ISRAEL JOURNAL OF MATHEMATICS, Vol. 59, No. 3, 1987

ON STRONGLY-NON-REFLEXIVE GROUPS

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ABSTRACT

A strong negative answer is given to the old question of whether every dual group is reflexive. Using $\diamondsuit_{\omega_1}$ a group A is constructed so that A, A^*, A^{**}, and A^{***} are weakly ω_1-separable groups of cardinality ω_1 and A^* is not isomorphic to A^{***}.

0. Introduction

If G is an abelian group, let G^* be the $\mathbf{Z}$-dual group of G, i.e., $G^* = \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{Z})$. There is a canonical homomorphism $\sigma : G \to G^{**}$, given by $\sigma(g)(y) = y(g)$, if $g \in G$ and $y \in G^*$. We say that G is *torsionless* if σ is one-one, and that G is *reflexive* if σ is an isomorphism. An old question, which goes back at least as far as Reid [R], is whether or not every dual group is reflexive, i.e., whether or not for all abelian groups A, A^* is reflexive. (It is well known that every A^* is always torsionless, and that the answer to the question is no if we allow A of measurable cardinality.) In this paper we shall give a strong negative answer to this question assuming $\diamondsuit_{\omega_1}$.

Say that G is *strongly-non-reflexive* if G is not isomorphic to G^{**} (by any isomorphism). By induction on $n \in \omega$, define G^{*n} : $G^{*0} = G$; $G^{*n+1} = (G^{*n})^*$. (So, for example, $G^{*3} = G^{***}$.) Say that G is *weakly ω_1-separable* if G is ω_1-free (i.e., every countable subgroup is free), and every countable subset of G is contained in a countable subgroup H such that G/H is ω_1-free. Say that G is *ω_1-separable* if G is ω_1-free and every countable subset of G is contained in a

Received May 5, 1986 and in revised form January 27, 1987

† Research partially supported by NSF Grant No. DMS-8400451.
†† Research partially supported by NSERC Grant No. A8948.

countable subgroup H which is a direct summand of G. Our main result is the following.

THEOREM ($\diamondsuit_{\omega_1}$). *There are abelian groups A_n $(n \in \omega)$ such that for all $n \in \omega$:*

(i) *A_n is weakly ω_1-separable of cardinality ω_1;*

(ii) *A_n is strongly-non-reflexive; in fact $\Gamma(A_n) \neq \Gamma(A_n^{**})$;*

(iii) $A_n^* = A_{n+1}$.

This theorem cannot be proved in ZFC. Indeed, it is a theorem of ZFC + MA + $\neg$CH (see [M]) that every weakly ω_1-separable group of cardinality ω_1 is ω_1-separable; however, if G is ω_1-separable, G^* is not (because $(\mathbf{Z}^{(\omega)} \oplus H)^* \cong \mathbf{Z}^\omega \oplus H^*$, and $\mathbf{Z}^\omega$ is not weakly ω_1-separable). Moreover, by a theorem of Huber [H; 5.5], MA + $\neg$CH implies that every ω_1-separable group of cardinality ω_1 is reflexive.

(By way of contrast, it should be noted that Mekler has shown that, assuming $\diamondsuit_{\omega_1}$, there is a group A such that A^* is ω_1-separable of cardinality ω_1 and strongly-non-reflexive — because A^{***} is not even weakly ω_1-separable. Also, Mekler and Shelah have shown, in ZFC, that there is a strongly-non-reflexive dual group, and recently Mekler has constructed one of cardinality $2^{\aleph_0}$.)

The history of the main theorem is rather complex. In early 1982, G. Sageev and S. Shelah announced the construction, assuming $\diamondsuit_{\omega_1}$, of a group A such that for all $n \in \omega$, A^{*n} is strongly-non-reflexive, and a manuscript [SaS] was circulated. (The A^{*n} were not weakly ω_1-separable.) Later in 1982 Shelah wrote a very brief sketch of a construction using a related method, which aimed at stronger results [S]; he transmitted this sketch to Eklof and Mekler in 1985 while Eklof was giving a course at Simon Fraser University on the structure of Hom. As part of that course Eklof presented, with the essential assistance of Mekler, a proof of the above main theorem (restricted to $n \in \{0, 1, 2, 3\}$), based on the methods of [SaS] and [S], but worked out in detail, and employing results of Chase (as in [EH]) to simplify the combinatorics. An important aspect of that presentation was the identification of the inductive condition (8), described in Section 1, which seems to be essential for the construction to work, but which was not explicitly given in [S] or [SaS]. Later, Eklof and Mekler completed the proof of the main theorem (constructing the A_n for all $n \in \omega$), using the additional condition (9).

At about the same time, in early 1985, Eda and Ohta announced the construction, in ZFC, of a group A of cardinality $2^{\aleph_0}$ such that for all $n \in \omega$, A^{*n} is (weakly) non-reflexive. (See [EDO].)

We assume that the reader has some familiarity with set-theoretic methods in abelian group theory. See, for example, [E] for the definitions of stationary sets and the invariant $\Gamma(A)$, and for a statement of $\diamondsuit_{\omega_1}$.

We have attempted to write the proof of the theorem so that it can be studied at various levels of detail. The general plan of the construction is outlined in Section 1. The actual construction is carried out in Section 3, with the help of various auxiliary results which are isolated and proved in Section 2. At a first reading the reader may wish to restrict himself to the construction of the A_n for $n = 0, 1, 2, 3$. This is enough to obtain a weakly ω_1-separable dual group (A_1) which is not isomorphic to its weakly ω_1-separable double dual (A_3); various simplifications then occur, e.g., in the proof of Lemma 2.3.

1. Outline of the construction

We are going to define by induction on $\alpha < \omega_1$ a directed system of countable free groups

$$\{A_{\alpha,n} \mid \alpha < \omega_1\} \cup \{i^{\beta}_{\alpha,n} : A_{\alpha,n} \rightarrow A_{\beta,n} \mid \alpha \leqq \beta < \omega_1\}$$

for each $n \in \omega$, such that $A_{\alpha,n+1}$ is a subgroup of $A^*_{\alpha,n}$. In the end we will let $A_n = \varinjlim\{A_{\alpha,n} \mid \alpha < \omega_1\}$. In order to prove the theorem we will need to do the construction so that for all $n \in \omega$:

(i) A_n is weakly ω_1-separable;

(ii) for $m \neq n$, $\Gamma(A_m) \neq \Gamma(A_n)$; and

(iii) $A_{n+1} = A^*_n$.

To achieve this we will impose a series of ten conditions on the stages of the construction. We now list those conditions, preceding each by a brief explanation of its purpose. (The conditions are required to hold for all $\alpha \leqq \beta \leqq \gamma < \omega_1$ and all $n \in \omega$.)

First, we require directed systems of pure embeddings:

(1) (a) $i^{\beta}_{\alpha,n} : A_{\alpha,n} \rightarrow A_{\beta,n}$ is a pure embedding;

(b) $i^{\gamma}_{\alpha,n} = i^{\gamma}_{\beta,n} \circ i^{\beta}_{\alpha,n}$;

(c) $i^{\alpha}_{\alpha,n}$ = the identity on $A_{\alpha,n}$.

