)$  be the  $\ell$ th member of  $\{\gamma_{\alpha,j} : j \in \text{Dom}(g_{\varepsilon(i)}^0) \setminus w_{\zeta^*}\}$ . Shrinking  $X$  without loss of generality  $\langle \gamma_{\alpha,j} : j \in w_{\zeta^*} \rangle$  does not depend on  $\alpha \in X$  (by  $\boxtimes$ );  $J = J_{\omega_1}^{\text{bd}}$  and  $X$  are as required.  $\square$

### Discussion 8.2.

- (1) Clearly we can replace  $\aleph_1, \aleph_{\omega_1+1}$  by any  $\theta, \lambda$  as in 7.4.
- (2) Normally if  $\mu$  is strong limit singular of cofinality  $\theta$ , (at least large enough), we can find long intervals  $\alpha_i$  of the  $\text{Reg} \cap \mu$  for  $i < \theta$ ,  $i < j \Rightarrow \sup(\alpha_i) < \min(\alpha_j)$  such that  $(\forall \bar{\lambda} \in \prod_i \alpha_i) [\max \text{pcf}(\text{Rang}(\bar{\lambda})) = \lambda^*]$  for some  $\lambda^* \in [\mu, 2^\mu]$ , usually  $\text{cf}(2^\mu)$ . This is a strong indication that  $\langle I_{\sup(\alpha_i), \min(\alpha_i)}^n : i < \theta \rangle$  will have a  $\lambda$ -sequence, so, for example, there is a  $(2^\theta)^+$ -c.c. Boolean algebra of cardinality  $\lambda$  having no independent subset of cardinality  $\lambda$ , for which even  $\lambda$ -Knaster property fails. To make this happen for no  $\mu$ , we need a very special pcf structure in the universe. But we do not know even if the following simple case is consistent.

**Question 8.3.** Is it consistent that

- (\*) for every set  $\mathfrak{a}$  of odd (or even) regular cardinals with  $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$  we have  $\max \text{pcf}(\mathfrak{a})$  is odd (or even respectively) (we may moreover ask  $(\forall \alpha) 2^{\aleph_\alpha} = \aleph_{\alpha+2}$ )?

Essentially by [26, §5]:

**Lemma 8.4.** Assume  $\mu > \theta = \text{cf}(\mu)$ ,  $\mu$  strong limit,  $\mu = \sum_{i < \theta} \mu_i$ ,  $\mu_i < \mu$  strong limit,  $\text{cf}(\mu_i) = \sigma_i$  and  $2^{\mu_i} = \mu_i^+$ ,  $\mu_i = \sum_{\zeta < \sigma_i} \mu_{i,\zeta}$ ,  $n_{i,\zeta} < \omega$ ,  $\lambda = \text{tcf}(\prod_{i < \theta} \mu_i^+ / J^*)$ ,  $J_\theta^{\text{bd}} \subseteq J^*$ .

Let  $I_{i,\zeta} = \text{ERI}_{\sup_{n_{i,\zeta}}(\mu_{i,\zeta})^+, \mu_{i,\zeta}^+}^{n_{i,\zeta}}$  and

$$J = \sum_{J^*} J_{\sigma_i}^{\text{bd}}.$$

Then

- (a) there is a  $(\lambda, J)$ -sequence  $\bar{\eta}$  for

$$\langle I_{i,\zeta} : i < \theta, \zeta < \sigma_i \rangle,$$

- (b) if  $i < \theta \Rightarrow \sigma_i = \theta$  then we can find  $\zeta(i) < \theta$  for  $i < \theta$  such that there is a  $\lambda$ -sequence  $\bar{\eta}$  for  $\langle I_{i,\zeta(i)} : i < \theta \rangle$ .

**Remark 8.5.** So if  $S = \{\mu: \mu \text{ strong limit, } \text{cf}(\mu) = \aleph_0, 2^\mu = \mu^+\}$  is unbounded, then for a class of cardinals  $\mu$  which is closed unbounded

- (\*) (a)  $\mu$  strong limit and  $\mu = \sup(S \cap \mu)$ ,  
 (b) if  $\text{cf}(\mu) = \aleph_0$  then we can find  $\lambda \in (\mu, 2^\mu] \cap \text{Reg}$  and  $\mu_n < \mu = \sum_n \mu_n$ ,  
 $\mu_n < \mu_{n+1}$  and there is a  $\lambda$ -sequence  $\bar{\eta}$  for  $\langle I_{\sum_n(\mu_n)^+, \mu_n^+}^n : n < \omega \rangle$ .

## 9. Having a $\lambda$ -sequence for a sequence of non-stationary ideals

**Lemma 9.1.** Assume

- (a)  $\mu$  is a strong limit singular of cofinality  $\theta$ ,  
 (b)  $\lambda = 2^\mu = \text{cf}(\lambda)$ ,  
 (c)  $\lambda_i$  regular increasing for  $i < \delta$  with limit  $\mu$ ,  $\delta < \mu$  (usually  $\delta = \theta$ ),  
 (d)  $J$  is an ideal on  $\delta$  extending  $J_\delta^{\text{bd}}$ ,  
 (e)  $\lambda = \text{tcf}(\prod_{i < \delta} \lambda_i / J)$ ,  
 (f)  $\langle A_\zeta : \zeta < \zeta(*) \rangle$  is a partition of  $\delta$  (so  $A_\delta$  pairwise disjoint) each  $A_\zeta$  in  $J^+$  (otherwise not interesting),  
 (g)  $|\delta| < \sigma = \text{cf}(\sigma) < \lambda_0$ .

Then there is a sequence  $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$ ,  $\eta_\alpha \in \prod_{i < \delta} \lambda_i$ ,  $\text{cf}(\eta_\alpha(i)) = \sigma$ , satisfying

- (\*) for any sequence  $\langle F_{\zeta, i} : \zeta < \zeta(*), i < \delta \rangle$  of functions, for every large enough  $\alpha < \lambda$  we have  
 (\*\*) if  $\zeta < \zeta(*)$ ,  $F_{\zeta, i}(\eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi)$  is a club of  $\lambda_i$  for  $i < \delta$  (really  $i \in A_\zeta$ ),  
 then

$$\left\{ i \in A_\zeta : \eta_\alpha(i) \notin F_{\zeta, i}(\eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi) \right\} \in J.$$

Moreover

- (\*\*)<sup>+</sup> if  $\zeta < \zeta(*)$ ,  $n < \omega$  and  $\beta_0, \dots, \beta_{n-1} < \alpha$ , and for each  $i \in A_\zeta$  we have:  
 $F_{\zeta, i}(\eta_{\beta_0}, \beta_0, \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi)$  is a club of  $\lambda_i$ , then

$$\left\{ i \in A_\zeta : \eta_\alpha(i) \notin F_{\zeta, i}(\eta_{\beta_0}, \beta_0, \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi) \right\} \in J.$$

**Discussion 9.2.** For a given  $\mu$  as in (a), clause (b) may fail, but then we will have another lemma. What about (e)?

If  $\theta > \aleph_0$  there are such  $\langle \lambda_i : i < \theta \rangle$  even for  $J = J_\theta^{\text{bd}}$  (see [25, VIII, §1]). If  $\theta = \aleph_0$  we do not know, but we know that the failures are “rare”. E.g.,

$$\{\delta < \omega_1 : \beth_\delta \text{ fails (e), i.e., } \neg[\beth_{\delta+1} =^+ \text{pp}(\beth_\delta)]\}$$

is not stationary. About  $\text{pp}_{J_\omega^{\text{bd}}}$ , e.g., if  $|\mathfrak{a}| \leq \aleph_0 \Rightarrow |\text{pcf}(\mathfrak{a})| \leq \aleph_0$  we then can get it, see [25, XI, §5].

**Remark 9.3.**

- (1) This can be rephrased as having a  $(\lambda, J)$ -sequence for  $\langle \prod J_{\lambda_{i,n}}^{\text{nst}, \sigma} : i < \delta \rangle$  with  $\lambda_{i,n}$  decreasing.

So compared to earlier theorems, the  $\lambda, \lambda_i$  for which the lemma applies are fewer, but the result is stronger: nonstationary ideal and we get also the “super” version see  $(**)^+$ .

- (2) Of course another variant is to start with  $I_i = J_{\lambda_i}^{\text{nst}, \sigma}$  and get  $J = J_{\lambda}^{\text{nst}, \sigma}$ .
- (3) Considering functions with finitely many  $\eta_{\beta}$ 's,  $\beta < \alpha$  as parameters (i.e.,  $(**)^+$ ); thinning  $\langle f_{\alpha}: \alpha < \lambda \rangle$  the conclusion follows.
- (4) In  $(**)^+$  instead of  $n < \omega$  we can ask  $n < \sigma$  if  $(\forall \alpha < \lambda)(|\alpha|^{<\sigma} < \lambda)$ .

**Proof of 9.1.** For simplicity we concentrate on  $(**)$  (in 10.1 we concentrate on the parallel of  $(**)^+$ ). List the possible  $\langle F_{\zeta, i}: i < \delta, \zeta < \zeta(*) \rangle$ , i.e., sequence with each  $F_{\zeta, i}$  having the “right” domain and range, which are clear from the statement, as  $\langle \langle F_{\zeta, i}^{\beta}: i < \delta, \zeta < \zeta(*) \rangle: \beta < \lambda \rangle$ . Let us define  $\eta_{\alpha} \in \prod_i \lambda_i$  by induction on  $\alpha$ .

For a given  $\alpha$  we choose  $\eta_{\alpha} \upharpoonright A_{\zeta}$  by induction on  $\zeta < \zeta(*)$ .

Define for  $i \in A_{\zeta}, \beta < \alpha$

$$C_i^{\beta} = \begin{cases} F_{\zeta, i}^{\beta}(\eta_{\alpha} \upharpoonright \bigcup_{\xi < \zeta} A_{\xi}) & \text{if this set is a club of } \lambda_i, \\ \lambda_i & \text{otherwise.} \end{cases}$$

So we need

**Fact.** *There is  $\eta \in \prod_{i \in A_{\zeta}} \lambda_i$  such that*

$$\bigwedge_{\beta < \alpha} \{i \in A_{\zeta}: \eta(i) \notin C_i^{\beta}\} \in J, \quad i \in A_{\zeta} \Rightarrow \text{cf}(\eta(i)) = \sigma.$$

**Proof of the Fact.** We shall choose by induction on  $\varepsilon < \sigma$  a function  $g_{\varepsilon} \in \prod_{i \in A_{\zeta}} \lambda_i$  such that  $\varepsilon_1 < \varepsilon \Rightarrow g_{\varepsilon_1} < g_{\varepsilon}$  (in all coordinates) and

$$(\forall \beta < \alpha)(\forall^J i \in A_{\zeta})[(g_{\varepsilon}(i), g_{\varepsilon+1}(i)) \cap C_i^{\beta} \neq \emptyset].$$

Why is this enough?

Let  $\nu = \eta \upharpoonright A_{\zeta}$  be defined by

$$\nu(i) = \bigcup_{\varepsilon < \sigma} g_{\varepsilon}(i).$$

Now  $\nu(i) < \lambda_i$  as  $g_{\varepsilon}(i) < \lambda_i$  and  $\sigma < \lambda_i = \text{cf}(\lambda_i)$ . (We can also say something for  $\sigma \geq \mu$ , but not now.) Also  $\langle g_{\varepsilon}(i): \varepsilon < \sigma \rangle$  is strictly increasing, so  $\text{cf}(\nu(i)) = \sigma$ .

Now let  $\beta < \alpha$  and define

$$B_{\beta}^* = \{i \in A_{\zeta}: \nu(i) \notin C_i^{\beta}\}.$$

We would like to have  $B_{\beta}^* \in J$ . For each  $i \in B_{\beta}^*$ , the sequence  $\langle g_{\varepsilon}(i): i < \sigma \rangle$  is a strictly increasing sequence of ordinals with limit not in  $C_i^{\beta}$ .

So for some  $\varepsilon_{\beta, i} < \sigma$

$$C_i^{\beta} \cap (g_{\varepsilon_{\beta, i}}(i), \nu(i)) = \emptyset.$$

So

$$\bigwedge_{\varepsilon \geq \varepsilon_{\beta,i}} (g_\varepsilon(i), g_{\varepsilon+1}(i)) \cap C_{\beta,i} = \emptyset.$$

Let  $\varepsilon_\beta = \sup_{i < \delta} \varepsilon_{\beta,i}$ .

Now  $\varepsilon_{\beta,i} < \sigma$  &  $\sigma = \text{cf}(\sigma) > |\delta| \geq |A_\zeta|$ , so  $\varepsilon_\beta < \sigma$ . So

$$\bigwedge_{i \in B_\beta^*} (g_{\varepsilon_\beta}(i), g_{\varepsilon_\beta+1}(i)) \cap C_i^\beta = \emptyset,$$

and hence  $B_\beta^* \in J$  as required, i.e.,  $\nu$  is the required  $\eta$ .

Why is the choice of the  $g_\varepsilon$  possible?

### Construction.

$\varepsilon = 0$ . Trivial.

$\varepsilon$  limit.  $g_\varepsilon(i) = \bigcup_{\varepsilon_1 < \varepsilon} g_{\varepsilon_1}(i) < \lambda_i$  (as  $\varepsilon < \sigma < \lambda_i = \text{cf}(\lambda_i)$ ).

$\varepsilon + 1$ . For  $\beta < \alpha$  define  $h_{\beta,\varepsilon} \in \prod_{i \in A_\zeta} \lambda_i$  by

$$h_{\beta,\varepsilon}(i) \stackrel{\text{def}}{=} \min\{\gamma < \lambda_i : (g_\varepsilon(i), \gamma) \cap C_i^\beta \neq \emptyset\}.$$

So  $\{h_{\beta,\varepsilon} : \beta < \alpha\}$  is a subset of  $\prod_{i \in A_\zeta} \lambda_i$  of cardinality  $< \lambda$ , but  $\prod_{i < \delta} \lambda_i / J$  hence  $\prod_{i \in A_\zeta} \lambda_i / (J \upharpoonright A_\zeta)$  has true cofinality  $\lambda$  (as if  $A_\zeta \in J$  there is nothing to prove). So there is  $g'_\varepsilon \in \prod_{i \in A_\zeta} \lambda_i$  which is a  $< J \upharpoonright A_\zeta$ -upper bound of  $\{h_{\beta,\varepsilon} : \beta < \alpha\}$ .

Let  $g_{\varepsilon+1}(i) = \max\{g'_\varepsilon(i), g_\varepsilon(i) + 1\}$ , clearly it is as required.  $\square$

### Claim 9.4.

(1) Assume

- (a)  $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$ , where  $\eta_\alpha \in \prod_{i \in \text{Dom}(J)} \text{Dom}(I_i)$  and  $J$  is an ideal on  $\delta$  extending  $J_\delta^{\text{bd}}$ , each  $I_i$  an ideal and  $I$  an ideal on  $\lambda$  extending  $J_\lambda^{\text{bd}}$ ,
- (b)  $\langle A_\zeta : \zeta < \zeta(*) \rangle$  is a partition of  $\text{Dom}(J)$ ,  $A_\zeta \notin J$ ,
- (c) for every  $\bar{F} = \langle F_i : i \in \text{Dom}(J) \rangle$ , for the  $I$ -majority of  $\alpha < \lambda$ , for every  $\zeta < \zeta(*)$  if  $F_i(\eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi) \in I_i$  for  $i \in A_\zeta$ , then

$$(\forall^J i \in A_\zeta) \left[ \eta_\alpha(i) \notin F_i \left( \eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi \right) \right].$$

- (d)  $I_j^* = \prod_{\ell < n_j} I_{i(j,\ell)}$  for  $j < \delta^*$ , where  $i(j,\ell) < \delta$ ,
- (e)  $J^* = \{A \subseteq \delta^* : \text{for some } B \subseteq \delta, \bigwedge_\zeta (B \cap A_\zeta) \in J \text{ and } \bigwedge_{i \in A} \bigvee_{\ell < n_j} i(j,\ell) \in B\}$  is an ideal on  $\delta^*$ ,
- (f)  $\eta_\alpha^*$  is defined by

$$\eta_\alpha^*(j) = \langle \eta_\alpha(i(j,\ell)) : \ell < n_j \rangle.$$

Then<sup>4</sup>  $\langle \eta_\alpha^* : \alpha < \lambda \rangle$  is a  $(\lambda, J^*, I)$ -sequence for  $\langle I_j^* : j < \delta \rangle$ .

<sup>4</sup> So we have dealt here with the case of  $J_\lambda^{\text{bd}}, \bar{\lambda}$  decreasing.

(2) If we strengthen clause (c) to the parallel of  $(**)^+$  in 9.1, then  $\langle \eta_\alpha^*: \alpha < \lambda \rangle$  is a super  $(\lambda, J^*, I)$ -sequence for  $\langle I_i^*: i < \delta \rangle$ .

**Proof.** Straightforward.  $\square$

**Conclusion 9.4A.** Assume (a)–(g) of 9.1 (see 9.2) and (a), (e) of 9.4. Then there is a super  $(\lambda, J^*)$ -sequence for  $\langle I_j^*: j < \delta \rangle$ .

**Conclusion 9.5.** Assume  $\mu > \text{cf}(\mu) = \aleph_0$  is a strong limit, and

$$\lambda = 2^\mu = \text{cf}(2^\mu) = \text{tcf}\left(\prod_{n < \omega} \lambda_n / J_\omega^{\text{bd}}\right),$$

$\lambda_n$  regular  $< \mu$ . Let  $\langle k_n: n < \omega \rangle$  be such that

$$(\forall k)(\exists^\infty n)(k_n = k),$$

and, e.g.,  $\theta = (2^{\aleph_0})^+$ .

For  $n < \omega$  and  $k < k_n$  let  $\ell(n, k) = \sum\{k_m: m < n\} + k$  and let

$$I_n = \prod_{k < k_n} J_{\lambda_{\ell(n, k)}}^{\text{nst}, \theta},$$

$$J = \{A \subseteq \omega: \sup_{n \in A} k_n < \omega\}.$$

Then there is a  $(\lambda, J)$ -sequence for  $\langle I_n: n < \omega \rangle$  (even a super one).

**Proof.** By Lemma 9.1 and Claim 9.4, we choose in 9.4 the parameters  $\delta = \omega$ ,  $\zeta(*) = \omega$  and let

$$A_\zeta = \left\{ \sum_{m \leq n} k_m - \zeta: k_n > \zeta \right\}. \quad \square$$

We may wonder on the “tcf” assumption; at the expense of using “some  $J$ ” this can be overcome:

**Claim 9.6.** Assume  $\mu > \text{cf}(\mu) = \aleph_0$  strong limit singular,

$$\lambda = 2^\mu = \text{cf}(2^\mu) \in \text{pcf}\{\lambda_n: n < \omega\},$$

$$\lambda_n = \text{cf}(\lambda_n) < \mu,$$

and  $\langle k_n: n < \omega \rangle$  is as in 9.5. Then we can find  $i(n, \ell) < \omega$ ,  $\ell < k_n$  with no repetitions,

$$i(n, 0) > i(n-1, k_{n-1}-1) > \dots > i(n-1, 0),$$

and letting

$$I_n = \prod_{\ell < k_n} J_{\lambda_{i(n, \ell)}}^{\text{nst}, \theta},$$

we have: for some ideal  $J \supseteq J_\omega^{\text{bd}}$  on  $\omega$ , there is a  $(\lambda, J)$ -sequence for  $\langle I_n: n < \omega \rangle$ .

**Proof.** Let

$$\text{pcf}_{J_\omega^{\text{bd}}}(\{\lambda_n: n < \omega\}) = \left\{ \chi: \text{cf}(\chi) = \chi = \text{tcf}\left(\prod_{n \in A} \lambda_n / J_A^{\text{bd}}\right) \right. \\ \left. \text{for some infinite } A \subseteq \omega \right\}.$$

By a pcf claim:

**Fact 9.6A.** We can find increasing  $\langle \chi_\varepsilon: \varepsilon < \varepsilon(*) \rangle$ ,  $\varepsilon(*) < \omega_1$ , a limit ordinal,  $J^*$  an ideal  $\supseteq J_{\varepsilon(*)}^{\text{bd}}$ , such that

$$\chi_\varepsilon \in \text{pcf}_{J_\omega^{\text{bd}}}(\{\lambda_n: n < \omega\}),$$

say

$$\chi_\varepsilon = \text{tcf}\left(\prod_{n \in B_\varepsilon} \lambda_n / J_{B_\varepsilon}^{\text{bd}}\right),$$

$\langle B_\varepsilon: \varepsilon < \varepsilon(*) \rangle$  is a partition of  $\omega$ , and

$$\lambda = \text{tcf}\left(\prod_{\varepsilon < \varepsilon(*)} \lambda_\varepsilon / J^*\right).$$

**Continuation of the proof of 9.6.** Let  $\langle k_n: n < \omega \rangle$  be as before. Choose

$$\langle i(n, \ell): \ell < k_n \rangle \text{ for each } n$$

such that

- (a)  $i(n, \ell) > i(n, \ell + 1)$ ,  $i(n, \ell_1) < i(n + 1, \ell_2)$ , and
- (b) for every  $k$  and  $\varepsilon_0, \dots, \varepsilon_{k-1}$ , for infinitely many  $n$  we have

$$k_n = k, \quad i(n, \ell) \in B_{\varepsilon_\ell}.$$

Let

$$A_\ell = \{i(n, \ell): n < \omega, k_n > \ell\}.$$

So

$\langle A_\ell: \ell < \omega \rangle$  is a sequence of pairwise disjoint subsets of  $\omega$  such that  $|A_\ell \cap B_{\varepsilon_\ell}| = \aleph_0$ .

We apply 9.1 for

$$\langle A_n: n < \omega \rangle, \quad \langle \lambda_n: n < \omega \rangle, \lambda, \mu. \quad \square$$

**Remark 9.6B.** If  $\mu > \text{cf}(\mu) > \aleph_0$ ,  $2^\mu$  regular, the parallel to 9.5 always occurs.  $\square$

If we use  $\bar{A} = \langle A_0 \rangle$ ,  $A_0 = \delta$  in 9.1:

**Conclusion 9.6C.** In 9.1 we get:

there is a  $(\lambda, J)$ -sequence for  $\langle I_i: i < \delta \rangle$ , even a super one.



**Remark 9.7.** By the proofs in [24, §1] we can replace  $\langle S_\theta^{\lambda_i}: i < \delta \rangle$ ,  $S_\theta^\lambda = \{\delta < \lambda_i: \text{cf}(\delta) = \theta\}$  by some large enough  $\bar{S} = \langle S_i: i < \delta \rangle$ , where  $S_i \in I[\lambda_i]$ , see below.

Also if  $\langle f_\alpha: \alpha < \lambda \rangle$  is  $<_J$ -increasing cofinal in  $\prod_{i < \delta} \lambda_i / J$ , continuous when it can be, then for some club  $E$  of  $\lambda$  we have  $\langle f_\delta: \delta \in E, \text{cf}(\delta) = \theta, \bar{f} \upharpoonright \delta$  has an exact least upper bound lub) is OK. Probably more interesting is to strengthen  $I_{\lambda_i}^{\text{nst}, \theta}$  by club guessing, as follows.

**Definition 9.8.** For  $\bar{C} = \langle C_\delta: \delta \in S \rangle$ ,  $S \subseteq \lambda$ , stationary

$$\text{id}^a(\bar{C}) = \{A \subseteq \lambda: \text{for some club } E \text{ of } \lambda \text{ the set} \\ \{\delta \in S: C_\delta \subseteq E\} \text{ is not stationary} \\ \text{(so as we can shrink } E, \text{ equivalently, empty)}\}.$$

**Lemma 9.9.** Assume

- (a)  $\mu$  is a strong limit singular  $\theta$ ,
- (b)  $\lambda = 2^\mu = \text{cf}(\lambda)$ ,
- (c)  $\lambda_i$  regular increasing for  $i < \delta$  with limit  $\mu$ ,  $\delta < \mu$  (usually  $\delta = \text{cf}(\mu)$ ),
- (d)  $J$  is an ideal on  $\delta$  extending  $J_\delta^{\text{bd}}$ ,
- (e)  $\lambda = \text{tcf}(\prod_{i < \delta} \lambda_i / J)$ ,
- (f)  $\langle A_\zeta: \zeta < \zeta(*) \rangle$  is a partition of  $\delta$  (so pairwise disjoint),
- (g)  $\sigma = \text{cf}(\sigma) < \mu$ , moreover  $\sigma < \lambda_0$  and satisfies  $\otimes_J^{\sigma, \delta}$  we have  $\sigma > \delta$  (or at least if  $A_\varepsilon \in J$  for  $\varepsilon < \sigma$  then  $\{i < \delta: i \in A_\varepsilon \text{ for every large enough } \varepsilon < \sigma\} \in J$ ).

