Algebra Univers. 62 (2009) 367–374
DOI 10.1007/s00012-010-0047-6
Published online May 29, 2010

Algebra Universalis

Large intervals in the clone lattice

Martin Goldstern and Saharon Shelah

Abstract. We give three examples of cofinal intervals in the lattice of (local) clones on an infinite set X, whose structure is on the one hand non-trivial but on the other hand reasonably well understood. Specifically, we will exhibit clones $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ such that

(1) the interval $[\mathscr{C}_1, \mathscr{O}]$ in the lattice of local clones is (as a lattice) isomorphic to $\{0, 1, 2, \dots\}$ under the divisibility relation,
(2) the interval $[\mathscr{C}_2, \mathscr{O}]$ in the lattice of local clones is isomorphic to the congruence lattice of an arbitrary semilattice,
(3) the interval $[\mathscr{C}_3, \mathscr{O}]$ in the lattice of all clones is isomorphic to the lattice of all filters on X.

1. Introduction

Definition 1.1. Let X be a nonempty set. The *full clone* on X, called $\mathscr{O}_X$ or just $\mathscr{O}$, is the set of all finitary functions (or "operations") on X: $\mathscr{O}_X = \bigcup_{n=1}^{\infty} \mathscr{O}^{(n)}$ where $\mathscr{O}^{(n)}$ is the set of all functions from X^n into X. A *clone* (on X) is a set $\mathscr{C} \subseteq \mathscr{O}$ which contains all projections and is closed under composition. Alternatively, $\mathscr{C}$ is a clone if $\mathscr{C}$ is the set of term functions of some universal algebra over X. Identifying a clone $\mathscr{C}$ with the algebra $(X, \mathscr{C})$ (whose term functions are the elements of $\mathscr{C}$) allows us to talk about subalgebras and automorphisms of $\mathscr{C}$.

The set of clones over X forms a complete algebraic lattice with largest element $\mathscr{O}$. The coatoms of this lattice are called "precomplete clones" or "maximal clones". (See also [Sz86], [PK79].)

Definition 1.2. A clone $\mathscr{C}$ is called a *local clone* iff each set $\mathscr{C}^{(k)} := \mathscr{C} \cap \mathscr{O}^{(k)}$ is closed in the product topology (Tychonoff topology) on X^{X^k}, where X is

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Presented by Á. Szendrei.
Received October 3, 2004; accepted in final form May 24, 2009.
2010 *Mathematics Subject Classification*: Primary: 08A40; Secondary: 08A05, 03E05.
Key words and phrases: precomplete clones, local clones, dually atomic, ternary discriminator, internal isomorphisms, ultrafilters.
The first author is grateful to the Hebrew University of Jerusalem for the hospitality during his visit, and to the Austrian Science foundation for supporting the joint research under FWF grant P13325-MAT. The second author is supported by the German-Israeli Foundation for Scientific Research & Development Grant No. G-294.081.06/93. Publication number 747.

taken to be discrete. In other words, $\mathscr{C}$ is local iff:

Whenever $f \in \mathscr{O}^{(k)} \setminus \mathscr{C}$, then there is a finite "witness" for it, i.e., there is a finite $A \subseteq X^k$ such that for all $g \in \mathscr{C}$: $g \restriction A \neq f \restriction A$.

The set of local clones over X forms again a complete lattice with largest element $\mathscr{O}$.

For any k-ary relation $\rho \subseteq X^k$, the set $\mathrm{Pol}(\rho)$ is the set of all functions preserving ρ. We will only need two special cases of this construction:

Definition 1.3. For any $A \subseteq X$, let $\mathrm{Pol}(A) := \bigcup_{n=1}^{\infty} \{f \in \mathscr{O}^{(n)} : f[A^n] \subseteq A\}$, and for any unary (partial) function h, let

$$\mathrm{Pol}(h) := \bigcup_{n=1}^{\infty} \{f \in \mathscr{O}^{(n)} : \forall \bar{x} = (x_1, \ldots, x_n) : f(h(x_1), \ldots, h(x_n)) = h(f(\bar{x}))\}.$$

Definition 1.4.

