On wide Aronszajn trees in the presence of MA

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Abstract

A wide Aronszajn tree is a tree of size and height ω_1 with no uncountable branches. We prove that under $MA(\omega_1)$ there is no wide Aronszajn tree which is universal under weak embeddings. This solves an open question of Mekler and Väänänen from 1994. We also prove that under the same assumption there is no universal Aronszajn tree, improving a result of Todorčević from 2007 who proved the same under the assumption of BPFA for posets of size $\aleph_1$. Finally, we prove that under $MA(\omega_1)$, every wide Aronszajn tree weakly embeds in an Aronszajn tree. [1]

1 Introduction

We study the class $\mathcal{T}$ of trees of height and size $\aleph_1$, but with no uncountable branch. We call such trees *wide Aronszajn trees*. A particular instance of such a tree is a classical Aronszajn tree, so the class $\mathcal{A}$ of Aronszajn trees satisfies $\mathcal{A} \subseteq \mathcal{T}$. Apart from their intrinsic interest in combinatorial set theory, these classes are also interesting from the topological point of view, since they give rise to a natural generalisations of metric spaces, ω_1-metric spaces introduced by Sikorski in [8] and further studied in [6], [10] or [3], for example. The ω_1-distance function in trees is given by the Δ-function, which is defined by $\Delta(x, y) = \mathrm{ht}(x \wedge_T y)$ for $x \neq y$ and $\Delta(x, x) = 0$. The function Δ is symmetric, takes values in ω_1 and satisfies the triangle inequality. Classes $\mathcal{T}$ and $\mathcal{A}$ can be quasi-ordered using the notion of *weak embedding*, which is defined as follows:

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Definition 1.1 *For two trees T_1, T_2, we say that T_1 is* weakly embeddable *in T_2 and we write $T_1 \leq T_2$, if there is $f : T_1 \to T_2$ such for all $x, y \in T_1$*

$$x <_{T_1} y \implies f(x) <_{T_2} f(y).$$

We are interested in the structure of $(\mathcal{T}, \leq)$ and $(\mathcal{A}, \leq)$. In particular, we address the question of the existence of a universal element in these classes. This is of special interest since among the many interesting and correct results of the paper [6] from 1993 there is also a claim that $MA(\omega_1)$ implies that there is a universal element in $(\mathcal{T}, \leq)$, the argument for which was soon after found to be faulty. Ever since, the status of the possible existence of a universal element in $(\mathcal{T}, \leq)$ under $MA(\omega_1)$ has remained an open question.

Our first result is Theorem 4.1, which proves that under $MA(\omega_1)$ there is no universal element in $(\mathcal{A}, \leq)$. [2] The second result is Theorem 5.1, which shows that under $MA(\omega_1)$ every wide Aronszajn tree embeds into an Aronszajn tree. Putting the two results together, we obtain the main result of the paper, Theorem 6.1, which shows that under $MA(\omega_1)$ the class $(\mathcal{T}, \leq)$ has no universal element. This resolves the question raised by [6].

2 Some facts about $(\mathcal{T}, \leq)$ and $(\mathcal{A}, \leq)$

Note that if there is a weak embedding from a tree to another, then there is one which preserves levels (see Observation 3.4), so we may restrict our attention to such embeddings. In [10], Todorčević studied level-preserving weak embeddings f which in addition satisfy the Lipshitz condition

$$\Delta_{T_1}(x, y) \geq \Delta_{T_2}(f(x), f(y)). \tag{1}$$

We may think of Lipshitz embeddings as contractions. This notion led Todorčević to introduce a subclass $\mathcal{L}$ of $\mathcal{A}$ which consists of those Aronszajn trees on which every level-preserving weak embedding from an uncountable subset of T has an uncountable Lipshitz restriction. He proved:

Theorem 2.1 *([10]) (1) There is a sequence $\langle T_z : z \in \mathbb{Z}\rangle$ of Lipshitz trees which is strictly increasing with respect to $\leq$ and moreover, for every $z \in \mathbb{Z}$, there is no Lipshitz tree S such that $T_z < S < T_{z+1}$.*
(2) There are $2^{\aleph_1}$ Aronszajn trees that are pairwise incomparable in the $\leq$ order.
(3) Under the assumption $BPFA^{\aleph_1}$, Lipshitz trees form a chain with respect to weak embeddings. This chain is both cofinal and coinitial in $\mathcal{A}$ and it has neither minimal nor maximal element.

[2] This improves a result of Todorčević from [10], who proved the same under the assumption of $BPFA^{\aleph_1}$, by which we mean BPFA for partial orders of size $\aleph_1$. See Corollary 2.2 below.

Corollary 2.2 *Assuming $BPFA^{\aleph_1}$, there is no universal element in $(\mathcal{A}, \leq)$.*

Our Corollary 5.8 shows that the conclusion of Theorem 2.1 (3) is true under $MA(\omega_1)$ and our Theorem 4.1 shows that the conclusion of Corollary 2.2 is true under $MA(\omega_1)$.

Many more results are known about $(\mathcal{A}, \leq)$, one can consult surveys [9] for earlier and [7] for more recent results. Much less is known about the full class $(\mathcal{T}, \leq)$. We cite the two results that we are aware of. The first one is a consistency result obtained by Mekler and Väänänen.

Theorem 2.3 *([6]) Assume CH holds and κ is a regular cardinal satisfying $\aleph_2 \leq \kappa$ and $\kappa \leq 2^{\aleph_1}$. Then there is a forcing notion that preserves cofinalities (hence cardinalities) and the value of 2^λ for all λ, and which forces the universality number of $(\mathcal{T}, \leq)$ and the universality number of $(\mathcal{A}, \leq)$ both to be κ.*

The next result, obtained by Džamonja and Väänänen, is in the presence of club guessing at ω_1 and the failure of CH. It concerns weak embeddings that satisfy a strengthening of the Lipshitz condition, called Δ-preserving and defined by

$$\Delta_{T_1}(x, y) = \Delta_{T_2}(f(x), f(y)). \tag{2}$$

Theorem 2.4 *([3]) Suppose that*

(a) there is a ladder system $\bar{C} = \langle c_\delta : \delta < \omega_1 \rangle$ which guesses clubs, i.e. satisfies that for any club $E \subseteq \omega_1$ there are stationarily many δ such that $c_\delta \subseteq E$,

(b) $\aleph_1 < 2^{\aleph_0}$.