Then

For  $\theta \in \text{Reg} \cap (\sigma, \lambda_0)$  we can find  $\langle S_i: i < \delta \rangle$ ,  $\langle \bar{C}^i: i < \delta \rangle$ ,  $\bar{I} = \langle I_i: i < \delta \rangle$ ,  $\bar{\eta} = \langle \eta_\alpha: \alpha < \lambda \rangle$  such that

- ( $\alpha$ )  $S_i \in I[\lambda_i]$  is stationary, and  $\delta \in S_i \Rightarrow \text{cf}(\delta) = \sigma$ ,
- ( $\beta$ )  $\bar{C}^i = \langle C_\delta^i: \delta \in S_i \rangle$ ,  $C_\delta^i$  a club of  $\delta$ ,
- ( $\gamma$ )  $I_i = \text{id}^a(\bar{C}^i) = \{A \subseteq \lambda_i: \text{for some club } E \text{ of } \lambda_i \text{ we have: } \delta \in S \cap A_i \text{ implies } \sup(C_\delta^i \setminus E) < \delta\}$ ,
- ( $\delta$ ) ( $*$ ) for any sequence  $\langle F_{\zeta, i}: \zeta < \zeta(*), i < \delta \rangle$  of functions, for every large enough  $\alpha < \lambda$  we have
- ( $**$ ) if  $\zeta < \zeta(*), F_{\zeta, i}(\eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi)$  a member of  $\text{id}^a(\bar{C}^i)$  for  $i < \delta$  (really  $i \in A_\zeta$ ), then

$$\left\{ i \in A_\zeta: \eta_\alpha(i) \in F_{\zeta, i} \left( \eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi \right) \right\} \in J.$$

Moreover

- ( $**$ )<sup>+</sup> if  $\zeta < \zeta(*), n < \omega$  and  $\beta_0, \dots, \beta_{n-1} < \alpha$  and for each  $i \in A_\zeta$  we have:  $F_{\zeta, i}(\eta_{\beta_0}, \beta_0, \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi)$  in a member of  $\text{id}^a(\bar{C}^i)$  then

$$\left\{ i \in A_\zeta: \eta_\alpha(i) \in F_{\zeta, i} \left( \eta_{\beta_0}, \beta_0, \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \upharpoonright \bigcup_{\xi < \zeta} A_\xi \right) \right\} \in J.$$

**Remark.**

- (1) Included in the proof are imitations of proofs from [24, §1] and of 9.1.
- (2) We have a bit of flexibility in the proof.
- (3) In  $(**)^+$ , we can replace  $n < \omega$  by  $n < \tau$  when  $(\forall \alpha < \lambda) (|\alpha|^{<\tau} < \lambda)$ .

**Proof.** Let  $\theta = 2^\sigma$ . By [24, §1] we can find  $\bar{e}^i$  such that:

- (i) for  $i < \delta$ ,  $\bar{e}^i = \langle e_\alpha^i: \alpha \in S_i \rangle$ ,  $S_i \in I[\lambda]$ ,
- (ii)  $e_\alpha^i$  a club of  $\alpha$  of order type  $\sigma$  such that  $\alpha \in S_i \Rightarrow \text{cf}(\alpha) = \sigma$ ,
- (iii) for  $\chi$  large enough,  $x \in \mathcal{H}(\chi)$ , we can find  $\langle N_i: i \leq \sigma \rangle$  such that  $x \in N_\varepsilon < (\mathcal{H}(\chi), \varepsilon, <_\chi^*)$ ,  $\langle N_\zeta: \zeta \leq \varepsilon \rangle \in N_{\varepsilon+1}$ ,  $N_\varepsilon$  increasing continuous,  $\|N_\varepsilon\| = \theta$ ,  $\theta + 1 \subseteq N_\varepsilon$  and

$$i < \delta \Rightarrow \sup e_{\sup(N_\sigma \cap \lambda_i)}^i \in S_i.$$

For  $\bar{d} \in \bigcup \{ \langle \prod_{i < \delta} e_i: e_i \text{ a club of } \sigma \rangle \}$  let  $\bar{e}^{i,\bar{d}} = \langle e_\alpha^{i,\bar{d}}: \alpha \in S_i \rangle$ ,  $e_\alpha^{i,\bar{d}} = \langle \beta \in e_\alpha^i: \text{otp}(e_\alpha^i \cap \beta) \in \bar{d}_i \rangle$ . For each such  $\bar{d}$  we repeat the proof of 9.1, so we choose  $\eta_\alpha = \eta_\alpha^{\bar{d}}$  by induction on  $\alpha < \lambda$ , and for each  $\alpha$ , choose  $\eta_\alpha \upharpoonright (\bigcup_{\varepsilon < \zeta} A_\varepsilon)$  by induction on  $\zeta \leq \zeta^*$ . If we succeed fine, so assume we fail. So for some  $\alpha = \alpha[\bar{d}]$ ,  $\zeta = \zeta[\bar{d}]$  the situation is:  $\langle \eta_\beta^{\bar{d}}: \beta < \alpha \rangle$  and  $\eta_\alpha^{\bar{d}} \upharpoonright (\bigcup_{\varepsilon < \zeta} A_\varepsilon)$  are defined, but we cannot define  $\eta_\alpha^{\bar{d}} \upharpoonright A_\zeta$  and as there we can compute a family  $\mathbf{E} = \mathbf{E}_{\bar{d}}^i$  of cardinality  $< \lambda$  whose members has the form  $\bar{B} = \langle B_i: i < \delta \rangle$ ,  $B_i \in \text{id}^a(\bar{e}^{i,d})$  and let  $E_{B_i}^i$  be a club of  $\lambda_i$  exemplifying  $B_i \in \text{id}^a(\bar{e}^{i,d})$ ; let  $\mathbf{E}_{\bar{d}}^i = \{ \langle E_{B_i}^i: i < \delta \rangle: \bar{B} = \langle B_i: i < \delta \rangle \in \mathbf{E} \}$ . Let  $\langle N_i: i \leq \sigma \rangle$  be as in  $\otimes$ (iii) for  $x = \{ \langle \langle E_{\bar{d}}^i, \bar{d} \rangle: d_i \subseteq \sigma \text{ a club for } i < \delta \rangle, \bar{\lambda}, \langle \bar{e}^i: i < \delta \rangle \}$ .

As in the proof of 9.1 quite easily:

$$\varepsilon \leq \sigma \ \& \ \bar{B} = \langle B_i: i < \delta \rangle \in \bigcup_{\bar{d}} \mathbf{E}_{\bar{d}}^i \Rightarrow \{ i < \delta: \sup(N_\varepsilon \cap \lambda_i) \notin E_{B_i}^i \} \in J.$$

Let  $d_i = \{ \text{otp}(e_{\sup(N_\sigma \cap \lambda_i)}^i \cap \sup(N_\varepsilon \cap \lambda_i)): \varepsilon < \sigma \text{ and } \sup(N_\varepsilon \cap \lambda_i) \in e_{\sup(N_\sigma \cap \lambda_i)}^i \}$ . Clearly  $d_i$  is a club of  $\sigma$  and let  $\bar{d} = \langle d_i: i < \delta \rangle$ . Now  $\langle \sup(N_\sigma \cap \lambda_i): i \in A_\zeta[\bar{d}] \rangle$  is as required.

**Conclusion 9.9A.**

- (1) In 9.9 we get:

for some function  $c: [\lambda]^2 \rightarrow \sigma$ , for every  $X, Y \in [\lambda]^\lambda$  and  $\zeta < \sigma$ , for some  $\alpha \in X$ ,  $\beta \in Y$  we have  $\alpha > \beta$  and  $c(\{\alpha, \beta\}) = \zeta$ .

- (2) In 9.9 we can add:

if, e.g.,  $\chi = (2^\lambda)^+$ , for every  $x \subseteq 2^\mu$  for every  $\alpha < \lambda$  large enough, for  $\zeta < \zeta^*$ , there is a sequence  $\langle N_\varepsilon: \varepsilon < \sigma \rangle$  as in the proof of 9.9 such that

$$(\boxtimes) \quad \{ i \in A_\zeta: \eta_\alpha(i) \neq \sup(N_\delta \cap \lambda_1) \} \in J.$$

**Remark 9.9B.** In 9.9A(1) we get even  $\text{Pr}_1(\lambda, \lambda, \sigma, \sigma)$ .

**Proof.** (1) We relay on part (2).

(2) For  $\alpha < \beta$  let  $c(\{\alpha, \beta\}) = \zeta$  if

$$\{i \in A_0: f_\beta(i) \geq f_\alpha(i) \text{ or } f_\beta(i) < f_\alpha(i) \ \& \ \zeta \neq \text{otp}(e_{f_\alpha(i)}^i \cap \beta)\} \in J,$$

and zero if there is no such  $\zeta$ .

Let  $X, Y \in [\lambda]^\lambda$ . Take  $\alpha \in X$  large enough, so that we can find  $\langle N_\varepsilon: \varepsilon \leq \sigma \rangle$  as there, with  $(\boxtimes)$  for part (2). We can find  $\beta \in N_{\zeta+1} \cap Y$  such that  $\langle \sup(N_\zeta \cap \lambda_i): i < \delta \rangle <_J \eta_\beta$  (as  $Y \cap N_{\zeta+1}$  is unbounded in  $\lambda \cap N_{\zeta+1}$ ). Now  $\alpha > \beta$  are as required.  $\square$

**Claim 9.10.** *In 9.1*

(1) *Instead of “ $\mu > \theta = \text{cf}(\theta) > |\delta|$ ” we can assume only*

$\otimes_1$   $\mu > \theta = \text{cf}(\theta)$  and if  $\langle u_\zeta: \zeta < \theta \rangle$  is a sequence of members of  $J$  then

$$\{i < \delta: \theta = \sup\{\zeta: i \notin u_\zeta\}\} = \delta \text{ mod } J.$$

(2) *Weakening the conclusion of 9.1 to “weak  $(J, \lambda)$ -sequence”, we can replace “ $\theta = \text{cf}(\theta) > |\delta|$ ” by*

$\otimes_2$   $\theta = \text{cf}(\theta)$  and if  $\langle u_\zeta: \zeta < \theta \rangle$  is a sequence of members of  $J$  then

$$\{i < \delta: \theta = \sup\{\zeta: i \notin u_\zeta\}\} \in J^+.$$

(3) *In part (1) and (2), if  $\theta > \aleph_0$ , then we can find  $\bar{C}^i = \langle C_\delta^i: \delta \in S_{\theta}^{\lambda_i} \rangle$  with  $C_\delta^i$  a club of  $\delta$  such that: we can replace  $I_{\lambda_i}^{\text{nst}, \theta}$  by  $\text{id}_{\lambda_i}^a(\bar{C}^i)$ , see 9.9 above.*

## 10. The power of a strong limit singular is itself singular: Existence

**Lemma 10.1.** *Assume*

- (a)  $\mu$  strong limit singular,
- (b)  $2^\mu$  is singular,  $\lambda = \text{cf}(2^\mu)$  (so  $2^\mu > \lambda > \mu$ ),
- (c)  $\mu > \sigma = \text{cf}(\sigma) > \text{cf}(\mu)$ ,
- (d)  $2^\mu = \text{pp}(\mu)$  (see discussion in §9).

Then

- ( $\alpha$ ) *we can find  $J, J^*, \bar{\theta}^i = \langle \theta_\zeta^i: \zeta < \text{cf}(\mu) \rangle$  for  $i < \lambda$  and  $\bar{\lambda}$  such that*
  - (i)  $\bar{\theta}^i$  is an increasing sequence of regular cardinals  $< \mu$  with limit  $\mu$  for  $i < \lambda$ ,
  - (ii)  $\bar{\lambda} = \langle \lambda_\alpha: \alpha < \lambda \rangle$  is an increasing sequence of regulars  $\in (\mu + \lambda, 2^\mu)$  with limit  $2^\mu$ ,
  - (iii)  $J \subseteq J^*$  are ideals on  $\text{cf}(\mu)$ ,  $\text{cf}(\mu)$ -complete,
  - (iv)  $\lambda_\alpha = \text{tcf}(\prod_\zeta \theta_\zeta^\alpha / J)$ ,
  - (v)  $\langle \bar{\theta}^\alpha: \alpha < \lambda \rangle$  is  $<_{J^*}$ -increasing, i.e.,  $\alpha < \beta \rightarrow \{\zeta < \text{cf}(\mu): \theta_\zeta^\alpha \geq \theta_\zeta^\beta\} \in J^*$ , with  $<_{J^*}$ -exact upper bound  $\langle \theta_\zeta^*: \zeta < \text{cf}(\mu) \rangle$  and  $(\theta_\zeta^*$  is a cardinal  $< \mu$ , normally singular)  $\mu = \lim(\theta_\zeta^*: \zeta < \text{cf}(\mu))$  and

$$\bigwedge_{\substack{\alpha < \lambda \\ \zeta < \text{cf}(\mu)}} \theta_\zeta^\alpha < \theta_\zeta^*,$$

- (vi) if  $J \neq J_{\text{cf}(\mu)}^{\text{bd}}$ , then  $\text{cf}(\mu) = \aleph_0$  and  $\text{pp}_{J_{\text{cf}(\mu)}^{\text{bd}}}(\mu) < 2^\mu$  and  $J$  as in 9.6 so for most such  $\mu$  we have the conclusion of (1), see [16] and §4.
- (β) If  $J, \bar{\theta}^\alpha (\alpha < \lambda), \bar{\lambda}$  are as in clause (α) then we can find  $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$  such that
- (i)  $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle, \eta_\alpha \in \prod_{\zeta < \text{cf}(\mu)} \theta_\zeta^* \subseteq {}^{\text{cf}(\mu)}\mu$ . Moreover,  $\eta_\alpha \in \prod_{\zeta < \text{cf}(\mu)} \theta_\zeta^\alpha$  and  $\sigma = \text{cf}(\eta_\alpha(i))$  for  $\alpha < \lambda, i < \text{cf}(\mu)$ .
- (ii) If  $\bar{C} = \langle C_\zeta : \zeta < \text{cf}(\mu) \rangle, \theta_\zeta^\alpha \cap C_\zeta$  a club of  $\theta_\zeta^\alpha$  for  $\alpha < \lambda, \zeta < \text{cf}(\mu)$ , then for some  $\alpha^* = \alpha_{\bar{C}}^*$  we have

$$\alpha \in [\alpha^*, \lambda] \Rightarrow (\forall^J \zeta < \text{cf}(\mu)) [\eta_\alpha(\zeta) \in C_\zeta].$$

- (γ) Assume  $\langle A_\varepsilon : \varepsilon < \varepsilon^* \rangle$  is a partition of  $\text{cf}(\mu)$  to sets not in  $J$ . Then we can add
- (ii)<sup>+</sup> For any sequence of functions

$$F = \langle F_\zeta : \zeta < \text{cf}(\mu) \rangle,$$

for some  $\alpha^* = \alpha_{\bar{F}}^*$ , for every  $\alpha \in [\alpha^*, \lambda]$  we have

(\*) if  $\varepsilon < \varepsilon^*, n < \omega, \beta_\ell < \alpha$  for  $\ell < n$  then

$$\{ \zeta < \text{cf}(\mu) : F_\zeta(\dots, \beta_\ell, \eta_{\beta_\ell}, \dots, \eta_\alpha \upharpoonright \bigcup_{\xi < \varepsilon} A_\xi) \cap \theta_\zeta^\alpha \text{ is a club of } \theta_\zeta^\alpha$$

$$\text{but } \eta_\alpha(\zeta) \notin F_\zeta(\beta_\ell, \eta_{\beta_\ell}, \dots, \eta_\alpha \upharpoonright \bigcup_{\xi < \varepsilon} A_\xi) \cap \theta_\zeta^\alpha \}$$

belongs to  $J$ . (If we use constant  $F$  this reduces to (ii).)

**Proof.** Of clause (α): First choose  $\langle \lambda_\alpha^0 : \alpha < \lambda \rangle$  as demanded in clause (ii) (but we will manipulate it later, possibly by clause (e)). Now as in 9.6, for each  $\alpha$  there are

$$J_\alpha, \bar{\theta}^\alpha = \langle \theta_\zeta^\alpha : \zeta < \text{cf}(\mu) \rangle$$

as there, so satisfying (i), (iii), (iv), (vi).

As  $\lambda = \text{cf}(\lambda) > \mu > 2^{\text{cf}(\mu)}$ , we can replace  $\bar{\lambda}$  by a subsequence, so without loss of generality  $J \subseteq J^*$ , so  $J^*$  is  $\text{cf}(\mu)$ -complete and  $\bar{\theta}^\alpha$  is  $<_J$ -increasing, see 7.0. So  $\langle \bar{\theta}^\alpha : \alpha < \lambda \rangle$  has  $<_{J^*}$ -exact upper bound  $\bar{\theta}^*$ , without loss of generality

$$\bigwedge_{\alpha, \zeta} \theta_\zeta^\alpha < \theta_\zeta^*.$$

So clause (v) holds.

*Note:* If  $\text{cf}(\mu) > \aleph_0$  we have  $J = J_\mu^{\text{bd}}$ .

(β) + (γ): (Here  $\text{cf}(\mu)$  can be replaced by any  $\delta \leq \mu$  such that  $\text{cf}(\delta) = \text{cf}(\mu)$ .)

List all relevant  $\bar{F} = \langle F_\zeta : \zeta < \delta \rangle$  with values subsets of  $\mu$ . So there are  $\leq 2^\mu$  of them, list them as  $\langle \bar{F}^i : i < 2^\mu \rangle$  with

$$\bar{F}^i = \langle F_\zeta^i : \zeta < \delta \rangle.$$

We choose  $\eta_\alpha \in \prod_{\zeta < \text{cf}(\mu)} \theta_\zeta^*$  by induction on  $\alpha$ .

For a given  $\alpha < \lambda$  we choose  $\eta_\alpha \upharpoonright A_\varepsilon$  by induction on  $\varepsilon < \varepsilon^*$ . We will choose  $\eta_\alpha \upharpoonright A_\varepsilon$  such that

(\*) if  $n < \omega, \beta_0, \beta_1, \dots, \beta_{n-1} < \alpha$  and  $i < \sup\{\lambda_\beta : \beta < \alpha\}$  (necessarily  $< \lambda_\alpha$ ),

$$\left\{ \zeta \in A_\varepsilon: F_\zeta^i(\dots \beta_\ell, \eta_{\beta_\ell}, \dots, \eta_\alpha \upharpoonright \bigcup_{\xi < \varepsilon} A_\xi) \cap \theta_\zeta^\alpha \text{ is a club of } \theta_\zeta^\alpha \text{ but} \right. \\ \left. \eta_\alpha(\zeta) \text{ does not belong to it} \right\} \in J.$$

But in 9.1's proof we have shown that this is possible.  $\square$

We have conclusions variants similar to the case  $2^\mu$  is regular.

## 11. Preliminaries to the construction of ccc Boolean algebras with no large independent sets

Monk [12] asks:

**Problem 33.** Assume  $\text{cf}(\mu) \leq \kappa < \mu < \lambda \leq \mu^{\text{cf}(\mu)}$ . Is it possible in ZFC that there is a Boolean algebra of cardinality  $\lambda$ , satisfying the  $\kappa$ -cc with no independent subset of cardinality  $\lambda$ ?

This is closely related to the problem of “is  $\lambda$  a free caliber of such Boolean algebra” (see also in Monk [12]).

Why in ZFC? Because of earlier results under “ $\mu$  strong limit,  $2^\mu = \mu^+$ ”, I think.

The real problem seems to me is for  $\lambda$  regular, and we shall prove that “almost always” there is such a Boolean algebra, so we prove the consistency of failure.

We shall use  $\langle J_{(\lambda_{i,0}, \lambda_{i,1})}^{\text{bd}}: i < \delta \rangle$  with regular  $\lambda_{i,0} > \lambda_{i,1}$ , but we use Boolean algebras whose existence is only consistent.

So we shall use  $\bar{\eta}$  a  $(\lambda, J)$ -sequence for  $\langle J_{(\lambda_{i,0}, \lambda_{i,1})}^{\text{bd}}: i < \delta \rangle$ , if  $\delta = \omega$  the Boolean algebra  $\mathbf{B}$  will have a dense subalgebra  $\mathbf{B}^*$  which will be the free product of  $\{\mathbf{B}_n: n < \omega\}$ ,  $x_t^-, x_t^+ \in \mathbf{B}_n$  for  $t \in \text{Dom}(I_n)$  and  $\mathbf{B} = \langle \mathbf{B}^*, y_\alpha: \alpha < \lambda \rangle$  where  $y_\alpha \in$  completion of  $\mathbf{B}^*$  is defined from  $\langle x_{\eta_\alpha(n)}^-, x_{\eta_\alpha(n)}^+ : n < \omega \rangle$ . We need special properties of  $\mathbf{B}_n$ ,  $x_t^-, x_t^+$  ( $t \in \text{Dom}(I_n)$ ). The construction continues [15, §3]. Concerning the parallel to 6.13 see later.

For the case  $\mu$  strong limit we can use instead subalgebras of the measure algebra. See §2. Now we have consistency (and independence) for  $\lambda$ ,  $\mu < \lambda \leq 2^\mu$ ,  $\mu$  strong limit singular, hence we concentrate on the other case where the behavior is different, i.e., when for some  $\chi$  we have  $\text{cf}(\mu) \leq \kappa < \chi = \chi^{<\kappa} < \mu < \lambda < \mu^{\text{cf}(\mu)} \leq 2^\chi$ . The proof here uses ideals which are “easier” and can be generalized to get “non- $n$ -independent subset of  $\mathbf{B}$  of cardinality  $\lambda$  for some specific  $n$ ”. For this we need to start with “there is a  $\lambda_n$ -complete uniform filter  $\mathcal{D}_n$  on  $\lambda_n^{+n}$ ”.

**Definition 11.1.** We say  $(\mathbf{B}_1, \bar{x}^+, \bar{x}^-)$  witness  $(I, \mathcal{T})$  if

- $\mathcal{T}$  is a set of Boolean terms written as  $\tau = \tau(x_1, \dots, x_{n_\tau})$ ,
- $I$  is an ideal,
- $\mathbf{B}$  is a Boolean algebra,
- $\bar{x}^+ = \langle x_t^+ : t \in \text{Dom}(I) \rangle$ ,  $x_t^+ \in \mathbf{B}$ ,
- $\bar{x}^- = \langle x_t^- : t \in \text{Dom}(I) \rangle$ ,  $x_t^- \in \mathbf{B}$ ,

- (f)  $x_t^- < x_t^+$ ,  
 (g) if  $X \in I^+$  and  $\mathbf{B} \subseteq \mathbf{B}'$  and

$$\mathbf{B}' \models x_t^- \leq y_t \leq x_t^+, \quad \text{for } t \in X,$$

then for some  $\tau(x_1, \dots, x_n) \in \mathcal{T}$  and pairwise distinct  $t_1, \dots, t_n \in X$  we have

$$\mathbf{B}' \models \tau(y_{t_1}, y_{t_2}, \dots, y_{t_n}) = 0.$$

**Explanation 11.2.** We think of having  $\bar{\eta}$  a  $(\lambda, J)$ -sequence for  $\langle I_i: i < \delta \rangle$ , and having  $(\mathbf{B}_i, \bar{x}_i^+, \bar{x}_i^-)$  witnessing  $(I_i, \mathcal{T})$  for  $i < \delta$  and using the sequence of intervals  $\langle (x_{i, \eta_\alpha(i)}^-, x_{i, \eta_\alpha(i)}^+): i < \delta \rangle$  as a sequence of approximations for an element  $x_\alpha$  of the desired Boolean algebra  $\mathbf{B}$  of cardinality  $\lambda$ .

