DIFFERENTIALLY CLOSED FIELDS

BY

SAHARON SHELAH

ABSTRACT

We prove that even the prime, differentially closed field of characteristic zero, is not minimal; that over every differential radical field of characteristic p, there is a closed prime one, and that the theory of closed differential radical fields is stable.

Introduction

Let T_d be the theory of differential fields, that is, the axioms of fields in addition to the following axioms on the (abstract) differentiation operator:

$$D(x + y) = Dx + Dy$$

$$D(xy) = (Dx)y + xDy.$$

Let an upper index indicate the characteristic of the field.

T_d is a natural generalization of the theory of fields which Ritt [4] invented. It is natural to look for an analog to the algebraic closure of a field. Seidenberg [7] has done algebraic work along these lines. Using his work, Robinson [5] showed that T_d^0 has a model completion T_{dc}^0 (that is, the theory of differentially closed fields of characteristic zero). Thus every T_d^0-field can be extended to a T_{dc}^0-field, however Robinson does not give an explicit set of axioms for T_{dc}^0. Blum [1] showed that the following axioms suffice:

(1) T_d^0

(2) For differential polynomials P_1, P_2 (in single variable y) of order m_1, m_2, for $m_1 > m_2$, there is a solution of $P_1 = 0$ which is not a solution of $P_2 = 0$, and there is a solution of $P_2 = 0$, provided that P_2 has degree greater than zero.

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Blum also showed that T_{dc}^0 is totally transcendental, and the maximal Morley rank is ω hence over every T_d^0-field there is a prime T_{dc}^0-field. (See Morely [3], or [6] for example.) By a general result of [10] (or [7], for example) this prime T_{dc}^0-field is unique. However here we answer a question of Blum (which appears in [6]) by showing that the prime T_{dc}^0-field is not necessarily minimal. This shows that the analogy with algebraically closed fields fails. The proof indicates to me (in contrast to Sacks [6, p. 307]) the following conjecture.

CONJECTURE 1.

(i) For every $m < \omega$ there is a $T_d{}^0$-field F, and a differential polynomial $P(y)$ of order m with coefficients in F such that the Morley rank of $P(y) = 0$ is 1, (or at least less than m) and that even P has integer coefficients.

(ii) Moreover, there is a differential polynomial of order 0 (less than m), $P_1(y) \neq 0$, such that if $y_1, \cdots, y_n$ are solutions of $P(y) = 0 \wedge P_1(y) \neq 0$ and $P_2(y_1, \cdots, D^i y_j, \cdots)_{i<m} = 0$, where P_2 is a polynomial with coefficients in F, then P_2 is the zero polynomial. (ii) implies (i).

Let us try to generalize to partial differentiation. Then we have a field with n differential operators, $D_1, \cdots, D_n$, satisfying in addition, that $D_i D_j y = D_j D_i y$. But nothing new results. When the characteristic of the field is zero, we obtain a model completion with elimination of quantifiers, which is totally transcendental and has maximal Morley rank ωn. I am quite sure that for characteristic p as well, this does not make any essential difference. If we add D_n, $n < \omega$, we arrive at a stable but not superstable theory.

We also show that although T_{dc}^0 is trivial in some aspects, when we allow cardinality quantifiers, it becomes complex. Hence T_{dc}^0 has 2^λ non-isomorphic models in every $\lambda > \aleph_0$.

CONJECTURE 2. T_{dc}^0 has $2^{\aleph_0}$ non-isomorphic models of power $\aleph_0$.

Wood [13], again using Seidenberg [7] deals with $T_d{}^p$ for $p > 0$. (Notice that here if an element a has a p-th root, then it is constant, that is, $Da = 0$.) Wood showed that T_d^p does not have the amalgamation property. However, if we add the axiom

$$[Dx = 0 \rightarrow (r(x)^p = y)] \wedge [Dx \neq 0 \rightarrow r(x) = 0]$$

and obtain T_{rd}^p (that is, the theory of radical differential fields of characteristic p), then it has the amalgamation property, and has a model completion T_{rdc}^p, which has elimination of quantifiers. (A $T_d{}^p$-field F can be expanded to a T_{rd}^p-field if

$Da = 0 \to (\exists x)(x^p = a)$ for $a \in F$, and the expansion is unique; thus we do not differentiate strictly between the field and its expansion.) Wood showed that, unlike F^0_{dc}, T^p_{rdc} is not totally transcendental, hence the existence of a prime T^p_{rdc}-field remains an open question. Wood and the author independently solved the question (the author proved it after [9] but before [14] were submitted, see [15]). We do not know however, whether Conjecture 3 holds.

NOTE. Some of the results of this paper were previously announced in [9].

CONJECTURE 3. The prime T^p_{rdc}-field over any T^p_{rdc} is unique.

By small changes in [13] it follows that T^p_{rdc} is not superstable (see [12]).

1. The non-minimality of the prime differentially closed field

Now we state the main lemma of this section.

LEMMA 4. *Let F be a differential field of characteristic zero, $y_1, \cdots y_n$ distinct nonzero solution (in F) of*

$$Dy = \frac{y}{1+y}.$$

If $P(x_1, \cdots, x_n)$ is a polynomial with rational coefficients and, in F, $P(y_1, \cdots, y_n) = 0$, then P is identically zero.

PROOF.