Then no family of size $< 2^{\aleph_0}$ of trees of size $\aleph_1$, even if we allow uncountable branches, can $\leq$-embed all members of $\mathcal{T}$ in a way that preserves Δ.

3 Specialising triples and their basic properties

Notation 3.1 *(1) For an ordinal $\gamma < \omega_1$ we denote by $\mathrm{ht}(\gamma)$ the unique α such that $\gamma \in [\omega\alpha, \omega\alpha + \omega)$.*

(2) Let $\mathcal{A}$ be the set of all normal rooted ω_1-trees with no uncountable branches whose α-th level of T is indexed by a subset of the ordinals in $[\omega\alpha, \omega\alpha + \omega)$, for $\alpha < \omega_1$. The root $\langle\rangle$ is considered of level -1.

(Recall that the requirement of being normal *for a rooted tree means that if $\gamma_0 \neq \gamma_1$ are of the same limit level, then there exists β with $\beta <_T \gamma_l$ for exactly one $l < 2$).*

(3) If $T \in \mathcal{A}$ and $s, t \in T$, we denote by $s \cap_T t$ the maximal ordinal γ such that $\gamma <_T s, t$. (Such an ordinal exists by the assumption in (1)).

If $\mathrm{ht}(x) = \alpha > \beta$, then by $x \restriction \beta$ we denote the unique ordinal y with $\mathrm{ht}(y) = \beta$ and $y <_T x$.

(4) For $T_1, T_2 \in \mathcal{A}$ and $(x, y) \in \bigcup_{\alpha < \omega_1} \mathrm{lev}_\alpha(T_1) \times \mathrm{lev}_\alpha(T_2)$, we let $\alpha = \mathrm{ht}(x, y)$ if $x \in \mathrm{lev}_\alpha(T_1)$ (and so $y \in \mathrm{lev}_\alpha(T_2)$).

Definition 3.2 *Let $\mathcal{A}_2^{\rm sp}$ be the set of all triples (T_1, T_2, c) where $T_1, T_2 \in \mathcal{A}$ and c is a function from $\bigcup_{\delta \text{ limit } <\omega_1} \mathrm{lev}_\delta(T_1) \times \mathrm{lev}_\delta(T_2)$ to ω such that*

- *if $c(x_1, y_1) = c(x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$, then $\alpha(x_1, y_1) \neq \alpha(x_2, y_2)$, $x_1 \perp_{T_1} x_2$, $y_1 \perp_{T_2} y_2$ and*
$$\Delta_{T_1}(x_1, x_2) > \Delta_{T_2}(y_1, y_2).$$

Remark 3.3 *Note that the function c in specialising triples satisfies a stronger condition than that of being Lipshitz from equation (1). By the definition of $\mathcal{A}$, we have that for any $T \in \mathcal{A}$ and any $\gamma \in T$, $\mathrm{ht}(\gamma)$ is the same as $\mathrm{ht}_T(\gamma)$. The defining condition of specialising triples could have therefore been written in termes of heights, $\mathrm{ht}(x_1 \cap x_2) > \mathrm{ht}(y_1 \cap y_2)$.*

Also note that a weak embedding is not required to be injective, but is injective on any branch of its domain. Finally, observe that every rooted Aronszajn tree is weakly bi-embeddable with a rooted normal one and hence that concentrating on rooted normal trees does not change anything from the point of view of universality results.

The following is well known, see for example Claim 6.1 of [2].

Observation 3.4 *If there exists a weak embedding from a tree T_1 to a tree T_2, then there exists one which preserves levels, namely satisfying $\mathrm{ht}_{T_1}(x) = \mathrm{ht}_{T_2}(f(x))$ for all $x \in T_1$.*

Proof. Let $f : T_1 \to T_2$ be a weak embedding. Define $g(t) = f(t) \restriction \mathrm{ht}(t)$ and note that if $s <_{T_1} t$, then $\mathrm{ht}(s) <_{T_1} \mathrm{ht}(t)$ and so $g(s) <_{T_2} g(t)$. $\bigstar_{3.4}$

Claim 3.5 *(1) If $(T_1, T_2, c) \in \mathcal{A}_2^{\rm sp}$ then both T_1 and T_2 are special Aronszajn trees.*
(2) If $(T_1, T_2, c) \in \mathcal{A}_2^{\rm sp}$ then T_1 is not weakly embeddable in T_2.
(3) Every rooted normal Aronszajn tree is isomorphic to a tree in $\mathcal{A}$.

Proof. (1) Clearly, every tree in $\mathcal{A}$ is an ω_1-tree, so T_1 and T_2 are ω_1-trees. Let us first show that T_1 is special, so we shall define a function $d : T_1 \to \omega$ which witnesses that.

Notice that by the assumption that T_2 is of height ω_1, we can choose z_δ of height $\delta \in T_2$, for every limit δ. Let $g : \omega \times \omega \times \omega \to \omega$ be a bijection. Every $x \in T_1$ is of the form $\omega\delta + \omega m + n$ for some limit ordinal δ and natural numbers m and n. For such x, define $d(x) = g(c(x \restriction \delta, z_\delta), m, n)$.

Suppose that $x = \omega\delta + \omega m + n$, $y = \omega\beta + \omega k + l$ and that $d(x) = d(y)$, while $x \neq y$. Therefore $g(c(x \restriction \delta, z_\delta), m, n) = g(c(y \restriction \beta, z_\beta), k, l)$ and we obtain $m = k$ and $n = l$ while $c(x \restriction \delta, z_\delta) = c(y \restriction \beta, z_\beta)$. Since $x \neq y$ we must have $\beta \neq \delta$ and therefore $x \restriction \delta \neq y \restriction \beta$. By the properties of c we obtain $x \restriction \delta \perp_{T_1} y \restriction \beta$ and therefore $x \perp y$. In conclusion, $d^{-1}(\{a\})$ is an antichain, for any $a < \omega$, and therefore d witnesses that T_1 is special. A similar proof shows that T_2 is special. As clearly every special ω_1-tree is Aronszajn, the claim is proved.