But we may think not only of “ $\{x_\alpha: \alpha < \lambda\}$  has no independent subset of cardinality  $\lambda$ ” but of other subsets of  $\mathbf{B}$ . So sometimes we use

**Definition 11.3.**

- (1) We say that  $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$  strongly witnesses  $(I, \mathcal{T})$  if: (a)–(f) as before, and  
 (g)<sup>+</sup> If  $\mathbf{B} \subseteq \mathbf{B}'$ ,

$$\mathbf{B}' \models x_t^- \leq y_t \leq x_t^+ \quad \text{for } t \in \text{Dom}(I),$$

$\langle b_\ell: \ell \leq m \rangle$  is a sequence of pairwise disjoint nonzero members of  $\mathbf{B}'$ ,  $m < \omega$   
 and

$$X \in \left( \prod_{\ell=1}^m I \right)^+,$$

and  $u \subseteq [1, m]$ , then we can find  $n$ ,  $\tau(x_1, \dots, x_n) \in \mathcal{T}$  and distinct  $\bar{t}^1, \dots, \bar{t}^n \in X$ , so  $\bar{t}^r = \langle t_\ell^r: \ell = 1, \dots, m \rangle$ , such that  $\tau(c_{\bar{t}^1}, \dots, c_{\bar{t}^n}) = 0$  where

$$c_{\bar{t}^r} = b_0 \cup \bigcup_{\substack{\ell \in [1, m] \\ \ell \in u}} (b_\ell \cap y_{t_\ell^r}) \cup \bigcup_{\substack{\ell \in [1, m] \\ \ell \notin u}} (b_\ell - y_{t_\ell^r}).$$

- (2) We say that  $(\mathbf{B}, \bar{x}^+, \bar{x}^-)$  witness  $(I, \mathcal{T})$   $m$ -strongly if we restrict ourselves to this  $m$ . Similarly  $[m_1, m_2]$ -strongly.

Next we need our specific  $(\mathbf{B}, \bar{x}^-, \bar{x}^+, I)$ . The following is essentially from [23, pp. 244–246].

**Claim 11.4.**

- (1) If  $\mu = 2^\lambda = \lambda^+$ , (or just  $\mu \rightarrow [\mu]_\mu^2$ ) and  $2^\mu = \mu^+$ , then we can find  $\bar{F} = \langle F_\alpha: \alpha < \mu^+ \rangle$  such that:

$$(*)_{\bar{F}}^\mu \quad \text{(a) } F_\alpha: [\mu]^2 \rightarrow \alpha \times \mu \text{ is one to one.}$$

- (b) If  $A \in (J_{(\mu^+, \mu)}^{\text{bd}})^+$ , then

for some  $(\alpha, i_0), (\alpha, i_1), (\beta, i_2) \in A$  we have

$$F_\alpha(\{i_0, i_1\}) = (\beta, i_2).$$

We write this also as

$$F(\{\alpha, i_0\}, \{\alpha, i_1\}) = (\beta, i_2).$$

We can add that for every  $\beta$  we have  $|\text{Rang}(F_\alpha) \cap (\{\beta\} \times \mu)| \leq 1$  for  $\alpha > \mu$ . We do not strictly distinguish  $\bar{F}$  from  $F$ .

- (2) The property  $(*)_F^\mu$  is preserved by forcing notions which have the  $(3, J_{(\mu^+, \mu)}^{\text{bd}})^+$ -c.c. (see 11.6 below).
- (3) Let  $\mathbf{B} = \mathbf{B}_{\bar{F}}$  be the Boolean algebra freely generated by

$$x_{\alpha, i}^+ = x_{(\alpha, i)}^+; x_{\alpha, i}^- = x_{(\alpha, i)}^- \quad (\text{for } (\alpha, i) \in \mu^+ \times \mu)$$

except that  $x_{\alpha, i}^- \leq x_{\alpha, i}^+$  and

$$x_{(\alpha, i)}^+ \cap x_{(\alpha, j)}^+ \cap x_{F_\alpha(i, j)}^+ = 0.$$

Then

- (i)  $(\mathbf{B}, \bar{x}^+, \bar{x}^-)$  witness  $(J_{(\mu^+, \mu)}^{\text{bd}}, \{x_0 \cap x_1 \cap x_2 = 0\})$ .
- (ii)  $\mathbf{B}$  satisfies the ccc.

**Remark 11.5.** On more general Boolean algebras generated by such equations see Hajnal et al. [7].

**Definition 11.6.** For an ideal  $J$  and forcing notion  $P$ , we say that  $P$  satisfies the  $(n, J)$ -c.c. if for  $\langle p_t : t \in A \rangle$ ,  $A \in J^+$ , there is  $B \subseteq A$ ,  $B \in J^+$  such that any  $n$  conditions in  $\{p_t : t \in B\}$  have a common upper bound.

**Fact 11.7.** If  $P$  is the forcing notion  $P_{\chi, \theta}$  of adding  $\chi$  Cohens for  $\theta$  and  $\lambda^{<\theta} = \lambda$  then  $P$  satisfies  $(n, J)$ -c.c. for  $n < \omega$ ,  $J = J_{(\lambda^{++}, \lambda^+)}$ .

**Proof of 11.4.** (1) Let  $\{A_\alpha : \alpha < \mu^+\}$  list all subsets  $A$  of  $\mu^+ \times \mu$  of cardinality  $\mu$  such that for every  $\beta < \mu^+$  we have  $|A \cap (\{\beta\} \times \mu)| \leq 1$ . For every  $\alpha$  such that  $\mu < \alpha < \mu^+$  choose  $H_\alpha : [\mu]^2 \rightarrow \alpha$  such that  $(\forall X \in [\mu]^\mu) [H_\alpha''([X]^2) = \alpha]$ . For each  $\alpha$ , choose  $F_\alpha(i, j) \in \{\beta_{[i, j]}^\alpha\} \times \mu$  by induction on  $<^\otimes$ , where  $\{i, j\} <^\otimes \{i', j'\}$  iff  $\max\{i, j\} < \max\{i', j'\} \vee (\max\{i, j\} = \max\{i', j'\} \ \& \ \min\{i, j\} < \min\{i', j'\})$ , with  $\beta_{i, j}^\alpha$  with no repetition so that

$$F_\alpha(i, j) \in \alpha \times \mu^+ \setminus \bigcup \{ \{\beta_{i', j'}^\alpha\} \times \mu : \{i', j'\} <^\otimes \{i, j\} \},$$

and if possible

$$F_\alpha(i, j) \in A_{H_\alpha(i, j)},$$

which occurs if  $A_{H_\alpha(i, j)} \subseteq \alpha \times \mu$ .

(2) Trivial. Let  $P$  be the forcing notion. Let  $p^* \Vdash \text{“} \mathcal{A} \in (J_{(\mu^+, \mu)}^{\text{bd}})^+ \text{”}$  and it exemplifies a contradiction to  $(*)_F^\mu$ . Let  $A \stackrel{\text{def}}{=} \{(\alpha, i) : p^* \Vdash (\alpha, i) \notin \mathcal{A}\}$ . So  $A \subseteq \mu^+ \times \mu$  and,

$$p^* \Vdash \text{“} A \supseteq \mathcal{A}, \quad \mathcal{A} \in (J_{(\mu^+, \mu)}^{\text{bd}})^+ \text{”},$$

hence

$$A \in (J_{\langle \mu^+, \mu \rangle}^{\text{bd}})^+.$$

For  $(\alpha, i) \in A$  there is  $p_{(\alpha, i)} \geq p^*$  such that

$$p_{(\alpha, i)} \Vdash “(\alpha, i) \in \mathcal{A}”.$$

Apply  $(3, J_{\langle \mu^+, \mu \rangle}^{\text{bd}})$ -cc to  $\langle p_{(\alpha, i)} : (\alpha, i) \in A \rangle$ , and obtain  $B$  as in Definition 11.6. As  $B \in (J_{\langle \mu^+, \mu \rangle}^{\text{bd}})^+$ , by  $(*)_F^\mu$  we can find  $(\alpha, i_0), (\alpha, i_1), (\beta, i_2) \in B$  such that

$$F_\alpha(\{i_0, i_1\}) = (\beta, i_2).$$

But by the choice of  $B$  there is  $q \in P$  such that

$$q \geq p_{(\alpha, i_0)}, p_{(\alpha, i_1)}, p_{(\beta, i_2)}$$

(hence  $q \geq p^*$ ). So

$$q \Vdash “(\alpha, i_0), (\alpha, i_1), (\beta, i_2) \in \mathcal{A} \quad \text{and} \quad F_\alpha(\{i_0, i_1\}) = (\beta, i_2)”.$$

But this contradicts the assumption on  $p^*, \mathcal{A}$ .

(3) For clause (i), read the definition. For clause (ii):

Call  $\mathcal{Z} \subseteq \mu^+ \times \mu$  closed if  $F(t_1, t_2) = t_3$  &  $|\{t_1, t_2, t_2\} \cap \mathcal{Z}| > 1 \Rightarrow \{t_1, t_2, t_3\} \subseteq \mathcal{Z}$ . Now,

(\*) if  $F(t_i, s_i) = r_i$  for  $i = 0, 1$  then

$$\{t_0, s_0, r_0\} \cap \{t_1, s_1, r_1\}$$

has  $\leq 1$  or 3 elements.

[Why? As each  $F_\alpha$  is one to one and

$$F = \bigcup_{\alpha < \mu^+} F_\alpha \upharpoonright (\{\alpha\} \times [\mu]^2)$$

and

$\{\{\alpha\} \times [\mu]^2 : \alpha < \mu^+\}$  are pairwise disjoint.]

(\*\*) If  $\mathcal{Z} \subseteq \mu^+ \times \mu$ , and  $\mathbf{B}_\mathcal{Z}$  is defined naturally: it is freely generated by  $\{x_t^+, x_t^- : t \in \mathcal{Z}\}$  except the equations explicitly demanded on those variables, then  $\mathbf{B}_\mathcal{Z} \subseteq \mathbf{B}$  (even if  $\mathcal{Z}$  is not closed).

[Why? If  $f : \{x_t^-, x_t^+ : t \in \mathcal{Z}\} \rightarrow \{0, 1\}$  preserves the equations, and we define

$$f^* : \{x_t^-, x_t^+ : t \in \mu^+ \times \mu\} \rightarrow \{0, 1\}$$

by

$$f^*(y) \stackrel{\text{def}}{=} \begin{cases} f(y) & \text{if } y = x_t^\pm, t \in \mathcal{Z}, \\ 0 & \text{if } y = x_t^\pm, t \notin \mathcal{Z}, \end{cases}$$

then  $f^*$  preserves the equations.]

(\*\*\*)  $\mathbf{B} \models \text{ccc}$ .



[Why? Let  $\langle a_\zeta: \zeta < \omega_1 \rangle$  be a sequence of nonzero elements. We can find finite  $Z_\zeta$  such that  $a_\zeta \in \mathbf{B}_{Z_\zeta}$ . Let  $f_\zeta: \mathbf{B}_{Z_\zeta} \rightarrow \{0, 1\}$  be a homomorphism such that  $f_\zeta(a_\zeta) = 1$ . Let

$$Z_\zeta^+ \stackrel{\text{def}}{=} Z_\zeta \cup \bigcup \{ \{t_1, t_2, t_3\}: F(t_1, t_2) = t_3, \text{ and } \{t_1, t_2, t_3\} \cap Z_\zeta > 1 \}.$$

Without loss of generality  $\langle Z_\zeta^+: \zeta < \omega_1 \rangle$  is a  $\Delta$ -system with heart  $Z^+$ .

Without loss of generality  $f_\zeta \upharpoonright \{x_t^+: t \in Z^+\}$  is constant.

Without loss of generality  $Z_\zeta \cap Z^+$  is constant.

So

(\*)<sub>4</sub> If  $\zeta \neq \xi < \omega_1$

$$F(t_1, t_2) = t_3 \quad \text{and} \quad \{t_1, t_2, t_3\} \subseteq Z_\xi \cup Z_\zeta,$$

then

$$\{t_1, t_2, t_3\} \subseteq Z_\zeta \quad \text{or} \quad \{t_1, t_2, t_3\} \subseteq Z_\xi.$$

[Why? Without loss of generality

$$|\{t_1, t_2, t_3\} \cap Z_\zeta| \geq 2.$$

So

$$\{t_1, t_2, t_3\} \subseteq Z_\zeta^+.$$

Now if  $t_i \in Z_\zeta^+ \setminus Z_\zeta$ , then  $t_i \notin Z_\xi$  (otherwise  $t_i \in Z_\zeta^+ \cap Z_\xi^+$ , hence  $t_i \in Z^+$ , but  $Z_\zeta \cap Z^+$  is constant). So  $\{t_1, t_2, t_3\} \subseteq Z_\zeta$ .)

Now  $f_\zeta \cup f_\xi$  preserves the equations on  $Z_\zeta \cup Z_\xi$  and by the homomorphism it induces,  $a_\zeta \cap a_\xi$  is mapped to 1, so  $\mathbf{B}_{Z_\zeta \cup Z_\xi} \models "a_\zeta \cap a_\xi \neq 0"$  hence by (\*\*) we have  $\mathbf{B} \models "a_\zeta \cap a_\xi \neq 0"$ .]  $\square$

**Fact 11.7A.** Assume

(a)  $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$  is a witness for  $(I, T)$ ,

(b)  $y_t^- = -x_t^+, y_t^+ = -x_t^-$  for  $t \in \text{Dom}(I)$ ,  $\bar{y}^- = \langle y_t^-: t \in \text{Dom}(I) \rangle$ ,  $\bar{y}^+ = \langle y_t^+: t \in \text{Dom}(I) \rangle$ ,

(c)  $T' = \{-\tau(-x_0, \dots, -x_{n-1}): \tau(x_0, \dots, x_{n-1}) \in T\}$ .

Then  $(\mathbf{B}, \bar{y}^-, \bar{y}^+)$  is a witness for  $(I, T')$  (and is called the dual of  $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$ ).

We may consider

**Definition 11.8.**

(1) Let  $(*)_{\overline{F}, \overline{H}}^\mu$  mean

(a)  $\overline{F} = \langle F_\alpha: \alpha < \mu^+ \rangle$ ,  $F_\alpha$  is a partial function from  $[\mu]^2$  into  $\alpha \times \mu$ ,

(b)  $\overline{H} = \langle H_\alpha: \alpha < \mu^+ \rangle$ ,  $H_\alpha$  is a partial function from  $[\mu]^2$  into  $\{0, 1\}$ ,

(c) if  $A \in (J_{(\mu^+, \mu)}^{\text{bd}})^+$  and  $\ell < 2$  then for some  $(\alpha, i_0), (\alpha, i_1) \in A$  we have  $F_\alpha(i_0, i_1) \in A$  and  $H_\alpha(i_0, i_1) = \ell$ ,

(d) the Boolean algebra  $\mathbf{B}_{\overline{F}, \overline{H}}$  defined below satisfies the c.c.c. We may write

$$F =: \bigcup_{\alpha < \mu^+} F_\alpha, H =: \bigcup_{\alpha < \mu^+} H_\alpha \text{ instead of } \overline{F}, \overline{H}, \text{ respectively.}$$

- (2)  $\mathbf{B}_{\overline{F}, \overline{H}}$  is the Boolean algebra generated freely by  $\{x_t^-, x_t^+ : t \in \mu^+ \times \mu\}$  except that  $x_t^- \leq x_t^+$  and  $x_{t_0}^+ \cap x_{t_1}^+ \cap x_{t_2}^+ = 0$  when  $F(t_0, t_1) = t_2$ ,  $H(t_0, t_1) = 0$  and  $(-x_{t_0}^-) \cap (-x_{t_1}^-) \cap (-x_{t_2}^-) = 0$  when  $F(t_0, t_1) = t_2$ ,  $H(t_0, t_1) = 1$ .

**Remark 11.9.** Of course  $\mathbf{B}_{\overline{F}, \overline{H}}$  is defined from two sets of triples, which are disjoint and no distinct two have  $> 1$  element in common.

**Claim 11.10.** Assume  $(*)_{F_0}^\mu$  of 11.4(1) and, e.g.,  $\mu = \lambda^+$ ,  $\lambda^{<\theta} = \lambda$ .

- (1) For some  $(\theta^{<\theta})^+$ -c.c.,  $\theta$ -complete, forcing notion  $P$  of cardinality  $\leq \mu^+$  we have

$$\Vdash_P "(*)_{F, H}^\mu \text{ for some } F, H".$$

- (2) If  $(*)_{F, H}^\mu$  and  $Q$  is a forcing notion satisfying the  $(3, J_{(\mu^+, \mu)}^{\text{bd}})$ -c.c. then in  $V^Q$  we have  $(*)_{F, H}^\mu$ . If  $V = V_0^P$ ,  $P$  as above it is enough that  $P * Q$  satisfies the  $(3, J_{(\mu^+, \mu)}^{\text{bd}})$ -c.c.

**Proof.** (1) Let

$$P = \{(f, h) : \text{for some } u = u_{(f, h)} \subseteq \mu^+ \times \mu \text{ of cardinality } < \theta \text{ we have: } f, h \text{ are partial functions, } \text{Dom}(f) = \text{Dom}(h) \subseteq (\text{Dom } F) \cap [u]^2, f \subseteq F_0 \text{ and } \text{Rang}(h) \subseteq \{0, 1\} \text{ and } \mathbf{B}_{f, h} \text{ satisfies the c.c.c.}\},$$

where  $\mathbf{B}_{f, h}$  is defined as in 11.8(2) (and see 11.9).

The order  $(f_1, h_1) \leq (f_2, h_2)$  iff

- (i)  $u_{(f_1, h_1)} \subseteq u_{(f_2, h_2)}$ ,
- (ii)  $f_1 = f_2 \upharpoonright [u_{(f_1, h_1)}]^2$ ,
- (iii)  $h_1 = h_2 \upharpoonright [u_{(f_1, h_1)}]^2$ ,
- (iv)  $\mathbf{B}_{(f_1, h_1)} \subseteq \mathbf{B}_{(f_2, h_2)}$  moreover  $\mathbf{B}_{(f_1, h_1)} \ll \mathbf{B}_{(f_2, h_2)}$ .

The reader can check.  $\square$

**Claim 11.11.** Assume  $2^{\lambda^\ell} = \lambda^{\ell+1}$  for  $\ell < n$  and let  $\lambda_\ell = \lambda^{n-\ell+1}$ .

- (1) We can find  $W$  such that
- (a)  $W \subseteq [\prod_{\ell < n} \lambda_\ell]^n$ ,
  - (b) if  $u_1 \neq u_2$  belongs to  $W$  then  $|u_1 \cap u_2| \leq 1$ ,
  - (c) if  $A \in (J_{(\lambda_\ell: \ell < n)}^{\text{bd}})^+$  then  $[A]^n \cap W \neq \emptyset$ ,
  - (d)  $\langle \lambda_\ell : \ell < n \rangle$  is a decreasing sequence of regulars.
- (2) There is a forcing notion  $Q$  of cardinality  $\lambda^+$ ,  $\lambda^+$ -complete satisfying the  $\lambda^+$ -c.c. and even the  $(n, J_{(\lambda_\ell: \ell < m)}^{\text{bd}})$ -c.c. and adding  $W$  satisfying (a)–(c) of part (1) and
- (e)  $W$  is locally finite: if  $A \subseteq \prod_{\ell < n} \lambda_\ell$  is finite, then for some finite  $B$ ,  $A \subseteq B \subseteq \prod_{\ell < n} \lambda_\ell$  and  $w \in W$  &  $|w \cap B| \geq 2 \Rightarrow w \subseteq B$ .
- (3) If  $P$  is adding  $\chi$  many  $\theta$ -Cohen reals,  $\lambda = \lambda^\theta$  and in  $V$ ,  $\overline{W}$  satisfies (a)–(e), then in  $V^P$  still clause (c) holds (and trivially the other demands on  $W$ ). (See [23].)

**Proof.** (1) We prove by induction on  $n$  that for any such  $\lambda$  satisfying  $\ell < n \Rightarrow 2^{\lambda^\ell} = \lambda^{\ell+1}$  we can find  $(W, F)$  such that (a)–(c) of 11.11(1) hold for  $W$ ,  $\langle \lambda^{+(\ell+1)} : \ell < n \rangle$  and

- (f)  $F : W \rightarrow \lambda^+$  satisfies: if  $A \in (J_{(\lambda^{+(\ell+1)}: \ell < n)}^{\text{bd}})^+$ , then  $\text{Rang}(F \upharpoonright [A]^n) = \lambda^+$ .

The induction step is as in the previous proof.

(2) Similar to the proof of 11.10.

(3) Because  $P$  satisfies the  $(n, J_{(\lambda_\ell: \ell < n)}^{\text{bd}})$ -c.c.  $\square$

**Claim 11.12.** Assume

(A)  $W, \langle \lambda_\ell: \ell < n \rangle$  satisfy (a)–(e) of Claim 11.11(1), (2).

(B)  $3 \leq m < n/2, n > 6$ .

(C)  $\mathbf{B}$  is the Boolean algebra generated by  $\{x_t^-, x_t^+ : t \in \prod_{\ell < n} \lambda_\ell\}$  freely except:

(\*)<sub>1</sub>  $x_t^- \leq x_t^+$ ,

(\*)<sub>2</sub> if  $w = \{t_0, \dots, t_{n-1}\} \in W$ , where  $t_\ell$  is increasing in the lexicographic order, and  $u \subseteq n, |u| \geq m$  and  $n - |u| > m$ , then

$$\bigcap_{\ell \in u} x_{t_\ell}^+ \cap \bigcap_{\ell < n, \ell \notin u} (-x_{t_\ell}^-) = 0.$$

(D)  $\mathcal{T} = \mathcal{T}_{n,m} = \{\bigcap_{\ell \in u} x_\ell \cap \bigcap_{\ell < n, \ell \notin u} (-x_\ell) : u \subseteq n, m \leq |u| \leq n - m\}$ .

Then

(i)  $\mathbf{B} \models "x_t^- < x_t^+ \ \& \ x_s^- \not\leq x_t^+"$  for  $t \neq s$  in  $\prod_{\ell < n} \lambda_\ell$ ,

(ii)  $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$  is a witness for  $(J_{\prod_{\ell < n} \lambda_\ell}^{\text{bd}}, \mathcal{T})$ ,

(iii)  $\mathbf{B}$  satisfies the c.c.c.

**Proof.** Clearly  $\mathbf{B} \models x_t^- \leq x_t^+$  by the equation in (\*<sub>1</sub>) and  $\mathbf{B} \models "x_t^- \neq x_t^+"$  because the function  $f_0$  given by,

$$f_0(x_s^-) = 0, \quad f_0(x_s^+) = \begin{cases} 1 & s = t, \\ 0 & s \neq t \end{cases}$$

preserves all the required equations (as  $2 \leq m$ ). Taken together,  $\mathbf{B} \models x_t^- < x_t^+$ . Also  $\mathbf{B} \models x_t^- \not\leq x_s^+$  when  $t \neq s$  using  $f_1$  defined by

$$f_1(x_r^+) = f_1(x_r^-) = \begin{cases} 1 & \text{if } r = t, \\ 0 & \text{if } r \neq t. \end{cases}$$

So clause (i) of the conclusion holds. Clause (ii) holds easily by the equation in (\*<sub>2</sub>) and assumption (A), i.e., (c) of 11.11(1).

We are left with verifying clause (iii), i.e., the c.c.c. So let  $a_\zeta \in \mathbf{B} \setminus \{0\}$  for  $\zeta < \omega_1$ . For every  $\zeta$  we can find a finite set  $Z_\zeta \subseteq \prod_{\ell < n} \lambda_\ell$  such that  $a_\zeta \in \langle x_t^-, x_t^+ : t \in Z_\zeta \rangle$ . By 11.11, i.e., by clause (A), without loss of generality

(\*) if  $w \in W$  &  $|w \cap Z_\zeta| \geq 2 \Rightarrow w \subseteq Z_\zeta$ .