Stage (i). Without loss of generality, assume F includes the field F_0 of algebraic numbers. We suppose $y_1, \cdots, y_{n+1} \in F$ are distinct and not zero, $P_0(y_1, \cdots, y_{n+1}) = 0$, $P_0(x_1, \cdots, x_{n+1})$ is a nontrivial polynomial with algebraic coefficients and we shall arrive at a contradiction. Without loss of generality, n is minimal; for this n the degree of P_0 in x_{n+1} is minimal and then the degree of P_0 is minimal. Hence $P_0(y_1, \cdots, y_n, x)$ is indecomposable over $F_0(y_1, \cdots, y_n)$, and $y_1, \cdots, y_n$ are transcendentally independent over F_0.

Stage (ii). Let us look at the function $x = y + \ln y$. Clearly, for real $y > 0$, $y + \ln y$ is an increasing function whose range is the set of real numbers; let its inverse be $y = f(x)$. Thus $f(x)$ is defined for every real x; it increases with x and as $x \to -\infty$, $f(x) \to 0$. As $f(x) + \ln f(x) = x$, $e^{f(x)} f(x) = e^x$, hence for $x \to -\infty$ $e^{o(1)} f(x) = e^x$ or $f(x) = e^x(1 + o(1))$. The function $f(x)$ is also defined for complex arguments, and then it is holomorphic but it is not single valued.

Stage (iii). Look at the differential equation $dy/dx = y/(1+y)$ (for $y \neq 0$) or equivalently $(1+y)/y \; dy = dx$ or $dy/y + dy = dx$ or $\ln y + y = x + c$ or $y = f(x+c)$. Thus if $y = g(t)$ is a function with complex values defined for all negative real numbers $t < t_0$, and if it is a solution of the equation, then for some complex c and branch of f, $g(t) = f(t+c)$ for every such t. Because, if $t_1 < t_0$, $g(t_1) \neq 0, -1$, choose $c_1 = g(t_1) + \ln g(t_1) - t_1$; then for a proper branch of f, $g(t_1) = f(t_1 + c_1)$. Hence by the uniqueness theorem $g(t) = f(t+c)$ for a neighborhood of t_1, and hence for all $t < t_0$. (We choose $g(t_1) \neq 0$ to make $f(g(t_1))$ well defined, and $g(t_1) \neq -1$ to avoid the branching point when $d/dy(y + \ln g(y)) = 0$). As $g(t)$ for $t < t_1$ cannot take always the values $0, -1$, we are through (remember that we assume $y \neq 0$).

Looking at $y + \ln y = x + c$, we know that

(3) $$\text{if } x \to -\infty \text{ then either } y \to -\infty \text{ or } y \to 0.$$

If $y \to -\infty$, $y = x + O(\ln x)$.

If $y \to 0$, then as before, $y = e^{x+c}(1 + o(1))$.

Stage (iv). Choose real negative numbers $a_1, \cdots, a_n$ such that $f(a_1), \cdots, f(a_n)$ will be algebraic numbers which are linearly independent over the rationals. By Lindemann's theorem (see [8])

$$e^{f(a_1)}, \cdots, e^{f(a_n)}$$

are transcendentally independent Since $f(a_i)$ are algebraic $\neq 0$, and $e^{f(x)}f(x) = e^x$; also $e^{a_1}, \cdots, e^{a_n}$ are transcendentally independent.

Stage (v). If $P(x_1, \cdots, x_n)$ is nontrivial, polynomial with algebraic coefficients, then for some $t_1 < 0$ $P(f(t + a_1), \cdots f(t + a_n)) \neq 0$ for all $t < t_1$. Suppose not. Let us see what will be the dominant term when $t \to -\infty$. If P has a free constant as $t \to -\infty$, $f(t + a_i) \to 0$ (see Stage (ii)). This is a contradiction. Now let $P(x_1, \cdots, x_n) = \sum_{\eta \in I} c_\eta \prod_{i=1}^n x_i^{\eta(i)}$, for c_η algebraic, $\eta(i)$ natural numbers.

Then $f(t + a_i) = \exp(t + a_i)(1 + o(1))$,

$$f(t + a_i)^n = \exp(nt + na_i)(1 + o(1))$$

$$\prod_{i=1}^{n} f(t + a_i)^{\eta(i)} = \exp([\Sigma\eta(i)]t)\exp(\Sigma\eta(i)a_i)\ (1 + o(1)).$$

As $t \to -\infty$, clearly the dominant terms will be those with minimal $\Sigma\eta(i)$, say m. Let $J = \{\eta \in I : \Sigma_{i=1}^n \eta(i) = m\}$ so for some $\eta' \in J$ $c_{\eta'} \neq 0$.

$$\begin{aligned} P(\cdots, f(t + a_i), \cdots) &= \sum_{\eta \in J} c_\eta \exp(mt)\exp(\Sigma\eta(i)a_i)(1 + o(1)) + O(\exp((m+1)t)) \\ &= \sum_{\eta \in J} c_\eta \prod_{i=1}^{n} \exp(\Sigma\eta(i)a_i))\exp(mt)(1 + o(1)) \\ &\quad + O(\exp(t(m+1)). \end{aligned}$$

For this to be zero for arbitrarily small $t < 0$, necessarily

$$0 = \sum_{\eta \in J} c_\eta \exp(\Sigma \eta(i) a_i) = \sum_{\eta \in J} c_\eta \prod_{i=1}^{n} (\exp(a_i))^{\eta(i)} .$$

As $\eta' \in J$, $c_{\eta'} \neq 0$, this contradicts the transcendental independence of the e^{a_i} (see Stage (iv)).