(2) Suppose for a contradiction that f is a week embedding from T_1 to T_2. By Observation 3.4, we can assume that f preserves levels. For each α limit $< \omega_1$ choose x_α on the α-th

level of T_1. Note that by the level preservation of f, the value $c(x_\alpha, f(x_\alpha))$ is well-defined. Consider $\{c(x_\alpha, f(x_\alpha)) : \alpha \text{ limit } < \omega_1\}$, which is necessarily a countable set since the range of c is ω. Hence, there must be $\alpha < \beta$ such that $c(x_\alpha, f(x_\alpha)) = c(x_\beta, f(x_\beta))$. By the defining property of c we have that $x_\alpha \perp_{T_1} x_\beta$.

Since f is strict-order preserving we have that $f(x_\alpha \cap_{T_1} x_\beta) <_{T_2} f(x_\alpha), f(x_\beta)$ and therefore $f(x_\alpha \cap_{T_1} x_\beta) \leq_{T_2} f(x_\alpha) \cap_{T_2} f(x_\beta)$. However, $\mathrm{ht}(f(x_\alpha \cap_{T_1} x_\beta)) = \mathrm{ht}(x_\alpha \cap_{T_1} x_\beta) > \mathrm{ht}(f(x_\alpha) \cap_{T_1} f(x_\beta))$, a contradiction.

(3) Obvious. $\bigstar_{3.5}$

4 Embeddings between Aronszajn trees and the non-existence of a universal element under MA

This section is devoted to the proof of the following theorem.

Theorem 4.1 *For every tree $T \in \mathcal{A}$, there is a ccc forcing which adds a tree in $\mathcal{A}$ not weakly embeddable into T. In particular, under the assumption of $MA(\omega_1)$ there is no Aronszajn tree universal under weak embeddings.*

We shall break the proof into the definition of the forcing and then several lemmas needed to make the desired conclusion.

Definition 4.2 *Suppose that $T \in \mathcal{A}$, we shall define a forcing notion $\mathbb{Q} = \mathbb{Q}(T)$ to consist of all $p = (u^p, v^p, <_p, c^p)$ such that:*

1. *$u^p \subseteq \omega_1 \cup \{\langle\rangle\}$, $v^p \subseteq T$ are finite and $\langle\rangle \in v^p$,*
2. *if $\alpha \in v^p$ then there is $\beta \in u^p$ with $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$,*
3. *$<_p$ is a partial order on u^p such that $\alpha <_p \beta$ implies $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ and which fixes $\alpha \cap_{<_p} \beta \in u^p$ for every two different elements α, β of u^p and fixes the root $\langle\rangle$ of u^p,*
4. *c^p is a function from $\bigcup_{\delta \text{ limit } <\omega_1} \mathrm{lev}_\delta(u^p) \times \mathrm{lev}_\delta(v^p)$ to ω such that the analogue of the requirement from 3.2)(4) holds, that is:*

   if $c(x_1, y_1) = c(x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$, then $\alpha(x_1, y_1) \neq \alpha(x_2, y_2)$, $x_1 \perp_{T_1} x_2$, $y_1 \perp_{T_2} y_2$ and
   $$\mathrm{ht}(x_1 \cap_{T_1} x_2) > \mathrm{ht}(y_1 \cap_{T_2} y_2).$$

The order $p \leq q$ on $\mathbb{Q}$ is given by inclusion $u^p \subseteq u^q, v^p \subseteq v^q, <_p \subseteq <_q, c^p \subseteq c^q$ with the requirement that if $p \leq q$, then the intersection and the root given by $<_p$ are preserved in $<_q$.

Lemma 4.3 *Suppose that G is $\mathbb{Q}$-generic and let*

$$T^* = \bigcup\{<_p\colon\, p \in G\} \text{ and } c = \bigcup\{c^p : p \in G\}.$$

Then $(T^*, T, c) \in \mathcal{A}_2^{\rm sp}$.

Proof. Clearly, T^* is a partial order on ω_1. For every $\alpha < \omega_1$ we have that $\mathrm{lev}_\alpha(T^*) \subseteq [\omega\alpha, \omega\alpha + \omega)$, since the same is true for every $<_p$ for $p \in G$. In particular, T^* is a tree. It is a rooted tree since every u^p for $p \in G$ has the same root. Let us observe that T^* is normal, using the following claim.

Claim 4.4 *Suppose that $\beta_0, \beta_1 \in [\omega\delta, \omega\delta + \omega) \cap T^*$, where δ is a limit ordinal. Then there is $\alpha \in T^*$ such $\alpha <^* \beta_l$ for exactly one $l < 2$.*

Proof. We can find $p \in G$ such that $\beta_0, \beta_1 \in u^p$. Therefore $<_p$ fixes $\alpha = \beta_0 \cap_{<_p} \beta_1$ and by the definition of the order in $\mathbb{Q}$ we must have $\alpha = \beta_0 \cap_{<^*} \beta_1$ $\bigstar_{4.4}$

We now show that T^* is of height ω_1.

Claim 4.5 *For every $\alpha < \omega_1$, the set $\mathcal{D}_\alpha$ of all p such that u^p has an element on level α is dense.*

Proof. Given $\alpha < \omega_1$, if u^p has no elements on level α, we shall first choose a $\gamma \in [\omega\alpha, \omega\alpha + \omega)$ and extend the order $<_p$ to $u^p \cup \{\gamma\}$ by letting γ be above the root $\langle\rangle$ of u^p but such that $\beta \cap_{<^p} \gamma = \langle\rangle$ for all $\beta \in u^p$. Since u^p did not have any elements on level α, neither does v^p, so we do not have to worry about extending c to include pairs whose first coordinate is γ. $\bigstar_{4.5}$

We can conclude that T^* is a normal ω_1-tree. The next density claim will show that c is defined on all $\bigcup_{\delta \text{ limit } <\omega_1} \mathrm{lev}_\delta(T^*) \times \mathrm{lev}_\delta(T)$ to ω and will therefore by Claim 3.5 (1) imply that $T^* \in \mathcal{A}$.

Claim 4.6 *Suppose that δ is a limit ordinal $< \omega_1$ and that there is x of height δ in u^p. If $y \in T$ is of height δ, then p has an extension q such that $y \in v^q$, in other words, the set $\mathcal{E}_y = \{q : y \in v^q\}$ is dense above p .*

Proof. It suffices to let $v^q = v^p \cup \{y\}$ and to extend c^p in a one-to-one way so that for any $x \in u^p$ of height δ, the value of $c^q(x, y)$ is different from any values taken by c^p. $\bigstar_{4.6}$

To finish the proof of Lemma 4.3 we have that c is as required, since every p satisfies the requirement from 4.2(4). $\bigstar_{4.3}$

Lemma 4.7 *The forcing $\mathbb{Q}(T)$ is ccc.*

Proof. Suppose that $\langle p_\zeta : \zeta < \omega_1\rangle$ is a given sequence of elements of $\mathbb{Q}(T)$. By extending each p_ζ if necessary, we can assume that for each ζ there is an element of v^{p_ζ} and hence of u^{p_ζ} of height ζ. Let $C = \{\zeta < \omega_1 : \omega\zeta = \zeta\}$, so a club of ω_1.