Let  $f_\zeta^*: \{x_t^-, x_t^+ : t \in Z_\zeta\} \rightarrow \{0, 1\}$  be such that it preserves all the equations (from (\*<sub>1</sub>) + (\*<sub>2</sub>)) on these variables and so the homomorphism it induces from  $\mathbf{B}_{Z_\zeta}$  to  $\{0, 1\}$ ,  $\hat{f}_\zeta^*$  maps  $a_\zeta$  to 1. Without loss of generality  $\langle Z_\zeta : \zeta < \omega_1 \rangle$  is a  $\Delta$ -system with heart  $Z$  and  $f_\zeta^* \upharpoonright \{x_t^-, x_t^+ : t \in Z\}$  is constant.

Let  $\zeta(1) < \zeta(2) < \omega_1$  and define  $f_2$

$$f_2(x_t^-) = \begin{cases} f_{\zeta(1)}^*(x_t^-) & \text{if } t \in Z_{\zeta(1)}, \\ f_{\zeta(2)}^*(x_t^-) & \text{if } t \in Z_{\zeta(2)}, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_2(x_t^+) = \begin{cases} f_{\zeta(1)}^*(x_t^+) & \text{if } t \in Z_{\zeta(1)}, \\ f_{\zeta(2)}^*(x_t^+) & \text{if } t \in Z_{\zeta(2)}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly it is well defined and with the right domain. Does  $f_2$  preserve all the equations?

*Case 1.*  $x_t^- \leq x_t^+$ . If  $t \notin Z_{\zeta(1)} \cup Z_{\zeta(2)}$  trivial (both get value zero), and if  $t \in Z_{\zeta(\ell)}$  then trivial (as  $f_{\zeta(\ell)}^*$  preserves this equation).

*Case 2.*  $\bigcap_{\ell \in u} x_{t_\ell}^+ \cap \bigcap_{\substack{\ell < n \\ \ell \notin u}} (-x_{t_\ell}^-) = 0$ . If  $\ell \in \{1, 2\}$  and  $\{t_0, \dots, t_{n-1}\} \subseteq Z_{\zeta(1)}$  this holds as  $f_{\zeta(\ell)}^*$  preserves this equation. So assume this fails for  $\ell = 1, 2$  so  $|\{t_0, \dots, t_{n-1}\} \cap Z_{\zeta(\ell)}| \leq 1$  hence  $2 \geq |\{t_0, \dots, t_{n-1}\} \cap (Z_{\zeta(1)} \cup Z_{\zeta(2)})|$  so  $\{\ell: t_\ell \notin Z_{\zeta(1)} \cup Z_{\zeta(2)}\}$  necessarily includes members of  $u$ , hence the equation holds.  $\square$

### Comment 11.13.

- (1) If in addition we have  $\kappa$ -complete maximal ideals  $I_{n,\ell}$  on  $\lambda_{n,\ell}$  extending  $J_{\lambda_{n,\ell}}^{\text{bd}}$  and  $\langle \lambda_{n,\ell}: \ell < n \rangle$  as above for  $\bar{\eta}$  a  $(\lambda, J)$ -sequence, e.g., for  $\langle I_n^*: n < \omega \rangle$  where  $I_n^* = \prod J_{\langle \lambda_{n,\ell}: \ell < n \rangle}$ , we are in a powerful situation as it can be applied to  $n$ -tuples rather than each one separately. But above we prepare the proof for not using it by having strong equations.
- (2) We can waive the “locally finite” demand proving as in the proof of (\*\*\*) in the proof of 11.4.

## 12. Constructing ccc Boolean algebras with no large independent sets

On such constructions see Roslanowski and Shelah [15, §3].

**Construction’s Hypothesis 12.1.** We assume

- (a)  $\bar{\eta}$  is a normal  $(\lambda, J)$ -sequence for  $\langle I_i: i < \delta \rangle$ ,
- (b)  $(\mathbf{B}_i, \bar{x}_i^-, \bar{x}_i^+)$  is a witness for  $(I_i, \mathcal{T}_i)$ ,  $\|\mathbf{B}_i\| = |\text{Dom}(I_i)|$ ,
- (c)  $\lambda = \text{cf}(\lambda)$ ,  $\sum_{i < \delta} |\text{Dom}(I_i)| < \lambda$ .

**Remark 12.2.** Actually  $\mathcal{T}_i$  do not influence the construction, only the properties of the Boolean algebra constructed. Similarly, the normality and the fact that  $\|\mathbf{B}_i\| = |\text{Dom}(I_i)|$ , as well as clause (c).

We define a Boolean algebra  $\mathbf{B}$  and  $y_\alpha \in \mathbf{B}$  ( $\alpha < \lambda$ ) as follows:

### Construction 12.3.

*Case 1.*  $\delta = \omega$ . Let  $\mathbf{B}_*$  be the free product of  $\{\mathbf{B}_i: i < \delta\}$  (so  $\mathbf{B}_n = *_{i < n} \mathbf{B}_i$ ,  $\mathbf{B}^n \subseteq \mathbf{B}^{n+1} \subseteq \mathbf{B}_*$ , so  $\mathbf{B}_* = \langle \bigcup_{n < \omega} \mathbf{B}_n \rangle_{\mathbf{B}_*}$ ).

Let  $\mathbf{B}_*^c$  be the completion of  $\mathbf{B}_*$ .

For each  $i < \delta$  and  $\eta \in \{\eta_\alpha \upharpoonright i: \alpha < \lambda\} \subseteq \prod_{j < i} \text{Dom}(I_j)$  we define  $y_\eta^- < y_\eta^+$  in  $\mathbf{B}^i$ . This is done by induction on  $i$ .

$$i = 0. y_\eta^- = 0, y_\eta^+ = 1.$$

$$i = j + 1. y_{\eta}^{-} = y_{\eta \uparrow j}^{-} \cup (y_{\eta \uparrow j}^{+} \cap x_{i, \eta(j)}^{-}), y_{\eta}^{+} = y_{\eta \uparrow j}^{-} \cup (y_{\eta \uparrow j}^{+} \cap x_{i, \eta(j)}^{+}).$$

So easily

$$j < i \Rightarrow y_{\eta \uparrow j}^{-} \leq y_{\eta \uparrow i}^{-} < y_{\eta \uparrow i}^{+} \leq y_{\eta \uparrow j}^{+}.$$

Now let  $y_{\alpha}$  be  $\text{lub}\{y_{\eta \uparrow i}^{-} : i < \delta\}$ . (Note: If  $\mathbf{B}_i \models "0 < x_{i,t}^{-} < x_{i,t}^{+} < 1"$  for  $t \in \text{Dom}(I_i)$ , then also  $y_{\alpha} = \text{maximal lower bound of } \{y_{\eta \uparrow i}^{+} : i < \delta\}$ . This will not be used.)

[Otherwise, the difference contains some member of  $\mathbf{B}_*$ , hence of some  $\mathbf{B}^i$  ( $i < \delta$ ), but there is none.]

Lastly  $\mathbf{B} = \mathbf{B}_{\bar{\eta}, \bar{I}, \langle (\mathbf{B}_i, \bar{x}_i^-, \bar{x}_i^+) : i < \delta \rangle}$  is the subalgebra of  $\mathbf{B}_*^c$  generated by  $\mathbf{B}_* \cup \{y_{\alpha} : \alpha < \lambda\}$  (by the finitary operations, so it is not complete).

Case 2.  $\delta > \omega$ . We find by induction on  $i < \delta$ ,  $\mathbf{B}^i$ ,  $\{(y_{\eta}^{-}, y_{\eta}^{+}) : \eta \in \{\eta_{\alpha} \uparrow i : \alpha < \lambda\}\}$  such that

- (i)  $\mathbf{B}^i$  increasing (by  $\subseteq$ , even  $\triangleleft$ ).
- (ii)  $\mathbf{B}^i \models y_{\eta}^{-} < y_{\eta}^{+}$  (when  $\bigvee_{\alpha} \eta = \eta_{\alpha} \uparrow i$ )

$$j < i \Rightarrow \mathbf{B}^i \models y_{\eta \uparrow j}^{-} \leq y_{\eta}^{-} \leq y_{\eta}^{+} \leq y_{\eta \uparrow j}^{+}.$$

(iii)  $\mathbf{B}^0$  is the trivial Boolean algebra.

(iv) If  $i = j + 1$  then  $\mathbf{B}^i = \mathbf{B}^j * \mathbf{B}_j$  (free product) and for  $\eta \in \{\eta_{\alpha} \uparrow i : \alpha < \lambda\}$

$$y_{\eta}^{-} = y_{\eta \uparrow j}^{-} \cup (y_{\eta \uparrow j}^{+} \cap x_{j, \eta(j)}^{-}),$$

$$y_{\eta}^{+} = y_{\eta \uparrow j}^{-} \cup (y_{\eta \uparrow j}^{+} \cap x_{j, \eta(j)}^{+}).$$

(v) For  $i$  limit,  $\mathbf{B}^i$  is generated freely by

$$\bigcup_{j < i} \mathbf{B}^j \cup \{y_{\eta}^{-}, y_{\eta}^{+} : \eta \in \{\eta_{\alpha} \uparrow i : \alpha < \lambda\}\}$$

except: the equations in  $\mathbf{B}$  and

$$y_{\eta \uparrow j}^{-} \leq y_{\eta}^{-} \leq y_{\eta}^{+} \leq y_{\eta \uparrow j}^{+} \quad \text{for } j < i, \eta \text{ as above.}$$

Lastly,  $\mathbf{B} \subseteq \text{completion}(\bigcup_{i < \delta} \mathbf{B}^i)$  is defined as in case 1 using  $y_{\alpha} \stackrel{\text{def}}{=} y_{\eta_{\alpha}}^{-}$ .

**Construction 12.3A.** A variant

$$\bar{x}_i^{\pm} = \langle x_{i, \eta}^{\pm} : \eta \in \{\eta_{\alpha} \uparrow (i+1) : \alpha < \lambda\} \rangle$$

so we use  $x_{i, \eta_{\alpha} \uparrow (i+1)}$  instead of  $x_{i, \eta_{\alpha}(i)}$ .

**Construction 12.3B.** A variant. It is like 12.3A but we are given  $(\mathbf{B}_i^{\pm}, \langle x_{i, \alpha} : \alpha < \lambda_i \rangle)$  and we define by induction on  $i$ ,  $\mathbf{B}^i$  (increasing with  $i$ ), and follows:

Case 1:  $i = 0$ :  $\mathbf{B}^i$  is the trivial Boolean algebra,  $y_{\eta_{\alpha} \uparrow i}^{-} = 0$ ,  $y_{\eta_{\alpha} \uparrow i}^{+} = 1$ .

Case 2:  $i = j + 1$ :  $\mathbf{B}^i$  is generated by  $\mathbf{B}^j \cup \{x_{\eta_{\alpha} \uparrow i}^{-}, x_{\eta_{\alpha} \uparrow i}^{+} : \alpha < \lambda\}$  freely except the equations in  $\mathbf{B}^j$  and

$$\tau(\dots, x_{\eta_{\alpha \ell} \uparrow i}^{-}, x_{\eta_{\alpha \ell} \uparrow i}^{+}, \dots)_{\ell < n} = 0$$

whenever  $\mathbf{B}_i \models \tau(\dots, x_{\eta_{\alpha_\ell}(j)}^-, x_{\eta_{\alpha_\ell}(j)}^+, \dots)_{\ell < n} = 0$ ; lastly defines

$$y_{\eta_\alpha \uparrow i}^- = y_{\eta_\alpha \uparrow j}^- \cup (y_{\eta_\alpha \uparrow j}^+ \cap x_{j, \eta_\alpha(j)}^-),$$

$$y_{\eta_\alpha \uparrow i}^+ = y_{\eta_\alpha \uparrow j}^+ \cup (y_{\eta_\alpha \uparrow j}^- \cap x_{j, \eta_\alpha(j)}^-).$$

Case 3:  $i$  limit.  $\mathbf{B}^i$  is generated by  $\bigcup_{j < i} \mathbf{B}^j \cup \{y_{\eta_\alpha \uparrow i}^-, y_{\eta_\alpha \uparrow i}^+ : \alpha < \lambda\}$  freely except the equations in  $\mathbf{B}^j$  for  $j < i$  and  $y_{\eta_\alpha \uparrow j}^- \leq y_{\eta_\alpha \uparrow i}^- \leq y_{\eta_\alpha \uparrow i}^+ \leq y_{\eta_\alpha \uparrow j}^+$  for  $\alpha < \lambda$ .

**Comment 12.4.** Clearly 12.3A includes 12.3B as a special case, but mostly there is no real difference in the uses. The reader may concentration on 12.3B.

**Discussion 12.5.** Usually the conclusions are of the form: among any  $\lambda$  elements of  $\mathbf{B}$ , something occurs. The first need is  $\|\mathbf{B}\| = \lambda$ , a trivial thing.

**Fact 12.6.**  $(*)_3 \Rightarrow (*)_2 \Rightarrow (*)_1$ , where

$$(*)_1 \quad \|\mathbf{B}\| = \lambda.$$

$$(*)_2 \quad \text{For every } \alpha < \beta < \lambda$$

$$\{i: \mathbf{B}_i \models \neg(\exists y)(x_{\eta_\alpha(i)}^- \leq y \leq x_{\eta_\alpha(i)}^+ \wedge x_{\eta_\beta(i)}^- \leq y \leq x_{\eta_\beta(i)}^+)\} \neq \emptyset,$$

i.e.,

$$\{i: \mathbf{B}_i \models x_{\eta_\alpha(i)}^- \not\leq x_{\eta_\beta(i)}^+ \vee x_{\eta_\beta(i)}^- \not\leq x_{\eta_\alpha(i)}^+\} \neq \emptyset.$$

$$(*)_3 \quad \text{If } t \neq s \text{ are in } \text{Dom}(I_i) \text{ for some } i < \delta, \text{ then}$$

$$\mathbf{B}_i \models x_t^- \not\leq x_s^+ \wedge x_s^- \not\leq x_t^+.$$

**Proof.** Easy.  $\square$

**Remark 12.7.** If not said otherwise, all examples satisfy  $(*)_3$ .

We will also be interested in stronger properties. In Section 15 we will be interested in the case  $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$  the pairs  $(x_\eta^-, x_\eta^+)$ ,  $(x_\nu^-, x_\nu^+)$  were independent.

**Claim 12.8.** Assume

$$(*) \quad a_\alpha \in \mathbf{B} \quad \text{for } \alpha < \lambda.$$

Then we can find in  $\mathbf{B}$  a sequence  $\langle b_\ell : \ell \leq m \rangle$  a  $\mathbf{B}$ -partition of 1 (i.e., a sequence of disjoint nonzero elements with union 1),  $m \geq 0$ , and  $X \in [\lambda]^\lambda$  and  $c \leq b_0$  in  $\mathbf{B}$  and  $n$ , and Boolean terms  $\tau_\ell$  for  $\ell = 1, \dots, m$  with  $n$  variables and ordinals  $\gamma_{\alpha, k} \in X$  for  $\alpha \in X$ ,  $k < n$  and  $\gamma_k$  for  $k \in [n, n^*)$ , where  $n^* \geq n$  and  $i^* < \delta$ ,  $v_k$  for  $k < n^*$  such that

- (i)  $n = 0$  iff  $m = 0$  iff  $\langle a_\alpha : \alpha \in X \rangle$  constant,
- (ii)  $\gamma_{\alpha, 0} < \gamma_{\alpha, 1} < \dots < \gamma_{\alpha, n-1}$  and  $\gamma_n < \gamma_{n+1} < \dots < \gamma_{n^*-1} < \gamma_{\alpha, 0}$ ,
- (iii) if  $\alpha < \beta$  are in  $X$  then  $\gamma_{\alpha, n-1} < \gamma_{\beta, 0}$ ,
- (iv) if  $\alpha \in X$  then  $a_\alpha \leq \bigcup_{\ell \leq m} b_\ell$ ,  $a_\alpha \cap b_0 = c$  and  $[\ell \in [1, m] \Rightarrow a_\alpha \cap b_\ell = \tau_\ell(y_{\gamma_{\alpha, 0}}, y_{\gamma_{\alpha, 1}}, \dots, y_{\gamma_{\alpha, n-1}})]$ , and  $[\ell \in [1, m] \Rightarrow 0 < a_\alpha \cap b_\ell < b_\ell]$  (so  $\tau_\ell$  nontrivial),

- (v)  $\eta_{\gamma_{\alpha,k}} \upharpoonright i^* = \nu_k$  for  $k < n$ ,
- (vi)  $\{b_\ell: \ell \leq m\} \subseteq (\mathbf{B}_i \cup \{y_{\gamma_k}: k \in [n, n^*]\})$  and  $\eta_{\gamma_k} \upharpoonright i^* = \nu_k$  for  $k \in [n, n^*]$ ,
- (vii)  $\langle \nu_k: k < n^* \rangle$  is with no repetition.

**Proof.** By the  $\Delta$ -system lemma and Boolean algebra manipulation.  $\square$

**Claim 12.9.** A sufficient condition to

$\otimes_0$   $\mathbf{B}$  has no independent subset of cardinality  $\lambda$

is

$\otimes_1$  if  $a_\alpha, X, n, m, \tau_\ell, \gamma_{\alpha,k}, b_\ell$  ( $\alpha \in X, k < n, \ell < n$ ) are as above in 12.8, and  $c_0 = 0, m = 1$ , then  $\{a_\alpha: \alpha \in X\}$  is not independent,

which follows from:

$\otimes_2$  if  $a_\ell, X, n, m, \tau_\ell, \gamma_{\alpha,k}$  ( $\alpha \in X, k < n$ ) are as above in 12.8,  $c_0 = 0, m = 1, i = i^*$ , then

$$\left\{ i: \begin{array}{l} \text{for every } A, \mathbf{B}' \text{ and } y_t, \text{ if } A \in ((I_i)^n)^+, \mathbf{B}_i \subseteq \mathbf{B}', \\ \mathbf{B}' \models x_t^- \leq y_t \leq x_t^+ \text{ for } t \in \text{Dom}(I_i), \\ \text{then } \langle \tau_1(y_{t_0}, \dots, y_{t_{n-1}}): \langle t_0, \dots, t_{n-1} \rangle \in A \rangle \\ \text{is not strongly independent} \end{array} \right\} \in J^+.$$

**Remark 12.10.** If we ask more on  $\bar{\eta}$ , we can weaken  $\otimes_2$ , like:

if  $n < \omega, \langle \gamma_{\alpha,k}: k < n \rangle$  increasing  $\alpha < \beta \Rightarrow \gamma_{\alpha,n-1} < \gamma_{\alpha,0}$ , then letting  $\eta'_\alpha = \langle \langle \eta_{\gamma_{\alpha,k}}(i): k < n \rangle: i < \delta \rangle$ , gives that  $\bar{\eta}' = \langle \eta'_\alpha: \alpha < \lambda \rangle$  is a  $(\lambda, J)$ -sequence for  $\langle (I_i)^n: i < \delta \rangle$  as well as some weaker versions.

**Proof of 12.9.**  $\otimes_1 \Rightarrow \otimes_0$ . We choose by induction on  $\ell \leq m$  a sequence  $\langle (\tau_\ell, \gamma_{\alpha,0}^\ell, \dots, \gamma_{\alpha, m(\ell)-1}^\ell): \alpha < \lambda \rangle$  such that

- (i)  $\tau_\ell = \tau_\ell(x_i, \dots, x_{m(\ell)-1})$  is a Boolean term, nontrivial,
- (ii)  $\gamma_{\alpha,0}^\ell < \gamma_{\alpha,1}^\ell < \dots < \gamma_{\alpha, m(\ell)-1}^\ell < \lambda$ ,
- (iii)  $\alpha < \beta < \lambda \Rightarrow \gamma_{\alpha, m(\ell)-1}^\ell < \gamma_{\beta,0}^\ell$  when they are well defined,
- (iv)  $\tau_\ell(a_{\gamma_{\alpha,0}^\ell}, \dots, a_{\gamma_{\alpha, m(\ell)-1}^\ell}) \cap \bigcup_{\ell_1 \leq \ell} b_{\ell_1} = 0$ .

For  $\ell = 0$ : Let  $\tau_\ell(x_0, x_1) = x_0 - x_1$ , so  $m(\ell) = 2$

$$\gamma_{\alpha,0}^0 = 2\alpha, \quad \gamma_{\alpha,1}^0 = 2\alpha + 1.$$

For  $\ell + 1$ : For each  $\alpha(*) < \lambda$ , apply  $\otimes_1$  with  $1 - b_{\ell+1}, b_{\ell+1}, \langle a_{\alpha(*)+\alpha}^\ell: \alpha < \lambda \rangle$ , where  $a_\alpha^\ell \stackrel{\text{def}}{=} \tau_\alpha^\ell(a_{\gamma_{\alpha,0}^\ell}, \dots, a_{\gamma_{\alpha, m(\ell)-1}^\ell})$  here standing for  $b_0, b_1, \langle a_\alpha: \alpha < \lambda \rangle$  there, and get a Boolean term  $\tau_{\alpha(*)}^{\ell+1}(x_0, \dots, x_{m(\ell+1, \alpha(*))-1})$ , and ordinals  $\beta_{\alpha(*)}^{\ell+1,0} < \dots < \beta_{\alpha(*)}^{\ell+1, m(\ell+1, \alpha(*))-1}$ , all in the interval  $[\alpha(*), \lambda)$ , such that

$$\tau_{\alpha(*)}^{\ell+1}(a_{\beta_{\alpha(*)}^{\ell+1,0}}^\ell, a_{\beta_{\alpha(*)}^{\ell+1,1}}^\ell, \dots, a_{\beta_{\alpha(*)}^{\ell+1, m(\ell+1, \alpha(*))-1}}^\ell) = 0.$$

Let  $X \in [\lambda]^\lambda$  be such that

- (a)  $\alpha \in X \Rightarrow \tau_\alpha^{\ell+1} = \tau_\ell^*, m(\ell, \alpha) = m(\ell, *)$ ,

(b)  $X$  is thin enough, i.e., if  $\alpha < \beta$  are in  $X$  then  $\beta_{\alpha,0}^\ell, \dots, \beta_{\alpha,m(\ell,*)}^\ell < \beta$ .

Now if  $\varepsilon$  is the  $\zeta$ th element of  $X$  we let

$$u_\zeta^{\ell+1} = \{\gamma_{\beta,m}^\ell: m < m(\ell) \text{ and } \beta \in \{\beta_{\varepsilon,0}^\ell, \dots, \beta_{\varepsilon,m(\ell,*)-1}^0\}\}.$$

So  $|u_\zeta^{\ell+1}| = m(\ell) \times m(\ell, *)$  let  $m(\ell+1) = m(\ell) \times m(\ell, *)$  let  $\gamma_{\zeta,0}^{\ell+1} < \gamma_{\zeta,1}^{\ell+1} < \dots < \zeta_{\zeta,m(\ell+1)-1}^{\ell+1}$  list  $u_\zeta^{\ell+1}$ , and it should be clear what is  $\tau_{\ell+1}$ . For  $\ell = m$  we have finished.

$\otimes_2 \Rightarrow \otimes_1$  Straight.  $\square$

**Fact 12.11.**

(1) In 12.8 we can add (so in  $\otimes_2$  of 12.9 we can assume) that

(viii)  $\tau_\ell(x_0, \dots, x_{n-1}) \in \{x_k, -x_k: k < n\}$

if

(\*) for a set of  $i < \delta$  from  $J^+$  we have  $\langle x_{i,t}^+ - x_{i,t}^-: t \in \text{Dom}(I_i) \rangle$  is a sequence of pairwise disjoint (nonzero) elements of  $\mathbf{B}_i$ .

(2) Assume

(\*)<sup>+</sup> for every  $i < \delta$  we have  $\langle x_{i,t}^+ - x_{i,t}^-: t \in \text{Dom}(I_i) \rangle$  is a sequence of pairwise disjoint (non zero) elements of  $\mathbf{B}_i$ .