- The ternary discriminator t (on the base set X) is defined to be the function $t\colon X^3 \to X$ satisfying $t(x, x, z) = z$ and $t(x, y, z) = x$ whenever $x \neq y$.
- An *internal isomorphism* of an algebra $(X, \mathscr{C})$ is a bijection $h\colon U \to V$ between two subalgebras of $(X, \mathscr{C})$ which is compatible with all operations of $\mathscr{C}$, i.e., $\mathscr{C} \subseteq \mathrm{Pol}(h)$. We write $\mathrm{Iso}(\mathscr{C})$ for the set of all internal isomorphisms.
- A local clone $\mathscr{C}$ is called *locally quasiprimal* iff the elements of $\mathscr{C}$ are exactly the operations which are compatible with all internal isomorphisms of $\mathscr{C}$, i.e., if $\mathscr{C} = \bigcap_{h \in \mathrm{Iso}(\mathscr{C})} \mathrm{Pol}(h)$. (Note that the inclusion $\subseteq$ holds by definition of $\mathrm{Iso}(\mathscr{C})$.)

Theorem 1.5 (Pixley's theorem, see [Px71] and [Px82])**.** *A local clone $\mathscr{C}$ is locally quasiprimal if $t \in \mathscr{C}$, where t is the ternary discriminator.*

In the following sections, we will use Pixley's theorem to describe intervals in the lattice of local clones. As a warmup, consider the following example:

Example 1.6. Fix an infinite set X, and let $s\colon X \to X$ be a 1–1 onto map without cycles. For $n > 0$, s^n is the n-th iterate of s, s^{-n} is the inverse of s^n, and s^0 is the identity function.

Then $\mathscr{C}_1 := \mathrm{Pol}(s)$ is a local clone, and the local clones containing $\mathrm{Pol}(s)$ are exactly the clones $\mathrm{Pol}(s^n)$ for $n \in \mathbb{Z}$; we have $\mathrm{Pol}(s^n) \subseteq \mathrm{Pol}(s^k)$ iff n divides k.

(These clones, and also the unbounded chain $\mathrm{Pol}(s^{2^n})$, were already considered in [RS84].)

Proof sketch. First, we note that $\mathrm{Pol}(s)$ contains the ternary discriminator; hence, $\mathrm{Pol}(s)$, as well as any local clone containing it, must be locally quasiprimal. Next, note that:

(∗) For all $a, b \in X$, there is a map $f \in \mathrm{Pol}(s)$ with $f(a) = b$.
[Why? Define $f(s^n(a)) := s^n(b)$ for all $n \in \mathbb{Z}$ and $f(x) = x$ for all x not of the form $s^n(a)$.]

Hence, $(X, \mathrm{Pol}(s))$ has no proper subalgebras, so the internal isomorphisms of $(X, \mathrm{Pol}(s))$ are exactly the automorphisms of $(X, \mathrm{Pol}(s))$. Clearly, s is an automorphism of this structure, and using $(*)$, it is easy to see that every automorphism must be of the form s^n for some $n \in \mathbb{Z}$.

Now let $\mathscr{D}$ be a local clone above $\mathrm{Pol}(s)$. Let I be the set of internal isomorphisms (=automorphisms) of $(X, \mathscr{D})$. Then I is a subset and even a subgroup of $\{s^n : n \in \mathbb{Z}\}$, say $I = \{s^{nk_0} : n \in \mathbb{Z}\}$ for some $k_0 \in \mathbb{Z}$.

Hence, $\mathscr{D}$ and $\mathrm{Pol}(s^{k_0})$ have the same set of internal isomorphisms; as both clones are locally quasiprimal, they must be equal. $\square$

2. A large interval of local clones

Theorem 2.1. *Let $(X, \vee)$ be a semilattice, and let $Con(X, \vee)$ be the lattice of congruences on $(X, \vee)$. Let $\mathscr{C}_2$ be the clone of all operations that are bounded by the* sup *function of the appropriate arity:*

$$\mathscr{C}_2 := \bigcup_{k=1}^{\infty} \{f \in \mathscr{O}^{(k)} : \forall x_1, \dots, x_k \; f(x_1, \dots, x_k) \leq x_1 \vee \cdots \vee x_k\}.$$

(Here, $x \leq y \;\Leftrightarrow\; x \vee y = y$ is the usual semilattice order.)