Stage (vi). By (v), $P_0(f(t+a_1), \cdots, f(t+a_n), y) = 0$ as an equation in y, has a solution $y = g(t)$ for each $t < t_0$, for some t_0 (make the leading coefficient nonzero). Also we can assume that for $t < t_0$, the resultant of this polynomial is not zero. (If it is identically zero as a polynomial in $f(t+a_i)$, $P_0(x_1, \cdots, x_{n+1})$ will be decomposable over $F_0(x_1, \cdots, x_n)$, contradicting the minimality of the degree of P_0 in x_{n+1}.)

Thus we can choose one branch of the solution $y = g(t)$ hence, clearly, g is an analytic function.

Stage (vii). g is a solution of $Dy = y/(1+y)$.

Let $P_0 = \Sigma_{\eta \in I} c_\eta \prod_{i=1}^{n+1} x_i^{\eta(i)}$ (c_η algebraic, $\eta(i)$ natural numbers). Note that if $h(t)$ solves $Dy = y/(1+y)$, then $(d/dt)\, h(t)^m = h(t)^m\, [m/(1+h(t))]$. Then

$$\begin{aligned} 0 &= \frac{d}{dt} P_0(f(t+a_1), \cdots, f(t+a_n), g(t)) \\ &= \sum_{\eta \in I} \left[c_\eta \prod_{i=1}^{n} f(t+a_i)^{\eta(i)} g(t)^{\eta(n+1)} \left(\sum_{i=1}^{n} \frac{\eta(i)}{1+f(t+a_i)} \right) \right] \\ &\quad + \sum_{\eta \in I} \left[c_\eta \prod_{i=1}^{n} f(t+a_i)^{\eta(i)} g(t)^{\eta(n+1)-1} \eta(n+1) \right] \frac{dg(t)}{dt} . \end{aligned}$$

The coefficient of $dg(t)/dt$ is $(d/dy)\, P_0(x_1, \cdots, x_n, y)$. As for all $t < t_0$ the resultant of $P_0(f(t+a_1), \cdots, f(t+a_n), y)$ is not zero, it has no common root with its derivative. So from the above-mentioned equality we can solve dg/dt (since the a_i's are real $f(t+a_i) > 0$, hence $1 + f(t+a_i) \neq 0$.) Thus, $dg/dt = P_1(\cdots f(t+a_i) \cdots, g(t)) / P_2(\cdots, f(t+a_i), \cdots, g(t))$. In the same way, in the differential field F,

$$Dy_{n+1} = P_1(\cdots, y_i, \cdots, y_{n+1}) / P_2(\cdots, y_i, \cdots, y_{n+1}).$$

On the other hand $Dy_{n+1} = y_{n+1}/(1+y_{n+1})$, so define

$$P_3(y_1, \cdots, y_{n+1}) \equiv P_1(y_1, \cdots, y_{n+1})(1+y_{n+1}) - P_2(y_1, \cdots, y_{n+1}) y_{n+1} = 0.$$

As n was minimal, $y_1, \cdots, y_n$ were transcendentally independent. Hence the polynomial $P_3(y_1, \cdots, y_n, x)$ is divisible by $P_0(y_1, \cdots, y_n, x)$. (The quotient has coefficients in $F_0(y_1 \cdots, y_n)$ and we can assume no denominator becomes zero when we replace y_i by $f(t + a_i)$ $t < t_0$.) So $P_3(f(t + a_i), \cdots, f(t + a_n), g(t)) = 0$ or equivalently $dg(t)/dt = g(t)/(1 + g(t))$.

Stage (viii). For some b and proper branch of f, $g(t) = f(t + b)$ for every $t < t_0$; and

(a) $f(t + b) = t + O(\ln|t|)$ for $t \to -\infty$

or

(b) $f(t + b) = e^{t+b}(1 + o(1))$ for $t \to -\infty$.

We obtain this result by combining stages (iii) and (vii) and (3).

We shall now contradict possibility (a). What will be the dominant part of $P_0(f(t + a_1), \cdots, f(t + a_n), f(t + b))$ (which is identically zero)?

If $P_0(x_1, \cdots, x_{n+1})$ has a term $c_1 x_{n+1}^m$, $m \geqq 0$ $c_1 \neq 0$, letting m be the maximal one, we obtain

$$P_0(f(t + a_1), \cdots f(t + a_n), f(t + b)) = c_1 t^m + O(t^{m-1}\ln|t|).$$

(Remember $f(t) = e^t(1 + o(n))$ for $t \to -\infty$). This goes to infinity when $t \to -\infty$ a contradiction, so there is no such term. Let

$$P_0(x_1, \cdots x_{n+1}) = \sum_{\eta \in I} c_\eta \prod_{i=1}^{n+1} x_i^{\eta(i)}$$

where c_η are algebraic. Then this equals

$$\sum_{\eta \in I} c_\eta \prod_{i=1}^{n} \exp((t + a_i)\eta(i))\, t^{\eta(n+1)}(1 + o(1)) \tag{4}$$

so the dominant terms are those with $\sum_{i=1}^n \eta(i)$ minimal, say m, and among them, those with maximal $\eta(n + 1)$, say k. So letting $J = \{\eta \in I \colon \Sigma\eta(i) = m,\ \eta(i + 1) = k\}$, (4) equals

$$\left(\sum_{\eta \in J} c_\eta \prod_{i=1}^{n} \exp(a_i \eta(i))\right) e^{mt} \cdot t^k (1 + o(1)).$$

Hence necessarily $\sum_{\eta \in J} c_\eta \prod_{i=1}^n (\exp(a_i))^{\eta(i)} = 0$, contradicting Stage (iv).