Then

(a) In 12.8 above we can add:

$$b_0, \dots, b_m = \bigcup_{i < \delta} B^i.$$

(b) Under 12.3B we can add: for  $k \in [1, m)$ , if  $i$  is large enough, if  $\alpha_0, \dots, \alpha_{n-1} \in X$  letting  $b_k^\ell$  be the projection of  $b_\alpha$  in  $\mathbf{B}^{i+1}$  (i.e., any element  $b$ ) satisfying

$$(\forall x \in \mathbf{B}^{i+1})(x \leq b_k \rightarrow b \leq b_k, x \geq b_k \rightarrow b \geq b_k)$$

(there is a minimal and maximal such  $b_k^i$  and they are in  $\langle \mathbf{B}^i \cup \{\rho: \rho = f_\alpha \upharpoonright (x+1) \text{ for some } i, \neg(v \triangleleft \rho) \} \rangle$ ,  $f_{\alpha_\ell} \upharpoonright i = f_{\alpha_0} \upharpoonright i$ ,  $\langle f_{\alpha_\ell}(i): \ell \leq s \rangle$  is with no repetitions and  $\tau(x_0, \dots, x_{s-1})$  is a Boolean term then

$$\mathbf{B} \upharpoonright b_k \models \tau(b_k \cap y_{\alpha_0}, \dots, b_k \cap y_{\alpha_{s-1}}) = 0 \Rightarrow$$

$$\mathbf{B}^{i+1} \models \tau(b_k^i \cap y_{\alpha_0}, \dots, b_k^i \cap y_{\alpha_{s-1}}) = 0$$

(we can even be more explicit).

**Proof.** Straightforward.  $\square$

We can now phrase sufficient conditions for having free caliber  $\lambda$  (for  $\mathcal{T}$ ) and for having no  $\mathcal{T}$ -free subset of  $\mathbf{B}$  of cardinality  $\lambda$ .

**Claim 12.12.** Sufficient conditions for “ $\mathbf{B}$  satisfies the  $\kappa$ -c.c.” are ( $\kappa$  is regular uncountable and):

(\*)<sub>1</sub>  $\delta = \omega$  and each  $\mathbf{B}_i$  satisfies the  $\kappa$ -Knaster condition,

(\*)<sub>2</sub> each  $\mathbf{B}_i$  satisfies the  $\kappa$ -Knaster condition and  $(\forall \alpha < \kappa) (|\alpha|^{|\delta|} < \kappa)$ ,



(\*)<sub>3</sub> each  $\mathbf{B}_i$  satisfies the  $\kappa$ -Knaster condition,  $\kappa > \delta$  and for every  $A \in [\lambda]^\kappa$ , and limit ordinal  $\delta' \leq \delta$  for some  $B \in [A]^\kappa$  and  $i < \delta$  we have

$$\alpha \in B, \beta \in B, \eta_\alpha \upharpoonright \delta' \neq \eta_\beta \upharpoonright \delta' \Rightarrow \text{lg}(\eta_\alpha \cap \eta_\beta) = i$$

(follows from “ $\bar{\eta}$  is  $\kappa^+$ -free”, see 1.17 and Definition 1.18).

**Claim 12.13.** Assume

- (A)  $\bar{\eta}$  is a normal  $(\lambda, J)$ -sequence for  $\langle I_i : i < \delta \rangle$ ,
- (B)  $(\mathbf{B}_i, \bar{x}_i^-, \bar{x}_i^+)$  is a witness for  $(I_i, \{x_0 \cap x_1 \cap x_2 = 0\})$ ,
- (C)  $\mathbf{B}$  is as constructed in 12.1, 12.3.

Then

- ( $\alpha$ )  $\lambda$  is not a free caliber of  $\mathbf{B}$ ,
- ( $\beta$ )  $\mathbf{B}$  has cardinality  $\lambda$  and satisfies the  $\kappa$ -c.c. if  $\kappa$  is as in 12.12.

**Proof.** Straightforward.  $\square$

**Conclusion 12.14.** Assume for simplicity that  $V \models \text{GCH}$ ,  $\theta = \theta^{<\theta} < \chi = \chi^{<\chi}$  and  $P$  is the forcing notion of adding  $\chi$   $\theta$ -Cohen subsets of  $\theta$ , i.e.,

$$P = \{f: f \text{ is a partial function from } \chi \text{ to } \{0, 1\} \\ \text{with domain of cardinality } < \theta\}.$$

Then (cardinal arithmetic on  $V^P$  is well known) and

- (\*) if  $\text{cf}(\mu) < \theta < \mu < \chi$  then there is a  $(2^{\text{cf}(\mu)})^+$ -c.c. Boolean algebra  $\mathbf{B}$  of cardinality  $\lambda = \mu^+$  such that  $\lambda$  is not a free caliber of  $\mathbf{B}$  (and even satisfying the  $\kappa$ -c.c. if  $\kappa$  is as in 12.12).

**Proof.** Use 12.13 and §11.  $\square$

The problem of “ $\mathbf{B}$  with no independent subset of cardinality  $\lambda$ ” is somewhat harder.

**Claim 12.15.** Assume

- (A)  $\bar{\eta}$  is a normal  $(\lambda, J)$ -sequence for  $\langle I_i : i < \delta \rangle$ .
- (B)  $(\mathbf{B}_i, \bar{x}_i^-, \bar{x}_i^+)$  is a witness for  $(I_i, \mathcal{T}_{n_i, m_i})$  (on  $\mathcal{T}_{n_i, m_i}$  see 11.12 clause (D)).
- (C)  $3 \leq m_i < n_i/2$ .
- (D) For every  $k < \omega$ ,  $\{i: km_i < n_i\} \in J^+$ .
- (E)  $\mathbf{B}$  is as in construction 12.1, 12.3.

Then

- (i)  $\mathbf{B}$  does not have a free subset of cardinality  $\lambda$ .
- (ii)  $\mathbf{B}$  has cardinality  $\lambda$  and satisfies the  $\kappa$ -c.c. if  $\kappa$  is as in 12.12.

**Proof.** Straightforward (using the criterion in 12.9).  $\square$

**Conclusion 12.16.** Assume for simplicity  $V \models \text{GCH}$ , and  $\theta = \theta^{<\theta} < \chi = \chi^{<\chi}$  is the forcing notion of adding  $\chi$   $\theta$ -Cohen reals. Then cardinal arithmetic in  $V^P$  is well known and

- (\*) if  $\text{cf}(\mu) < \theta < \mu < \chi$  then there is a  $(2^{\text{cf}(\mu)})^+$ -c.c. Boolean algebra  $\mathbf{B}$  of cardinality  $\lambda = \mu^+$  without an independent subset of cardinality  $\lambda$ .
- (\*\*) we can demand that  $\mathbf{B}$  satisfies the  $(\text{cf}(\mu)^+)$ -c.c. if  $\text{cf}(\mu) = \aleph_0$  or

$$V \models \{ \delta < \mu^+ : \text{cf}(\delta) = \text{cf}(\mu) \} \in I[\lambda].$$

**Proof.** By 12.15, where  $(\mathbf{B}_i, \bar{x}_i^-, \bar{x}_i^+)$  is provided by 11.12 (and  $W$  for it by 11.11). For (\*\*) see 1.17(2).  $\square$

We would also like sufficient condition for inequalities, for simplicity  $n = 2$ .

**Claim 12.17.**

- (1) Assume 12.1, 12.3 and (\*) of 12.11 and  $n < \omega$  and  $\tau^0 = \tau^0(x_0, \dots, x_{n-1})$  a Boolean term and  $\tau^1 = \tau^0(-x_0, \dots, -x_{n-1})$ . Then  $(*)_1 \Rightarrow (*)_2$ , where
  - (\*)<sub>1</sub> if  $\ell < 2$  for a set of  $i < \delta$  from  $J^+$  we have: if  $X \in I_i^+$  then for some  $t_0, \dots, t_{n-1} \in X$ , pairwise distinct, we have

$$\mathbf{B}_i \models \tau^\ell(x_{i,t_0}, \dots, x_{i,t_{n-1}}) = 0,$$

- (\*)<sub>2</sub> if  $a_\alpha \in \mathbf{B}$  for  $\alpha < \lambda$  then for some  $k < \omega$  and  $\alpha_{\ell,m} < \lambda$  for  $\ell < k$ ,  $m < n$  we have  $\alpha_{\ell,0} < \alpha_{\ell,1} < \dots < \alpha_{\ell,m-1} < \alpha_{\ell+1,0}$  (for  $\ell < k$ ) and for some  $i(\ell) \in \{0, 1\}$  for  $\ell < k$  we have

$$\mathbf{B}_i \models \bigcap_{\ell < k} \tau^{i(\ell)}(a_{\ell,0}, \dots, a_{\ell,m-1}) = 0.$$

- (2) Assume 12.1, 12.3 (using 12.3B) and (\*) of 12.8 and for simplicity  $I_i = J_{\lambda_i}^{\text{bd}}$  and assume further  $n < \omega$ ,  $\mathbf{t}$  a function from  $\{0, \dots, n-1\}$  to  $\{+1, -1\}$  and  $\tau^0 = \tau^0(x_0, \dots, x_{n-1})$  a Boolean term, increasing in  $x_\ell$  if  $\mathbf{t}(\ell) = +1$ , decreasing with  $x_\ell$  if  $\mathbf{t}(\ell) = -1$ . Let  $\tau^1(x_0, \dots, x_{n-1}) = \tau^0(-x_0, \dots, -x_{n-1})$ . Assume also  $\tau_0(-x_0, \dots, -x_{n-1}) = 0$  if  $x_\ell \in \{0, 1\}$  and  $\bigwedge_\ell (x_\ell = 1 \equiv \mathbf{t}(\ell) = 1)$  or  $\bigwedge_\ell x_\ell = 1 \equiv \mathbf{t}(\ell) = -1$ . Then  $(*)_3 \Rightarrow (*)_4$ , where

- (\*)<sub>3</sub> for a set of  $i < \delta$  which belongs to  $J^+$  the following holds: if  $\gamma_{\alpha,\ell} < \lambda_i$  and

$$\alpha < \beta < \lambda_i \Rightarrow \max_{\ell < n} \gamma_{\alpha,\ell} < \min_{\ell < n} \gamma_{\alpha,\ell}$$

then for some  $\alpha(0) < \dots < \alpha(n-1)$  we have, for every  $\ell < n$ :

$$\tau^0(x_{\gamma_{\alpha(0),\ell}}^{\mathbf{t}(\ell)}, x_{\gamma_{\alpha(1),\ell}}^{\mathbf{t}(\ell)}, \dots, x_{\gamma_{\alpha(n-1),\ell}}^{\mathbf{t}(\ell)}) = 0,$$

$$\tau^1(x_{\gamma_{\alpha(0),\ell}}^{-\mathbf{t}(\ell)}, x_{\gamma_{\alpha(1),\ell}}^{-\mathbf{t}(\ell)}, \dots, x_{\gamma_{\alpha(n-1),\ell}}^{-\mathbf{t}(\ell)}) = 0,$$

- (\*)<sub>4</sub> if  $a_\alpha \in \mathbf{B}$  for  $\alpha < \lambda$  then for some  $\alpha_0 < \dots < \alpha_{n-1}$  we have  $\tau^0(a_{\alpha_0}, \dots, a_{\alpha_{n-1}}) = 0$ .

**Proof.** Easy.  $\square$

**Comments 12.18.**

- (1) This concludes the proof of the consistency of the existence, answering a part of Monk's Problem 33.
- (2) We can get " $\mathbf{B} \models (\text{cf}(\mu))^+$ -c.c." when 12.12 provides one.
- (3) We may still like to get "no  $k$ -independent set" for some specific  $k$  as done in 12.17. Probably also 11.13 will help but we have not really looked into it.

Clearly it is supposed to have, for a  $J^+$ -set of  $i$ 's:

- (\*) <sub>$i$</sub>  for some function  $F$ , if  $m < \omega$ , and  $X \subseteq (\text{Dom} I_i)^m$  is  $F$ -large (i.e., if  $k < \omega$ ,  $\bar{i}^0, \dots, \bar{i}^{k-1} \in X$  and  $F(\bar{i}^0, \dots, \bar{i}^{k-1}) \in I$  then for some  $\bar{i} \in X$ ,  $\text{Rang } \bar{i} \cap F(\bar{i}^0, \dots, \bar{i}^{k-1}) = \emptyset$ ).

Then for some distinct  $\bar{i}^0, \dots, \bar{i}^{n-1} \in X$ , we have

$$\ell < m \Rightarrow \tau(t_\ell^0, t_\ell^1, \dots, t_\ell^{n-1}) = 0.$$

See more in 15.11, 15.12.

**13. The singular case**

We continue to deal with Problem 33 of Monk [12]. This time we concentrate on the case  $\lambda$  is singular. Though a priori this looked to be the side issue, we can get quite a coherent picture.

**Note.** If  $\kappa > \text{cf}(\lambda)$  there is such a Boolean algebras (the disjoint sum of  $\text{cf}(\lambda)$  Boolean algebras each of cardinality  $< \lambda$ ). Moreover

**Claim 13.1.** Assume

- (\*)  $\lambda > \text{cf}(\lambda) = \theta$  and  $(\forall \alpha < \lambda) (|\alpha|^{<\kappa} < \lambda)$  and  $\lambda > \kappa = \text{cf}(\kappa) > \aleph_0$ .

(1) The following conditions are equivalent:

(A) There are  $\mathbf{B}$  and  $a_\zeta$  such that

- (a)  $\mathbf{B}$  is a  $\kappa$ -c.c. Boolean algebra,
- (b)  $a_\zeta \in \mathbf{B} \setminus \{0\}$  for  $\zeta < \theta$ ,
- (c) if  $\langle w_\zeta : \zeta < \theta \rangle$  is a sequence of pairwise disjoint finite subsets of  $\theta$  then for some finite  $u \subseteq \theta$  we have

$$\bigcap_{\zeta \in u} \bigcup_{\xi \in w_\zeta} a_\xi = 0.$$

(B) There is a Boolean algebra  $\mathbf{B}$  of cardinality  $\lambda$  with no independent subset of cardinality  $\lambda$ .

(2) The following conditions are equivalent:

(A)' There are  $\mathbf{B}$ ,  $a_\zeta$  such that

- (a)  $\mathbf{B}$  is a  $\kappa$ -c.c. Boolean algebra,
- (b)  $a_\zeta \in \mathbf{B} \setminus \{0\}$  for  $\zeta < \theta$ ,
- (c) for any  $X \in [\theta]^\theta$  for some finite  $w \subseteq X$  we have  $\bigcap_{\zeta \in w} a_\zeta = 0$ .

(B)' There is a Boolean algebra  $\mathbf{B}$  of cardinality  $\geq \lambda$  which does not have  $\lambda$  as a free caliber.

**Proof.** (1) (A)  $\Rightarrow$  (B) The case  $\theta = \aleph_0$  is easier, so we leave it to the reader.

Without loss of generality  $\mathbf{B}$  has cardinality  $\theta$ . Let  $\lambda = \kappa + \theta + \sum_{\zeta < \theta} \lambda_\zeta$  where  $\lambda > \lambda_\zeta > \kappa + \theta + \sum_{\xi < \zeta} \lambda_\xi$ . Let  $\mathbf{B}^*$  be the Boolean algebra freely generated by  $\mathbf{B} \cup \{x_{\zeta,\alpha} : \zeta < \theta, \alpha < \lambda_\zeta^+\}$  except for the equations in  $\mathbf{B}$  and

$$x_{\zeta,\alpha} \leq a_\zeta \quad (\text{for } \zeta < \theta, \alpha < \lambda_\zeta^+).$$

Clearly  $\mathbf{B} \subseteq \mathbf{B}^*$  and assume that  $\{b_\gamma : \gamma < \lambda\} \subseteq \mathbf{B}^*$  is independent. Then for each  $\gamma$  there are  $n(\gamma) < \omega$  and Boolean terms  $\tau_\gamma$  and  $\zeta_{\gamma,\ell} < \theta, \alpha_{\gamma,\ell} < \lambda_{\zeta_{\gamma,\ell}}$  for  $\ell < n_\zeta$  and  $c_{\gamma,\ell} \in \mathbf{B}$  for  $\ell < m(\gamma)$  such that  $b_\gamma = \tau_\gamma(x_{\zeta_{\gamma,0},\alpha_{\gamma,0}}, \dots, x_{\zeta_{\gamma,n(\gamma)-1},\alpha_{\gamma,n(\gamma)-1}}, c_{\gamma,0}, \dots, c_{\gamma,m(\gamma)-1})$ . As  $\text{cf}(\lambda) = \theta > \aleph_0$ , without loss of generality  $\tau_\gamma = \tau, n(\gamma) = n(*)$  and  $m(\gamma) = m(*)$ . Also for each  $\varepsilon < \theta$  there is  $X_\varepsilon \in [\lambda_\varepsilon^+]^{\lambda_\varepsilon^+}$  such that

$$(*) \quad \gamma \in X_\varepsilon \text{ implies } \zeta_{\gamma,\ell} = \zeta_{\varepsilon,\ell} < \theta, c_{\gamma,\ell} = c_{\varepsilon,\ell}^* \in \mathbf{B}.$$

Without loss of generality,  $\langle \zeta_{\varepsilon,\ell} : \ell < n(*) \rangle$  is nondecreasing. We can find  $Y \in [\theta]^\theta$  such that  $\langle \langle \zeta_{\varepsilon,\ell} : \ell < n(*) \rangle : \varepsilon \in Y \rangle$  is a  $\Delta$ -system. In fact for some  $n'(*) \leq n(*)$  we have

$$(*)_1 \quad \varepsilon \in Y \ \& \ \ell < n'(*) \Rightarrow \zeta_{\varepsilon,\ell} = \zeta_{\ell}.$$

$$(*)_2 \quad \varepsilon_1 \in Y \ \& \ \varepsilon_2 \in Y \ \& \ \varepsilon_1 < \varepsilon_2 \Rightarrow \zeta_{\varepsilon_1,n(*)-1} < \zeta_{\varepsilon_2,n(*)}.$$

By renaming, without loss of generality  $X_\varepsilon = [\lambda_\varepsilon, \lambda_\varepsilon^+]$  for  $\varepsilon \in Y$ . Let

$$w_\varepsilon = \{ \zeta_{\varepsilon,\ell} : n'(*) \leq \ell < n(*) \},$$

so let  $u$  be as required in clause (A)(c), so  $u \subseteq \theta$  is finite.

Let for  $\varepsilon \in u, \gamma_{\varepsilon,1} < \gamma_{\varepsilon,2}$  be members of  $X_\varepsilon$ .

Clearly

$$b_{\gamma_{\varepsilon,1}} \Delta b_{\gamma_{\varepsilon,2}} \leq \bigcup_{\ell \in [n'(*), n(*)]} a_{\zeta_{\varepsilon,\ell}},$$

hence

$$\begin{aligned} \bigcap_{\varepsilon \in u} (b_{\gamma_{\varepsilon,1}} \Delta b_{\gamma_{\varepsilon,2}}) &\leq \bigcap_{\varepsilon \in u} \left( \bigcup_{\ell \in [n'(*), n(*)]} a_{\zeta_{\varepsilon,\ell}} \right) \\ &= \bigcup_{\varepsilon \in u} \bigcap_{\xi \in w_\varepsilon} a_\xi = \emptyset, \end{aligned}$$

so  $\langle b_\gamma : \gamma < \lambda \rangle$  is not independent.

$\neg(\text{A}) \Rightarrow \neg(\text{B})$  Like [22]. In short: Let  $\lambda = \sum_{\zeta < \theta} \lambda_\zeta, (\forall \alpha < \lambda) (|\alpha|^{<\kappa} < \lambda_\zeta), \lambda_\zeta = \text{cf}(\lambda_\zeta) > \kappa + \theta + \sum_{\xi < \zeta} \lambda_\xi$ . Let  $S_\zeta = \{\delta < \lambda_\zeta : \text{cf}(\delta) \geq \kappa\}$ . Remember that by [22]:

$\boxtimes_{\lambda_\zeta} [\mathbf{B}]$  Let  $\mathbf{B}$  be a  $\kappa$ -c.c. Boolean algebra. Then:

(\*) for any  $\bar{x} = \langle x_\alpha : \alpha < \lambda_\zeta \rangle$  pairwise distinct  $x_\alpha \in \mathbf{B}$ , there are  $a^- < a^+$  in  $\mathbf{B} \setminus \{0\}$ , such that: if  $\langle \mathbf{B}_\alpha : \alpha < \lambda_\zeta \rangle$  is an increasing continuous sequence

of subalgebras of  $\mathbf{B}$  of cardinality  $< \lambda_\zeta$  satisfying  $x_\alpha \in \mathbf{B}_{\alpha+1}$ ,  $\{a^-, a^+\} \subseteq \mathbf{B}_0$ , we have

$$\left\{ \begin{array}{l} a^- \leq x_\delta \leq a^+ \text{ and} \\ \delta \in S_\zeta: (\forall y)[0 < y \leq a^+ - a^- \ \& \ y \in \mathbf{B}_\delta \rightarrow \\ (x_\delta - a^-) \cap y \neq 0 \ \& \ (a^+ - x_\delta) \cap y \neq 0] \end{array} \right\}$$

is stationary.

So fix  $\bar{x} = \langle x_\gamma: \gamma < \lambda \rangle$ , sequence of distinct elements of  $\mathbf{B}$ , for each  $\zeta < \theta$  let  $a_\zeta^-, a_\zeta^+$  be as in (\*) (for  $\bar{x} \upharpoonright \lambda_\zeta$ ), and let  $a_\zeta = a_\zeta^+ - a_\zeta^- \in \mathbf{B}^+$ . Let  $\mathbf{B}_\alpha^\zeta$  be the subalgebra generated by  $\{x_\gamma: \gamma < \max\{\alpha, \bigcup_{\xi < \zeta} \lambda_\xi\}\} \cup \{a_\xi: \xi < \theta\}$  for  $\alpha < \lambda_\zeta$  and for each  $\zeta < \theta$  let  $S_\zeta$  be as above.

As  $\neg(A)$ , necessarily there is a sequence of pairwise disjoint finite subsets of  $\theta$ , say  $\bar{u} = \langle u_\varepsilon: \varepsilon < \theta \rangle$  with any finite intersection of members  $\langle \bigcup_{\xi \in u_\varepsilon} a_\xi: \varepsilon < \theta \rangle$  is not zero.

Now we can manipulate, choosing by induction on  $\varepsilon < \theta$ ,  $\bar{t}^{\varepsilon, \alpha} \in \prod_{\zeta \in u_\varepsilon} S_\zeta$  and defining

$$a_{\varepsilon, \alpha}^* = \bigcup_{\zeta \in u_\varepsilon} \left( \left( a_\zeta - \bigcup_{\xi \in u_\varepsilon \setminus (\zeta+1)} a_\xi \right) \cap x_{t_\zeta^{\varepsilon, \alpha}} \right).$$

(2) Similarly.  $\square$

### Discussion 13.2.

- (1) Note: if  $\theta < \kappa$ , clearly  $(A)_\theta$  &  $(A)'_\theta$ .
- (2) Note if  $(\forall \alpha < \theta) (|\alpha|^{<\kappa} < \theta)$ , then  $\neg(A)_\theta$  &  $\neg(A)'_\theta$ .
- (3) Note that if  $\chi = \chi^{<\chi} < \chi(*) = \chi(*)^{<\chi(*)}$  then for some  $\chi^+$ -c.c. ( $< \chi$ )-complete forcing notion of cardinality  $\chi(*)$  in  $V^P$  we have  $\neg(A)_\theta$  &  $\neg(A)'_\theta$  when  $\theta = \text{cf}(\theta) \in (\chi, \chi(*))$ .
- (4) It is natural to get  $\text{CON}(\kappa < \chi = \chi^{<\chi} < \theta = \text{cf}(\theta) < 2^\chi + (A)_\theta$  &  $\neg(A)'_\theta$ ). This is well connected to our problems but we have not looked at it.

**Claim 13.3.** In 11.3 the condition

$$(*) \ (\forall \alpha < \lambda) (|\alpha|^{<\kappa} < \lambda)$$

can be replaced by the weaker one

$$(*)^- \text{ for arbitrarily large regular } \lambda' < \lambda \text{ we have } \boxtimes_{\lambda'}[\mathbf{B}] \text{ for any } \kappa\text{-c.c. Boolean algebra (see 13.1's proof).}$$

## 14. Getting free caliber for regular cardinals

Remember that  $\lambda$  is a free caliber of a Boolean algebra  $\mathbf{B}$  if for any  $X \in [\mathbf{B}]^\lambda$  there is an independent  $Y \in [X]^\lambda$ ; of course we can replace a Boolean algebra by a locally compact topological space (which is a slightly more general case, but the proof is not really affected).