For $\zeta \in C$ let us define $q_\zeta = p_\zeta \restriction \zeta$, by which we mean:

1. $u^{q_\zeta} = u^{p_\zeta} \cap (\zeta \cup \{\langle\rangle\}), v^{q_\zeta} = v^{p_\zeta} \cap (\zeta \cup \{\langle\rangle\})$,
2. $<_{q_\zeta} = <_{p_\zeta} \restriction u^{q_\zeta}$ and
3. $c^{q_\zeta} = c^{p_\zeta} \restriction (u^{q_\zeta} \times v^{q_\zeta})$.

There is a stationary set $S \subseteq C$, a condition q^* and integers $n^*, m^* < \omega$ such that for every $\zeta \in S$ we have:

1. $q_\zeta = q^*$,
2. the size of $u^{p_\zeta} \setminus u^{q^*}$ is n^* and the size of $v^{p_\zeta} \setminus v^{q^*}$ is m^*. We enumerate them increasingly as ordinals in the form $\langle x_i^\zeta : i < n^*\rangle$ and $\langle y_j^\zeta : j < m^*\rangle$,
3. the value of $c^{p_\zeta}(x_i^\zeta, y_j^\zeta)$ and the fact that it is defined or not depends only on i and j and not on ζ, and
4. letting $\gamma^* = \max(u^{q^*} \cup v^{q^*})$, we have $\min(u^{p_\zeta} \setminus u^{q^*}) > \gamma^* + \omega$ and similarly for $v^{p_\zeta} \setminus v^{q^*}$.

By thinning further, we may assume that for for every $\varepsilon < \zeta$ in S,

- $u^{p_\varepsilon} \cup v^{p_\varepsilon} \subseteq \zeta$,
- the unique ordinal-order-preserving functions $f_{\varepsilon,\zeta}$ from u^{p_ε} to u^{p_ζ} and $g_{\varepsilon,\zeta}$ from v^{p_ε} to v^{p_ζ} give rise to an isomorphism between p_ε to p_ζ which fixes q^*. In particular, it maps $<_{p_\varepsilon}$ to $<_{p_\zeta}$ fixing u^{q^*} and similarly for $<_T\restriction v^{p_\varepsilon}$ and $<_T\restriction v^{p_\zeta}$.
- for every $\alpha \in v^{p_\zeta} \setminus v^{p_\varepsilon}$ we have that $\alpha \restriction_T (\gamma^* + \omega) = g^{-1}_{\varepsilon,\zeta}(\alpha) \restriction_T (\gamma^* + \omega)$.

Let us now consider what could render two conditions p_ε and p_ζ for ε and ζ in S, incompatible. The minimum requirement on a condition q with $q \geq p_\varepsilon, p_\zeta$ would be that $u^q \supseteq u^{p_\varepsilon} \cup u^{p_\zeta}$ and $v^q \supseteq v^{p_\varepsilon} \cup v^{p_\zeta}$. It may happen that there are $i < n^*$ and $j < m^*$ such that $x_i^\varepsilon \in u^{p_\varepsilon} \setminus \varepsilon$ and $y_j^\varepsilon \in v^{p_\varepsilon} \setminus \varepsilon$, so $x_i^\zeta \in u^{p_\zeta} \setminus \zeta$ and $y_j^\zeta \in v^{p_\zeta} \setminus \zeta$, such that $c(x_i^\varepsilon, y_j^\varepsilon)$ is defined, and hence $c(x_i^\zeta, y_j^\zeta)$ is defined and $c(x_i^\zeta, y_j^\zeta) = c(x_i^\varepsilon, y_j^\varepsilon)$. However, for all we know, y_j^ε and y_j^ζ might be compatible in T and therefore we run into a problem with the requirement (6) of Definition 4.2 of the forcing. We shall solve this difficulty by invoking the following lemma, essentially due to Baumgartner, Malitz and Reindhardt [1], here taken from Jech's book [4], where one can find a proof. In fact, although the book states the Claim in terms of Aronszajn trees, the same proof works for any tree of height and cardinality ω_1, as long as the tree does not have an uncountable branch. We shall use that fact in §5, so we state the claim in these terms.

Claim 4.8 ([4], Lemma 16.18) *If* $\mathbf{T}$ *is tree of height and cardinality* ω_1 *with no uncountable branches and* W *is an uncountable collection of finite pairwise disjoint subsets of* $\mathbf{T}$*, then there exist* $s, s' \in W$ *such that any* $x \in s$ *is incomparable with any* $y \in s'$*.*

We can now apply Claim 4.8 to find $\varepsilon < \zeta$ both in S such that any y^ε_j is incomparable with any $y^\zeta_{j'}$. Now we claim that p_ε and p_ζ are compatible. Let us start by defining $v = v^{p_\varepsilon} \cup v^{p_\zeta}$ and $u' = u^{p_\varepsilon} \cup u^{p_\zeta}$. In order to get a condition we shall have to extend u' and also define $<_p$, but note already that if $\alpha \in v$, then there is an element of height $\mathrm{ht}(\alpha)$ in u', since the analogue is true about u^{p_ε} and u^{p_ζ}. So conditions 1. and 2. of Definition 4.2 are easy to fulfil and it is condition 4. that is difficult. Once we fulfil it, that Condition 3. will follow from the proof.

Our choices so far imply that $c = c^{p_\varepsilon} \cup c^{p_\zeta}$ is a well defined function. In order to use it to fulfil condition 4. of Definition 4.2, we have to check through all the pairs $(x_1, y_1) \neq (x_2, y_2)$ in $\bigcup_{\delta \text{ limit } <\omega_1} \mathrm{lev}_\delta(u') \times \mathrm{lev}_\delta(v)$ such that $c(x_1, y_1) = c(x_2, y_2)$. If $(x_1, y_1), (x_2, y_2)$ are both in $\mathrm{dom}(c^{p_\varepsilon})$ or both are in $\mathrm{dom}(c^{p_\zeta})$, then the condition 4. is satisfied for them, so the interesting case is when they are not.