Second, in order to apply Chase's results on dual bases (Lemma 2.1), we require:

(2) (a) $A_{\alpha,n}$ is a free group of rank ω;

(b) $A_{\alpha,n+1}$ is a pure dense subgroup of $A^*_{\alpha,n}$.

Sh:169

If $y \in G^*$, $x \in G$ we denote by $\langle y, x \rangle$ or $\langle x, y \rangle$, interchangeably, the result of applying the function y to x. We want the maps $i^{\beta}_{\alpha,n}$ to be compatible with the inclusion of $A_{\alpha,n+1}$ in $A^*_{\alpha,n}$:

(3) $\quad$ for all $x \in A_{\alpha,n}$, $\quad y \in A_{\alpha,n+1}$, $\qquad \langle y, x \rangle = \langle i^{\beta}_{\alpha,n+1}(y), i^{\beta}_{\alpha,n}(x) \rangle$.

In order to achieve (i), we want the $A_{\alpha,n}$, for a fixed n, to form an ω_1-filtration such that all successor stages are ω_1-pure:

(4) (a) if γ is a limit ordinal, then $A_{\gamma,n} = \varinjlim \{A_{\alpha,n} \mid \alpha < \gamma\}$ and the $i^{\gamma}_{\alpha,n}$ are the associated canonical injections; moreover $A_{\gamma+1,n}/i^{\gamma+1}_{\gamma,n}(A_{\gamma,n})$ is divisible or zero.

(b) if α is a successor ordinal, then $A_{\beta,n}/i^{\beta}_{\alpha,n}(A_{\alpha,n})$ is free.

Fix a decomposition of $\lim(\omega_1)$ as a disjoint union of stationary sets:

$$\lim(\omega_1) = \coprod_{k \in \omega} S_k \amalg \coprod_{k \geqq 1} U_k$$

such that $\diamondsuit_{\omega_1}(S_k)$ holds for all $k \in \omega$. In order to achieve (ii), we impose a condition which will insure that $\Gamma(A_n) \supseteq \tilde{U}_k$ only if $n \geqq k$.

$$\text{(5)} \qquad \begin{gathered} \text{If } \alpha \in U_k, \text{ then } A_{\alpha+1,n}/i^{\alpha+1}_{\alpha,n}(A_{\alpha,n}) \text{ is} \\ \text{non-zero iff } n = k + 2m \text{ for some } m \geqq 0. \end{gathered}$$

There remains the achievement of (iii). For this we make use of $\diamondsuit_{\omega_1}(S_k)$ in order to insure that $A_{k+1} = A^*_k$. At stage $\alpha \in S_k$ we shall consider the homomorphism $h_\alpha : A_{\alpha,n} \to \mathbf{Z}$ given us by $\diamondsuit_{\omega_1}(S_k)$ — see Remark 2.7. If h_α does not belong to $A_{\alpha,k+1}$, then we will desire to define $A_{\alpha+1,k}$ so as to "kill" h_α, i.e., h_α *does not* extend to a homomorphism: $A_{\alpha+1,k} \to \mathbf{Z}$. At the same time, in order to be able to define $i^{\alpha+1}_{\alpha,k+1}$, we must insure that each $y \in A_{\alpha,k+1} (\subseteq A^*_{\alpha,k})$ *does* extend to an element of $A^*_{\alpha+1,k}$. In order to achieve simultaneously these two objectives, we will need to impose some closure conditions on the stages of the construction.

Consider first the canonical map $\sigma_{\alpha,k-1} : A_{\alpha,k-1} \to A^{**}_{\alpha,k}$ and suppose that $h_\alpha = \sigma_{\alpha,k-1}(x)$ for some $x \in A_{\alpha,k-1}$, and that $h_\alpha \notin A_{\alpha,k+1}$. We cannot hope to kill h_α, because we can always define its extension, h'_α, to $A_{\alpha+1,k}$ by the rule: $\langle h'_\alpha, y \rangle = \langle y, i^{\alpha+1}_{\alpha,k-1}(x) \rangle$ for all $y \in A_{\alpha+1,k} (\subseteq A^*_{\alpha+1,k-1})$. Thus, we require:

Sh:169

(6) (a) $A_{\alpha,n+2}$ contains the image of $A_{\alpha,n}$ under $\sigma_{\alpha,n}$ (where $\sigma_{\alpha,n}$ is defined by: $\langle \sigma_{\alpha,n}(x), y \rangle = \langle y, x \rangle$ for all $y \in A_{\alpha,n+1}$, $x \in A_{\alpha,n}$); so we can regard $\sigma_{\alpha,n}$ as a map: $A_{\alpha,n} \to A_{\alpha,n+2}$;
(b) $\sigma_{\beta,n} \circ i^{\beta}_{\alpha,n} = i^{\beta}_{\alpha,n+2} \circ \sigma_{\alpha,n}$.

For each $n \geqq 1$, we also have a canonical map $\rho_{\alpha,n+2} : A_{\alpha,n+2} \to A^*_{\alpha,n-1}$, which is a splitting of $\sigma_{\alpha,n}$. Each $A_{\alpha,n}$ for $n \geqq 1$ must contain the image of $A_{\alpha,n+2}$ under $\rho_{\alpha,n+2}$, because we cannot kill $\rho_{\alpha,n+2}(y)$ while extending $y \in A_{\alpha,n+2}$ to an element of $A^*_{\alpha+1,n+1}$. For the sake of uniformity we also require a splitting $\rho_{\alpha,2}$ of $\sigma_{\alpha,0}$.

(7) (a) if $n \geqq 1$, $A_{\alpha,n}$ contains the image of $A_{\alpha,n+2}$ under $\rho_{\alpha,n+2}$ (where $\rho_{\alpha,n+2}$ is defined by: $\langle \rho_{\alpha,n+2}(y), x \rangle = \langle y, \sigma_{\alpha,n-1}(x) \rangle$ for all $y \in A_{\alpha,n+2}$, $x \in A_{\alpha,n-1}$); so we can regard $\rho_{\alpha,n+2}$ as a map: $A_{\alpha,n+2} \to A_{\alpha,n}$;
(b) there is a homomorphism $\rho_{\alpha,2} : A_{\alpha,2} \to A_{\alpha,0}$ such that $\rho_{\alpha,2} \circ \sigma_{\alpha,0} =$ the identity on $A_{\alpha,0}$;
(c) $\rho_{\beta,n+2} \circ i^{\beta}_{\alpha,n+2} = i^{\beta}_{\alpha,n} \circ \rho_{\alpha,n+2}$.