Then $[\mathscr{C}_2, \mathscr{O}_X] \simeq Con(X, \vee)$. That is, there is a lattice isomorphism between the set of local clones above $\mathscr{C}_2$ and the set of congruences of $(X, \vee)$.

Remark 2.2. If $\emptyset \subsetneq I \subsetneq X$ is an ideal, then the partition $\{I, X \setminus I\}$ corresponds to a congruence relation which is a coatom in $Con(X, \vee)$. In fact, all coatoms are obtained in this form. It is clear that $Con(X, \vee)$ is dually atomic.

It will be notationally more convenient to deal with congruence orders rather than congruence relations.

Definition 2.3. Let $(X, \vee)$ be a semilattice. We call $\preccurlyeq \subseteq X \times X$ a *congruence order* on $(X, \vee)$ if $\preccurlyeq$ is transitive, extends the semilattice order $\leq$ and satisfies

$$\forall x, y, z \in X : \; x \preccurlyeq z \;\&\; y \preccurlyeq z \;\Rightarrow\; (x \vee y) \preccurlyeq z. \tag{**}$$

The following fact is easy to check:

Fact 2.4. The maps

$$\begin{aligned} \preccurlyeq \;&\mapsto\; \{(x, y) : x \preccurlyeq y \;\&\; y \preccurlyeq x\} \\ \theta \;&\mapsto\; \{(x, y) : (x \vee y, y) \in \theta\} \end{aligned}$$

are monotone bijections between the congruence relations θ and congruence orders $\preccurlyeq$ on $(X, \vee)$, and they are inverses of each other.

Definition 2.5. For any clone $\mathscr{C}$ on the set X and any subset $E \subseteq X$, we write $\langle E \rangle_{\mathscr{C}}$ for the subalgebra of $(X, \mathscr{C})$ that is generated by E. In other words: $\langle E \rangle_{\mathscr{C}} = \bigcup_{k=1}^{\infty} \{ f(\bar{a}) : \bar{a} \in E^k, \, f \in \mathscr{C}^{(k)} \}$.

Definition 2.6. We define a correspondence between clones on X and preorders (quasiorders) on X through two maps $\mathscr{C} \mapsto R_{\mathscr{C}}$ and $\preccurlyeq \mapsto \mathscr{E}(\preccurlyeq)$.

- For any clone $\mathscr{C}$ on X, let $R_{\mathscr{C}}$ be the preorder on X defined by

$$x \; R_{\mathscr{C}} \; y \;\Leftrightarrow\; x \in \langle y \rangle_{\mathscr{C}}.$$

  The associated equivalence relation $\sim_{\mathscr{C}}$ is then given by $\langle x \rangle_{\mathscr{C}} = \langle y \rangle_{\mathscr{C}}$, and the algebra $\langle y \rangle_{\mathscr{C}}$ generated by y is just the half-open interval

$$(y]_{R_{\mathscr{C}}} := \{x \in X : x \; R_{\mathscr{C}} \; y\}.$$

- For any preorder $\preccurlyeq$ on X, let the clone $\mathscr{E}(\preccurlyeq)$ be defined by

$$\mathscr{E}(\preccurlyeq) = \bigcap_{a \in X} \mathrm{Pol}(\,(a]_{\preccurlyeq}).$$

Lemma 2.7. *Let $\leq_1$ and $\leq_2$ be congruence orders on $(X, \vee)$, and assume $x \leq_1 y \;\rightarrow\; x \leq_2 y$. Then $\mathscr{E}(\leq_1) \subseteq \mathscr{E}(\leq_2)$.*

Proof. Let f be a k-ary function in $\mathscr{E}(\leq_1)$. Let $a \in X$, $b_1, \ldots, b_k \leq_2 a$. We have to show $f(b_1, \ldots, b_k) \leq_2 a$.