So, necessarily, (b) holds.

Stage (ix). Let $a_{n+1} = b$; by the last stage $f(t + a_i) = \exp(t + a_i)(1 + o(1))$ for $1 \leqq i \leqq n + 1$.

As $P_0(\cdots,f(t+a_i),\cdots)=0$ and the dominant part of it for $t\to-\infty$ is

$$\left(\sum_{\eta\in J} c_\eta \prod_{i=1}^{n} (\exp(a_i)^{\eta(i)}\right)\exp(t\Sigma\eta(i))$$

(J is the set of $\eta\in I$ with minimal $\Sigma\eta(i)$ so

$$P_4(x_1,\cdots,x_{n+1})=\sum_{\eta\in J} c_\eta \prod_{i=1}^{n+1} x_i^{\eta(i)}$$

is homegeneous) then necessarily $P_4(\cdots,e^{a_i},\cdots)=0$, that is, $e^{a_1},\cdots,e^{a_{n+1}}$ are transcendentally dependent. As P_4 is homogeneous, for every t,

$$P_4(\cdots,e^{t+a_i},\cdots)=0 \text{ or}$$

(a) $P_4(\cdots,f(t+a_i)\exp(f(t+a_i)),\cdots)=0$; but also

(b) $P_0(\cdots,f(t+a_i),\cdots)=0$.

Stage (x). We choose $a_1,\cdots,a_n$, only so that $P^i(\cdots,a_i,\cdots,\cdots,e^{a_i},\cdots)\neq 0$ for a specific finite set of polynomials P^i with algebraic coefficients. Thus there is an $\varepsilon>0$ and t_0' so that every $a_i\in(a_i-\varepsilon,\ a_i'+\varepsilon)$ will satisfy the same demands for $t<t_0'$, hence all our conclusions, in particular the existence of a_{n+1}'. Hence for $t<t_0'$ (by (a), (b) from stage (ix))

(a) $P_4(\cdots,f(t+a_i')\exp(f(t+a_i')),\cdots)=0$ and

(b) $P_0(\cdots,f(t+a_i'),\cdots)=0$.

Let k_1 be the degree of $P_0(x_1,\cdots x_{n+1})$, and k_2 be the dimension of the field F_1 generated by the coefficients of P_0 over the rationals.

Now choose $t^*<t_0'$ so that $t^*+a_i+\varepsilon<t_0'$; and choose a_i' in $(a_i'-\varepsilon,a_i+\varepsilon)$ so that $f(t^*+a_i')$, $i=1,n$ are algebraic but not linearly dependent over the rationals and moreover $f(t^*+a_i')=q_i^1+q_i^2a^i$, q_i^1,q_i^2 rationals, $q_i^2\neq 0$ and a^i is the $p^{(i)}$-root of 2 where $p^{(1)}>k_1k_2$, $p^{(i+1)}>\prod_{j\leq i}p^{(j)}k_1$, $p^{(i)}$ natural numbers.

By (b) $f(t^*+a_{n+1}')$ is algebraic over $f(t^*+a_i')$, $i=1,n$; hence algebraic, and $\exp(f(t^*+a_i'))$, $i=1$, n are transcendentally independent by Lindman theorem, but $\exp(f(t+a_{n+1}'))$ depends on them, by (a).

By (a) and Lindman's theorem (see [8]), $f(t^*+a_{n+1}')$ is linearly dependent on $a_1',\cdots,a_n'$ over the rationals, hence for rationals q_i, $f(t^*+a_{n+1}')=\sum_{i=1}^n q_if(t^*+a_i')$. We can substitute this in $P_0(\cdots,f(t^*+a_i'),\cdots)=0$ and obtain $P_5(f(t^*+a_1'),\cdots,f(t^*+a_n'))=0$, where P_5 is a polynomial over F_1, and the degree of P_5 is $\leq k_1$. This implies that P_5 is identically zero by dimensional consideration, and the condition on the set of $p^{(i)}$.

If we substitute in $P_0(x_1,\cdots,x_{n+1})$ $x_{n+1} = \Sigma q_i x_i$, we obtain the zero polynomial. By the minimality of the degree of P_0 in x_{n+1}, and in general, we can assume $P_0(x_1,\cdots,x_{n+1}) = x_{n+1} - \Sigma q_i x_i$.

Stage (xi). Now

$$y_{n+1} = \sum_{i=1}^{n} q_i y_i \text{ for } q_i \text{ complex rationals.}$$

Hence

$$Dy_{n+1} = \sum_{i=1}^{n} q_i D y_i = \sum_{i=1}^{n} q_i \frac{y_i}{1+y_i} = Dy_{n+1} = \frac{y_{n+1}}{1+y_{n+1}}$$

$$= \sum_{i=1}^{n} q_i \frac{y_i}{1+y_i} = \sum_{i=1}^{n} q_i y_i \Big/ \left(1 + \sum_{i=1}^{n} q_i y_i\right).$$

As $y_1,\cdots,y_n$ are transcendentally independent, this is an identity so it holds if we substitute for the set of y_i complex numbers. If $i \neq j$, $q_i \neq 0$, $q_j \neq 0$ set $y_i = -1 + \varepsilon$, $y_j \neq -1, -(1+q_i y_i)1/q_j$ and $y_k = 0$ for $k \neq i,j$. Then we obtain a contradiction as $\varepsilon \to 0$. Thus $n = 1$, $y_2 = y_{n+1} = q_1 y_1$, and

$$q_1 \frac{y_1}{1+y_1} = \frac{q_1 y_1}{1+q_1 y_1}.$$

For $y_1 \neq 0$ we obtain $q_1 = 0$ or $q_1 = 1$. If $q_1 = 0$, $y_2 = 0$; if $q_1 = 1$, $y_2 = y_1$, a contradiction in any case.