Now, if the A_n are not to be reflexive, $\sigma_{\alpha,n}$ and $\rho_{\alpha,n+2}$ will not be isomorphisms (for sufficiently large α). Hence, the endomorphism $\sigma_{\alpha,n} \circ \rho_{\alpha,n+2}$ of $A_{\alpha,n+2}$ is not the identity. This endomorphism in turn induces a non-trivial idempotent endomorphism θ of $A^*_{\alpha,n+2}$; if $y \in A_{\alpha,n+3} (\subseteq A^*_{\alpha,n+2})$ we cannot hope to kill $\theta(y)$ while extending y to an element of $A^*_{\alpha+1,n+2}$. Thus, we must require:

(8) (a) if $n \geqq 3$, then $A_{\alpha,n}$ contains the image of $A_{\alpha,n}$ under the endomorphism $\theta^{(0)}_{\alpha,n}$ of $A^*_{\alpha,n-1}$ defined by: $\langle \theta^{(0)}_{\alpha,n}(y), x \rangle = \langle y, \sigma_{\alpha,n-3}\rho_{\alpha,n-1}(x) \rangle$ for all $y \in A^*_{\alpha,n-1}$, $x \in A_{\alpha,n-1}$; so we can regard $\theta^{(0)}_{\alpha,n}$ as an endomorphism of $A_{\alpha,n}$;
(b) if $n \geqq 3$, $i^{\beta}_{\alpha,n} \circ \theta^{(0)}_{\alpha,n} = \theta^{(0)}_{\beta,n} \circ i^{\beta}_{\alpha,n}$.

The endomorphism $\theta^{(0)}_{\alpha,n}$ in turn induces an idempotent endomorphism $\theta^{(1)}_{\alpha,n+1}$ of $A^*_{\alpha,n}$, under which $A_{\alpha,n+1}$ must be closed. Like falling dominos, one closure condition leads to another:

(9) (a) if $n \geqq 4$, then $A_{\alpha,n}$ contains the image of $A_{\alpha,n}$ under the endomorphisms $\theta^{(t)}_{\alpha,n}$ of $A^*_{\alpha,n-1}$ for all $t = 1, \ldots, n-3$, where the $\theta^{(t)}_{\alpha,n}$ are defined by induction on n: $\langle \theta^{(t)}_{\alpha,n}(y), x \rangle = \langle y, \theta^{(t-1)}_{\alpha,n-1}(x) \rangle$ for all $y \in A^*_{\alpha,n-1}$, $x \in A_{\alpha,n-1}$; so we can regard the $\theta^{(t)}_{\alpha,n}$ as endomorphisms of $A_{\alpha,n}$;
(b) if $n \geqq 4$, $i^{\beta}_{\alpha,n} \circ \theta^{(t)}_{\alpha,n} = \theta^{(t)}_{\beta,n} \circ i^{\beta}_{\alpha,n}$ for all $t = 1, \ldots, n-3$.

We will conclude with one technical condition, but first we need a definition. A *model*, B, is a sequence of groups B_n ($n \in \omega$) together with a map $\rho_2 : B_2 \to B_0$, satisfying the conditions (2), 6(a), 7(a), (b), 8(a) and 9(a) (when we replace $A_{\alpha,n}$ by B_n, $\sigma_{\alpha,n}$ by $\sigma_n : B_n \to B^*_{n+1}$, etc.). For example, for a fixed $\alpha < \omega_1$, the groups $A_{\alpha,n}$ and map $\rho_{\alpha,2}$ that we construct will form a model, which we denote by A_α. If B is another model, the direct sum $A_\alpha \oplus B$ is defined to be the model C such that for all n, $C_n = A_{\alpha,n} \oplus B_n$; we regard C_{n+1} as a subgroup of $C^*_n = (A_{\alpha,n} \oplus B_n)^*$ by identifying the latter with $A^*_{\alpha,n} \oplus B^*_n$ where the elements of $A^*_{\alpha,n}$ (resp. B^*_n) are regarded as functions on $A_{\alpha,n} \oplus B_n$ which are zero on B_n (resp. $A_{\alpha,n}$).

Now we are ready to state the final condition, which describes the successor of successor stages of the construction. The conditions imposed will enable us to carry out the construction at successor of limit stages. (An ordinal α is called *even* (resp. *odd*) if $\alpha = \lambda + 2n (\alpha = \lambda + 2n + 1)$ for some $\lambda \in \lim(\omega_1)$, $n \in \omega$.)

(10) if $\gamma = \alpha + 1$ where α is a successor ordinal, then there exists a model B_α such that $A_\gamma = A_\alpha \oplus B_\alpha$, and:
   (a) if α is even, then there is an isomorphism $\psi_\alpha : A_\alpha \to B_\alpha$;
   (b) if α is odd, then for every $n \geqq 2$ there exists $u_{\alpha,n} \in B_{\alpha,n} - \{0\}$ such that $\rho_{\gamma,n}(u_{\alpha,n}) = 0$ and for all $t = 0, \ldots, n-3$, $\theta^{(t)}_{\gamma,n}(u_{\alpha,n}) = 0$.

Now suppose we have constructed directed systems satisfying the above conditions, where we have employed diamond as indicated. Let $A_n = \varinjlim\{A_{\alpha,n} : \alpha < \omega_1\}$, and let $i^{\omega_1}_{\alpha,n}$ be the canonical injection: $A_{\alpha,n} \to A_n$. Then by (2)(b) and (3), A_{n+1} is naturally a subgroup of A^*_n. As we have already noted, (4) and (5) imply, respectively, that (i) and (ii) hold, so it remains only to show that $A^*_n \subseteq A_{n+1}$ for all $n \in \omega$. Suppose not, i.e., there exists $k \in \omega$ and $h \in A^*_k - A_{k+1}$. By $\diamondsuit_{\omega_1}(S_k)$ (see Remark 2.7) there exists a stationary set $E \subseteq S_k$ such that for all $\alpha \in E$, $h \restriction A_{\alpha,k}$ is the function, h_α, given us by $\diamondsuit_{\omega_1}(S_k)$ at stage α. Since h_α was not killed, it must have been that $h_\alpha \in A_{\alpha,k+1} = \varinjlim\{A_{\beta,k+1} : \beta < \alpha\}$. Thus there exists $\beta < \alpha$ such that $h \restriction A_{\alpha,k}$ belongs to $i^\alpha_{\beta,k+1}(A_{\beta,k+1})$. By Fodor's Theorem [J; Thm. 22, p. 59] there exists $\beta \in \omega_1$ such that for uncountably many $\alpha \in E$, $h \restriction A_{\alpha,k} \in i^\alpha_{\beta,k+1}(A_{\beta,k+1})$. Since $A_{\beta,k+1}$ is countable, there exists $z \in A_{\beta,k+1}$ such that for uncountably many $\alpha \in E$, $h \restriction A_{\alpha,k} = i^\alpha_{\beta,k+1}(z)$. Hence, clearly, $h = i^{\omega_1}_{\beta,k+1}(z)$, i.e., $h \in A_{k+1}$, a contradiction.

This completes the outline of the construction. The construction will be carried out in Section 3.

2. Auxiliary results

The reader is advised to read as far as Lemma 2.2, and then skip to Section 3, returning here as needed in the course of reading Section 3.

In the next lemma E and F are free groups of rank ω such that E is a subgroup of F^*. We say that E is *dense* in F^*, if it is dense in the $\sigma(F)$-topology on F^*, i.e., for all $x_1, \ldots, x_n \in F$ and all $y \in F^*$ there exists $z \in E$ such that $\langle z, x_i \rangle = \langle y, x_i \rangle$ for $i = 1, \ldots, n$. The following is due to Chase (cf. §1 and Theorem 3.2 of [C]).