Let $b^* := b_1 \vee \cdots \vee b_k$. Then

- As $b_i \leq_1 b^*$ for all i, we have $f(b_1, \ldots, b_k) \leq_1 b^*$; hence $f(b_1, \ldots, b_k) \leq_2 b^*$.
- As $\leq_2$ is a congruence order, we conclude from $b_i \leq_2 a$ that also $b^* \leq_2 a$ holds.
- Hence, $f(b_1, \ldots, b_k) \leq_2 a$. □

Lemma 2.8. *Let $\preccurlyeq$ be a preorder on X. Then the following are equivalent for all $a, b \in X$:*

(i) $a \preccurlyeq b$.
(ii) $\chi_{a,b} \in \mathscr{E}(\preccurlyeq)$, *where $\chi_{a,b}$ maps b to a, and is the identity otherwise.*
(iii) *There is a unary function $f \in \mathscr{E}(\preccurlyeq)$ with $f(b) = a$.*
(iv) $a \in \langle b \rangle_{\mathscr{E}(\preccurlyeq)}$, *i.e., $a \; R_{\mathscr{E}(\preccurlyeq)} \; b$.*

Proof. The implications (i)⇒(ii)⇒(iii)⇒(i) and (iii)⇔(iv) are all easy. □

Lemma 2.9. *Let $\mathscr{C}$ be a clone on X and $\mathscr{D} := \mathscr{E}(R_{\mathscr{C}})$. Then $\mathscr{C}$ and $\mathscr{D}$ have the same 1-generated subalgebras: $\forall b \in X : \; \langle b \rangle_{\mathscr{C}} = \langle b \rangle_{\mathscr{D}}$.*

Proof. By the equivalence (i)⇔(iv) in Lemma 2.8, the relations $R_{\mathscr{C}}$ and $R_{\mathscr{D}}$ coincide. Now $\langle b \rangle_{\mathscr{C}} = (\,b\,]_{R_{\mathscr{C}}} = (\,b\,]_{R_{\mathscr{D}}} = \langle b \rangle_{\mathscr{D}}$. □

The relation $\langle x \rangle_{\mathscr{C}} \subseteq \langle y \rangle_{\mathscr{C}}$ carries information only about the unary functions of $\mathscr{C}$; in our context, however, this is sufficient, because our clones $\mathscr{C}$ are generated by $\mathscr{C}^{(1)} \cup \{\vee\}$. The "encoding" property defined below will help us to reduce questions about subalgebras to questions about 1-generated subalgebras.

Definition 2.10. Let $\mathscr{C}$ be a clone on X, $*$ a binary function in $\mathscr{C}$. We say that $(\mathscr{C}, *)$ *encodes pairs* iff $\langle x, y \rangle_{\mathscr{C}} = \langle x * y \rangle_{\mathscr{C}}$ for all $x, y \in X$.

Fact 2.11. Assume that both $(\mathscr{C}, *)$ and $(\mathscr{D}, *)$ encode pairs. Then the following are equivalent:

(1) $\mathscr{C}$ and $\mathscr{D}$ have the same subalgebras.

(2) $\mathscr{C}$ and $\mathscr{D}$ have the same finitely generated subalgebras.
(3) $\langle x\rangle_{\mathscr{C}} = \langle x\rangle_{\mathscr{D}}$ for all $x \in X$.

Lemma 2.12. *Let $(X, \vee)$ be a semilattice, and let $x \le y$ iff $x \vee y = y$. Then*

(1) *$\mathscr{E}(\le)$ is a local clone containing the binary function $\vee$ as well as the ternary discriminator. In fact,* $\mathscr{E}(\le) = \bigcup_{k=1}^{\infty}\{ f \in \mathscr{O}^{(k)} : \forall x_1, \dots, x_k\, f(x_1, \dots, x_k) \le x_1 \vee \dots \vee x_k \} = \mathscr{C}_2$.
(2) $\langle x, y\rangle_{\mathscr{C}_2} = \langle x \vee y\rangle_{\mathscr{C}_2}$, *and similarly* $\langle x, y\rangle_{\mathscr{D}} = \langle x \vee y\rangle_{\mathscr{D}}$ *for all clones* $\mathscr{D} \supseteq \mathscr{C}_2$.
(3) *If $U, V \le (X, \mathscr{C}_2)$, and $h\colon U \to V$ is an isomorphism with respect to the operations in $\mathscr{C}_2$, then one of the following holds:*
   - *$U = V$, and h is the identity on U.*
   - *U and V are singleton subalgebras.*

Proof. (1) and (2) are obvious.