Monk asks whether there is a  $\kappa$ -cc Boolean algebra  $\mathbf{B}$  of cardinality  $\geq \lambda$  with no independent subset of cardinality  $\lambda$ , and  $\mu$  such that

$$\mu < \lambda < \mu^\kappa, \quad (\forall \alpha < \mu)(|\alpha|^\kappa < \lambda).$$

Here we deal with the case of  $\lambda$  regular and give a sufficient set-theoretic condition on  $\kappa$  such that any  $\kappa$ -cc Boolean algebra of cardinality  $\geq \lambda$  has  $\lambda$  as a free caliber, so the consistency of a negative answer follows, but we do not directly force. So this section is complementary to Sections 12 and 11.

**Hypothesis 14.1.**

(a)  $\lambda = \text{cf}(\lambda) > 2^\kappa$ , but for simplicity we assume

$$\lambda = \mu^+, \quad \mu = \sum_{i < \text{cf}(\mu)} \lambda_i, \quad \lambda_i = \lambda_i^{<\kappa}, \quad \text{cf}(\mu) < \kappa.$$

We shall use it to shorten proofs when helpful, and, later, will show what can be done without it.

(b)  $\mathbf{B}^*$  is a  $\kappa$ -cc. Boolean algebra,  $a_\alpha \in \mathbf{B}$  for  $\alpha < \lambda$  are pairwise distinct.

Let  $\bar{a} \stackrel{\text{def}}{=} \langle a_\alpha : \alpha < \lambda \rangle$ . We would like to find  $X \in [\lambda]^\lambda$  such that  $\{a_\alpha : \alpha \in X\}$  is independent.

**Definition 14.2.** For  $\mathbf{B} \subseteq \mathbf{B}^*$ ,  $x \in \mathbf{B}^*$  let

$$\text{Proj}^0(x, \mathbf{B}) \stackrel{\text{def}}{=} \{y \in \mathbf{B} : y \leq x\},$$

$$\text{Proj}^1(x, \mathbf{B}) \stackrel{\text{def}}{=} \{y \in \mathbf{B} : y \cap x = 0\},$$

$$\text{Proj}^2(x, \mathbf{B}) \stackrel{\text{def}}{=} \{y \in \mathbf{B} : y = 0 \text{ or } (\forall z)(0 < z \leq y \ \& \ z \in \mathbf{B} \Rightarrow 0 < z \cap x < z)\}.$$

**Fact 14.3.** Let  $\mathbf{B} \subseteq \mathbf{B}^*$ ,  $x \in \mathbf{B}^*$

- (1) If  $y_\ell \in \text{Proj}^\ell(x, \mathbf{B})$  for  $\ell < 3$ , then  $\langle y_\ell : \ell < 3 \rangle$  are pairwise disjoint.
- (2)  $\bigcup_{\ell < 3} \text{Proj}^\ell(x, \mathbf{B})$  is dense in  $\mathbf{B}$ .
- (3)  $\text{Proj}^\ell(x, \mathbf{B})$  is an ideal on  $\mathbf{B}$ .
- (4)  $\text{Proj}^\ell(x, \mathbf{B})$  is complete inside  $\mathbf{B}^*$ , i.e., if in  $\mathbf{B}^*$  we have  $x$  is  $\leq$  lub of  $\{x_\alpha : \alpha < \alpha^*\}$  and  $\{x_\alpha : \alpha < \alpha^*\} \subseteq \text{Proj}^\ell(x, \mathbf{B})$  and  $x \in \mathbf{B}$  then  $x \in \text{Proj}^\ell(x, \mathbf{B})$ .

**Definition 14.4.**

$$\chi = \chi_{\bar{a}} = \text{Min}\{\|\mathbf{B}\| : \mathbf{B} \subseteq \mathbf{B}^*, |W_{\mathbf{B}}| = \lambda\},$$

where

$$W_{\mathbf{B}} = W_{\mathbf{B}, \bar{a}} = \{\alpha : \text{Proj}^2(a_\alpha, \langle \mathbf{B} \cup \{a_\beta : \beta < \alpha\} \rangle_{\mathbf{B}^*}) = \{0\}, \text{ and } \text{Proj}^\ell(a_\alpha, \mathbf{B}) \text{ is predense in } \text{Proj}^\ell(a_\alpha, \langle \mathbf{B} \cup \{a_\beta : \beta < \alpha\} \rangle_{\mathbf{B}^*}) \text{ for } \ell = 0, 1\}.$$

**Remark 14.5.**

- (1)  $\text{Proj}^2(a_\alpha, \mathbf{B}) = \{0\}$  is close to saying,  $a_\alpha =$  the lub in  $\mathbf{B}^*$  of  $\text{Proj}^0(a_\alpha, \mathbf{B})$ , but not the same (holds if  $\mathbf{B} \triangleleft \mathbf{B}^*$ ).  
Could have worked with a variant as indicated.

(2) Trivially  $\chi \leq \lambda$ , use  $\mathbf{B} = \langle a_\alpha : \alpha < \lambda \rangle_{\mathbf{B}^*}$ .

**Fact 14.6.** *If  $\chi = \lambda$ , then for some  $X \in [\lambda]^\lambda$ ,  $\langle a_\alpha : \alpha \in X \rangle$  is independent.*

**Proof.** Let  $\mathbf{B}_\alpha \stackrel{\text{def}}{=} \langle a_\beta : \beta < \alpha \rangle_{\mathbf{B}^*}$ , so  $\mathbf{B}_\alpha$  are increasing continuous in  $\alpha$ ,  $\|\mathbf{B}_\alpha\| \leq \aleph_0 + |\alpha| < \lambda$ . Let

$$S \stackrel{\text{def}}{=} \{\alpha < \lambda : \text{Proj}^2(a_\alpha, \mathbf{B}_\alpha) = \{0\}\},$$

$$S' \stackrel{\text{def}}{=} \{\alpha \in S : \text{cf}(\alpha) \geq \kappa\}.$$

Now

(\*)  $S'$  is not stationary.

[Why? For  $\delta \in S'$ ,  $\ell < 2$  let  $\mathcal{I}_{\delta, \ell} \subseteq \text{Proj}^\ell(a_\delta, \mathbf{B}_\delta) \setminus \{0\}$  be an antichain, maximal under the conditions defining  $\text{Proj}^\ell$ . So  $|\mathcal{I}_{\delta, \ell}| < \kappa$ , as  $\mathbf{B}^* \models \kappa\text{-cc}$ . Hence for some  $f(\delta) < \delta$  we have

$$\mathcal{I}_{\delta, 0} \cup \mathcal{I}_{\delta, 1} \subseteq \mathbf{B}_{f(\delta)}.$$

So if  $S'$  is stationary, by Fodor's lemma, for some  $\alpha^* < \lambda$ ,  $S^* = \{\delta \in S' : f(\delta) = \alpha^*\}$  is stationary.

We would like to show:

$$(**) \quad \delta \in S^* \Rightarrow \text{Proj}^2(a_\delta, \mathbf{B}_{\alpha^*}) = \{0\}.$$

If so, we have gotten that  $\mathbf{B}_{\alpha^*}$ ,  $S^*$  exemplify  $\chi \leq \|\mathbf{B}_{\alpha^*}\|$ , contradiction. For proving (\*\*), let  $\delta \in S^*$ , assume  $b \in \text{Proj}^2(a_\delta, \mathbf{B}_{\alpha^*}) \setminus \{0\}$ .

So, by 14.3, (for  $\mathbf{B}_{\alpha^*}$ ,  $a_\delta$ ) we have  $(\forall x \in \mathcal{I}_{\delta, 0} \cup \mathcal{I}_{\delta, 1}) x \cap b = 0$ .

Now,  $b \notin \text{Proj}^2(a_\delta, \mathbf{B}_\delta)$ , as the latter is  $\{0\}$ . So, there is  $c$  such that  $\mathbf{B}_\delta \models "0 < c \leq b$  and  $c \cap a_\delta = 0 \vee c \leq a_\delta"$ , that is  $c \in \text{Proj}^0(a_\delta, \mathbf{B}_\delta) \cup \text{Proj}^1(a_\delta, \mathbf{B}_\delta)$ , but as  $c \leq b$  we have

$$(\forall x \in \mathcal{I}_{\delta, 0} \cup \mathcal{I}_{\delta, 1})(x \cap c = 0).$$

So  $c$  contradicts the maximality of  $\mathcal{I}_{\delta, 0}$  (if  $c \in \text{Proj}^0(a_\delta, \mathbf{B}_\delta)$ ) or of  $\mathcal{I}_{\delta, 1}$  (if  $c \in \text{Proj}^1(a_\delta, \mathbf{B}_\delta)$ ).

The contradiction proves (\*\*) and (\*).]

So  $\lambda \setminus S$  is stationary. For  $\delta \in \lambda \setminus S$  choose  $b_\delta \in \text{Proj}^2(a_\delta, \mathbf{B}_\delta) \setminus \{0\}$ . So by Fodor's lemma, for some  $b^* \in \bigcup_{\alpha < \lambda} \mathbf{B}_\alpha$  we have

$$S^{**} \stackrel{\text{def}}{=} \{\delta : \delta \in \lambda \setminus S, b_\delta = b^*\} \text{ is stationary.}$$

Now we know that  $\langle a_\delta : \delta \in S^* \rangle$  is independent.  $\square$

**Remark 14.6A.** In the characteristic case,  $\mathbf{B}^*$  is the completion of a Boolean algebra of smaller cardinality  $\mathbf{B}'$ , so  $\chi \leq \|\mathbf{B}'\|$ .

**Claim 14.7.** *Now, without loss of generality*

$$\boxtimes \quad \mathbf{B}^* = \langle \mathbf{B} \cup \{a_\alpha : \alpha \in W_{\mathbf{B}}\} \rangle \text{ for some } \mathbf{B} \subseteq \mathbf{B}^*, \\ \|\mathbf{B}\| = \chi, \quad W_{\mathbf{B}} = \lambda.$$

**Proof.**  $\mathbf{B} \subseteq \mathbf{B}^*$  exemplifies the value of  $\chi$ , let  $\mathbf{B}^c$  be the completion of  $\mathbf{B}$ , and we can let for  $\alpha \in W_{\mathbf{B}}$

$$a'_\alpha = \text{lub in } \mathbf{B}^c \text{ of } \text{Proj}^0(a_\alpha, \mathbf{B}).$$

Now if  $Y \in [W_{\mathbf{B}}]^\lambda$ ,  $\langle a'_\alpha : \alpha \in Y \rangle$  is independent in  $\mathbf{B}^c$  then  $\{a_\alpha : \alpha \in Y\}$  is independent in  $\mathbf{B}^*$ . Alternatively use  $\langle \mathbf{B} \cup \{a_i : \alpha \in W_{\mathbf{B}}\} \rangle_{\mathbf{B}^*}$ .

(Remember:  $\mathbf{B}$  is not necessarily a complete subalgebra of  $\mathbf{B}^*$ .)

**Definition 14.8.** Let

$$K \stackrel{\text{def}}{=} \{ \overline{\mathbf{B}} : \overline{\mathbf{B}} = \langle \mathbf{B}_i : i \leq \chi \rangle \text{ is an increasing continuous sequence of subalgebras of } \mathbf{B}^*, \|\mathbf{B}_i\| < \aleph_0 + |i|^+, \text{ and } W_{\mathbf{B}_\chi} \in [\lambda]^\lambda, \mathbf{B}_\chi \supseteq \mathbf{B} \text{ (of } \boxtimes \text{ of 14.7)} \}$$

(so  $W_{\mathbf{B}_\chi}$  is co-bounded in  $\lambda$ , in fact if  $\mathbf{B}_\chi \subseteq \langle \mathbf{B} \cup \{a_\beta : \beta < \alpha\} \rangle_{\mathbf{B}^*}$  then  $|W_{\mathbf{B}_\chi} \supseteq (\alpha, \lambda)|$ ).

**Fact 14.9.**

- (1)  $\text{cf}(\chi) < \kappa$ .
- (2)  $\text{cov}(\chi, \chi, \kappa, 2) \geq \lambda$ , meaning:

$$\lambda \leq \min\{ |\mathcal{P}| : \mathcal{P} \subseteq [\chi]^{<\chi} \ \& \ (\forall A \in [\chi]^{<\kappa}) (\exists B \in \mathcal{P}) (A \subseteq B) \}.$$

**Proof.** (1) By (2).

(2) Assume not. Remember  $\mathbf{B} \subseteq \mathbf{B}^*$ ,  $|W_{\mathbf{B}}| = \lambda$ ,  $\|\mathbf{B}\| = \chi$ .

For each  $\alpha \in W_{\mathbf{B}}$  choose  $\mathcal{I}_{\alpha, \ell} \subseteq \text{Proj}^\ell(a_\alpha, \mathbf{B})$  for  $\ell < 2$  as in the proof of 14.6. Let  $\mathcal{P} \subseteq [\mathbf{B}]^{<\chi}$ ,  $|\mathcal{P}| < \lambda$  and

$$(\forall A \in [\chi]^{<\kappa}) (\exists B \in \mathcal{P}) (A \subseteq B).$$

So for each  $\alpha \in W_{\mathbf{B}}$ , there is  $A_\alpha \in \mathcal{P}$  such that  $\mathcal{I}_{\alpha, 0} \cup \mathcal{I}_{\alpha, 1} \subseteq A_\alpha$ . So for some  $A^* \in \mathcal{P}$

$$W = \{ \alpha \in W_{\mathbf{B}} : \mathcal{I}_{\alpha, 0} \cup \mathcal{I}_{\alpha, 1} \subseteq A^* \} \in [\lambda]^\lambda$$

(exists as we divide  $W_{\mathbf{B}}$  into  $|\mathcal{P}|$  sets, so at least one has size  $\lambda$ , as  $|\mathcal{P}| < \lambda = \text{cf}(\lambda)$ ). Now  $\chi \leq |(A^*)_{\mathbf{B}}|$ , contradiction, as in the proof of 14.6 (to the definition of  $\chi$ ).  $\square$

**Definition 14.10.** For  $\overline{\mathbf{B}} \in K$  and  $\alpha \in W_{\mathbf{B}_\chi}$  let

$$u(\alpha, \overline{\mathbf{B}}) \stackrel{\text{def}}{=} \{ i < \chi : \text{for some } \ell < 2, \text{Proj}^\ell(a_\alpha, \mathbf{B}_i) \text{ is not a predense subset of } \text{Proj}^\ell(a_\alpha, \mathbf{B}_{i+1}) \}.$$

**Discussion 14.10A.** We may consider  $\overline{\mathbf{B}}' = \langle \mathbf{B}'_i : i \leq \chi \rangle \in \kappa$  when

$$\mathbf{B}'_i = \langle \mathbf{B}_i \cup X \rangle, \quad X \text{ fixed countable } \subseteq \mathbf{B}^*.$$

Possibly

$$u(\alpha, \overline{\mathbf{B}}) \neq u(\alpha, \overline{\mathbf{B}}')$$



or just for some  $i$ ,  $\text{Proj}^\ell(a_\alpha, \mathbf{B}_i)$  is not dense in  $\text{Proj}^\ell(a_\alpha, \mathbf{B}'_i)$ . We think of the set of such  $\alpha$  as bad, and put them all in one  $\lambda$ -complete ideal. But maybe  $\lambda$  belongs to it. So we will try to find some  $\overline{\mathbf{B}}$  for which this does not occur.

This will help in that we eventually try to choose  $\alpha_\zeta \in W_{\overline{\mathbf{B}}}$  for  $\zeta < \lambda$  by induction on  $\zeta$  such that  $\langle a_{\alpha_\zeta} : \zeta < \lambda \rangle$  is independent.

So in stage  $\zeta$  we consider all

$$X \in [\{\alpha_\xi : \xi < \zeta\}]^{<\aleph_0}.$$

The existence of  $\overline{\mathbf{B}}$  requires some properties of  $\lambda$  which certainly hold in the main case (with  $\lambda = \mu^+ \dots$ ).

So to ease the proof instead of every  $i < \chi$ , we use “every  $i < \chi$  large enough”.

**Definition 14.11.**

(1) We define a partial order on  $K$ :  $\overline{\mathbf{B}}^1 \leq \overline{\mathbf{B}}^2$  if for every  $i$  large enough

$$i \leq \chi \Rightarrow \mathbf{B}_i^1 \subseteq \mathbf{B}_i^2.$$

(2) We say  $\overline{\mathbf{B}}^2$  is finitely generated over  $\overline{\mathbf{B}}^1$  if for some finite  $X$

$$\mathbf{B}_i^2 = \langle \mathbf{B}_i^1 \cup X \rangle_{\mathbf{B}^*} \quad \text{for } i < \chi \text{ large enough.}$$

In this case we let  $\overline{\mathbf{B}}^1[X] = \langle \mathbf{B}_i^1[X] : i \leq \chi \rangle$  be  $\overline{\mathbf{B}}^2$ .

(3) For  $\overline{\mathbf{B}}^1 \leq \overline{\mathbf{B}}^2$  let

$$\begin{aligned} \text{Bad}(\mathbf{B}^1, \mathbf{B}^2) = \{ \alpha : & \text{if } \alpha \in W_{\mathbf{B}_\chi^1} \cap W_{\mathbf{B}_\chi^2}, \\ & \text{then for arbitrarily large } i < \chi, \text{ for some } \ell < 2, \\ & \text{Proj}^\ell(a_\alpha, \mathbf{B}_i^1) \text{ is not dense in } \text{Proj}^\ell(a_\alpha, \mathbf{B}_i^2) \}. \end{aligned}$$

(4)  $J_{\overline{\mathbf{B}}^1}$  is the  $\lambda$ -complete ideal on  $\lambda$  generated by all  $\text{Bad}(\overline{\mathbf{B}}^1, \overline{\mathbf{B}}^2)$ , where  $\overline{\mathbf{B}}^1 \leq \overline{\mathbf{B}}^2$  and  $\overline{\mathbf{B}}^2$  is finitely generated over  $\overline{\mathbf{B}}^1$ .

What do we need to carry a proof?

**Lemma 14.12.** *There is  $\overline{\mathbf{B}}^\otimes \in K$  such that  $\lambda \notin J_{\overline{\mathbf{B}}^\otimes}$ .*

**Remark 14.12A.** We may like to have  $J \supseteq J_{\overline{\mathbf{B}}^\otimes}$  normal extending  $I_\lambda^{\text{nst}, \theta}$  (and  $\lambda \notin J$ ), then we need more work.

**Proof in the case  $\lambda = \chi^+$ .** (Enough, see 14.1(a).) Assume there is no such  $\overline{\mathbf{B}} = \overline{\mathbf{B}}^\otimes$ . We choose by induction on  $\zeta < \chi$ ,  $\overline{\mathbf{B}}^\zeta \in K$ , such that  $\overline{\mathbf{B}}^\zeta$  is increasing with  $\zeta$  and: for each  $\zeta$ , as  $\lambda \in J_{\overline{\mathbf{B}}^\zeta}$  we can find  $\langle X_{\zeta, \varepsilon} : \varepsilon < \varepsilon_\zeta \rangle$  witnessing it, i.e.,  $X_{\zeta, \varepsilon} \in [\mathbf{B}^*]^{<\aleph_0}$ ,  $\varepsilon_\zeta < \lambda$  (so without loss of generality  $\varepsilon_\zeta \leq \chi$ )

$$\lambda = \bigcup_{\varepsilon < \varepsilon_\zeta} \text{Bad}(\overline{\mathbf{B}}^\zeta, \overline{\mathbf{B}}^\zeta[X_{\zeta, \varepsilon}]),$$

where

$$\mathbf{B}_i^\zeta[X_{\zeta,\varepsilon}] = \langle \mathbf{B}_i^\zeta \cup X_{\zeta,\varepsilon} \rangle_{\mathbf{B}^*}.$$

Now easily  $(K, \leq)$  is  $\chi^+$ -directed, so we demand

$$\bigwedge_{\varepsilon < \varepsilon_\zeta} \overline{\mathbf{B}}^\zeta \leq \overline{\mathbf{B}}^\zeta[X_{\zeta,\varepsilon}] \leq \overline{\mathbf{B}}^{\zeta+1}.$$

Also  $i \in [\zeta, \lambda)$  &  $\zeta < \xi \leq \chi \Rightarrow \mathbf{B}_i^\zeta \subseteq \mathbf{B}_i^\xi$ . Let  $\delta^* < \lambda$  be such that

$$\bigwedge_{\zeta < \chi} \mathbf{B}_\chi^\zeta \subseteq \langle \mathbf{B} \cup \{a_\alpha : \alpha < \delta^*\} \rangle_{\mathbf{B}^*}.$$

So for each  $\zeta < \chi$  we have

$$\delta^* \in \bigcup_{\varepsilon < \varepsilon_\zeta} \text{Bad}(\overline{\mathbf{B}}^\zeta, \overline{\mathbf{B}}^\zeta[X_{\zeta,\varepsilon}]),$$

hence there is  $\xi(\zeta) < \varepsilon_\zeta$  such that

$$\delta^* \in \text{Bad}(\overline{\mathbf{B}}^\zeta, \overline{\mathbf{B}}^\zeta[X_{\zeta,\xi(\zeta)}]).$$

For each  $\zeta < \chi$ , there is  $i(\zeta) < \chi$  such that  $X_{\zeta,\xi(\zeta)} \subseteq \mathbf{B}_{i(\zeta)}^{\zeta+1}$ ,  $\zeta < i(\zeta)$  hence

$$(\forall i)[i(\zeta) \leq i \leq \chi] \Rightarrow \mathbf{B}_i^\zeta[X_{\zeta,\xi(\zeta)}] \subseteq \mathbf{B}_i^{\zeta+1}$$

because

$$X_{\zeta,\xi(\zeta)} \subseteq \mathbf{B}_{i(\zeta)}^{\zeta+1} \subseteq \mathbf{B}_i^{\zeta+1}.$$

We restrict ourselves to  $\xi < \kappa$ . So without loss of generality

$$\bigwedge_{\zeta_1 < \zeta_2 \leq \kappa} \bigwedge_{\alpha \in [\kappa^+, \chi]} \mathbf{B}_\alpha^{\zeta_1} \subseteq \mathbf{B}_\alpha^{\zeta_2},$$

and if  $\zeta$  is a limit and  $\alpha \in [\kappa^+, \chi]$ , then  $\mathbf{B}_\alpha^\zeta = \bigcup_{\xi < \zeta} \mathbf{B}_\alpha^\xi$ . As  $\text{cf}(\chi) < \kappa$ , there is  $i(*) < \chi$  such that  $Z = \{\zeta < \kappa : i(\zeta) \leq i(*)\}$  is unbounded (we can demand more).

Now the set  $u(\delta^*, \overline{\mathbf{B}}^\kappa)$  has cardinality  $< \kappa$  because  $\mathbf{B}^*$  satisfies the c.c.c.

Remember,

$$u(\delta^*, \overline{\mathbf{B}}^\kappa) = \left\{ i < \chi : \bigcup_{\ell=0,1} \text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\kappa) \right. \\ \left. \text{is not predense in } \bigcup_{\ell=0,1} \text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_{i+1}^\kappa) \right\}.$$

Choose for  $i \in u(\delta^*, \overline{\mathbf{B}}^\kappa) \cup \{\kappa^+\}$  and  $\ell = 0, 1$  a predense subset  $\mathcal{I}_{\kappa,i}^{\delta^*,\ell}$  of  $\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_{i+1}^\kappa)$  of cardinality  $< \kappa$ .

Now, for  $i \in u(\delta^*, \overline{\mathbf{B}}^\kappa) \cup \{\kappa^+\} \setminus \kappa^+$  the sequence  $\langle \mathbf{B}_{i+1}^\zeta : \zeta \leq \kappa \rangle$  is increasing continuous. So for some  $\zeta_i < \kappa$

$$\mathcal{I}_{\kappa,i}^{\delta^*,0} \cup \mathcal{I}_{\kappa,i}^{\delta^*,1} \subseteq \mathbf{B}_{i+1}^{\zeta_i}.$$

Let

$$\zeta(\delta^*) \stackrel{\text{def}}{=} \sup_i \zeta_i < \kappa.$$

So clearly

(\*) if  $i \in [\kappa^+, \chi]$ ,  $\ell < 2$ , then

$$\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^{\zeta[\delta^*]}) = \text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\kappa) \cap \mathbf{B}_i^{\zeta[\delta^*]}$$

is a predense subset of  $\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\kappa)$ .