2.1. Lemma.

(a) *E is dense in F^* iff for every $x \in F$ such that $\langle x \rangle$ (the subgroup generated x) is pure in F, there exists $y \in E$ such that $\langle y, x \rangle = 1$;*

(b) *E is pure in F^* iff for every $y \in E$ such that $\langle y \rangle$ is pure in E, there exists $x \in F$ such that $\langle y, x \rangle = 1$;*

(c) *E is a pure and dense subgroup of F^* iff there exist dual bases of E and F, i.e., bases $\{x_i : i \in \omega\}$ and $\{y_j : j \in \omega\}$ of F and E respectively such that for all i, j, $\langle y_j, x_i \rangle = \delta_{ij}$.* □

We shall also need the following elementary facts:

2.2. Lemma.

(a) *Let F be a subgroup of F' such that F'/F is divisible. The inclusion map $\iota : F \to F'$ induces a map $\iota^* : (F')^* \to F^*$ which is one-one, i.e., every element of F^* has at most one extension to an element of $(F')^*$;*

(b) *Suppose $F \subseteq F'$ are subgroups of a torsion-free group G such that F is pure in G and F'/F is divisible. Then F' is pure in G.*

Proof.

(a) ι induces an exact sequence

$$0 \to \mathrm{Hom}(F'/F, \mathbf{Z}) \to \mathrm{Hom}(F', \mathbf{Z}) \xrightarrow{\iota^*} \mathrm{Hom}(F, \mathbf{Z}).$$

By hypothesis, $\mathrm{Hom}(F'/F, \mathbf{Z}) = 0$.

(b) There is a short exact sequence $0 \to F'/F \to G/F \to G/F' \to 0$ which splits because F'/F is divisible. Hence G/F' is isomorphic to a subgroup of G/F; but G/F is torsion-free by hypotheses; so F' is pure in G. □

Now two result on models. Let B be a model (see before (10) in Section 1). Denote by σ_n (resp. ρ_{n+2}, $\theta_n^{(t)}$) the map: $B_n \to B_{n+2}$ (resp.: $B_{n+2} \to B_n$; $B_n \to B_n$) defined in condition 6(a) (resp. 7(a); 8(a) and 9(a)). Let $\theta_n^{(-1)}$ be $\sigma_{n-2} \circ \rho_n$.

Let T'_n denote the set of all formal products $\theta_n^{(t_1)} \theta_n^{(t_2)} \cdots \theta_n^{(t_l)}$ where each t_i is $\geqq 0$ and $\leqq n-3$; we shall call these products *n-terms* (we include the empty product, ϕ, which we identify with the identity function on B_n). By definition of a model, if $b \in B_n$ and τ is an n-term, then $\tau(b) \in B_n$. The next lemma will enable us to construct a model by induction on n; it will guarantee that the closure of B_n under the n-terms will not require a change in the choice of B_m for $m < n$. Let T_n denote the set of all formal products $\theta_n^{(t_1)}\theta_n^{(t_2)}\cdots\theta_n^{(t_r)}$ where now $t_i \in \{-1, 0, 1, \ldots, n-3\}$. If $\tau \in T'_{n+1}$, where $\tau = \theta_{n+1}^{(t_1)}\theta_{n+1}^{(t_2)}\cdots\theta_{n+1}^{(t_r)}$, let $\tau' = \theta_n^{(t_r-1)}\cdots\theta_n^{(t_1-1)} \in T_n$.

2.3 Lemma. *Let $n \geqq 2$. For every $\tau \in T'_{n+1}$ there is a $\hat{\tau} \in T_{n-1}$ such that: whenever $B_0, \ldots, B_n$ satisfy, as far as they are defined, the requirements for a model, then for all $b \in B_{n-1}$, $\rho_{n+1}\tau(\sigma_{n-1}(b)) = \hat{\tau}(b)$.*

Proof. The proof is by induction on n. For $n = 2$, $T'_{n+1} = \{(\theta_2^{(0)})^k : k \in \omega\}$; clearly for $\tau = (\theta_2^{(0)})^k$, we can let $\hat{\tau} = \phi$. Now let $n > 2$. If $\tau \in T'_{n+1}$, we can write

$$\tau = [\theta_{n+1}^{(0)}]\tau_1\theta_{n+1}^{(0)}\tau_2\cdots\theta_{n+1}^{(0)}\tau_s[\theta_{n+1}^{(0)}],$$

where each $\tau_j \in T'_{n+1}$ is a formal product of $\theta_{n+1}^{(t)}$'s for $t > 0$, and the brackets indicate that the first and last $\theta_{n+1}^{(0)}$'s may or may not appear. In order to avoid unwieldy notation, we shall do a representative computation, assuming $\tau = \tau_1\theta_{n+1}^{(0)}\tau_2$. For all B_{n-1}, $y \in B_{n-2}$ we have the following (where the numbers above the equality sign indicate the conditions used):

$$\begin{aligned}
\langle \rho_{n+1}\tau\sigma_{n-1}(b), y\rangle &\overset{(7)}{=} \langle \tau\sigma_{n-1}(b), \sigma_{n-2}(y)\rangle \\
&\overset{(9)}{=} \langle \theta_{n+1}^{(0)}\tau_2\sigma_{n-1}(b), \tau_1'\sigma_{n-2}(y)\rangle \\
&\overset{(8)}{=} \langle \tau_2\sigma_{n-1}(b), \sigma_{n-2}\rho_n\tau_1'\sigma_{n-2}(y)\rangle.
\end{aligned}$$

By induction the latter equals

$$\begin{aligned}
\langle \tau_2\sigma_{n-1}(b), \sigma_{n-2}\hat{\tau}_1'(y)\rangle &\overset{(9)}{=} \langle \sigma_{n-1}(b), \tau_2'\sigma_{n-2}\hat{\tau}_1'(y)\rangle \\
&\overset{(6)}{=} \langle b, \tau_2'\sigma_{n-2}\hat{\tau}_1'(y)\rangle \\
&\overset{(8,9)}{=} \langle \tau_2''(b), \sigma_{n-2}\hat{\tau}_1'(y)\rangle \\
&\overset{(6)}{=} \langle \tau_2''(b), \hat{\tau}_1'(y)\rangle \\
&\overset{(8,9)}{=} \langle \bar{\tau}_1\tau_2''(b), y\rangle
\end{aligned}$$

where if $\hat{\tau}_1' = \theta_{n-2}^{(k_1)}\cdots\theta_{n-2}^{(k_s)} \in T_{n-2}$, then $\bar{\tau}_1 = \theta_{n-1}^{(k_s+1)}\cdots\theta_{n-1}^{(k_1+1)} \in T'_{n-1}$. So let $\hat{\tau} = \bar{\tau}_1\tau_2''$. $\square$

REMARK. For example (for $n \geqq 5$), we can compute:

$$\text{if } \tau = \theta^{(1)}_{n+1}\theta^{(3)}_{n+1}\theta^{(2)}_{n+1}\theta^{(0)}_{n+1}, \quad \text{then } \hat{\tau} = \phi;$$

$$\text{if } \tau = \theta^{(3)}_{n+1}\theta^{(1)}_{n+1}\theta^{(2)}_{n+1}\theta^{(0)}_{n+1}, \quad \text{then } \hat{\tau} = \theta^{(1)}_{n-1};$$

$$\text{if } \tau = \theta^{(2)}_{n+1}\theta^{(3)}_{n+1}\theta^{(1)}_{n+1}\theta^{(0)}_{n+1}, \quad \text{then } \hat{\tau} = \theta^{(0)}_{n-1}\theta^{(1)}_{n-1};$$

$$\text{if } \tau = \theta^{(2)}_{n+1}\theta^{(3)}_{n+1}\theta^{(0)}_{n+1}\theta^{(1)}_{n+1}, \quad \text{then } \hat{\tau} = \theta^{(0)}_{n-1}\theta^{(1)}_{n-1}\theta^{(-1)}_{n-1}.$$

(Fortunately, we do not need this remark, or the precise algorithm for computing $\hat{\tau}$!)