(3): For any a, b, define $\psi_{a,b}(a, b) = a$, and $\psi_{a,b}(x, y) = y$ otherwise. Clearly, $\psi_{a,b} \in \mathscr{C}_2$.

Assume that U contains at least 2 elements, and let $u \in U$ with $h(u) \ne u$. If there is some element $a < u$ in U, then the inequality

$$h(\psi_{a,u}(a, u)) = h(a) \ne h(u) = \psi_{a,u}(h(a), h(u))$$

shows that h is not an internal isomorphism. Otherwise, let $u' \in U \setminus \{u\}$ and $b := u \vee u' \in U$, then $u < b$, and we get

$$h(\psi_{u,b}(u, b)) = h(u) \ne h(b) = \psi_{u,b}(h(u), h(b)).$$ □

Proof of Theorem 2.1. We just need to collect a few implications:

(1) For any local clone $\mathscr{D} \supseteq \mathscr{C}_2$, the relation $R_{\mathscr{D}}$ is a congruence order. [The main property to check is 2.3(∗∗): If $\langle x\rangle_{\mathscr{D}} \subseteq \langle z\rangle_{\mathscr{D}}$ and $\langle y\rangle_{\mathscr{D}} \subseteq \langle z\rangle_{\mathscr{D}}$, then $\langle x \vee y\rangle_{\mathscr{D}} = \langle x, y\rangle_{\mathscr{D}} \subseteq \langle z\rangle_{\mathscr{D}}$.]
(2) The map $\mathscr{D} \mapsto R_{\mathscr{D}}$ is monotone on the set of all clones $\mathscr{D} \supseteq \mathscr{C}_2$. [Obvious.]
(3) For any congruence order $\preccurlyeq$, the clone $\mathscr{E}(\preccurlyeq)$ is a local clone extending $\mathscr{C}_2$. [Obvious.]
(4) The map $\preccurlyeq\ \mapsto \mathscr{E}(\preccurlyeq)$ is monotone on the set of congruence orders. [By 2.7.]
(5) Any congruence order $\preccurlyeq$ coincides with $R_{\mathscr{E}(\preccurlyeq)}$. [By 2.8.]
(6) Finally, we claim that each local clone $\mathscr{D} \supseteq \mathscr{C}_2$ coincides with $\mathscr{D}' := \mathscr{E}(R_{\mathscr{D}})$: From 2.9, we know that $\mathscr{D}$ and $\mathscr{D}'$ have the same 1-generated subalgebras, so from 2.11, we conclude that they have the same subalgebras. By 2.12, they have the same internal isomorphisms, so by Pixley's theorem they must be equal. □

Example 2.13. Let $(X, <)$ be a linearly ordered set. Then the congruence relations on $(X, \max)$ are exactly the equivalence relations with convex classes. As a special case, consider the semilattice $(\mathbb{N}, \max)$. A congruence relation is just a partition of $\mathbb{N}$ into disjoint intervals.

The map $\theta \mapsto A_\theta := \{\max E : E \text{ is a finite congruence class}\}$ is an antitone 1–1 map from the congruence relations onto $\mathscr{P}(\mathbb{N})$, the power set of $\mathbb{N}$. The empty set corresponds to $\mathscr{O}$, or to the equivalence relation with a single class; the set $\mathbb{N}$ itself corresponds to $\mathscr{C}_2$, or to the equivalence with singleton classes.

3. A large interval of clones

On any infinite set X, we will define a clone $\mathscr{C}_3$ such that the interval $[\mathscr{C}_3, \mathscr{O}]$ in the full clone lattice is very large (with $2^{2^{|X|}}$ precomplete elements) but still reasonably well understood.

Definition 3.1.