Therefore $\alpha(x_1, y_1) \neq \alpha(x_2, y_2)$, and let us suppose, without loss of generality, that $\alpha(x_1, y_1) < \alpha(x_2, y_2)$. Then necessarily $(x_1, y_1) \in \mathrm{dom}(c^{p_\varepsilon}) \setminus \mathrm{dom}(c^{p_\zeta})$ and $(x_2, y_2) \in \mathrm{dom}(c^{p_\zeta}) \setminus \mathrm{dom}(c^{p_\varepsilon})$. We have assured that this implies that y_1 and y_2 are incompatible in T. Let $\gamma = \mathrm{ht}(y_1 \cap_T y_2)$, so $\gamma < \alpha(x_1, y_1)$. So far we know nothing about $x_1 \cap x_2$ since neither $<_{p_\varepsilon}$ nor $<_{p_\zeta}$ have the pair (x_1, x_2) in its domain. Knowing that $\alpha(x_1, y_1)$ is a limit ordinal, we are going to choose a successor ordinal β_{x_1,x_2} above $\max(\gamma, \gamma^*)$ and below $\alpha(x_1, y_1)$ and an ordinal w_{x_1,x_2} of height β_{x_1,x_2} which is not $<_{p_\varepsilon}$ above any element of u^{p^ε}. We shall add w_{x_1,x_2} to u' and declare $w_{x_1,x_2} = x_1 \cap_< x_2$. We do this for all pairs relevant to condition 4., by induction on the number of such pairs, each time avoiding all interaction with what we have already chosen. At the end let u be the union of u' and the set of all such w_{x_1,x_2}. Since the new elements are all of successor height, this will not bring us in danger of creating new instances of condition 4. Finally, to fulfil condition 3. we need to extend $<_{p_\varepsilon} \cup <_{p_\zeta}$ to a partial order $<$ on u which will respect the commitments on $\cap_<$ which we have just made, which is possible by the way we chose β_{x_1,x_2}.

Then the condition $q = (u, v, <, c)$ is a common extension of p_ε, p_ζ.$\bigstar_{4.7}$

Proof. (of Theorem 4.1) To finish the proof, we suppose that we are in a model of $MA(\omega_1)$ and that T is an Aronszajn tree. Without generality, passing to a weakly bi-embeddable copy and adding a root if necessary, we can assume that T is rooted and normal.Then by forcing by the ccc forcing $\mathbb{Q}(T)$ (Lemma 4.7) and intersecting $\aleph_1$ many dense set $\mathcal{D}_\alpha$ for $\alpha < \omega_1$ (Claim 4.5) and $\mathcal{E}_y$ for $y \in T$ (Claim 4.6), we obtain that the generic Aronszajn tree T^* does not weakly embed into T (Lemma 4.3 and Claim 3.5(2)). Therefore, T is not universal, and since T is arbitrary, the theorem is proved. $\bigstar_{4.1}$

Remark 4.9 *Theorem 4.1 gives another proof of the main result of [1], which is that under* $MA(\omega_1)$ *all Aronszajn trees are special.*

5 Embedding wide Aronszajn trees into Aronszajn trees

This section is devoted to the proof of the following theorem:

Theorem 5.1 *For every tree $T \in \mathcal{T}$, there is a ccc forcing which adds a tree in $\mathcal{A}$ into which T weakly embeds. In particular, under the assumption of $MA(\omega_1)$ the class $\mathcal{A}$ is cofinal in the class $(\mathcal{T}, \leq)$.*

Following the pattern from Section §4, we shall break the proof into the definition of the forcing and then several lemmas needed to make the desired conclusion. The forcing is dual to the one in §4, in the sense that we now start with a tree T in $\mathcal{T}$ and generically add an Aronszajn tree that T weakly embeds to. We use the control function c to make sure that the generic tree does not have an uncountable branch.

For the definition of the forcing, we represent every $T \in \mathcal{T}$ by an isomorphic copy which is a subtree of ${}^{\omega_1>}\omega_1$.

Definition 5.2 *Suppose that $T \subseteq {}^{\omega_1>}\omega_1$ is a tree of size $\aleph_1$ and with no uncountable branches, we define a forcing notion $\mathbb{P} = \mathbb{P}(T)$ to consist of all $p = (u^p, v^p, <_p, f^p, c^p)$ such that:*

1. *$u^p \subseteq T$, $v^p \subseteq \omega_1$ are finite and $\langle\rangle \in u^p$,*
2. *u^p is closed under intersections,*
3. *$<_p$ is a partial order on v^p,*
4. *f^p is a surjective weak embedding from $(u^p, \subset)$ onto $(v^p, <_p)$,*
5. *for every $\eta \in u^p$, we have $\mathrm{ht}(f^p(\eta)) = \mathrm{lg}(\eta)$,*
6. *c^p is a function from v^p into ω such that*

$$\alpha <_p \beta \implies c^p(\alpha) \neq c^p(\beta).$$

The order $p \leq q$ on $\mathbb{P}$ is given by inclusion $u^p \subseteq u^q, v^p \subseteq v^q, <_p \subseteq <_q$ and $c^p \subseteq c^q$.

Lemma 5.3 *Suppose that G is $\mathbb{P}$-generic and let*

$$T^* = \bigcup\{<_p : p \in G\}, f = \bigcup\{f^p : p \in G\}, \text{ and } c = \bigcup\{c^p : p \in G\}.$$

Then T^* is an Aronszajn tree, f is a level-preserving weak embedding of T into T^*, $c : T^* \to \omega$ and $\alpha <_{T^*} \beta \implies c(\alpha) \neq c(\beta)$.

Proof. Clearly, T^* is a partial order on a subset of ω_1, c is a well defined function into ω and f is a function from a subset of T into T^* which is a weak embedding of its domain into its range. In addition, f is level-preserving in the sense that for all $\eta \in \mathrm{dom}(f)$ we have $\mathrm{ht}(f(\eta)) = \lg(\eta)$ and c satisfies $\alpha <_{T^*} \beta \implies c(\alpha) \neq c(\beta)$.