[Why? By induction on  $i$ . If  $i = \kappa^+$  directly. If  $i$  is a limit—trivial. If  $i = j + 1 \geq \kappa^+$ ,  $j \notin u(\delta^*, \overline{\mathbf{B}}^\kappa)$ , then by transitivity of being predense in. If  $i = j + 1$ ,  $j \in u(\delta^*, \overline{\mathbf{B}}^\kappa)$ , using  $\mathcal{I}_j^{\delta^*, \ell}$ .]

Now, clearly

$$\zeta \in [\zeta(\delta^*), \kappa) \Rightarrow \bigwedge_{\ell < 2} \bigwedge_{i \in [\kappa^+, \chi]} (\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\zeta) \text{ is predense in } \text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^{\zeta+1})).$$

This follows from (\*). Choose  $\zeta \in Z \setminus \zeta(\delta^*)$  so we contradict the choice of  $\overline{\mathbf{B}}^{\zeta+1}$ .  $\square$

**Convention 14.13.** We fix  $\overline{\mathbf{B}}^\otimes \in K$  such that  $\lambda \notin J_{\overline{\mathbf{B}}^\otimes}$ .

**Fact 14.14.**  $\{\alpha < \lambda: u(\alpha, \overline{\mathbf{B}}^\otimes) \text{ bounded in } \chi\}$  is bounded in  $\lambda$ .

**Proof.** By the choice of  $\chi$  as minimal.  $\square$

**Convention 14.15.** Let  $f_\alpha$  be an increasing function from  $\text{otp}(u(\alpha, \overline{\mathbf{B}}^\otimes))$  onto  $u(\alpha, \overline{\mathbf{B}}^\otimes)$ .

**Fact 14.16.** For some  $j^* < \kappa$

$$Y_0 = \{\alpha < \lambda: \text{Dom}(f_\alpha) = j^*\} \in (J_{\overline{\mathbf{B}}^\otimes})^+.$$

So without loss of generality  $(\forall \alpha)[\text{Dom}(f_\alpha) = j^*]$ .

**Claim 14.17.** We can find  $\langle \gamma_j^*: j < j^* \rangle$ ,  $w^* \subseteq j^*$  such that:

(\*)<sub>1</sub> if  $\bar{\gamma} = \langle \gamma_j: j < j^* \rangle$ ,  $\gamma_j \leq \gamma_j^*$ ,

$$\gamma_j = \gamma_j^* \Leftrightarrow j \in w^*,$$

then the set of  $\alpha \in Y_0$  satisfying the following, is in  $(J_{\overline{\mathbf{B}}^\otimes})^+$ :

$$j \in w^* \Rightarrow f_\alpha(j) = \gamma_j^*,$$

$$j \in j^* \setminus w^* \Rightarrow \gamma_j < f_\alpha(j) < \gamma_j^*.$$

Also

(\*)<sub>2</sub>  $j \in j^* \setminus w^* \Rightarrow \text{cf}(\gamma_j^*) > 2^\kappa$  and

$$\lambda = \max \text{pcf}\{\text{cf}(\gamma_j^*): j \in j^* \setminus w^*\}.$$

(\*)<sub>3</sub> Moreover if we fix  $\mu = \mu^{<\kappa} < \lambda$  we can demand

$$j \in j^* \setminus w^* \Rightarrow \text{cf}(\gamma_j^*) > \mu.$$

(\*)<sub>4</sub> If  $j^* = \sup(J^* \setminus w^*)$ , and  $E$  is the equivalence relation on  $j^* \setminus w^*$  defined by  $j_1 E j_2 \Leftrightarrow \gamma_{j_1}^* = \gamma_{j_2}^*$  (so the equivalence classes are convex) then  $J$  is an ideal on  $j^*$  such that  $J_{j^*}^{\text{bd}} \subseteq J$ ,  $w^* \in J$ ,

$$A \in J \Rightarrow \bigcup \{j/E : j \in A\} \in J, \quad \text{and}$$

$$(\alpha) \quad \prod_{j < j^*} \gamma_j^* / J \text{ has true cofinality } \lambda, \text{ so possibly shrinking } Y_0, \text{ for} \\ \alpha < \beta \text{ in } Y_0, f_\alpha <_J f_\beta.$$

**Proof.** By 7.0(0) (or [26, 6.6D] or [20, 6.1]), as  $j^* < \kappa$ , so  $2^{|j^*|} < \lambda$ .  $\square$

**Observation 14.18.**  $\langle \gamma_j^* : j < j^* \rangle$  is non-decreasing, with limit  $\chi$ , and  $\gamma_j^* < \chi$  and of course,  $\text{cf}(j^*) = \text{cf}(\chi)$ .

**Proof.** As  $\text{Rang}(f_\alpha) \subseteq \chi$ , and the fact,  $\gamma_j^* < \chi$  if  $j \in w^*$ ,  $\gamma_j^* \leq \chi$  if  $j \notin w^*$ , but then

$$\text{cf}(\gamma_j^*) \geq 2^\kappa > \kappa > \text{cf}(\chi). \quad \square$$

**Comment on the Claim 14.19.**

- (1) For it, possibly  $\bigwedge_\alpha f_\alpha = f^*$ , so then we get  $w^* = j^*$ . Also possibly  $f_\alpha(j) < \alpha$ , so  $w^* = \emptyset$  and  $J = \{\emptyset\}$ .
- (2) If the ideal  $J_{\overline{B}}^\otimes$  is normal enough, for some  $X \in (J_{\overline{B}}^\otimes)^+$ ,  $\langle f_\alpha : \alpha \in X \rangle$  is  $<_J$ -increasing.
- (3) If  $(\forall \alpha < \lambda)(|\alpha|^{|j^*|} < \lambda)$ , then necessarily

$$j \in j^* \setminus w^*, \quad \text{cf}(\gamma_j^*) = \lambda$$

(like the  $\Delta$ -system lemma). *BUT* for the interesting case, and in particular by our assumptions, this is not the case: as  $\gamma_j^* \leq \chi < \lambda$ , hence  $J \supseteq [j^*]^{<\aleph_0}$ .

**Hypothesis 14.20.** Each  $B_i^\otimes$  is the union of  $\mu$  filters  $\langle \mathcal{D}_{i,\beta} : \beta < \mu \rangle$ ,  $\mu = \mu^{<\kappa}$  (we can use somewhat less), this of course is only a consistent assumption.

**Claim 14.21.** For some

$$\bar{t} = \langle t_j : j < j^* \rangle \in j^* \mu$$

we can restrict ourselves to

$$Y_1 = \left\{ \begin{array}{l} j \in w^* \Rightarrow f_\alpha(j) = \gamma_j^*, \\ \alpha < \lambda : j \in j^* \setminus w^* \Rightarrow \gamma_j^{**} < f_\alpha(j) < \gamma_j^* \text{ and} \\ \bigwedge_{j < j^*} (\text{Proj}^2(a_\alpha, \overline{B}_{\gamma_j^{**}}^\otimes) \cap \mathcal{D}_{\gamma_j^{**}, t_j} \neq \{0\}) \end{array} \right\},$$

where

$$\gamma_j^{**} = \begin{cases} \gamma_j^* & \text{if } j \in w^*, \\ \bigcup \{\gamma_i^* : \gamma_i^* < \gamma_j^*\} & \text{otherwise} \end{cases}$$

in particular  $Y_1 \notin J_{\overline{\mathbf{B}}^\otimes}$ .

**Proof.** As  $\mu^{<\kappa} < \lambda$ ,  $J_{\overline{\mathbf{B}}^\otimes}$   $\lambda$ -complete and  $j \in j^* \setminus w^* \Rightarrow \text{cf}(\gamma_j^*) > \mu$ .  $\square$

**Claim 14.22.** For some  $X \in [X]^\lambda$ , the sequence  $\langle a_\alpha : \alpha \in X \rangle$  is independent.

**Proof.**

Case 1.  $w^*$  is unbounded in  $j^*$ : We choose by induction on  $\beta < \lambda$ ,

$$N_\beta < (\mathcal{H}((2^\lambda)^+), \in, <_{(2^\lambda)^+}^*)$$

increasing continuous,  $\|N_\beta\| < \lambda$ ,  $N_\beta \cap \lambda \in \lambda$ ,  $\langle N_{\beta_1} : \beta_1 \leq \beta \rangle \in N_{\beta+1}$  and  $\overline{\mathbf{B}}^\otimes$ ,  $\mathbf{B}^*$ ,  $\bar{a} \in N_0$ . Let  $\alpha_\beta = \alpha(\beta)$  be the first  $\alpha < \lambda$  such that  $\alpha \in Y_1$ ,  $\alpha \notin \bigcup (J_{\overline{\mathbf{B}}^\otimes} \cap N_\beta)$

$$\left( \text{so } \bigwedge_{j \in w^*} f_\alpha(j) = \gamma_j^* \right).$$

Clearly  $\alpha_\beta \in \lambda \cap N_{\beta+1} \setminus N_\beta$ ,  $\langle \alpha_{\beta_1} : \beta_1 < \beta \rangle \in N_{\beta+1}$ . Let  $n < \omega$ ,  $\beta_1 < \dots < \beta_n$  and we will prove that  $\langle \alpha_{\alpha(\beta_\ell)} : \ell = 1, \dots, n \rangle$  is independent.

Now

$$j \in w^* \Rightarrow \text{there is } b_j \in \bigcap_{\ell=1}^n \text{Proj}^2(a_{\alpha_{\beta_\ell}}, \overline{\mathbf{B}}_{\gamma_j^*}^\otimes) \setminus \{0\}.$$

[Why? As  $\alpha_{\beta_1}, \dots, \alpha_{\beta_n} \in Y_1$ , so  $\mathcal{D}_{\gamma_j^*, \iota_j} \cap \text{Proj}^2(a_{\alpha_{\beta_\ell}}, \mathbf{B}_{\gamma_j^*}^\otimes) \neq \emptyset$ . Choose  $b_{j,\ell}$  there, so  $b_j = \bigcap_{\ell=1}^n b_{j,\ell}$  is OK.]

Consider

$$\text{Bad}(\mathbf{B}^\otimes, \overline{\mathbf{B}}^\otimes [\{a_{\alpha(\beta_1)}, \dots, a_{\alpha(\beta_\ell)}\}]) \in J_{\overline{\mathbf{B}}^\otimes},$$

it belongs to  $N_{\beta_{\ell+1}}$ . So

$$\alpha_{\beta_{\ell+1}} \notin \text{Bad}(\mathbf{B}^\otimes, \mathbf{B}^\otimes [\{a_{\alpha(\beta_1)}, \dots, a_{\alpha(\beta_\ell)}\}]).$$

So for each  $\ell$  for some  $i_\ell < \chi$ ,  $k < 2$  &  $i \in [i_\ell, \chi) \Rightarrow \text{Proj}^k(a_{\alpha_{\beta_{\ell+1}}}, \mathbf{B}_i^\otimes)$  is predense in  $\text{Proj}^k(a_{\alpha_{\beta_{\ell+1}}}, \langle \mathbf{B}_i^\otimes \cup \{a_{\alpha(\beta_1)}, \dots, a_{\alpha(\beta_\ell)}\} \rangle)$ .

So if  $j \in w^*$ ,  $\gamma_j^* > \sup_{\ell=1, \dots, n} i_\ell$  (exists) and  $\eta \in [1, n]2$ , we prove by induction on  $\ell$  that

$$b_j^\ell = b_j \cap \bigcap_{k=1}^{\ell} (a_{\alpha_{\beta_k}})^{[\eta(k)]}.$$

For  $\ell = 0$  trivial.

For  $\ell > 0$ ,  $b_j^{\ell-1} \in \langle \mathbf{B}_{\gamma_j^*}^\otimes \cup \{a_{\alpha_{\beta_1}}, \dots, a_{\alpha_{\beta_{\ell-1}}}\} \rangle$  is  $> 0$ , is in

$$\text{Proj}^2(a_{\alpha_\beta}, \langle \mathbf{B}_{\gamma_j^*}^\otimes \cup \{a_{\alpha(\beta_1)}, \dots, a_{\alpha(\beta_{\ell-1})}\} \rangle)$$

as it is below  $b_j$  and  $b_j \in \text{Proj}^2(a_{\alpha(\beta_\ell)}, \mathbf{B}_{\gamma_j}^{\otimes})$  by its choice and  $j$  is  $> i_\ell$ , so  $b_j \in \text{Proj}^2(a_{\alpha\beta_\ell}, \langle \overline{\mathbf{B}}^{\otimes} \cup \{a_{\alpha\beta_1}, \dots, a_{\alpha\beta_{\ell-1}}\} \rangle)$ . We use implicitly

**Fact 14.23.** For  $\alpha < \lambda$  large enough,

$$i < \chi \Rightarrow \text{Proj}^2(a_\alpha, \overline{\mathbf{B}}_i^{\otimes}) \neq \{0\}.$$

**Proof.** By  $\chi$ 's minimality.  $\square$

*Case 2.* Not 1, i.e.,  $w^*$  bounded in  $j^*$  or just  $j^* = \sup(J^* \setminus w)$ . Similarly using  $(*)_2$  of 14.17 find  $j \in j^* \setminus w^*$  such that if  $j_\ell \in j/E$  for  $\ell = 1, \dots, n$  then  $f_{\alpha\beta_1}(j_1) < f_{\alpha\beta_2}(j_2) < \dots < f_{\alpha\beta_n}(j_n)$ .  $\square$

**Conclusion 14.24.** If  $\mu = \mu^{<\mu} < \theta = \theta^{<\theta}$  then for some  $\mu$ -complete  $\mu^+$ -c.c. forcing notion of cardinality  $\theta$ , in  $V^P$ :

If  $\mathbf{B}$  is a  $\kappa$ -c.c. Boolean algebra of cardinality  $\geq \lambda$ ,  $\mu = \mu^{<\kappa}$ ,  $\lambda = \text{cf}(\lambda) \in (\mu, \theta]$  then  $\lambda$  is a free caliber of  $\mathbf{B}$ .

**Proof.** By 14.1–14.23 above and [21].  $\square$

**Claim 14.25.** The following implications hold:  $(*)_1 \Rightarrow (*)_2 \Rightarrow (*)_3 \Rightarrow (*)_4$  where

- $(*)_1$  (a)  $\mu^{2^{<\kappa}} = \mu < \lambda = \text{cf}(\lambda)$ ,  
 (b) if a Boolean algebra  $\mathbf{B}$  satisfies the  $(2^{<\kappa})^+$ -c.c. and  $|\mathbf{B}| < \lambda$ , then  $\mathbf{B}$  is the union of  $\mu$  filters.
- $(*)_2$  (a)  $\kappa < \lambda = \text{cf}(\lambda)$ ,  
 (b) if a Boolean algebra  $\mathbf{B}$  satisfies the  $\kappa$ -c.c., for  $i < \lambda$ ,  $F_i \subseteq \mathbf{B} \setminus \{0\}$  is a set of  $< \kappa$  members closed under intersection then we can find  $< \lambda$  filters  $\mathcal{D}_\alpha$  ( $\alpha < \alpha^* < \lambda$ ) of  $\mathbf{B}$  such that  $(\forall i < \lambda) (F_i \subseteq \mathcal{D}_\alpha)$ .
- $(*)_3$  (a)  $\kappa < \lambda = \text{cf}(\lambda)$ ,  
 (b) if a Boolean algebra  $\mathbf{B}$  satisfies the  $\kappa$ -c.c.,  $\mathcal{D}$  a  $\lambda$ -complete uniform filter on  $\lambda$ ,  $\theta = \text{cf}(\theta) < \kappa$  and for  $i < \lambda$ ,  $F_i$  is a decreasing sequence of elements of  $\mathbf{B} \setminus \{0\}$  of length  $\theta$  then for some  $X \in \mathcal{D}^+$ ,  $\bigcup_{i \in X} F_i$  belongs to some ultrafilter on  $\mathbf{B}$ .
- $(*)_4$  (a)  $\kappa < \lambda = \text{cf}(\lambda)$ ,  
 (b) if  $\mathbf{B}$  is a  $\kappa$ -c.c. Boolean algebra of cardinality  $\geq \lambda$  then  $\lambda$  is a free caliber of  $\mathbf{B}$ .

**Proof.** Should be clear from the proof in §14.  $\square$

## 15. On irr: The invariant of the ultraproduct bigger than the ultraproduct of invariants

We solve here some of the questions of Monk [12] on the possibility that

$$\text{inv} \left( \prod_{\zeta < \kappa} \mathbf{B}_\zeta / \mathcal{D} \right) > \prod_{\zeta < \kappa} \text{inv}(\mathbf{B}_\zeta) / \mathcal{D}.$$

In 15.1–15.10A we deal with the irredundance number  $\text{irr}$  (getting consistency of the above and solving [12, Problem 26]). We then prove the existence of such examples in ZFC (improving Rosłanowski and Shelah [15]) for  $\text{inv} = s, \text{hd}, \text{hL}, \text{Length}$  solving [12, Problems 46, 51, 55, 22], respectively. See more in [19].

**Hypothesis 15.1.**  $\lambda = \lambda^{<\lambda}, n(*) < \omega$ .

**Definition 15.2.**  $P = P_\lambda^{n(*)}$  is the set of  $p = (u, \mathbf{B}, \overline{F}) = (u^p, \mathbf{B}^p, \overline{F}^p)$  such that

- (a)  $u \in [\lambda^+]^{<\lambda}$ ,
- (b)  $\mathbf{B}$  is a Boolean algebra generated by  $\{x_\alpha: \alpha \in u\}$ ,
- (c)  $\alpha \in u \Rightarrow x_\alpha \notin \langle \{x_\beta: \beta \in u \cap \alpha\} \rangle_{\mathbf{B}}$ ,
- (d) in  $\mathbf{B}$ ,  $\{x_\alpha: \alpha \in u\}$  is  $n(*)$ -independent, i.e., any nontrivial Boolean combination of  $\leq n(*)$  members of  $\{x_\alpha: \alpha \in u\}$  is not zero (in  $\mathbf{B}$ ),
- (e)  $\overline{F} = \langle F_\ell: \ell \leq n(*) \rangle$  and  $F_{\ell+1} \subseteq F_\ell$ ,
- (f)  $F_\ell$  is a family of functions from  $\{x_\alpha: \alpha \in u\}$  to  $\{0, 1\}$  respecting the equations holding in  $\mathbf{B}$ . Call the homomorphism (from  $\mathbf{B}$  to  $\{0, 1\}$ ) that  $f$  induces,  $\hat{f}$ ,
- (g) if  $f \in F_{\ell+1}$ ,  $\ell < n(*)$  and  $\alpha \in u$  then for some  $f' \in F_\ell$  we have

$$f' \upharpoonright (\alpha \cap u) = f \upharpoonright (\alpha \cap u), \quad f'(\alpha) \neq f(\alpha),$$

- (h) if  $f: u \rightarrow \{0, 1\}$  and  $(\forall v \in [u]^{<\aleph_0})(f \upharpoonright v \in F_\ell)$  then  $f \in F_\ell$ ,
- (i) if  $a \in \mathbf{B} \setminus \{0\}$  then for some  $f \in F_0$ , we have  $\hat{f}(a) = 1$ .

The order is  $p \leq q$  iff

- ( $\alpha$ )  $u^p \subseteq u^q$ ,
- ( $\beta$ )  $\mathbf{B}^p$  is a subalgebra of  $\mathbf{B}^q$ ,
- ( $\gamma$ )  $F_\ell^p = \{f \upharpoonright u^p: f \in F_\ell^q\}$ .

Let  $\mathbf{B} =$  the direct limit of  $\{\mathbf{B}^p: p \in \mathcal{G}_P\}$ .

**Note.** We can ignore  $\mathbf{B}^p$  as it is reconstructible from  $F_0^p$ . Also clause (d) follows from the rest.

**Notation 15.3.** We let  $p \upharpoonright \alpha = (u^p \cap \alpha, \langle x_\beta: \beta \in u^p \cap \alpha \rangle_{\mathbf{B}}, \langle F_\ell \upharpoonright \alpha: \ell \leq n(*) \rangle)$  where  $F_\ell \upharpoonright \alpha = \{f \upharpoonright \alpha: f \in F_\ell\}$ .

**Fact 15.4.**  $(p \upharpoonright \alpha) \leq p$  for  $p \in P$ .

**Fact 15.5.** In  $P$ , every increasing sequence of length  $< \lambda$  has a lub: essentially the union.

**Proof.** Trivial (use compactness and clause (h) of Definition 15.2).  $\square$

**Fact 15.6.** For  $\alpha < \lambda$ ,  $\{p \in P: \alpha \in u^p\}$  is dense open.

**Proof.** If  $p \in P$  let us define  $q = (u^q, \mathbf{B}^q, F^q)$ ,  $u^q = u^p \cup \{\alpha\}$ ,  $\mathbf{B}^q$  is  $\mathbf{B}^p$  if  $\alpha \in u^p$ , and is the free extension of  $\mathbf{B}$  by  $x_\alpha$  otherwise,  $F_\ell^q = \{f \in F_\ell^p: f \upharpoonright u^p \in F_\ell^p\}$ .  $\square$

**Fact 15.7.**

- (1) If  $p \in P$ ,  $p \upharpoonright \alpha \leq q$  and  $u^q \subseteq \alpha$  then  $p, q$  are compatible.  
 (2)  $P$  satisfies the  $\lambda^+$ -c.c. and even in  $\lambda^+$ -Knaster.

**Proof.** (1) Let us define  $r = (u^r, \mathbf{B}^r, \overline{F}^r)$  by:

$$u^r = u^p \cup u^q, \quad F_\ell^r = \{f: f \in u^r \text{ and } f \upharpoonright u^p \in F_\ell^p, f \upharpoonright u^q \in F_\ell^q\}.$$

Now

$$(*)_1 \quad F_\ell^p = F_\ell^r \upharpoonright u^p.$$

[Why? if  $f \in F_\ell^p$ , then  $f \upharpoonright \alpha = f \upharpoonright (\alpha \cap u) \in F_\ell^p \upharpoonright \alpha$  but  $p \upharpoonright \alpha \leq q$ . Hence there is  $g \in F_\ell^q$  such that  $f \upharpoonright \alpha \subseteq g$ , so  $f \cup g \in F_\ell^r$ ,  $(f \cup g) \upharpoonright u^p = f$ , so  $F_\ell^p \subseteq F_\ell^r \upharpoonright u^p$ . The other direction holds by the choice of  $F_\ell^r$ .]

$$(*)_2 \quad F_\ell^q = F_\ell^r \upharpoonright u^p.$$

[Why? Similarly using 15.4.]

$$(*)_3 \quad F_{\ell+1}^r \subseteq F_\ell^r.$$

[Why? As  $F_{\ell+1}^p \subseteq F_\ell^p$ ,  $F_{\ell+1}^q \subseteq F_\ell^q$ .]

$$(*)_4 \quad \text{if } f \in F_{\ell+1}^r, \beta \in u^r \text{ then for some } g \in F_\ell^r \text{ we have } f \upharpoonright \beta \subseteq g, f(\beta) \neq g(\beta).$$

[Why? The proof splits into two cases:

*Case 1.*  $\beta \in u^q$ . So  $f \upharpoonright \alpha \in F_{\ell+1}^q \upharpoonright \alpha$  but  $q \in P$  so there is  $g_0 \in F_\ell^q$  such that  $(f \upharpoonright \alpha) \upharpoonright \beta \subseteq g_0$ ,  $(f \upharpoonright \alpha)(\beta) \neq g_0(\beta)$  so  $g_0 \in F_\ell^q = F_\ell^p \upharpoonright \alpha$  so  $g_0 \upharpoonright (u^p \cap \alpha) \in F_\ell^p \upharpoonright \alpha$  so there is  $g_1$  such that

$$g_0 \upharpoonright (u^p \cap \alpha) \subseteq g_1 \in F_\ell^p.$$

So  $g_0 \cup g_1 \in F_\ell^r$  is as required.