(1) For $A \subseteq X$ and $n \geq 1$, let $\Delta_n(A) := \{(a, \dots, a) \in X^n : a \in A\}$.
(2) For any function $f \in \mathscr{O}^{(n)}$, let $f^{(1)} \in \mathscr{O}^{(1)}$ be defined by $f^{(1)}(x) = f(x, \dots, x)$.
(3) For any function $f \in \mathscr{O}^{(n)}$, we let $\text{fix}(f) = \{x : f^{(1)}(x) = x\}$, and $\text{nix}(f) = \{x : f^{(1)}(x) \neq x\}$.
(4) For any clone $\mathscr{C}$, we define $\text{fix}(\mathscr{C}) := \{\text{fix}(f) : f \in \mathscr{C}\}$, and $\text{nix}(\mathscr{C}) := \{\text{nix}(f) : f \in \mathscr{C}\}$.
(5) For any family $T \subseteq \mathscr{P}(X)$, we define

$$\mathscr{E}_T := \{f \in \mathscr{O} : \text{fix}(f) \in T\} = \bigcup_{A \in T} \bigcap_{a \in A} \text{Pol}(\{a\}).$$

Definition 3.2. Let $\mathscr{C}_3 := \mathscr{E}_{\{X\}}$ be the clone of *idempotent* functions, i.e., of all functions satisfying $f(x, \dots, x) = x$ for all x.

Theorem 3.3. *The map $T \to \mathscr{E}_T$ is an order isomorphism between the set of all filters on X (including the improper filter $\mathscr{P}(X)$) and the set of all clones above $\mathscr{C}_3$.*

In particular, the precomplete clones above $\mathscr{C}_3$ are exactly the clones of the form $\mathscr{E}_U$, where U is an ultrafilter on X.

Remark 3.4. The subalgebras of $\mathscr{C}_3$ are exactly all singleton sets, and $\mathscr{C}_3$ contains the ternary discriminator. Hence, every local clone above $\mathscr{C}_3$ is determined by its subalgebras. For every $A \subseteq X$, the clone

$$\mathscr{E}_{\{A\}} := \bigcap_{a \in A} \text{Pol}(\{a\}) = \{\, f : f(a, a, \dots, a) = a \text{ for all } a \in A \,\}$$

is a local clone whose subalgebras are exactly the singleton sets $\{a\}$ with $a \in A$. Hence, the local clones above $\mathscr{C}_3$ are exactly the clones of the form $\mathscr{E}_{\{A\}}$ for $A \subseteq X$. In the language of Theorem 3.3: the local clones above $\mathscr{C}_3$ are exactly the clones corresponding to principal filters.

Lemma 3.5. *Assume that $\mathscr{D} \supseteq \mathscr{C}_3$ is a clone, $f \in \mathscr{D}$, and* $\text{fix}(f) \subseteq \text{fix}(g)$. *Then $g \in \mathscr{D}$. Hence, every clone $\mathscr{D} \supseteq \mathscr{C}_3$ is determined by* $\text{nix}(\mathscr{D})$*: $f \in \mathscr{D} \Leftrightarrow$* $\text{nix}(f) \in \text{nix}(\mathscr{D})$.

Proof. For $\bar{x} = (x_1, \dots, x_n)$, define

$$G(\bar{x}, y) = \begin{cases} x_1 & \text{if } x_1 = \dots = x_n = y, \\ g(\bar{x}) & \text{otherwise.} \end{cases}$$

Clearly, $G \in \mathscr{C}_3$. Let $\bar{x} \in \Delta_n(\mathrm{fix}(f)) \subseteq \Delta_n(\mathrm{fix}(g))$; then $g(\bar{x}) = G(\bar{x}, f^{(1)}(x_1)) = x_1$. If $\bar{x} \in \Delta_n(X) \setminus \Delta_n(\mathrm{fix}(f))$, then $f^{(1)}(x_1) \neq x_1$, so $G(\bar{x}, f^{(1)}(x_1)) = g(\bar{x})$ by definition; the same holds for $\bar{x} \notin \Delta_n(X)$. Hence, $g(\bar{x}) = G(\bar{x}, f^{(1)}(x_1))$ for all $\bar{x} \in X^n$. □

Lemma 3.6. *Let $\mathscr{D}$ be a clone with $\mathscr{C}_3 \subseteq \mathscr{D}$, $\mathscr{D} \neq \mathscr{O}$. Then* $\mathrm{nix}(\mathscr{D})$ *is an ideal, and* $\mathrm{fix}(\mathscr{D})$ *is a filter.*

Proof. Lemma 3.5 shows that $\mathrm{nix}(\mathscr{D})$ is downward closed. Now let $A_\ell = \mathrm{nix}(f_\ell)$, $f_\ell \in \mathscr{D}$ for $\ell = 1, 2$, and assume that $A_1 \cap A_2 = \emptyset$.