Claim 5.4 *Domain of f^* is T.*

Proof. Let $\rho \in T$, we shall show that $\mathcal{E}_\rho = \{p \in \mathbb{P} : \rho \in \mathrm{dom}(f^p)\}$ is dense. Suppose that $p \in P$ is given and suppose that $p \notin \mathcal{E}_\rho$. We shall define an extension q of p which is in $\mathcal{E}_\rho$. Let us define $u_0^q = u^p \cup \{\rho\}$. Let $\alpha = \lg(\rho)$. We shall first extend f^p to u_0^q. For the ease of reading, we divide the proof into steps.

(1) The first case is that either there is no $\tau \in u^p$ with $\rho \subset \tau$, or that there are such τ but there is no $\tau', \rho' \in u^p$ such that $\lg(\rho') = \alpha$, $\rho' \subset \tau'$ and $f^p(\tau') = f^p(\tau)$. In this case choose $\gamma \in [\omega\alpha, \omega\alpha + \omega) \setminus v^p$ and define $v_0^q = v^p \cup \{\gamma\}$, $f^q(\rho) = \gamma$. Let $\gamma >_q \beta$ for any $\beta = f^p(\sigma)$ for some $\sigma \subset \rho$ and $\gamma <_q \delta$ for any $\delta = f^p(\tau)$ for $\rho \subset \tau$ and $\tau \in u^p$. Then the relation $<_q$ is a partial order. We let $c^q(\gamma)$ be any value in ω not taken by c^p.

(2) This step is **the main point**. It is that there is $\tau \in u^p$ with $\rho \subset \tau$ and $\tau', \rho' \in u^p$ such that $\lg(\rho') = \alpha$, $\rho' \subset \tau'$ and $f^p(\tau') = f^p(\tau)$. In this case we shall have $v_0^q = v^p, <_q^0 = <_p$ and $c^q = c^p$, so let us show how to extend f^p to f^q. Let τ be of the least length among all τs as in the assumption of this case. We are then obliged to let $f^q(\rho) = f^p(\rho')$, since $f^p(\tau)$ can have only one restriction to the level α and $f^p(\rho')$ is already such a restriction. Note that for any $\tau'', \rho'' \in u^p$ such that $\lg(\rho'') = \alpha$, $\rho'' \subset \tau''$, $f^p(\tau'') = f^p(\tau)$, we must have $f^p(\rho'') = f^p(\rho')$ since f^p is a weak embedding. However, there is a *possible problem*: there could be σ, σ' and ρ'' such that $\lg(\rho'') = \alpha$, $\rho \subset \sigma$, $\rho'' \subset \sigma'$, $f^p(\sigma) = f^p(\sigma')$, which would force us to have $f^p(\rho) = f^p(\rho'')$, but maybe $f^p(\rho'') \neq f^p(\rho')$. Luckily, this cannot happen since u^p *is closed under intersections*, so for any such σ we would have $\rho = \sigma \cap \tau \in u^p$, which is not the case. In fact, any $\sigma \in u^p$ with $\rho \subset \sigma$ must satisfy $\tau \subseteq \sigma$.

(3) Now we know what $f^q(\rho)$ is and we have to discuss the closure under intersections. If there is $\tau \in u^p$ with $\rho \subset \tau$, then taking such τ of minimal length, we have that for every $\sigma \in u^p$, $\rho \cap \sigma = \tau \cap \sigma$, by the minimality of the length of τ and the fact that u^p is closed under intersections. In this case we let $u^q = u_0^q$ and $v^q = v_0^q$ and we are done. So suppose that there is no such τ. Let $\sigma \in u^p$ be the longest initial segment of ρ which is in u^p, which exists since u^p is finite and it contains $\langle\rangle$. Then, if there are intersections of the elements of u_0^q which are not already be in u_0^q, they must be of the form $\tau \cap \rho$ for some $\tau \in u^p$ with $\sigma \subset \tau$. We add all such $\tau \cap \rho$ to u_0^q to form u^q and we note that this set is now closed under intersections. Moreover, for each β such that there is an element of $u^q \setminus u_0^q$ of length β, we choose an ordinal $\gamma_\beta \in [\omega, \omega\beta + \omega) \setminus \mathrm{ran}(f^p)$ and we let $f^p(\sigma) <_q \gamma_{\beta_1} <_q \gamma_{\beta_2} <_q f^q(\rho)$ for any such $\beta_1 < \beta_2$. We extend $<_q$ by transitivity. Finally we choose distinct element $c_\beta \in \omega \setminus \mathrm{ran}(c_0^q)$ for each such β and let $c^q(\gamma_\beta) = c_\beta$. $\bigstar_{5.4}$

Claim 5.5 *For every $\alpha < \omega_1$ we have that* $\mathrm{lev}_\alpha(T^*) \subseteq [\omega\alpha, \omega\alpha + \omega)$ *and* T^* *has size* $\aleph_1$.

Proof. It follows from the definition of the forcing that $\mathrm{ran}(f^p \restriction (\mathrm{lev}_\alpha(T)) \subseteq [\omega\alpha, \omega\alpha+\omega)$ for every $p \in \mathbb{P}$. That every $\mathrm{lev}_\alpha(T)$ is non-empty follows from Claim 5.4. $\bigstar_{5.5}$

It follows that T^* is an ω_1-tree. By genericity we have that the domain of c is T^* and that $c\colon T^* \to \omega$ satisfies $\alpha <_p \beta \implies c^p(\alpha) \neq c^p(\beta)$.

Claim 5.6 *T^* has no uncountable branch.*

Proof. This is an easy consequence of the properties of c, namely c is 1-1 on any branch, and its range is ω. $\bigstar_{5.6}$

Therefore T^* is an Aronszajn tree. To finish the proof of the lemma, it remains to verify that $f : T \to T^*$ is a weak embedding, which follows from the genericity. $\bigstar_{5.3}$

Lemma 5.7 *The forcing $\mathbb{P}(T)$ is ccc.*

Proof. Suppose that $\langle p_\zeta : \zeta < \omega_1\rangle$ is a given sequence of elements of $\mathbb{P}(T)$. By extending each p_ζ if necessary, using the density of the sets $\mathcal{E}_\rho$ from Claim 5.4, we can assume that for each $\zeta < \omega_1$:

(a) there is an element of u^{p_ζ} and hence of v^{p_ζ} of height ζ, and that

(b) for every $\rho \in u^{p_\zeta}$ and every $\beta < \lg(\rho)$ such that there is an element of u^{p_ζ} of height β, the point $\rho \restriction \beta$ is in u^{p_ζ}.