THEOREM 5. *The prime differentially closed field is not minimal. (It is the prime T^0_{dc}-field over the field of rational numbers.)*

PROOF. Let F be that field. The equation $Dy = y/(1+y)$ is not an algebraic formula since in some T^0_{dc}-field (of functions) it has infinitely many solutions. Hence it has infinitely many nonzero solutions $y_i \in F$, $i < \omega$. Since the theory T^0_{dc} has elimination of quantifiers, clearly the $\{y_i : i < \omega\}$ is an indiscernible set, hence by [10] (or see for example [6]), F is not minimal. (The elaboration for this particular case is easy: there is a field $F' \subseteq F$ prime over the field generated by $\{y_{2i} : i < \omega\}$, and $F' \neq F$ as $y_{2i+1} \notin F'$).

LEMMA 6. *Let F be a differential field; $\{f,g\}$ differentially independent elements of F. Let $y_1,\cdots,y_n$, be distinct nonzero solutions of $Dy = yf/(1+y)$; $y^1,\cdots,y^m$ be distinct nonzero solutions of $Dy = (y/(1+y))g$. Then for no non-trivial polynomial P with rational coefficients, $P(y_1,\cdots,y_n, y^1,\cdots,y^m) = 0$.*

PROOF. Similar to that of Theorem 5.

REMARK. No doubt the restrictions on f, g can be weakened.

THEOREM 7. *For every $\lambda > \aleph_0$, T_{dc}^0 has 2^λ non-isomorphic fields of power λ.*

PROOF. Let F be a differentially closed field of power λ, f_i, $g_i \in F$, and $\{f_i : i < \lambda\} \cup \{g_i : i < \lambda\}$ a differentially independent set with F prime over it.

Let $\phi(x_1, x_2) = [Dx_1 = (x_1/(1 + x_1))x_2]$. By Lemma 6, if y is a new element satisfying $\phi(y, f_i g_j)$, F' the prime differentially closed field over $F(y)$, and $\langle h, l\rangle \neq \langle i, j\rangle$ then no $y' \in F' - F$ satisfies $\phi(y', f_h g_l)$. By repeating, we can obtain for any binary relation R over λ a field F_R such that

$$|\{y \in F_R : \phi(y, f_i g_j)\}| = \aleph_1 \text{ iff } \langle i, j\rangle \in R \text{ iff}$$

$$|\{y \in F_R : \phi(y, f_i g_j)\}| \neq \aleph_0.$$

Then by [11] the result follows easily.

2. On the existence of T^p_{rdc}-prime field over T^p_{rd}-field

THEOREM 8. *Over every differential radical field of characteristic p (= T^p_{rd}-field) there is a prime differentially closed radical field (= T_{rdc}-field).*

PROOF.

Stage (i). By Morley [3] (or see [6]) it suffices to prove the following. (Remember that by Wood [13], T^p_{rdc} has elimination of quantifiers.)

Let F be a T^p_{rd}-field and let $\phi(x)$ be a consistent formula with parameters from F. Then there is a consistent formula $\psi(x)$ with parameters from F such that $\psi(x) \rightarrow \phi(x)$ and $\psi(x)$ defines an isolated type, that is, if y satisfies ψ, then the structure of $F_{rd}(y)$ (the T^p_{rd}-field generated by F, y) is uniquely defined. Without loss of generality, ϕ is a quantifier-free formula and moreover it is a conjunction of atomic formulas and negation of action formulas.

We can also assume without loss of generality that F is separately closed.

Stage (ii). Let $F' \supseteq F$ be a T^p_{rd}-field in which y satisfies $\phi(x)$. Let $\tau_0 = \tau_0(y) = y$ and $\tau_1 = \tau_1(y), \cdots, \tau_n = \tau_n(y)$ be the terms appearing (maybe as subterms)in $\phi(y)$ which are of the form $r(\cdots)$. (Remember r is the pth root.) Let $n(i)$ be the highest n such that $D^n\tau_i$ appear in ϕ. We can assume without loss of generality that in ϕ there appears no term of the form $D(\sigma_1 + \sigma_2)$ or $D(\sigma_1\sigma_2)$ (since then we could simplify it); and that if $r(\sigma)$ appears in it, then one of the conjuncts of ϕ is $D\sigma = 0$

Thus if $F \subseteq F'' \subseteq F'$, F'' is a T^p-field and $D^j\tau_i(y) \in F''$ for $j \leqq n(i)$, then $\phi(y)$ is meaningful in F''.

Stage (iii). We derive ϕ' from ϕ by adding to it for each $i \leqq n$ a conjunct as follows:

(a) If there is an $m = m(i)$ such that $D^m\tau_i(y)$ is in the separable closure of $F'_i = F(\cdots, D^j\tau_k(y), \cdots, D^l\tau_i(y), \cdots)_{k<i\, l<m}$ then let $P_i(x) = \sum_l \sigma^i_l x^l$ be an indecomposable polynomial over F'_i of which $D^m\tau_i(y)$ is a root. Then the conjunct will be $\sum_l \sigma^i_l [D^m\tau_i(y)]^l = 0 \wedge \sigma \neq 0$ where σ is the resultant of $P_i(x)$.