*Case 2.*  $\beta \notin u^q$ . So  $\beta \in u^p \setminus \alpha$ . Now  $f \upharpoonright u^p \in F_{\ell+1}^p$  hence there is  $f' \in F_\ell^p$  such that

$$f' \upharpoonright (u^p \cap \beta) = f \upharpoonright (u^p \cap \beta), \quad f'(\beta) \neq f(\beta).$$

Now  $f \upharpoonright \alpha \in F_{\ell+1}^q$  hence  $f \upharpoonright \alpha \in F_\ell^q$  hence

$$(f \upharpoonright \alpha) \cup f' \in F_\ell^r \text{ is as required.}$$

By  $F_\ell^r$  we can define  $\mathbf{B}^r$  and is as required.]

(2) Follows from (1).  $\square$

**Claim 15.8.** If  $k > 2n(*) + 1$ ,  $\langle \delta_\ell: \ell < k \rangle$  is increasing,  $\delta_\ell < \lambda$ ; we stipulate  $\delta_k = \lambda^+$ , for  $\ell < k$ ,  $p_\ell \in P$ ,  $p_\ell \upharpoonright \delta_\ell = p^*$ ,  $u^{p_\ell} \subseteq \delta_{\ell+1}$  and for  $\ell, m < k$ ,  $\text{OP}_{u^{p_m}, u^{p_\ell}}: u^{p_\ell} \rightarrow u^{p_m}$  maps  $p_\ell$  to  $p_m$  (the natural meaning  $\text{otp}(u^{p_\ell}) = \text{otp}(u^{p_m})$ ) and

$$F_n^{p_\ell} = \{f \circ \text{OP}_{u^{p_\ell}, u^{p_m}}: f \in F_n^{p_m}\}$$

so  $\text{OP}_{u^{p_\ell}, u^{p_m}}$  induces an isomorphism  $\text{OP}_{p_\ell, p_m}$  from  $\mathbf{B}^{p_\ell}$  onto  $\mathbf{B}^{p_m}$ , then there is  $q \in P$  such that

$$(a) \quad \bigwedge_{m < k} p_m \leq q,$$

$$(b) \quad \text{if } b \in \mathbf{B}^{p_0} \text{ then } \mathbf{B}^q \models \text{“} b = \bigcup_{\substack{u \subseteq (0, k) \\ |u| > n(*)}} (\bigcap_{m \in u} \text{OP}_{p_m, p_0}(b)) \text{”}.$$



**Proof.** (1) Let us define  $q$ : put  $u^q = \bigcup_{m < k} u^{p_m}$  and

$$F_\ell^q = \{f \in {}^{(u^q)}2: n(*) - \ell \geq |\{m \in [0, k): (\exists \alpha \in u^{p_0} \setminus u^{p_m^*}) \\ [f(\text{OP}_{u^{p_m}, u^{p_0}}(\alpha)) \neq f(\alpha)]\}| \\ \text{and } f \upharpoonright u^{p_m} \in F_\ell^{p_m} \text{ for } m < k\}.$$

Now note

$$(*)_1 \quad F_\ell^{p_m} = F_\ell^q \upharpoonright u^{p_m}.$$

[Why? If  $f \in F_\ell^q$  then  $f \upharpoonright u^{p_m} \in F_\ell^{p_m}$  by the definition of  $F_\ell^q$ . If  $f \in F_\ell^{p_m}$ , for  $m_1 < k$  we let  $f_{m_1} = f \circ \text{OP}_{u^{p_{m_1}}, u^{p_m}}$ , so  $\bigcup_{m_1 < k} f_{m_1} \in F_\ell^q$  and we are done.]

$$(*)_2 \quad \text{if } f \in F_{\ell+1}^q, \alpha \in u^q \text{ then for some } g \in F_\ell^q$$

$$g \upharpoonright \alpha = f \upharpoonright \alpha, \quad g(\alpha) \neq f(\alpha).$$

[Why? If  $\alpha \in u^{p_0}$  we have  $f \upharpoonright u^{p_0} \in F_{\ell+1}^{p_0}$  and there is  $g_0 \in F_\ell^{p_0}$ , such that  $g_0 \upharpoonright \alpha = f \upharpoonright \alpha$ ,  $g_0(\alpha) \neq f(\alpha)$ . Let  $g_m = \text{OP}_{u^{p_0}, u^{p_m}} \circ g_0$ . Then  $g = \bigcup_{m < k} g_m$  is as required.

If not,  $\alpha \in u^{p_m} \setminus u^{p_m^*}$  for some  $m > 0$ , so  $\alpha \geq \delta_m$  and  $f \upharpoonright u^{p_m} \in F_{\ell+1}^{p_m}$  so there is  $g \in F_{\ell+1}^{p_m}$ ,  $g \upharpoonright \alpha = f \upharpoonright \alpha$ ,  $g(\alpha) \neq f(\alpha)$ . Now  $g^* = g \cup (f \upharpoonright (\bigcup_{\substack{m_1 < k \\ m_1 \neq m}} u^{p_{m_1}}))$  is as required.]

So

$$(*)_3 \quad q \in P \text{ and } p_m \leq q.$$

So (a) of the conclusion holds. By clause (i) of Definition 15.2 and the choice of  $q$  also clause (b) holds.  $\square$

**Conclusion 15.9.**  $\Vdash_{P_x^{n(*)}}$  “ $\mathcal{B}$  is a Boolean algebra generated by  $\{x_\alpha: \alpha < \lambda^+\}$ , which is  $n(*)$ -free hence  $\text{irr}_{n(*)}(\mathcal{B}) \geq \lambda^+$  but  $\text{irr}_{2n(*)+1}(\mathcal{B}) = \lambda$ ”.

**Proof.** Putting together the claims.  $\square$

**Conclusion 15.10.** If  $\lambda = \lambda^{<\lambda} > \aleph_0$  and the forcing notion  $P$  is  $P = \prod_n P_\lambda^n$  (where  $P_\lambda^n$  is from Definition 15.2) then

(\*)  $P$  is a  $\lambda^+$ -complete  $\lambda^+$ -c.c. forcing notion, and in  $V^P$  for some Boolean algebras  $\mathcal{B}_n$  ( $n < \omega$ ) we have

$$(a) \quad \text{irr}_n(\mathcal{B}_n) = \lambda^+, \text{irr}_{2n+1}(\mathcal{B}_n) = \lambda,$$

(b) for  $\mathcal{D}$  a nonprincipal ultrafilter on  $\omega$ ,

$$\lambda^+ \leq \text{irr}\left(\prod_{n < \omega} \mathcal{B}_n / \mathcal{D}\right), \quad \prod_{n < \omega} \text{irr}(\mathcal{B}_n) / \mathcal{D} = \lambda^\omega / \mathcal{D} = \lambda,$$

(c) so  $\text{irr}(\prod_{n < \omega} \mathcal{B}_n / \mathcal{D}) > \prod_{n < \omega} \text{irr}(\mathcal{B}_n) / \mathcal{D}$ .

**Proof.** The  $\lambda^+$ -c.c. follows from 15.7(2). The  $\mathcal{B}_n$  are from 15.2. The proof that  $\Vdash_P$  “ $\text{irr}_n(\mathcal{B}_n) = \lambda^+$  but  $\text{irr}_{2n+2}(\mathcal{B}_n) = \lambda'$ ” is like the proof of 15.9.

Concerning  $\text{irr}(\prod_{n < \omega} \mathcal{B}_n / \mathcal{D}) = \lambda^+$  use  $x_\alpha^* = \langle x_\alpha^n: n < \omega \rangle / \mathcal{D} \in \prod_{n < \omega} \mathcal{B}_n / \mathcal{D}$ .  $\square$

**Comment 15.10A.** Surely in 15.9 we can fix exactly the  $n$  such that  $\text{irr}_n(\mathbf{B}) = \lambda^+$ ,  $\text{irr}_{n+1}(\mathbf{B}) = \lambda$ . For this it suffices to demand in 15.8 that  $k = n(*) + 2$ . Let  $a_\ell \in \mathbf{B}^{p_\ell}$  (for  $\ell < k$ ) be such that  $\text{OP}_{p_\ell, p_0}(a_0) = a_\ell$  and replace (b) in the conclusion by

(b)' for some Boolean term  $\tau$ ,

$$\mathbf{B}^a \models "a_0 = \tau(a_1, \dots, a_{n(*)+1})".$$

In fact,

$$\tau(x_1, \dots, x_{n(*)+1}) = \bigcup_{i=1}^{n(*)} \left[ \bigcap_{m=1}^{i-1} (x_m \Delta x_{m+1}) \cap (-(x_i \Delta x_{i+1})) \right].$$

In the proof we let  $u^q = \bigcup_{m < k} u^{p_m}$  and

$$F_\ell^q = \{f \in {}^{(u^p)}2: \text{ for } m < k \text{ we have } f_m =: f \upharpoonright u^{p_m} \text{ belongs to } F_\ell^{p_m} \text{ and for some } i \in \{1, \dots, n(*) - \ell\} \text{ we have: } j \in \{1, \dots, i - 1\} \Rightarrow [\hat{f}_j(a_j) = 0 \Leftrightarrow \hat{f}_{j+1}(a_{j+1}) = 1] \text{ and } \hat{f}_i(a_i) = \hat{f}_{i+1}(a_{i+1}) = \hat{f}_0(a_0)\}$$

(where for  $f \in F_\ell^{p_m}$ ,  $\hat{f}$  is the homomorphism from  $\mathbf{B}^q$  into  $\{0, 1\}$  which  $f$  induces).

**Claim 15.11.** Assume

- (A)  $\lambda = \text{tcf}(\prod_{i < \delta} \lambda_i / J)$ ,
- (B)  $\bar{\lambda} = \langle \lambda_i: i < \delta \rangle$  is a sequence of regular cardinals  $> |\delta|$ ,
- (C)  $\lambda_i > \max \text{pcf}\{\lambda_j: j < i\}$ , so necessarily  $J_\delta^{\text{bd}} \subseteq J$ ,
- (D)  $\langle A_\zeta: \zeta < \kappa \rangle$  is a sequence of pairwise disjoint members of  $J^+$ ,
- (E)  $\mathcal{D}$  is a uniform ultrafilter on  $\kappa$ .

Then, we can find a Boolean algebra  $\mathbf{B}_\zeta$  for  $\zeta < \kappa$  such that for  $\text{inv} \in \{s, \text{hd}, \text{hL}\}$  (see Monk [12])

- (a)  $\text{inv}^+(\mathbf{B}_\zeta) \leq \lambda$  so  $\lambda = \chi^+ \Rightarrow \text{inv}(\mathbf{B}_\zeta) \leq \chi$  (moreover  $\text{inv}_2^+(\mathbf{B}_3) \leq \lambda$ ; see [15]),
- (b)  $\text{inv}^+(\prod_{\zeta < \kappa} \mathbf{B}_\zeta / \mathcal{D}) > \lambda$  (so if  $\lambda = \chi^+$  then  $\text{inv}(\prod_{\zeta < \kappa} \mathbf{B}_\zeta / \mathcal{D}) \geq \lambda$ ).

**Proof.** Let  $\bar{\eta} = \langle \eta_\alpha: \alpha < \lambda \rangle$  be a  $<_J$ -increasing cofinal sequence of members of  $\prod_{i < \delta} \lambda_i$  such that

$$\zeta < \kappa \Rightarrow \lambda_\zeta > |\{\eta_\alpha \upharpoonright \zeta: \alpha < \lambda\}|$$

(such  $\bar{\eta}$  exists by [25, II 3.5]). We define  $(\mathbf{B}_{\zeta,i}^*; \bar{x}_{\zeta,i}^-, \bar{x}_{\zeta,i}^+)$  for  $\zeta < \kappa$ ,  $i < \delta$  as follows. Let  $I_i = J_{\lambda_i}^{\text{bd}}$ , so  $\bar{x}_{\zeta,i}^- = \langle x_{\zeta,i,\alpha}^-: \alpha < \lambda_i \rangle$ ,  $\bar{x}_{\zeta,i}^+ = \langle x_{\zeta,i,\alpha}^+: \alpha < \lambda_i \rangle$ .

Case 1.  $i \notin \bigcup \{A_\varepsilon: \varepsilon \in [\zeta, \kappa)\}$ . Let  $\mathbf{B}_{\zeta,i}$  be the Boolean algebra generated by  $\{x_{\zeta,i,\alpha}^-, x_{\zeta,i,\alpha}^+: \alpha < \lambda_i\}$  freely except that  $x_{\zeta,i,\alpha}^- \leq x_{\zeta,i,\alpha}^+$ , and  $(x_{\zeta,i,\alpha}^+ - x_{\zeta,i,\alpha}^-) \cap (x_{\zeta,i,\beta}^+ - x_{\zeta,i,\beta}^-) = 0$  when  $\alpha < \beta < \lambda_i$ .

Case 2.  $i \in \bigcup \{A_\varepsilon: \varepsilon \in [\zeta, \kappa)\}$ . Let  $\mathbf{B}_{\zeta,i}$  be the Boolean algebra generated by  $\{x_{\zeta,i,\alpha}^-, x_{\zeta,i,\alpha}^+: \alpha < \lambda_i\}$  freely except that

$$\alpha < \beta \Rightarrow x_{\zeta,i,\alpha}^- \leq x_{\zeta,i,\alpha}^+ \leq x_{\zeta,i,\beta}^- \leq x_{\zeta,i,\beta}^+$$

(e.g.,  $\mathbf{B}_{\zeta,i} \subseteq \mathcal{P}(\lambda_i)$ ,  $x_{\zeta,i,\alpha}^- = [0, 4\alpha + 1)$ ,  $x_{\zeta,i,\alpha}^+ = [0, 4\alpha + 2)$ ).

Let  $\mathbf{B}_\zeta$  be constructed as in 12.1, 12.3 from  $\bar{\lambda}$ ,  $\langle I_i: i < \delta \rangle$ ,  $(\mathbf{B}_{\zeta,i}, \bar{x}_{\zeta,i}^-, \bar{x}_{\zeta,i}^+)$  for  $i < \delta$ , and let  $y_\alpha^\zeta, y_\eta^\zeta$  be as there.

Now  $\text{inv}^+(\prod_{\zeta < \kappa} \mathbf{B}_\zeta / \mathcal{D}) > \lambda$  is exemplified by  $\langle y_\alpha^*: \alpha < \lambda \rangle$  where  $y_\alpha^* = \langle y_\alpha^\zeta: \zeta < \kappa \rangle / \mathcal{D}$ , because for  $\alpha < \lambda$ ,  $u \subseteq \lambda \setminus \{\alpha\}$  finite, for some  $\zeta^* < \kappa$ , we have  $\beta \in u \Rightarrow \ell g(\eta_\alpha \cap \eta_\beta) \in \delta \setminus \bigcup_{\varepsilon \in [\zeta, \kappa]} A_\varepsilon$ , hence

$$\zeta \in [\zeta_{\alpha,\beta}, \kappa) \Rightarrow \mathbf{B}_\zeta \models y_\alpha^\zeta - \bigcap_{\beta \in u} y_\beta^\zeta > 0.$$

Hence  $\{\zeta < \kappa: \mathbf{B}_\zeta \models y_\alpha^\zeta \cap y_\beta^\zeta = 0\} \supseteq [\zeta_{\alpha,\beta}, \kappa) \in \mathcal{D}$  and therefore  $\prod_{\zeta < \kappa} \mathbf{B}_\zeta / \mathcal{D} \models "y_\alpha^* - \bigcap_{\beta \in u} y_\beta^* > 0"$ .

Lastly  $\text{inv}_2^+(\mathbf{B}_\zeta) \leq \lambda$  follows by 12.17(2) for  $\tau(x_0, x_1, x_2) = (x_1 - x_0 \cup x_2)$  with the variables permuted according to the particular  $\text{inv}$ .  $\square$

**Claim 15.12.** *Claim 15.11 holds for Length too.*

**Proof.** We repeat the proof of 15.11, but in the definition of  $\mathbf{B}_{\zeta,i}$  just interchange the two cases.

*Case 1.*  $i \notin \bigcup \{A_\varepsilon: \varepsilon \in [\zeta, \kappa]\}$ . Let  $\mathbf{B}_{\zeta,i}$  be as  $\mathbf{B}_{\zeta,i}$  in case 2 in the proof of 15.11.

*Case 2.*  $i \in \bigcup \{A_\varepsilon: \varepsilon \in [\zeta, \kappa]\}$ . As in case 1 in the proof of 15.11 or just let  $\mathbf{B}_{i,\zeta}$  be generated by  $\{x_\alpha^-, x_\alpha^+: \alpha < \lambda_i\}$ ,  $\{x_{\zeta,i,\alpha}^-, x_{\zeta,i,\alpha}^+: \alpha < \lambda_i\}$  freely except  $x_{\zeta,i,\alpha}^- \leq x_{\zeta,i,\alpha}^+$ .

Now for  $\alpha < \beta < \lambda$ , letting  $i(\alpha, \beta) = \text{Min}\{i: \eta_\alpha(i) \neq \eta_\beta(i)\}$  and  $\zeta_{\alpha,\beta} = \text{Min}\{\zeta: i(\alpha, \beta) \notin \bigcup \{A_\varepsilon: \varepsilon \in [\zeta, \kappa]\}$  we have

$$\zeta \in [\zeta_{\alpha,\beta}, \kappa) \Rightarrow \mathbf{B}_{\zeta,i} \models "y_\alpha^\zeta \leq y_\beta^\zeta \text{ or } y_\beta^\zeta < y_\alpha^\zeta",$$

hence

$$\prod_{\zeta < \kappa} \mathbf{B}_\zeta \models "y_\alpha^* < y_\beta^* \text{ or } y_\beta^* < y_\alpha^*",$$

where  $y_\alpha^* = \langle y_\alpha^\zeta: \zeta < \kappa \rangle / \mathcal{D}$ .

As for  $\text{Length}^+(\mathbf{B}_\zeta) \leq \lambda$ , it is by 12.7(1).  $\square$

**Conclusion 15.13.**

(1) If  $\mathcal{D}$  is a uniform ultrafilter on  $\kappa$ , then for a class of cardinals  $\chi = \chi^\kappa$  and Boolean algebras  $\mathbf{B}_i$  for  $i < \kappa$  such that, for  $\text{inv} \in \{s, \text{hL}, \text{hd}\}$  we have:

- (a)  $\text{inv}(\mathbf{B}_i) \leq \chi$  hence  $\prod_{i < \kappa} \text{inv}(\mathbf{B}_i) \leq \chi$ , or
- (b)  $\text{inv}(\prod_{i < \kappa} \mathbf{B}_i / \mathcal{D}) = \chi^+$ .

(2) Similarly with  $\text{inv} = \text{Length}$ .

**Proof.** Let  $\chi$  be any strong limit singular cardinal of cofinality  $> \kappa$ . So by [25, VIII, §1] we can find  $\langle \lambda_i: i < \text{cf}(\chi) \rangle$ , strictly increasing sequence of regular cardinals  $< \chi$  with  $\text{tcf}(\prod_{i < \text{cf}(\chi)} \lambda_i / \mathcal{J}_{\text{cf}(\chi)}^{\text{bd}}) = \chi^+$ . Without loss of generality  $\prod_{i < j} \lambda_i < \lambda_j$  and let for  $i < \kappa$ ,  $A_i = \{\alpha\kappa + i: \alpha < \text{cf}(\chi)\}$ . So we can apply 15.11 (for part (1)) or 15.12 (for part (2)).  $\square$

**Remark.** For cellularity similar results hold (in ZFC), i.e.,  $c(\mathbf{B}_n) \leq \lambda$ ,  $c(\prod_{n < \omega} \mathbf{B}_n) > \lambda$ , see on it in Monk [12, pp. 61–62]; by [25, III 4.11, p. 181, 4.12] so this applies to  $\lambda = \mu^+$  for  $\lambda > \aleph_1$  by [25, II, 4.1], [27], to  $\lambda$  inaccessible not Mahlo by [25, III 4.8(2), p. 177] and for many Mahlo cardinals (see [25, III, 4.10A, p. 178]. For incomparability number (Inc) similar results are proved “almost in ZFC”, see [28].

## References

- [1] B. Dushnik, E.W. Miller, Partially ordered sets, *Amer. J. Math.* 63 (1941) 600–610.
- [2] M. Džamonja, K. Kunen, Measures on HS spaces, *Fund. Math.* 143 (1993) 41–54.
- [3] M. Džamonja, K. Kunen, Properties of the class of measure separable spaces, *Fund. Math.* 147 (1995) 261–277.
- [4] R. Engelking, M. Karłowicz, Some theorems of set theory and their topological consequences, *Fund. Math.* 57 (1965) 275–285.
- [5] D. Fremlin, *Consequences of Martin’s Axiom*, Cambridge Univ. Press, Cambridge, 1984.
- [6] M. Gitik, S. Shelah, On densities of box products, *Topology Appl.* 88 (1998) 219–237. [GiSh:597]; math.LO/9603206.<sup>5</sup>
- [7] A. Hajnal, I. Juhasz, Z. Szentmiklossy, On the structure of CCC partial orders, *Algebra Universalis*, to appear.
- [8] R.G. Haydon, On Banach spaces which contain  $\ell^1(\tau)$  and types of measures on compact spaces, *Israel J. Math.* 28 (1977) 313–324.
- [9] R.G. Haydon, On dual  $L^1$ -spaces and injective bidual Banach spaces, *Israel J. Math.* 31 (1978) 142–152.
- [10] K. Kunen, A compact  $l$ -space under CH, *Topology Appl.* 12 (1981) 283–287.
- [11] D. Monk, *Cardinal Invariants of Boolean Algebras*, Lectures in Math., ETH Zurich, Birkhauser, Basel, 1990.
- [12] D. Monk, *Cardinal Invariants of Boolean Algebras*, Progress in Mathematics, Vol. 142, Birkhäuser, Basel, 1996.
- [13] G. Plebanek, Non-separable Radon measures on small compact spaces, Preprint, 1996.
- [14] G. Plebanek, Radon measures and mappings onto Tychonoff cubes, Preprint, 1996.
- [15] A. Roslanowski, S. Shelah, Cardinal invariants of ultraproducts of Boolean algebras, *Fund. Math.* 155 (1998) 101–151. [RoSh:534]; math.LO/9703218.
- [16] S. Shelah, *Analytical Guide and Corrections to “S. Shelah, Cardinal Arithmetic*, Oxford University Press, 1994. [Sh:g]”. [Sh:E12]; math.LO/9906022.
- [17] S. Shelah, Applications of PCF theory, *J. Symbolic Logic*, accepted. [Sh:589]; math.LO/9804155.
- [18] S. Shelah, Cellularity of free products of Boolean algebras (or topologies), *Fund. Math.*, accepted. [Sh:575]; math.LO/9508221.
- [19] S. Shelah, Constructing Boolean algebras for cardinal invariants, *Algebra Universalis*, submitted. [Sh:641]; math.LO/9712286.
- [20] S. Shelah, PCF and infinite free subsets, *Archive for Mathematical Logic*, accepted. [Sh:513]; math.LO/9807177.
- [21] S. Shelah, A weak generalization of MA to higher cardinals, *Israel J. Math.* 30 (1978) 297–306. [Sh:80].
- [22] S. Shelah, Remarks on Boolean algebras, *Algebra Universalis* 11 (1980) 77–89. [Sh:92].
- [23] S. Shelah, On saturation for a predicate, *Notre Dame J. of Formal Logic* 22 (1981) 239–248. [Sh:126].

<sup>5</sup>References of the form math.XX/... refer to the xxx.lanl.gov archive.

- [24] S. Shelah, Advances in cardinal arithmetic, in: N.W. Sauer et al. (Eds.), *Finite and Infinite Combinatorics in Sets and Logic*, Kluwer Academic Publishers, Dordrecht, 1993, pp. 355–383. [Sh:420].
- [25] S. Shelah, *Cardinal Arithmetic*, Oxford Logic Guides, Vol. 29, Oxford University Press, 1994. [Sh:g].
- [26] S. Shelah, Further cardinal arithmetic, *Israel J. Math.* 95 (1996) 61–114. [Sh:430]; math.LO/9610226.
- [27] S. Shelah, Colouring and non-productivity of  $\aleph_2$ -cc, *Ann. of Pure Applied Logic* 84 (1997) 153–174. [Sh:572]; math.LO/9609218.
- [28] S. Shelah, On  $\sigma$ -entangled linear orders, *Fund. Math.* 153 (1997) 199–275. [Sh:462]; math.LO/9609216.