Let

$$C = \big\{\zeta < \omega_1 : \omega\zeta = \zeta \text{ and } (\forall\varepsilon < \zeta)\max\{\lg(\rho), \rho(\alpha) : \rho \in u^{p_\varepsilon}, \alpha \in \mathrm{dom}(\rho)\} \ < \zeta\big\},$$

so C is a club of ω_1. By extending again if necessary, we shall require that for every $\zeta \in C$, there is an element in u^{p_ζ} of height in $(0, \zeta)$. For $\zeta \in C$ let us define $q_\zeta = p_\zeta \restriction \zeta$, by which we mean:

1. $u^{q_\zeta} = \{\tau \restriction \zeta : \tau \in u^{p_\zeta}\}, v^{q_\zeta} = v^{p_\zeta} \cap \zeta$,
2. $<_{q_\zeta} = <_{p_\zeta} \restriction v^{q_\zeta}$ and
3. $f^{q^\zeta} = f^{p^\zeta} \restriction u^{q_\zeta}$, $c^{q_\zeta} = c^{p_\zeta} \restriction v^{q_\zeta}$.

There is a stationary set $S \subseteq C$ such that:

1. for every $\zeta \in S$ we have: $v^{q_\zeta} = v^*$, $<_{q_\zeta} = <^*$, $c^{p_\zeta} = c^*$ are fixed,
2. the sets u^{q_ζ} form a Δ-system with root u^*,
3. for every $\varepsilon < \zeta \in S$ there is a level-preserving order isomorphism $\varphi_{\varepsilon,\zeta}$ from u^{q_ε} to u^{q_ζ} which is identity on u^*,

4. for every $\varepsilon < \zeta \in S$, $f^{q_\varepsilon} = f^{q_\zeta} \circ \varphi_{\varepsilon,\zeta}$,
5. for every $\varepsilon < \zeta \in S$, there is an order preserving isomorphism $\psi_{\varepsilon,\zeta}$ from $(u^{p_\varepsilon}, <_{p_\varepsilon})$ to $(u^{p_\zeta}, <_{p_\zeta})$ which extends $\varphi_{\varepsilon,\zeta}$ and such that $f^{p_\varepsilon} = f^{p_\zeta} \circ \varphi_{\varepsilon,\zeta}$[3],
6. for every $\varepsilon < \zeta \in S$, there is an order preserving isomorphism $i_{\varepsilon,\zeta}$ from $(v^{p_\varepsilon}, <_{p_\varepsilon})$ to $(v^{p_\zeta}, <_{q_\zeta})$ which is identity on v^*.

By the fact that there is an element of height ζ in u^{p_ζ}, we have that each $u^{p_\zeta} \setminus u^{q_\zeta} \neq \emptyset$. Since $\langle\rangle \in u^{q_\zeta}$ we have that $u^{q_\zeta} \neq \emptyset$ for all ζ, but even more so, u^{q_ζ} has an element of height in $(0, \zeta)$. Let $\alpha_1 = \max\{\lg(\rho) : \rho \in u^{q_\zeta}\}$ and $\alpha_0 = \min\{\lg(\rho) : \rho \neq \langle\rangle \in u^{q_\zeta}\}$. By the choice of $\varphi_{\varepsilon,\zeta}$, the choice of α_0 and α_1 does not depend on ζ. Finally let $\delta = \min(C) \setminus \alpha_1$.

Our requirements imply that we can use Claim 4.8 to find $\varepsilon < \zeta \in S \setminus \delta$ such that for every $\rho \in u^{p_\varepsilon} \setminus u^*$ and $\sigma \in u^{p_\zeta} \setminus u^*$, ρ and σ are incomparable. We shall find a common extension of p_ε and p_ζ.

We first define $u_0 = u^{p_\varepsilon} \cup u^{p_\zeta}$. We also define $f_0 = f^{p_\varepsilon} \cup f^{p_\zeta}$, which is well defined by the assumptions of the Δ-system and similarly $c_0 = c^{p_\varepsilon} \cup c^{p_\zeta}$. We also simply let $<_0 = <_{p_\varepsilon} \cup <_{p_\zeta}$, which still gives a partial order by the choice of ε and ζ. The only problems is that u_0 is not necessarily closed under intersections. Let us analyse what type of intersection can occur and what we need to add to make u_0 closed under intersections.

Let $\rho, \tau \in u_0$. If $\rho, \tau \in u^{p_\varepsilon}$ or $\rho, \tau \in u^{p_\zeta}$ then $\rho \cap \tau \in u^*$. Let us now suppose that $\rho \in u^{p_\varepsilon} \setminus u^{p_\zeta}$ and $\tau \in u^{p_\zeta} \setminus u^{p_\varepsilon}$, the other case is symmetric.

<u>Case 1.</u> $\lg(\rho \cap \tau) < \alpha_0$.

Using that $\rho \restriction \alpha_0 \in u^{p_\varepsilon}$ and $\tau \restriction \alpha_0 \in u^{p_\zeta}$, it suffices to consider the case $\lg(\rho) = \lg(\tau) = \alpha_0$.

Let $\sigma_0, \ldots, \sigma_n$ be all $\sigma = \rho \cap \tau$ obtained in this way. We choose for each of them distinct $f(\sigma_i)$ with $\mathrm{ht}(f(\sigma_i)) = \lg(\sigma_i)$ (note that necessarily $f(\sigma_i) \in \omega_1 \setminus \mathrm{ran}(f_0)$) and distinct c_i in $\omega \setminus \mathrm{ran}(c_0)$. Extend u_0 by adding all $\sigma_0, \ldots, \sigma_n$ and v_0 by adding all $f(\sigma_i)$, $<_0$ to a transitive order on v_0 which satisfies $f(\sigma_i) <_0 f(\rho), f(\tau)$ when $\sigma_i = \rho \cap \tau$ for some $\rho_i \in u^{p_\varepsilon}$, $\tau_i \in u^{p_\zeta}$. Extend c_0 to include the values $c_i = c(f(\sigma_i))$ as above. Call the resulting tuple $(u_1, v_1, <_1, f_1, c_1)$.

<u>Case 2.</u> $\lg(\rho \cap \tau) \in [\alpha_0, \alpha_1)$.