2.4 LEMMA. *There is a model B such that for $n \geqq 2$ there exists $u_n \in B_n$ such that $\rho_n(u_n) = 0$ and for all $t = 0, \ldots, n-3$, $\theta^{(t)}_n(u_n) = 0$.*

PROOF. We shall define the B_n by induction on n so that in addition, for all $n \geqq 2 : B_n = C_n \oplus D_n$ where C_n is dense in B^*_{n-1}; $\sigma_{n-2}(B_{n-2}) \subseteq C_n$; D_n has rank ω; $\rho_n(D_n) = 0$; and for all $n \geqq 3$ and $t = 0, \ldots, n-3$, $\theta^{(t)}_n(B_n) \subseteq C_n$ and $\theta^{(t)}_n(D_n) = 0$. Let B_0, $B_1 \subseteq B^*_0$ be free groups of rank ω such that B_1 is pure and dense in B^*_0. Let $\sigma_0 : B_0 \to B^*_1$ be the canonical map and let $C_2 = \sigma_0(B_0)$. Let $S = \{y_i : i \in I\}$ be an uncountable pure-independent subset of B^*_1; we claim that there exists $y_i \in S$ such that $C_2 + \mathbf{Z}y_i = C_2 \oplus \mathbf{Z}y_i$ and is pure in B^*_1.

Indeed, otherwise, by a counting argument there exist $m, d \in \mathbf{Z}$, $i \neq j \in I$ and $b \in C_2$ such that m and d are relatively prime, and m divides both $dy_i + b$ and $dy_j + b$ in B^*_1. But then m divides $d(y_i - y_j)$, which contradicts the pure-independence of S. By repeating the argument we can find a subgroup D_2 of B^*_1 of rank ω such that $C_2 + D_2 = C_2 \oplus D_2$ and is pure in B^*_1. Let $B_2 = C_2 \oplus D_2$, and define ρ_2 to be zero on D_2.

Now suppose B_i has been defined for all $i \leqq n$ (for some $n \geqq 2$) such that the inductive conditions are satisfied and $B_0, \ldots, B_n$ satisfy the conditions for a model as far as they are defined. Let

$$\sigma_{n-1} : B_{n-1} \to B^*_n, \quad \rho_{n-1} : B^*_n \to B^*_{n-2} \quad \text{and} \quad \theta^{(t)}_{n+1} : B^*_n \to B^*_n \quad (\text{for } t \leqq n-3)$$

be the maps defined in (6), (7), (8) and (9). Let C'_{n+1} be the closure in $B^*_n = C^*_n \oplus D^*_n$ of $\sigma_{n-1}(B_{n-1})$ under the $\theta^{(t)}_{n+1}$, and let C_{n+1} be the pure closure of C'_{n+1} in B^*_n. Note that Lemma 2.3 implies that $\rho_{n+1}(C_{n+1}) \subseteq B_{n-1}$. Moreover, by Lemma 2.1, $\sigma_{n-1}(B_{n-1})$, and hence C_{n+1}, is dense in B^*_n because B_n is pure in B^*_{n-1}. As in the case $n = 1$, we can, by a counting argument, find $D_{n+1} \subset D^*_n$ of rank ω such that $C_{n+1} + D_{n+1} = C_{n+1} \oplus D_{n+1}$ and it is pure in B^*_n. Let $B_{n+1} = C_{n+1} \oplus D_{n+1}$. Then $\rho_{n+1}(D_{n+1}) = 0$ since $D_{n+1} \subseteq D^*_n$ and

$\sigma_{n-2}(B_{n-2}) \subseteq C_n$. Also, for all $t = 1, \ldots, n-2$, and all $y \in D_{n+1}$, $x \in B_n$, $\langle \theta_{n+1}^{(t)}(y), x \rangle = \langle y, \theta_n^{(t-1)}(x) \rangle = 0$ because $\theta_n^{(t-1)}(x) \in C_n$; thus $\theta_{n+1}^{(t)}(D_{n+1}) = 0$. Hence $\theta_{n+1}^{(t)}(B_{n+1}) = \theta_{n+1}^{(t)}(C_{n+1}) \subseteq C_{n+1}$. Finally, notice that $\rho_{n+1}(B_{n+1}) = \rho_{n+1}(C_{n+1}) \subseteq B_{n-1}$. □

2.5 Lemma. *Let G be a countable free group such that $G = \bigcup_{n \in \omega} G_n$ where for all $n \in \omega$ $G_n \subseteq G_{n+1}$, and G_{n+1}/G_n is free; say $G_{n+1} = G_n \oplus C_n$. For each $n \in \omega$ let $r_n \in \mathbf{Z}$ be such that $(n+1)r_n$ divides r_{n+1}. Also for each $n \in \omega$ let $\{a_{n,i} : i \in I\}$ be a countable set such that for all $i \in I$, $a_{n,i} \in C_n - \{0\}$. Then there is a countable free group G' containing G such that*

(i) *for all $n \in \omega$, G'/G_n is free;*
(ii) *G'/G is divisible and non-zero;*
(iii) *there exist elements $z_{n,i}$ of G' $(n \in \omega, i \in I)$ which generate G' over G and satisfy:*

$$(\Delta) \qquad r_{n+1} z_{n+1,i} = r_n(z_{n,i} - a_{n,i}).$$

(*We denote $z_{n,i}$, suggestively by $\Sigma_{j=n}^{\infty} (r_j/r_n)a_{j,i}$.*)

Proof. Notice that $G = G_0 \oplus \bigoplus_{j \in \omega} C_j$. Let $P = \Pi_{j \in \omega} C_j$. Let $z_{n,i}$ be the element of P given by:

$$z_{n,i}(j) = \begin{cases} 0 & \text{if } j < n, \\ \dfrac{r_j}{r_n} a_{j,i} & \text{if } j \geqq n. \end{cases}$$

Let G' be the subgroup of $G_0 \oplus P$ generated by $G \cup \{z_{n,i} : n \in \omega, i \in I\}$. Then clearly (iii) holds and consequently G'/G is divisible. Since the $a_{n,i}$ are non-zero, G'/G is non-zero. Moreover G' is free because it is a countable subgroup of an ω_1-free group; and G'/G_n is free because it is isomorphic to a countable subgroup of $\Pi_{j \geqq n} C_j$. □