Let $B = X \setminus (A_1 \cup A_2)$. We may assume that either $|A_1| \geq 2$, or $B \neq \emptyset$ (or both). In either case, there is a unary function f'_1 with $\mathrm{nix}(f'_1) = A_1$, and f'_1 maps A_1 into $A_1 \cup B$. By Lemma 3.5, $f'_1 \in \mathscr{D}$. So, $f_2 \circ f'_1 \in \mathscr{D}$. Since $\mathrm{nix}(f_2 \circ f'_1) = A_1 \cup A_2$, we see that $\mathrm{nix}(\mathscr{D})$ is closed under $\cup$. □

Fact 3.7. For all filters T and all $A \in T$, there is a unary function f satisfying $\mathrm{fix}(f) = A$.

Proof of Theorem 3.3. Again, we just collect some implications.

(1) The maps $\mathscr{D} \mapsto \mathrm{fix}(\mathscr{D})$ and $T \mapsto \mathscr{E}_T$ are clearly monotone with respect to set inclusion.
(2) For every clone $\mathscr{C} \supseteq \mathscr{C}_3$, the set $\mathrm{fix}(\mathscr{C})$ is a filter (by Lemma 3.6).
(3) For every filter T, the set $\mathscr{E}_T$ is a clone. (Obvious.)
(4) For every filter T, we have $T = \mathrm{fix}(\mathscr{E}_T)$. (By Fact 3.7.)
(5) For every clone $\mathscr{D} \supseteq \mathscr{C}_3$, we have $\mathscr{D} = \mathscr{E}_{\mathrm{fix}(\mathscr{D})}$: Let $\mathscr{D}' = \mathscr{E}_{\mathrm{fix}(\mathscr{D})}$. From the previous item we conclude $\mathrm{fix}(\mathscr{D}') = \mathrm{fix}(\mathscr{D})$.

Now by Lemma 3.5 we see $\mathscr{D} = \mathscr{D}'$. □

Remark 3.8. If we regard the set X as a discrete topological space, then the Stone–Cech compactification of X is

$$\beta X = \{U : U \text{ is an ultrafilter on } X\}.$$

There is a canonical 1–1 order-preserving correspondence between the filters on X (ordered by $\subseteq$) and the closed subsets of βX (ordered by $\supseteq$).

So, the interval $[\mathscr{C}_3, \mathscr{O}]$ in the full clone lattice is isomorphic (as a complete lattice) to the family of closed subsets of βX, ordered by reverse inclusion: $\mathscr{O}$ corresponds to the empty set, each precomplete clone in $[\mathscr{C}_3, \mathscr{O}]$ corresponds to a singleton set.

Note that for any closed subset $F \subseteq \beta X$ and any $p \in \beta X \setminus F$, also $F \cup \{p\}$ is closed, and moreover,

F covers G (i.e., $F \supset G$, and the interval (G, F) is empty) iff $G = F \cup \{p\}$ for some $p \in \beta X \setminus F$

In particular, let $\mathscr{C}_{\rm bd} \supseteq \mathscr{C}_3$ be the clone corresponding to the ideal of small sets, i.e.,

$$\mathscr{C}_{\rm bd} := \{f \in \mathscr{O} : \exists B \subseteq X, |B| < |X|, \forall x \in X \setminus B : f(x, \dots, x) = x\}.$$

Then every clone $\mathscr{C} \supsetneq \mathscr{C}_{\rm bd}$ has exactly $2^{2^{|X|}}$ lower neighbors in the clone lattice; the clone corresponds to a closed set F, and the lower neighbors correspond to closed sets $F \cup \{p\}$. This is a special case of a theorem of [Ma81].

Acknowledgement. We are grateful to the referee for pointing out Pixley's theorem, which led to essential simplifications of our proofs.

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Martin Goldstern

DMG/Algebra, TU Wien, Wiedner Hauptstraße 8-10/104.1, 1040 Wien, Austria
e-mail: Martin.Goldstern@tuwien.ac.at
URL: http://www.tuwien.ac.at/goldstern/

Saharon Shelah

Mathematics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel
e-mail: shelah@math.huji.ac.il
URL: http://shelah.logic.at/