Let $\sigma = \rho \cap \tau$. By our assumption **(b)** we can assume that $\rho \in u^{q^\varepsilon} \setminus u^*$ and $\tau \in u^{q^\zeta} \setminus u^*$ are of the least possible length with the intersection σ. By the fact that $\varphi_{\varepsilon,\zeta}$ preserves both order and height, another application of **(b)** lets us assume that $\mathrm{ht}(\rho) = \mathrm{ht}(\tau)$. The possible **dangerous configuration** is that there are $\rho' \in u^{q^\varepsilon} \setminus u^*$, $\tau' \in u^{q^\zeta} \setminus u^*$ of length $\lg(\rho)$ and $\sigma' \in u^{q^\varepsilon} \setminus u^*$, $\sigma'' \in u^{q^\zeta} \setminus u^*$ of length $\lg(\sigma)$ such that $\sigma' \subset \rho'$ and $\sigma'' \subset \tau'$, $f^{p_\varepsilon}(\rho') = f^{p_\varepsilon}(\rho) = f^{p_\zeta}(\tau) = f^{p_\zeta}(\tau')$, yet $f^{p_\varepsilon}(\sigma') \neq f^{p_\zeta}(\sigma'')$. If there were such points we would not be able to extend f_1 to σ and keep it a weak embedding. Luckily, this cannot happen since if there were to be any elements η of u^{q^ε} of length $\lg(\sigma)$, then by the fact that u^{p^ε} satisfies the assumption **(b)**, $\sigma = \rho \restriction \lg(\eta))$ is already in u^{p^ε}, so in u^{q^ε}.

[3] since u^{p_ε} and u^{p_ζ} are closed under intersections, $\varphi_{\varepsilon,\zeta}$ necessarily preserves intersections

This analysis shows that we can proceed as in Case 1 to extend $(u_1, v_1, <_1, f_1, c_1)$ to $(u_2, v_2, <_2, f_2, c_2)$ which is closed under all intersections of Case 2 and satisfies other requirements of being a condition. Note that $(u_2, v_2, <_2, f_2, c_2)$ remains closed under the intersections of length $< \alpha_0$.

Case 3. $\lg(\rho \cap \tau) = \alpha_1$.

Let $\sigma = \rho \cap \tau$. We have that $\sigma = \rho \restriction \alpha_1 \in u^{p^\varepsilon}$ and $\sigma = \tau \restriction \alpha_1 \in u^{p^\zeta}$ and hence $\sigma \in u^*$, a contradiction.

Case 4. $\lg(\rho \cap \tau) > \alpha_1$.

Let $\sigma = \rho \cap \tau$. By the choice of S, we have that u^{p^ζ} does not have any elements of length $\lg(\sigma)$ and by the fact that u^{p^ε} is closed under restrictions, since $\sigma = \rho \restriction \lg(\sigma)$, we have that there are no elements of u^{p^ε} of length $\lg(\sigma)$ either. Hence we can proceed like in Case 1. Once we are done closing under intersections of this type, we finally obtain a common extension of p_ε, p_ζ. $\bigstar_{5.7}$

Proof. (of Theorem 5.1) The proof follows by putting the lemmas together. $\bigstar_{5.1}$

We remark that putting Theorem 5.1 together with the results of [10], gives a nice consequence about the class of Lipshitz trees, as follows.

Corollary 5.8 *Under $BPFA^{\aleph_1}$ the class $\mathcal{L}$ of Lipshitz trees is cofinal in the class $(\mathcal{T}, \leq)$.*

Proof. Clearly $BPFA^{\aleph_1}$ implies $MA(\omega_1)$ for partial orders of size $\aleph_1$, but it is well-known (see Theorem II.3.1 in [5]) that this is equivalent to $MA(\omega_1)$. Hence by Theorem 5.1 the class $\mathcal{A}$ is cofinal in $(\mathcal{T}, \leq)$. But by Lemma 7.1 in [10], $\mathcal{L}$ is cofinal in $\mathcal{A}$ and therefore the conclusion follows. $\bigstar_{5.8}$

6 Conclusion

Putting the results of Section §4 and Section §5 together, we obtain our main theorem, as follows.

Theorem 6.1 *Under $MA(\omega_1)$, there is no wide Aronszajn tree universal under weak embeddings.*

Proof. Assume $MA(\omega_1)$ and suppose for a contradiction that T is a universal element in $(\mathcal{T}, \leq)$. By Theorem 5.1, there is an Aronszajn tree T' such that $T \leq T'$, so T' is universal in $(\mathcal{T}, \leq)$ and so in $(\mathcal{A}, \leq)$. However, by Theorem 4.1 $(\mathcal{A}, \leq)$ does not have a universal element, a contradiction. $\bigstar_{6.1}$

References

[1] J. Baumgartner, J. Malitz, and W. Reinhardt. Embedding trees in the rationals. *Proceedings of the National Academy of Sciences*, 67(4):1748–1753, 1970.

[2] Mirna Džamonja and Jouko Väänänen. Chain models, trees of singular cardinality and dynamic EF-games. *J. Math. Log.*, 11(1):61–85, 2011.

[3] Mirna Džamonja and Jouko Väänänen. A family of trees with no uncountable branches. *Topology Proc.*, 28(1):113–132, 2004. Spring Topology and Dynamical Systems Conference.

[4] Thomas Jech. *Set Theory*. Springer-Verlag, Berlin Heidelberg, 3rd millenium edition, 2003.

[5] Kenneth Kunen. *Set Theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., 1980.

[6] A. Mekler and J. Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *The Journal of Symbolic Logic*, 58(3):1052–1070, 1993.

[7] Justin Tatch Moore. Structural analysis of Aronszajn trees. In Costas Dimitracopoulos, Ludomir Newelski, Dag Normann, and John Steel, editors, *Proceedings of the 2005 Logic Colloquium in Athens, Greece*, volume 28 of *Lecture Notes in Logic*, pages 85–107, 2006.

[8] Roman Sikorski. Remarks on some topological spaces of high power. *Fundamenta Mathematicae*, 37:125–136, 1949.

[9] Stevo Todorčević. Trees and linearly ordered sets. In Kenneth Kunen and Jerry E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 235–293. North Holland, Amsterdam, 1984.

[10] Stevo Todorčević. Lipschitz maps on trees. *Journal of the Institute of Mathematics of Jussieu*, 6(3):527–556, 2007.