2.6. Corollary. *Let G, G', $a_{n,i}$ be as in 2.5. Suppose $f \in G^*$ such that for all i, $f(a_{n,i}) = 0$ for almost all $n \in \omega$. Then f has a unique extension to a homomorphism $f' : G' \to \mathbf{Z}$.*

Proof. Say $f(a_{n,i}) = 0$ for $n > m_i$. For a fixed $i \in I$, let $m = m_i$ and define $f'(z_{k,i}) = 0$ for $k > m$. Let

$$f'(z_{m,i}) = f(a_{m,i}), \quad f'(z_{m-1,i}) = \frac{r_m}{r_{m-1}} f'(z_{m,i}) + f(a_{m-1,i}), \quad \text{etc.}$$

(using (Δ) in 2.5). □

2.7 Remark. Finally, in this section, we discuss the use of diamond. We can assume without loss of generality that the $A_{\alpha,n}$ are constructed so that the underlying set of $A_{\alpha,n}$ is $\omega\alpha$. Now $\diamondsuit_{\omega_1}(S_k)$ gives us a set $\{X_\alpha : \alpha \in S_k\}$ such that each X_α is a subset of $\omega\alpha \times \mathbf{Z}$, and for every $Y \subseteq \omega_1 \times \mathbf{Z}$, $\{\alpha \in S_k : Y \cap (\omega\alpha \times \mathbf{Z}) = X_\alpha\}$ is stationary in ω_1. (See [E, p. 21].) In our construction, if we have defined the groups $A_{\alpha,n}$ $(n \in \omega)$ for some $\alpha \in S_k$, let $h_\alpha : A_{\alpha,k} \to \mathbf{Z}$ be X_α if X_α is a homomorphism; otherwise let $h_\alpha = 0$.

3. The construction

To begin the construction, let $A_{0,n}$ $(n \in \omega)$ be free groups of rank ω satisfying (2) such that the maps $\sigma_{0,n} : A_{0,n} \to A_{0,n+2}$ are isomorphisms; let $\rho_{0,n+2}$ be the inverse of $\sigma_{0,n}$; and let $\theta_{0,n}^{(t)}$ = the identity on $A_{\alpha,n}$ for $n \geqq 3 + t$, $t \geqq 0$.

Now suppose that for some $\gamma > 0$ we have constructed $A_{\alpha,n}$, $i_{\alpha,n}^{\beta}$, etc., satisfying (1)–(10) for all $\alpha \leqq \beta < \gamma$, $n \in \omega$. Our construction of the $A_{\gamma,n}$, $i_{\alpha,n}^{\gamma}$, etc. will divide into four cases.

Case I. γ is a limit ordinal

In this case, let $A_{\gamma,0} = \bigcup_{\beta<\gamma} A_{\beta,0}$. Let $i_{\beta,0}^{\gamma}$ be inclusion of $A_{\beta,0}$ into $A_{\gamma,0}$. Now suppose that $A_{\gamma,m}$, $i_{\beta,m}^{\gamma}$ $(\beta < \gamma)$ have been defined for all $m \leqq n$ for some $n \geqq 0$, such that

$$A_{\gamma,n} = \varinjlim \{A_{\beta,n} : \beta < \gamma\}.$$

(Cf. 4(a).) For every $\beta < \gamma$ and $y \in A_{\beta,n+1}$ define $i_{\beta,n+1}^{\gamma}(y) \in A_{\gamma,n}^{*}$ as follows: if $x \in A_{\alpha,n}$ for some $\alpha < \gamma$,

$$\langle i_{\beta,n+1}^{\gamma}(y), i_{\alpha,n}^{\gamma}(x)\rangle = \begin{cases} \langle y, i_{\alpha,n}^{\beta}(x)\rangle & \text{if } \alpha \leqq \beta, \\ \langle i_{\beta,n+1}^{\alpha}(y), x\rangle & \text{if } \beta < \alpha. \end{cases}$$

This is well-defined by (3). Let

$$A_{\gamma,n+1} = \{i_{\beta,n+1}^{\gamma}(y) : \beta < \gamma, y \in A_{\beta,n+1}\} \subseteq A_{\gamma,n}^{*}.$$

One can easily check, using Lemma 2.1, that all of the conditions (1)–(10) are satisfied for all $\alpha \leqq \beta \leqq \gamma$, $n \in \omega$.

Case II. $\gamma = \alpha + 1$, where α is a successor ordinal

In this case, we must satisfy condition (10). If α is even, let B_α be an isomorphic copy of the model A_α (see before (10) in Section 1). Let $\psi_\alpha : A_\alpha \to B_\alpha$ be an isomorphism. Define A_γ to be $A_\alpha \oplus B_\alpha$. For every $n \in \omega$, let $i^\gamma_{\alpha,n}$ be the canonical injection of $A_{\alpha,n}$ into $(A_\alpha \oplus B_\alpha)_n = A_{\alpha,n} \oplus B_{\alpha,n}$.

If α is odd, we use Lemma 2.4: let B_α be the model constructed in that lemma; let $A_\gamma = A_\alpha \oplus B_\alpha$; let $u_{\alpha,n}$ be any element of $D_n - \{0\}$.

For future reference, observe that in both cases, by construction, if $y \in A_{\alpha,n+1}$, then $\langle y, b \rangle = 0$ for all $b \in B_n$. This will enable us to apply Corollary 2.6 in Cases III and IV.

Case III. $\gamma = \alpha + 1$, where $\alpha \in U_k$ for some $k \geqq 1$

In this case we must satisfy condition (5). Let $A_{\gamma,m} = A_{\alpha,m}$ for $m < k$. Choose a *ladder*, η, on α, i.e., a strictly increasing function $\eta : \omega \to \alpha$ such that $\alpha = \sup\{\eta(n) : n \in \omega\}$. Moreover choose η such that, for all n, $\eta(n)$ is odd. Let $u_n = u_{\eta(n),k} \in A_{\eta(n)+1,k}$ (cf. condition (10)(b)). In an abuse of notation, let us identify $A_{\beta,k}$ (for $\beta < \alpha$) with $i^\alpha_{\beta,k}(A_{\beta,\kappa}) \subseteq A_{\alpha,k}$; with this identification $A_{\alpha,k} = \bigcup_{n\in\omega} A_{\eta(n),k}$ (cf. (4)(a)). Then by Lemma 2.5 — with $|I| = 1$, $G = A_{\alpha,k}$, $G_n = A_{\eta(n),k}$, $r_n = n!$, $a_n = u_n$ — we can construct $A_{\gamma,k} (= G')$ such that there exists $\{z_n : n \in \omega\} \subseteq A_{\gamma,k}$ which generate $A_{\gamma,k}$ over $A_{\alpha,k}$ such that $(n+1)z_{n+1} = z_n - u_n$ for all $n \in \omega$; moreover $A_{\gamma,k}/A_{\eta(n),k}$ is free for all $n \in \omega$. We shall denote z_n by $\Sigma_{j=n}^{\infty} (j!/n!)u_j$.