This guarantees that $D^l\tau_i(y)$, $l \geqq m$ is in $F_i(D^m\tau_i(y))$ and that $D^m\tau_i(y)$ is separably algebraic over F_i'.

(b) If there is not such an m, we add nothing.

Stage (iv). Let $F'' \subseteq F'$ be the T^p-field generated by F and $D^j\tau_i(y)$ for $j \leqq n(i)$ (that is, generated only by the field operations). Supplement it by defining $D(D^{n(i)}\tau_i(y)) = 0$, if i satisfies (b) above; we obtain a T^p_d-field F^* and by [7] there is a T^p_{rd}-field $F^{**} \supseteq F^*$. Add to ϕ', for each such i, the conjunct $D^{n(i)+1}\tau_i(x) = 0$ to obtain ϕ''.

Stage (v). Now case (a) of Stage (iii) always occurs, hence we can express each $D^j\tau_i(y)$ $(j > n(i))$ by a polynomial in $\{D_k\tau_l(y) : k \leqq n(l), l \leqq n(i)\}$ with coefficients in F. Add to ϕ'' conjuncts so that the trancendence rank of $F(\cdots, D^k\tau_e(y), \cdots) = F_d(\tau_0(y), \cdots, \tau_n(y))$ is minimal. For each $j \leqq n(i)$, if $D^j\tau_i(y)$ is algebraically dependent on $\{D^l\tau_k(y) : k < i \text{ or } k = i, l < j\}$, then we obtain ϕ''' by adding conjuncts to ϕ to make the degree of the polynomial it solves as small as possible.

Without loss of generality let y in F' satisfy $\psi(y) \equiv \phi'''(y)$.

Now ψ completely determines the structure of

$$F'' =^{df} F(\cdots, D^j\tau_i(y), \cdots)_{j \leqq n(i), i \leqq n} = F_d(\tau_0(y), \cdots, \tau_n(y)).$$

If F'' is a T^p_{rd}-field, then we are through. This is equivalent to saying that $c \in F'' - F$, $Dc = 0$ implies c has a p root in F''.

Stage (vi). Suppose $c \in F'' - F$, $Dc = 0$ but c has no p root in F''. We arrive at a contradiction.

Let $c = P_0(\cdots, D^j\tau_i(y), \cdots)$, where P_0 is a polynomial over F. Now if in Stage (v) we had also added $P_0(\cdots, D^j\tau_i(x), \cdots) = b$ for any $b \in F$ to $\phi''(x)$, the transcendence rank of $F_d(\tau_0(y), \cdots)$ would have become smaller. We have not done it because it is impossible. In other words, letting

$$\theta_0(x_1) = (\exists x)(x_1 = P_0(\cdots, D^j\tau_i(x), \cdots) \wedge \psi(x))$$

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and $F^c \supseteq F''$ be a T^p_{rdc}-field, then for no $b \in F$, $F^c \models \theta_0(b)$. As T^p_{rdc} has elimination of quantifiers for some quantifier-free $\theta_1(x_1)$, $T^p_{rdc} \vdash (\forall x_1)\ [\theta_1(x_1) \equiv \theta_0(x_1)]$. Without loss of generality, F^c is $|F|^+$-saturated.

Stage (vii). Let $F^0 \subseteq F$ be the prime field (that is, the one generated by 1) and let $a_n \in F$, for $n < \omega$, be distinct elements which are in the separable closure of F^0 in F. Clearly $F \models \neg\,\theta_1(a_n) \wedge Da_n = 0$. By the compactness theorem there is an element $a \in F^c - F$, $F^c \models \neg\,\theta_1(a) \wedge D(a) = 0$. Let F^1 be the separable closure of $F^0_{rd}(a)$ in F^c and let F^2 be the separable closure of $F^0(c)$ in F''. Clearly for $b \in F^2$ $Db = 0$, and there is an embedding $f: F^2 \to F^1, f(c) = a$ which is the identity on F^* (see below). Let F^3 be the closure of F^1 to a T_{rd}-subfield of F^c. Notice that $F^* = \{b \in F^c: b$ is separably algebraic over $F^0\}$ is a T_{rd}-field; hence F^* is algebraically closed. The diagram is shown in Fig. 1 (arrows denote inclusion).

$$\begin{tikzcd}
 & F^1 \arrow[rr] \arrow[d] & & F^3 \arrow[dr] & \\
F^0 \arrow[r] & F^* \arrow[r] \arrow[drr] & F \arrow[r] & F'' \arrow[r] & F^C \\
 & & & F^2 \arrow[u] &
\end{tikzcd}$$

Fig. 1

Notice that:

(a) ([7]) although the amalgamation property does not hold for T^p_d-fields in general, if

1. $g_1: F^\delta \to F_\alpha$, $g_2: F^\delta \to F_\beta$ are embeddings of T^p_d-fields, and

2. $b \in F^\delta$, $Db = 0$ but has no p-th root in F^δ implies $g_1(b)$ has no p-th root in F_α,

3. no $b \in F_\alpha - g_1(F^\delta)$ is the root of a separable polynomial over $g_1(F^\delta)$, then there is a T^j_d-field F_γ, and embeddings $f_1: F_\alpha \to F_\gamma$, $f_2 = F_\beta \to F_\gamma$ such that $f_1 g_1 = f_2 g_2$, and without loss of generality for example f_1 is the identity.