We must realize $A_{\gamma,k}$ as a pure, dense subgroup of $A^*_{\gamma,k-1}$ $(= A^*_{\alpha,k-1})$ containing $A_{\alpha,k}$. For any $y \in A_{\beta,k+1}$, $\langle y, u_n \rangle = 0$ if $\eta(n) > \beta$; hence by Corollary 2.6, every $y \in A_{\alpha,k+1}$ has a unique extension, y', to an element of $A^*_{\gamma,k}$. Now if $z \in A_{\gamma,k}$ and $x \in A_{\alpha,k-1}$ define $\langle z, x \rangle = \langle \sigma_{\alpha,k-1}(x)', z \rangle$. Clearly this mapping of $A_{\gamma,k+1}$ into $A^*_{\alpha,k-1}$ is a homomorphism which is the identity on $A_{\alpha,k}$. We must check that it is one-one: suppose that $z \neq 0$; then there exists $n \in \omega$ such that n does not divide z; but there exists $a \in A_{\alpha,k}$ such that n divides $z - a$; so n does not divide a, and hence there exists $x \in A_{\alpha,k-1}$ such that n does not divide $\langle a, x \rangle$; since n divides $\langle z - a, x \rangle$ we must have that $\langle z, x \rangle \neq 0$. Notice also that, under this identification, $A_{\gamma,k}$ is a pure subgroup of $A^*_{\gamma,k-1}$, by Lemma 2.2(b); and of course $A_{\gamma,k}$ is dense in $A^*_{\gamma,k-1}$ since it includes $A_{\alpha,k}$. Notice also that $\rho_{\gamma,k}(z_n) = 0$ and $\theta^{(t)}_{\gamma,k}(z_n) = 0$ for all $n \in \omega$ and all $t \leqq n-3$ by choice of the u_n.

Now we must define the $A_{\gamma,m}$ for $m > k$. Let

$$A_{\gamma,k+1} = \{y' : y \in A_{\alpha,k+1}\} \subseteq A^*_{\gamma,k},$$

and let $i^\gamma_{\alpha,k+1}$ be the canonical isomorphism: $y \to y'$. Notice that for $x \in A_{\gamma,k-1} = A_{\alpha,k-1}$, $\sigma_{\gamma,k-1}(x) = (\sigma_{\alpha,k-1}(x))'$; hence $\sigma_{\gamma,k-1}(A_{\gamma,k-1}) \subseteq A_{\gamma,k+1}$, so (by

2.1) $A_{\gamma,k+1}$ is dense in $A^*_{\gamma,k}$ because $A_{\gamma,k}$ is pure in $A^*_{\gamma,k-1}$. Moreover, it's easy to check, using 2.1(b), that $A_{\gamma,k+1}$ is pure in $A^*_{\gamma,k}$.

We can regard the elements of $A_{\alpha,k+2}$ as members of $A^*_{\gamma,k+1}$. Let $A_{\gamma,k+2}$ be the closure under the $\theta^{(l)}_{\gamma,k+2}$ of

$$A_{\alpha,k+2} \cup \left\{ \sum_{j=n}^{\infty} \frac{j!}{n!} \sigma_j(u_j) : n \in \omega \right\}$$

(where $\sigma_j(u_j) = i^{\alpha}_{\eta(j)+1,k+2}(\sigma_{\eta(j)+1,k}(u_j))$). Notice that by Lemma 2.3, $\rho_{\gamma,k+2}(A_{\gamma,k+2}) = A_{\gamma,k}$. Again, we can apply Lemma 2.5 to see that (4) holds. Now by Corollary 2.6 each $y \in A_{\alpha,k+3}$ extends uniquely to an element of $A^*_{\gamma,k+2}$; so, as before, we can identify $A_{\gamma,k+2}$ with a pure, dense subgroup of $A^*_{\gamma,k+1}$. we continue in this manner to define the $A_{\gamma,m}$ for $m \geqq k+3$. In particular we have for all $m \geqq k$:

$$A_{\gamma,m}/i^{\gamma}_{\alpha,m}(A_{\gamma,m}) = \begin{cases} 0 & \text{if } m \equiv k+1 \pmod 2, \\ \text{divisible and non-zero} & \text{if } m \equiv k \pmod 2. \end{cases}$$

This completes the construction in case III.

Before undertaking the last case,we need to introduce a more general notion of a term. We shall use general symbols σ_n, ρ_n, $\theta^{(l)}_n$ for the functions in a model B. For a fixed k, and for any m such that $m \equiv k \pmod 2$ we define $T_{k,m}$ to be the set of all formal products of the functions σ_i, ρ_j, $\theta^{(l)}_{(l)}$ which define a function from B_k to B_m; the elements of $T_{k,m}$ are called (k,m)-*terms*. (We include ϕ in $T_{k,k}$.) For example, the following is a (3,7)-term

$$\tau_0 := \theta^{(4)}_7 \sigma_5 \theta^{(1)}_5 \theta^{(2)}_5 \sigma_3 \theta^{(0)}_3.$$

If $a \in A_{\beta,k}$ and $\tau \in T_{k,m}$ then there is an obvious definition of the element $\tau(a) \in A_{m,k}$; for example, if $a \in A_{\beta,3}$

$$\tau_0(a) = [\theta^{(4)}_{\beta,7} \sigma_{\beta,5} \theta^{(1)}_{\beta,5} \theta^{(2)}_{\beta,5} \sigma_{\beta,3} \theta^{(0)}_{\beta,3}](a).$$

Notice that by 6(b), 7(c), 8(b), and 9(b), if $\beta < \delta$ then $\tau_0(i^{\delta}_{\beta,k}(a)) = i^{\delta}_{\beta,m}(\tau_0(a))$.

Case IV. $\gamma = \alpha + 1$, where $\alpha \in S_k$ for some $k \in \omega$

In this case, $\diamondsuit_{\omega_1}(S_k)$ gives us an element $h_\alpha \in A^*_{\alpha,k}$ (see Remark 2.7). If $h_\alpha \in A_{\alpha,k+1}$, let $A_{\gamma,m} = A_{\alpha,m}$ for all $m \in \omega$. Otherwise, choose a ladder $\eta : \omega \to \alpha$ such that for all $n \in \omega$, $\eta(n)$ is an *even* successor ordinal.

Let $\{x_i : i \in \omega\}$ be a basis of $A_{\alpha,k}$ dual to a basis $\{y_j : j \in \omega\}$ of $A_{\alpha,k+1}$ (cf.

Lemma 2.1(c)). Then $\langle h_\alpha, x_i\rangle \neq 0$ for infinitely many i because otherwise $h_\alpha = \Sigma_i \langle h_\alpha, x_i\rangle y_i \in A_{\alpha,k+1}$. So there is a strictly increasing function $f: \omega \to \omega$ such that for all n, $\langle h_\alpha, x_{f(n)}\rangle \neq 0$; let $\bar{x}_n = x_{f(n)}$. Without loss of generality, $\bar{x}_n \in A_{\eta(n),k}$.

We now choose integers r_n and elements $a_n \in A_{\alpha,k}$ by induction on $n \in \omega$. Let $\zeta : \omega \to \mathbf{Z}$ be a bijection. Let $r_0 = 1$. Suppose that $r_0, \ldots, r_n, a_0, \ldots, a_{n-1}$ have been chosen. If $n = 0$, let $q_0 = \zeta(0)$. If $n \geqq 1$ consider the following system of n equations in $n-1$ unknowns, $\mu_1, \ldots, \mu_n$:

$$r_{i+1}\mu_{i+1} = r_i(\mu_i - \langle h_\alpha, a_i\rangle) \quad \text{for } i = 1, \ldots, n-1,$$

$$r_1\mu_1 = r_0(\zeta(n) - \langle h_\alpha, a_0\rangle).$$

This system has at most one solution in $\mathbf{Z}$. If it has none, let $r_{n+1} = (n+1)r_n$, and let $a_n = \bar{x}_n - v_n$, where

$$v_n = \psi_{\eta(n)}(\bar{x}_n) \in B_{\eta(n),k}.$$

(See 10(a) for $\psi_{\eta(n)}$.)

If the system has a solution, let q_n be the value of μ_n in this solution. Then either

(i) $q_n - \langle h_\alpha, \bar{x}_n - v_n\rangle \neq 0$

or

(ii) $q_n - \langle h_\alpha, 2\bar{x}_n - v_n\rangle \neq 0$

since $\langle h_\alpha, \bar{x}_n\rangle \neq 0$. If case (i) holds, let $a_n = \bar{x}_n - v_n$; otherwise let $a_n = 2\bar{x}_n - v_n$; then choose r_{n+1} so that $(n+1)r_n$ divides r_{n+1}, and r_{n+1} does not divide $r_n(q_n - \langle h_\alpha, a_n\rangle)$.

Now let

$$z_n = \sum_{j=n}^{\infty} \frac{r_j}{r_n} a_j.$$

We claim that if $\{z_n : n \in \omega\} \subseteq A_{\gamma,k}$, then h_α does not extend to $A_{\gamma,k}$. (In a harmless abuse of notation, we are pretending that the $i^\alpha_{\beta,m}$ are inclusion maps) Indeed, for the possible value, $\zeta(n)$, of $\langle h_\alpha, z_0\rangle$ we have chosen a_n and r_{n+1} so that $\langle h_\alpha, z_{n+1}\rangle$ is not defined.

Fix m such that $m \equiv k \pmod 2$, and for each $\tau \in T_{k,m}$, let $a_{n,\tau} = \tau(a_n)$. Apply Lemma 2.5 with $G = A_{\alpha,m}$, $G_n = A_{\eta(n),m}$, and $\{a_{n,i} : i \in I\} = \{a_{n,\tau} : \tau \in T_{k,m}\}$. The lemma is applicable because, for a fixed n, all the $a_{n,\tau}$ lie in the same complementary summand of G_n in G_{n+1}; namely, in case (i) (respectively case (ii)) the $a_{n,\tau}$ lie in the summand generated by $(w - \psi_{\eta(n)}(w): w \in G_n\}$ (respecti-

Sh:169

vely, by $\{2w - \psi_{\eta(n)}(w): w \in G_n\})$ plus a complementary summand of $A_{\eta(n)+1,m}$ in $A_{\eta(n+1),m}$. So we obtain a group $A_{\gamma,m}$ generated over $A_{\alpha,m}$ by the elements $\{\tau(z_n): n \in \omega, \tau \in T_{k,m}\}$ where

$$\tau(z_n) = \sum_{j=n}^{\infty} \frac{r_j}{r_n} \tau(a_j).$$

Then $A_{\gamma,m}/A_{\alpha,m}$ is divisible and for all $\beta < \alpha$, $A_{\gamma,m}/A_{\beta,m}$ is free. Notice that, by construction, the $A_{\gamma,m}$'s are closed under the $\theta^{(t)}_{\gamma,m}$'s, $\sigma_{\gamma,m}$'s, $\rho_{\gamma,m}$'s.

Next, we must show that every $y \in A_{\alpha,m+1}$ extends (uniquely) to a homomorphism y' on $A_{\gamma,m}$. It suffices to show that for all terms $\tau \in T_{k,m}$, $\langle y, \tau(a_j)\rangle = 0$ for almost all j, because then we can define

$$\langle y, \tau(z_n)\rangle = \sum_{j=n}^{\infty} \frac{r_j}{r_n} \langle y, \tau(a_j)\rangle$$

(cf. Corollary 2.6). Now a_j is either $\bar{x}_j - v_j$ or $2\bar{x}_j - v_j$, and by the construction in case II, $\langle y, \tau(v_j)\rangle = 0$ for almost all j, so the problem is to show that

$$(*) \qquad \langle y, \tau(\bar{x}_j)\rangle = 0 \quad \text{for almost all } j.$$

The proof of (*) is by induction on the length of τ, simultaneously for all $m \equiv k$ (mod 2). If the length of τ is 0, then $\tau = \phi$ and $\tau(\bar{x}_j) = \bar{x}_j$; then (*) follows from the choice of the $\bar{x}_j$ as members of a dual basis. If the length of τ is greater than 0, the proof divides into 3 cases; it is here that we use — as we must — our closure conditions (6)–(9).

Case A. $\tau = \sigma_{\alpha,m-2}\tau'$, for some $\tau' \in T_{k,m-2}$

Then $\langle y, \tau(\bar{x}_j)\rangle = \langle \rho_{\alpha,m+1}(y), \tau'(\bar{x}_j)\rangle = 0$, for almost all j by induction because $\rho_{\alpha,m+1}(y) \in A_{\alpha,m-1}$. (Here we use condition (7).)

Case B. $\tau = \rho_{\alpha,m+2}\tau'$, for some $\tau' \in T_{k,m+2}$

Then

$$\langle y, \tau(\bar{x}_j)\rangle = \langle \sigma_{\alpha,m+1}(y), \sigma_{\alpha,m}\rho_{\alpha,m+2}\tau'(\bar{x}_j)\rangle = \langle \theta^{(0)}_{\alpha,m+3}(\sigma_{\alpha,m+1}(y)), \tau'(\bar{x}_j)\rangle = 0$$

for almost all j, by induction because $\theta^{(0)}_{\alpha,m+3}(\sigma_{\alpha,m+1}(y)) \in A_{\alpha,m+3}$. (Here we use conditions (6) and (8).)

Case C. $\tau = \theta^{(t)}_{\alpha,m}\tau'$, for some $t \geqq 0$, $\tau' \in T_{k,m}$

Then $\langle y, \tau(\bar{x}_j)\rangle = \langle \theta^{(t+1)}_{\alpha,m+1}(y), \tau'(\bar{x}_j)\rangle = 0$ for almost all j, by induction because $\theta^{(t+1)}_{\alpha,m+1}(y) \in A_{\alpha,m+1}$. (Here we use condition (9).)

This completes the proof of (*). Hence, as in Case III, we can define $A_{\gamma,m+1}$ for every $m \equiv k \pmod 2$ so that $i^{\gamma}_{\alpha,m+1}$ is in isomorphism: $y \rightarrow y'$. Also, as in Case II, we can realize each $A_{\gamma,m}$ as a pure, dense subgroup of $A^{*}_{\gamma,m-1}$.

This completes the construction, and therefore completes the proof of the theorem. □

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