(b) If $b \in F^2$, and b has no p-th root in F^2 then b has no p-th root in F''. Because, without loss of generality, $b \notin F^*$. Suppose b has a p-th root in F''. Then $\sum_{j<n}(\sum_{i<n(j)} t_{ij}c^i)b^j = 0$, $t_{ij} \in F^0$ where $\sum_{j<n}(\sum_{i<n(j)} t_{ij}c^i)x^j$ is indecomposable, and for some $j \neq 0 \pmod p$ $\sum_i t_{ij}c^i \neq 0$ (because $b \in F^2$). We can assume n, $n(j)$ are minimal. As c, $b \notin F^*$, they are transcendental over F_0, hence $\sum_{i,j} t_{ij}x^i y^j$ is indecomposable over $F^0_;$ and $\sum_i(\sum_j t_{ij}b^j)x^i$ is indecomposable over F^0. Then in

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F^c, $\Sigma_{i\,j} t_{ij} r(c)^i r(b)^j = 0$ (remember $r(t_{ij}) = t_{ij}$ as $t_{ij} \in F^0$). Since $r(b) \in F''$, $\Sigma_j t_{ij} r(b)^j \in F''$ but $r(c)$ cannot be separably algebraic over F''. Now $i \neq 0 \pmod p$ implies $\Sigma_j t_{ij} r(b)^j = 0$, hence $\Sigma_j t_{ij} b^j = 0$ and $t_{ij} = 0$ (as b is not algebraic over F^0). Thus $\Sigma_{ij} t_{pi\,j} c^{pi} b^j = 0$, and in F'', $\Sigma_{i.j} t_{pi\,j} c^i r(b)^j = 0$, so $r(b)$ is separably algebraic over F^2 and $r(b) \in F'' - F^2$, and we have finished.

(c) No $b \in F'' - F^2$ is the root of a separable polynomial over F^2, because F^2 is the separable closure of $F(c)$ in F''.

Stage (viii). Combine (*a*), (*b*), (*c*), and $f: F^2 \to F^1$ from Stage (v).

Let $F^\delta = F^2$, $F_\alpha = F''$, $F_\beta = F^3$, $g_1 =$ the identity, $g_2 = f$. Then by (b), (c), (2) and (3) of (a) hold. Hence there are *a* T^p_{rdc}-field $F_\gamma \supseteq F''$ and an embedding $g: F^3 \to F_\gamma$ such that $gf =$ identity, hence $g(a) = c$. Now $F^3 \models \neg\, \theta_1(a)$ (we chose a in this way) hence $F_\gamma \models \neg\, \theta_1(c)$, hence $F_\gamma \models \neg\, \theta_0(c)$. But $F_\gamma \supseteq F''$, so $F_\gamma \models \theta_0(c)$, a contradiction.

Q.E.D.

3. Stability of T^p_{rdc}

THEOREM 9. *T^p_{rdc} is stable.*

PROOF.

Stage (i). Suppose $F^1 \subseteq F^2$ are T^p_{rdc}-fields, $|F^1| \leqq \lambda$. We should prove that the set of types elements of F^2 realized over F^1 is $\leqq \lambda^{\aleph_0}$. For each $y \in F^2$ choose a countable field $F_y \subseteq F^2$ such that

(a) F_y is a countable T^p_{rdc}-field, $y \in F_y$,

(b) $F_y \cap F^1$ is a T^p_{rdc}-field,

(c) if $a_1, \cdots, a_n \in F_y$ are linearly dependent over F_1, then they are linearly dependent over $F_y \cap F^1$.

Let F^y be the field ($= T^p_d$-field) generated by $F_y \cup F^1$.

Stage (ii). Now define an equivalence relation over F^2:

$$y_1 \sim y_2 \text{ iff } F_{y_1} \cap F^1 = F_{y_2} \cap F^1,$$

and there is an isomorphism f from F_{y_1} onto F_{y_2}, $f(y_1) = y_2$, f restricted to $(F_{y_1} \cap F^1) =$ identity.

Clearly $\sim$ has $\leqq \lambda^{\aleph_0}$ equivalence classes; if $y_1 \sim y_2$ then we can extend the corresponding f to an isomorphism from F^{y_1} onto F^{y_1} which is the identity over F^1. If F^{y_1} is a T^p_{rd}-field this implies (as T^p_{rdc} has elimination of quantifiers) that y_1, y_2 realize the same type over F^1. Hence it suffices to prove

(5) $$F^y \text{ is } T^p_{rd}\text{-field}.$$

Let $F = F_y \cap F^1$, $F_1 = F_y$, $F_2 = F^1$.

REMARK. In fact we have more than the needed information to prove that the T^p_d-field F^y generated by F_1, F_2, is a T^p_{rd}-field.

Stage (iii). Suppose $c^* \in F^y$, $Dc = 0$ but c has no p-th root in F^y. Thus $c^* = \sum a_i b_i / \sum a^i b^i$, a_i, $a^i \in F_1$, $b_i, b^i \in F_2$. Then $c = \sum a''_i b''_i / (\sum^n a'_i b'_i)^p$, and clearly $D(\sum_i a''_i b''_i) = 0$. So without loss of generality $c = \sum_{i=1}^n a_i b_i$, $a_i \in F_1$, $b_t \in F_2$. Choose the sets a_i, b_i so that n is minimal. This implies that

(a) $\{a_1, \cdots, a_n\}$ are linearly independent over F,

(b) $b_1, \cdots, b_n$ are linearly independent over F. Hence

(c) $\{a_i b_j : i, j \leqq n\}$ are linearly independent over F.

PROOF OF (c). *If* $\sum_{i,j} t_{i,j} a_i b_i = 0$, $t_{i,j} \in F$ then $\sum_i a_i (\sum_j t_{i,j} b_j) = 0$. Since the $a_i \in F_1$ are linearly independent over F they are also linearly independent over F_2 (by Stages (a)–(c)); thus $\sum_j t_{i,j} b_j = 0$ and hence $t_{i,j} = 0$.

Stage (iv).

(a) Da_i is linearly dependent on $\{a_1, \cdots, a_n\}$ over F;

(b) Db_i is linearly dependent on $\{b_1, \cdots, b_n\}$ over F.

PROOF. Choose $1 \leqq i_1 < \cdots < i_l \leqq n$ such that $\{a_1, \cdots, a_n, Da_{i_1}, \cdots, Da_{i_l}\}$ is linearly independent over F, and each Da_i depends on it over F. Choose similarly $1 \leqq j_1 < \cdots < j_k \leqq n$ such that $\{b_1, \cdots, b_n, Db_{j_1}, \cdots, Db_{j_k}\}$ is linearly independent over F but each Db_j depends on it over F.

$$0 = Dc = \sum_i a_i Db_i + \sum_i (Da_i) b_i.$$

Substitute the expressions of Da_i, $i \notin \{i_1, \cdots, i_j\}$, and for $Db_j, j \notin \{j_1, \cdots, j_k\}$, and collect the terms. Then as in (iii) the coefficient of each $a_i b_j$, $a_i Db_j$, $(Da_{i_m}) b_j$ is zero. If $l > 0$ the coefficient of $(Da_{i_1}) b_1$ is 1, a contradiction. Thus $l = 0$, and similarly $k = 0$. Hence for some $t^l_j \in F$, $s^i_j \in F$, $Da_l = \sum_j t^l_j a_j$ $Db_i = \sum s^i_j b_j$.

Stage (v). Let

(6) $$Dx_l = \sum_{j=1}^n u^j_l x_j, \ i < n; \ u^l_j \in F$$

or, in short, $D\bar{x} = U\bar{x}$, $\bar{x}$ is a vector of length n, U an $n \times n$ matrix. Then there are solutions $\bar{a}_0, \cdots, \bar{a}_m$ (for $m < n$) in F such that for any other solution $\bar{a}$ from F^2 there are $d_0, \cdots, d_m \in F^2$, $Dd_i = 0$ such that $\bar{a} = \sum_{1 \leq m} d_i \bar{a}_i$. Let $\bar{a}_0, \cdots, \bar{a}_m$ be a maximal set of solutions of (6) which are linearly independent over F (as vectors). Let $\bar{a}$ be any other solution from F. (This is sufficient as F is an elementary submodel of F^2.) Then

$$\begin{aligned}
\bar{a} &= \sum_{i\leq m} d_i\bar{a}_i \text{ for some } d_i \in F^2 \text{ and} \\
U\bar{a} &= D\bar{a} = D\left(\sum_i d_i\bar{a}_i\right) = \sum_i D(d_i)\bar{a}_i + \sum_i d_iD\bar{a}_i \\
&= \sum_i (Dd_i)\bar{a}_i + \sum_i d_i(U\bar{a}_i) = \sum_i (Dd_i)\bar{a}_i + U\left(\sum_i d_i\bar{a}_i\right) \\
&= \sum (Dd_i)\bar{a}_i + U\bar{a}.
\end{aligned}$$

Thus $\Sigma_i(Dd_i)\bar{a}_i = 0$, $Dd_i \in F$. Since the set of $\bar{a}_i$ was linearly independent in F, and F is an elementary submodel of F^2, and since T^p_{rdc} is model-complete, the set of $\bar{a}_i$ is linearly independent in F^2. Hence $Dd_i = 0$. The same holds for F_1, F_2 instead of F^2.

Stage (vi). Combining the conclusions of (iv), (v), we arrive at the following representations:

$$\begin{aligned}
a_i &= \sum_j \alpha^i_j d_j \text{ for } \alpha^i_j \in F,\ d_j \in F_1,\ Dd_j = 0,\ j < n_1. \\
b_i &= \sum_j \beta^i_j e_j \text{ for } \beta^i_j \in F,\ e_j \in F_2,\ De_j = 0,\ j < n_2.
\end{aligned}$$

Hence

$$c = \sum \gamma^i_j d_i e_j \text{ for } \gamma^i_j \in F,\ d_i \in F_1,\ e_j \in F_2,\ Dd_i = De_j = 0.$$

Choose such representation with minimal n_1; among those with minimal n_1, choose a representation with a minimal n_2. Hence the set of d_i is linearly independent over F and also the e_j are linearly independent over F.

Hence, as in Stage (iii), $\{d_ie_j : i < n_1,\ j < n_2\}$ is linearly independent over F. Now since

$$0 = Dc = \sum_{i,j} (D\gamma^i_j)d_ie_j \text{ (as } Dd_i = De_j = 0)$$

and $D\gamma^i_j \in F$, clearly $D\gamma^i_j = 0$. Thus γ^i_j have a p-th root in F, d_i has a p-th root in F_1, and e_j has a p-th root in F_2. Thus

$$r(c) = \sum_{i,j} r(\gamma^i_j)r(d_i)r(e_j) \in (\text{the field generated by } F_1, F_2).$$

Q.E.D.

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ADDED IN PROOF

1. The non-minimality of the prime T^0_{dc}-field was also proved, independently by Rosenlicht [5a].
2. Wood [14], [15] also gives a nice set of axioms of T^p_{rdc}.
3. The answer to Conjecture 3 is positive.

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THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL