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KEISLER'S ORDER IS NOT SIMPLE (AND SIMPLE THEORIES MAY NOT BE EITHER)

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Abstract. Solving a decades-old problem we show that Keisler's 1967 order on theories has the maximum number of classes. In fact, it embeds $\mathcal{P}(\omega)/\text{fin}$. The theories we build are simple unstable with no nontrivial forking, and reflect growth rates of sequences which may be thought of as densities of certain regular pairs, in the sense of Szemerédi's regularity lemma. The proof involves ideas from model theory, set theory, and finite combinatorics.

Keisler's order is a longstanding classification problem in model theory, introduced in 1967 [9] as a possible way of comparing the complexity of theories. Informally, say $T_1 \trianglelefteq T_2$ if the regular ultrapowers of models of T_1 are more likely to be saturated than those of T_2. Keisler's paper established that there was a minimum class, containing algebraically closed fields of fixed characteristic, and a maximum class, containing Peano arithmetic. By work of the second author in the seventies, see [31] Chapter VI, the union of the first two classes in Keisler's order gives an independent characterization of the *stable theories*, which are fundamental to modern model theory. Recently there has been much progress; for an account of work in the last decade, and some applications, see e.g. [20] or [17].

Among the questions raised by Keisler (see e.g. [10] p. 13) were how many classes the order had, whether it was linear, and what were syntactic characterizations of the minimum and maximum classes.

As of 1978 [31], the number was at least four, linearly ordered. Several years ago we discovered infinitely many classes, in fact an infinite descending chain [25], using certain hypergraphs first studied by Hrushovski [7]. Building on that construction, one can find conditional instances of nonlinearity (i.e. assuming a supercompact cardinal), as observed independently by Ulrich [35] and the authors [21]. Recently, we found unconditional (ZFC) instances of nonlinearity [27]. It would be consistent with these papers to conjecture that instances of nonlinearity were few, and that the number of equivalence classes was countable.

In the present paper we prove, in ZFC, that Keisler's order has the maximum number of classes (continuum many), by constructing a new family of simple unstable theories with no nontrivial forking which reflect growth rates of certain sequences of densities of finite graphs, and by developing new methods for building ultrafilters on Boolean algebras which carefully reflect these theories. (Both constructions seem quite flexible. Perhaps one reason some major structural conjectures about simple theories have remained stalled for decades is that simple unstable theories may have a much richer structure than previous examples suggest.)

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The rough idea of our construction is as follows.

We first build pairs of infinite, finitely branching trees with an edge relation beween nodes of corresponding height which thins out in an appropriate way as the height grows. In our main case[1], whenever two nodes of height k connect, their sets of immediate successors form a bipartite graph which, depending on the level, is either complete, or sparse and random (with a size and edge probability which is a function of the height). Associated to each height is a notion of "small" and "large" and the sparse graphs in question have the property that every small set of vertices has a common neighbor and no large set of vertices does. These structures, called parameters, can be thought of as encoding reduced graphs for the models of simple theories we then construct. The *level function* of a parameter is the set of heights at which we use sparse random (as opposed to complete) graphs; these are our choice of a way to track growth rates. To any such parameter we then associate a simple theory, essentially a kind of bipartite random graph filtered through unary predicates, which is simple unstable with no nontrivial forking. We prove that as the sequences of finite densities in the parameters vary sufficiently, as measured by the level functions, the associated theories have wildly different saturation behavior (a fundamentally infinitary phenomenon).

What happens on the ultrafilter side? For κ regular and uncountable, we define a new chain condition to match the simple theories and which says, very roughly speaking, if we are given κ positive elements of our Boolean algebra, then after moving to a subset $\mathcal{U}$ also of size κ, for any finite n not in some ideal (of which more soon), any finite $u \subseteq \mathcal{U}$ which is "large" in the sense of level n has a subset v which is still "large" and whose elements are all compatible. The precise sense in which we choose our family of theories to be orthogonal to each other has to do with the fact that for any partition of our final set of parameters into $\mathcal{M}$ and $\mathcal{N}$, the ideal of subsets of ω generated by the subsets where the level functions of elements of $\mathcal{M}$ are 1, does not contain (mod finite) the set where the level function of $\mathfrak{n}$ is 1, for any $\mathfrak{n} \in \mathcal{N}$. (The idea is that if we preserve the chain condition using the ideal coming from $\mathcal{M}$, any future ultrafilter will omit at least one type in any theory from $\mathcal{N}$, since given any purported solution, the ultrafilter can concentrate too many of its conditions at points where they cannot all be satisfied.) Essentially this allows us to build by induction a Boolean algebra and a filter (eventually an ultrafilter) on it, adding formal solutions to problems coming from theories from $\mathcal{M}$ at suitable inductive stages, while preserving the chain condition using the ideal coming from $\mathcal{M}$ which ensures omission of a type for any theory with a parameter from $\mathcal{N}$. Previous model-theoretic constructions of ultrafilters were focused exclusively on free Boolean algebras; for details, see §8.

§11 contains the statements of the main theorems.

These results suggest that not only do model theoretic dividing lines predict jumps in the complexity of theorems in finite combinatorics (as e.g. in stable regularity [18], or stable arithmetic regularity [34]), but also densities in the sense of finite combinatorics can control behavior of infinite models tightly enough that the resulting changes in complexity are detected by ultrafilters, so are candidates for model theoretic dividing lines.

The model theoretic, set theoretic, and combinatorial aspects of these constructions admit natural variations and raise interesting open questions, see §13.

[1]This sketch describes theories with additional input from §6; the frame in §2 is more basic.

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Is this the end of a certain line of work on this problem? We think inversely: it tells us where to look.

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Contents

1. Notation and conventions

Convention 1.1. *Unless otherwise stated, all graphs are simple graphs: no loops and no multiple edges.*

Convention 1.2. *We will often write bipartite graphs as triples (V, W, E), where V, W are the sets of vertices and the $E \subseteq V \times W$ is the edges. We will call a bipartite graph* complete *if $E = V \times W$, so in this case E is asymmetric.*

In this paper we will have both finite and infinite (possibly uncountable) random graphs; the infinite ones are random in the sense of model theory, which should not cause confusion. The next two definitions, ordinary 1.3 and bipartite 1.4, explain what this means.

Convention 1.3 (The model-theoretic random graph)**.** *"The theory of the random graph" means the set of first-order axioms in the language with a binary relation symbol E, and equality, which say that E is symmetric irreflexive, that there are infinitely many elements, and for any two finite disjoint sets v, w, there is a vertex a such that $E(a, b)$ for all $b \in v$ and $\neg E(a, c)$ for all $c \in w$.*

Convention 1.4 (Infinite *bipartite* random graphs)**.** *If (A, B, E) is a bipartite graph and A, B are infinite, we may call it a* bipartite random graph *to mean that the following two conditions hold: for any two finite disjoint $u, v \subseteq B$, there is $a \in A$ such that $\bigwedge_{b \in u} E(a, b) \wedge \bigwedge_{b \in v} \neg E(a, b)$, and conversely, for any two finite disjoint $u, v \subseteq A$, there is $b \in B$ such that $\bigwedge_{a \in u} E(a, b) \wedge \bigwedge_{a \in v} \neg E(a, b)$.*

Convention 1.5 (Trees)**.** *Recall that a tree is a partially ordered set such that the set of predecessors of any given node is well ordered, so in particular linearly ordered. In this paper, the partial order will always be given by initial segment, denoted* $\trianglelefteq$. *In this paper, all trees will be finitely branching, and all nodes of all trees will have finite height, so any tree will be either of finite or countable height.*

Remark 1.6. *The symbol* $\trianglelefteq$ *is used in this paper to denote two unrelated kinds of partial orders: to denote the partial order on elements of a given tree and to denote Keisler's ordering on theories. Since these two contexts never overlap, this should not cause any confusion.*

Definition 1.7. *As usual, we denote by* $\mathfrak{B}^0_{\alpha,\mu,\aleph_0}$ *the free Boolean algebra generated by* α *independent partitions each of size* μ, *and* $\mathfrak{B}^1_{\alpha,\mu,\aleph_0}$ *is its completion.*

The last subscript, $\aleph_0$, in 1.7 refers to the fact that any intersection of $< \aleph_0$ elements from distinct partitions is nonempty. Often it is understood and so not written. One could ask for a larger depth of intersection, as was used in e.g. [27]. On the existence of $\mathfrak{B}^0_{2^\lambda,\mu}$, i.e. $\mathfrak{B}^0_{2^\lambda,\mu,\aleph_0}$, when $\lambda \geq \mu$ see e.g. Fichtenholz-Kantorovich, or Hausdorff, or [31] Appendix Theorem 1.5. When $\theta > \aleph_0$, the existence theorem requires $\lambda = \lambda^{<\theta} \geq \mu$. In this paper, to find $D_0, \mathcal{D}_*, \mathbf{j}$ as in 7.1 below, we use the completion.

Recall that for a family of functions $\mathcal{G}$, $\mathrm{FI}_{\aleph_0}(\mathcal{G})$ is the set of functions h whose domain is a finite subset of $\mathcal{G}$, and such that for each $g \in \mathrm{dom}(h)$, $h(g) \in \mathrm{range}(g)$ (see e.g. [31] Definition 3.6 p. 358). In the case of our Boolean algebra generated by α many independent partitions of size μ, the analogous objects are finite functions h with domain $\subseteq \alpha$ such that $h(\eta) \in \mu$ for each $\eta \in \mathrm{dom}(h)$, the idea being roughly that for any such h the element $\mathbf{x}_h$ is the finite intersection of the $h(\alpha)$-th piece from the α-th partition, for $\alpha \in \mathrm{dom}(h)$. Since h is a function, and with finite domain, each such $\mathbf{x}_h$ is nonzero. For a detailed explanation of this notation, see [27] §1. As the generators are dense in the completion, we have:

Definition 1.8. *Let* $\mathfrak{B} = \mathfrak{B}^1_{\alpha,\mu,\aleph_0}$. *Then, in our notation, the elements of the form* $\mathbf{x}_f$ *for* $f \in \mathrm{FI}_{\aleph_0}(\alpha)$ *are dense in* $\mathfrak{B}$.

Fact 1.9 (Δ-system lemma, see e.g. Kunen [12] III.6.15)**.** *Let* ν *and* κ *be regular cardinals such that* $\aleph_0 \leq \nu < \kappa$. *Assume that* $(\forall \alpha < \kappa)(\alpha^{<\nu} < \kappa)$. *Let* $\mathcal{A}$ *be a family of sets with* $|\mathcal{A}| = \kappa$, *such that* $|A| < \nu$ *for all* $A \in \mathcal{A}$. *Then there is a* $\mathcal{B} \subseteq \mathcal{A}$ *of size* κ *such that* $\mathcal{B}$ *forms a* Δ-*system.*

Note that the family of sets need not be subsets of κ; we place no restriction on their provenance, only restrictions on size of the family and size of the sets. We will mostly use the case $\nu = \aleph_0$:

Corollary 1.10. *If* κ *is an uncountable regular cardinal and* $\mathcal{A}$ *is a family of* κ *sets, all of them finite, there is* $\mathcal{B} \subseteq \mathcal{A}$ *of size* κ *which forms a* Δ-*system.*

A central definition in this paper will be *Keisler's order*. For more on the order, see [9], or for example the extended introduction to [24]. Some key points:

Definition 1.11 (Keisler's order, [9])**.** *Let* T_1, T_2 *be complete countable first-order theories. We say* $T_1 \trianglelefteq T_2$ *if for every infinite* λ, *every regular ultrafilter* $\mathcal{D}$ *on* λ, *every model* $M_1 \models T_1$, *and every model* $M_2 \models T_2$, *if* $(M_2)^\lambda/\mathcal{D}$ *is* λ^+-*saturated, then* $(M_1)^\lambda/\mathcal{D}$ *is* λ^+-*saturated.*

Recall that the ultrafilter $\mathcal{D}$ on λ is *regular* if there exists a regularizing family, meaning $X = \{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{D}$ such that the intersection of any infinitely many elements of X is empty. By a lemma of Keisler [9, 2.1], if $\mathcal{D}$ is regular, then the choice of M_1, M_2 in 1.11 does not matter, up to elementary equivalence. For more on regular ultrafilters, see [2] §4.3 and §6.1.

Recall that a regular ultrafilter $\mathcal{D}$ on λ is λ^+*-good* if for every $f : [\lambda]^{<\aleph_0} \to \mathcal{D}$ which is monotonic has a multiplicative refinement, that is, if $u \subseteq v$ implies $f(u) \supseteq f(v)$ for all $u, v \in [\lambda]^{<\aleph_0}$, then there exists $g : [\lambda]^{<\aleph_0} \to \mathcal{D}$ such that $g(u) \subseteq f(u)$ for all $u \in [\lambda]^{<\aleph_0}$ and $g(u) \cap f(v) = g(u \cup v)$ for all $u, v \in [\lambda]^{<\aleph_0}$.

Keisler [9] proved that good ultrafilters characterize the maximum class in Keisler's order: if $\mathcal{D}$ is a regular ultrafilter on λ, then $\mathcal{D}$ is λ^+-good if and only if $M^\lambda/\mathcal{D}$ is λ^+-saturated for some, equivalently every, model of every complete countable theory T. By extension,

Convention 1.12. *If $\mathcal{D}$ is a regular ultrafilter on λ and $\kappa \leq \lambda$ and T is a complete countable theory, we may say*

$$\mathcal{D} \textit{ is } (\kappa^+, T)\textit{-good}$$

if for some, equivalently every, $M \models T$ we have that $M^\lambda/\mathcal{D}$ is κ^+-saturated. When $\kappa = \lambda$, we may just say "$\mathcal{D}$ is good for T.*" Note that the negation, "$\mathcal{D}$ is not (κ^+, T)-good" means that for some, equivalently every, $M \models T$, the ultrapower $M^\lambda/\mathcal{D}$ is not κ^+-saturated.*

2. New theories

This section defines a new family of simple theories. (The reader may prefer to begin with the exposition in §3.) Recall the convention on trees, 1.5.

Notation 2.1 (Notation for trees)**.**

(1) *In this section, a tree will always denote a subset of ${}^{\omega>}\omega$, closed under initial segments, and partially ordered by intial segment, denoted $\trianglelefteq$.*

(2) *For $\mathcal{T}_i$ a tree and $k < \omega$, let $\mathcal{T}_{i,k}$ denote the kth level of $\mathcal{T}_i$, i.e*

$$\mathcal{T}_{i,k} = \mathcal{T}_i \cap {}^k\omega.$$

That is, that any $\eta \in \mathcal{T}_{i,k}$ has length k and[2] *is a function from $\{0, \dots, k-1\}$ to ω. We may write $\eta(t)$ for the value of η at $t \in \operatorname{dom}(\eta)$.*

(3) *Let $\mathcal{T}_{i,\leq k}$ denote $\bigcup_{\ell \leq k} \mathcal{T}_{i,\ell}$.*

(4) *For $\mathcal{T}_i$ a tree and $\eta \in \mathcal{T}_{i,k}$, denote the immediate successors of η in $\mathcal{T}_i$ by*

$$\operatorname{ims}_{\mathcal{T}_i}(\eta) = \{\eta' \in \mathcal{T}_{i,k+1} \ : \ \eta \trianglelefteq \eta'\}.$$

(5) *For $\mathcal{T}_i$ a tree, denote the leaves of $\mathcal{T}_i$ by*

$$\lim(\mathcal{T}_i) = \{\eta \in {}^\omega\omega : \omega \restriction k \in \mathcal{T}_{i,k} \textit{ for all } k < \omega\}.$$

Definition 2.2. *Call $\xi : \omega \to \{0,1\}$ a* level function *if $\{i < \omega : \xi(i) = 1\}$ is infinite, and (for convenience) $\xi(0) = 1$.*

[2]note that under this setup the "0th level" is empty, i.e $\mathcal{T}_{1,0} = \mathcal{T}_{2.0} = \emptyset$.

The idea of a level function, 2.2, will be that level i of the tree is active if $\xi(i) = 1$, and not if $\xi(i) = 0$ (the "lazy levels"), explained presently.

The first main ingredient is that of a *parameter* (basic parameter 2.3, parameter 2.5) which will give us the blueprint on which a theory can be based.

Definition 2.3. *A* basic parameter $\mathfrak{m}$ *consists of a pair of trees $\mathcal{T}_1$, $\mathcal{T}_2$, a sequence of binary relations $\mathcal{R}_k$ for $k < \omega$, and a level function ξ, all satisfying the following.*

(1) $\mathcal{T}_1$ and $\mathcal{T}_2$ are subtrees of ${}^{\omega>}\omega$ with finite splitting and no maximal node.
(2) For $k < \omega$, $\mathcal{R}_k \subseteq \mathcal{T}_{1,k} \times \mathcal{T}_{2,k}$, and for $k = 0, 1$ we have equality.
(3) If $(\eta_1, \eta_2) \in \mathcal{R}_{k+1}$ then $(\eta_1 \upharpoonright k, \eta_2 \upharpoonright k) \in \mathcal{R}_k$.
(4) If $(\eta_1, \eta_2) \in \mathcal{R}_k$, $\eta_2' \in \operatorname{ims}_{\mathcal{T}_2}(\eta_2)$, then for at least two distinct $\eta_1', \eta_{1,}'' \in \operatorname{ims}_{\mathcal{T}_1}(\eta_1)$ we have $(\eta_1', \eta_2') \in \mathcal{R}_{k+1}$ and $(\eta_1'', \eta_2') \in \mathcal{R}_{k+1}$, and the parallel for the trees reversed. [Informally, if two elements at one level are connected, then every immediate successor of one of them is connected to at least two immediate successors of the other.]
(5) If $\xi(k) = 0$, then $(\mathcal{T}_{1,k+1}, \mathcal{T}_{2,k+1}, \mathcal{R}_{k+1})$ adds no new constraints meaning: if $(\eta_1, \eta_2) \in \mathcal{T}_{1,k+1} \times \mathcal{T}_{2,k+1}$ and $(\eta_1 \upharpoonright k, \eta_2 \upharpoonright k) \in \mathcal{R}_k$ then $(\eta_1, \eta_2) \in \mathcal{R}_{k+1}$.
(6) Let $\mathcal{R} = \bigcup_{k<\omega} \mathcal{R}_k$.

Remark 2.4. *On the level functions.* Conditions 2.3(5)-(6) tell us essentially that if $\xi(i) = 0$, $\mathcal{R}_{i+1}$ is set by (2.3)(5) and contributes no new constraints: if two elements connect in $\mathcal{R}_i$, then $\mathcal{R}_{i+1}$ is a complete bipartite graph on their immediate successors, whereas if two elements don't connect in $\mathcal{R}_i$, $\mathcal{R}_{i+1}$ is an empty graph on their successors. We call i a "lazy level" (we chose to say this about i, although we could have said this about $i + 1$). In contrast if $\xi(i) = 1$, we will have a lot of freedom in choosing $\mathcal{R}_{i+1}$, subject to 2.3(6) and 2.5. The usefulness of this feature, the level function, will be more apparent starting in §6 when we pattern the $\mathcal{R}_i$'s on tailor-made sequences of bipartite random graphs, and start comparing theories whose level functions are in some natural sense independent.

Since we will be interested in varying the edge families $\mathcal{R}_k$, the following conditions will ensure there are a minimum of edges and edge coherence to define a model completion. In the rest of this paper, we will always assume them to be true. We could have included them in 2.3.

Definition 2.5. *We say the basic parameter $\mathfrak{m}$ is a* parameter *when, in addition:*[3]

(1) Fullness: For every $\eta \in \lim(\mathcal{T}_1)$, there are continuum many $\rho \in \lim(\mathcal{T}_2)$ such that $(\eta \upharpoonright k, \rho \upharpoonright k) \in \mathcal{R}_k$ for all $k < \omega$.
[note by 2.3(4), at each stage there are at least two distinct successors]
Likewise, for every $\rho \in \lim(\mathcal{T}_2)$, there are continuum many $\eta \in \lim(\mathcal{T}_1)$ such that $(\eta \upharpoonright k, \rho \upharpoonright k) \in \mathcal{R}_k$ for all $k < \omega$.
(2) Left extension: if $k < \omega$, $\nu \in \mathcal{T}_{2,k}$, $u \subseteq \mathcal{T}_{1,k+1}$, $|u| \leq k$ satisfy
$$(\forall \eta \in u)[(\eta \upharpoonright k, \nu) \in \mathcal{R}_k]$$
then there are $\geq k + 1$ elements $\rho \in \operatorname{ims}_{\mathcal{T}_2}(\nu)$ such that
$$(\forall \eta \in u)[(\eta, \rho) \in \mathcal{R}_{k+1}].$$
(3) Right extension: if $k < \omega$, $\nu \in \mathcal{T}_{1,k}$, $u \subseteq \mathcal{T}_{2,k+1}$, $|u| \leq k$ satisfy
$$(\forall \rho \in u)[(\nu, \rho \upharpoonright k) \in \mathcal{R}_k$$

[3] We repeat the conditions for both sides since $\mathcal{R}_k$ is not required to be symmetric.

then there are $\geq k+1$ *elements* $\eta \in \mathrm{ims}_{\mathcal{T}_1}(\nu)$ *such that*

$$(\forall \rho \in u)[(\eta, \rho) \in \mathcal{R}_{k+1}].$$

Remark 2.6. *Together,* 2.3 *and the extension axioms of* 2.5 *imply that the branching of each* $\mathcal{T}_\ell$ *at height* k *is at least* $k+1$.

Note that extension does not require the elements in the set u to have an immediate common predecessor.

As the results of this paper indicate it may be interesting to further investigate theories in this region, we include two comments on alternative definitions.

Discussion 2.7. *We could have weakened left and right extension by asking, e.g.: for every* $k_1 < \omega$ *there is* $k_2 > k_1$ *such that if* $k_3 \geq k_2$, $\nu \in \mathcal{T}_{3-\ell}, k_3$, $u \subseteq \mathcal{T}_{\ell,k_3+1}$, $|u| \leq k_1$ *satisfy, etc. With this we would gain a little in some places, e.g.* 2.11, *and lose a little in others, e.g.* 2.17. *The clean formulation in* 2.5 *is sufficient for our purposes here. Informally, rather than working with a fixed branching and letting the number of connections be arbitrarily slow-growing, we encode* $f(k) \geq k+1$ *in our extension axioms and in the construction allow branching to be arbitrarily large.*

Discussion 2.8. *Another variation we do not use here would be to say* $\mathfrak{m}$ *is* very nice *when we may add a non-connection clause to the extension axioms, e.g. if* $k < \omega$, $\nu \in \mathcal{T}_{2,k}$, $u, v \subseteq \mathcal{T}_{1,k+1}$, *are disjoint,* $|u \cup v| \leq k$ *satisfy*

$$(\forall \eta \in u)[(\eta \restriction k, \nu) \in \mathcal{R}_k$$

then there are $\geq k+1$ *elements* $\rho \in \mathrm{ims}_{\mathcal{T}_2}(\nu)$ *such that* $(\forall \eta \in u)[(\eta, \rho) \in \mathcal{R}_{k+1}]$ *and* $(\forall \eta \in v)[(\eta, \rho) \notin \mathcal{R}_{k+1}]$ – *and similarly for* $\mathcal{T}_1, \mathcal{T}_2$ *reversed.*[4]

Returning to the main line of the construction, an important feature of this setup is its potential for symmetry, which will help in our proofs.

Definition 2.9. *For any parameter* $\mathfrak{m}_1$, *the dual* $\mathfrak{m}_2 = \mathrm{dual}(\mathfrak{m}_1)$ *is defined by:*

(1) $(\mathcal{T}_2^{\mathfrak{m}_2}, \mathcal{T}_1^{\mathfrak{m}_2}) = (\mathcal{T}_1^{\mathfrak{m}_1}, \mathcal{T}_2^{\mathfrak{m}_1})$.
(2) $\mathcal{R}_n^{\mathfrak{m}_2} = \{(\eta_2, \eta_1) : (\eta_1, \eta_2) \in \mathcal{R}_n^{\mathfrak{m}_1}\}$.

Observation 2.10. *If* $\mathfrak{m}$ *is a basic parameter, so is* $\mathrm{dual}(\mathfrak{m})$, *and* $\mathrm{dual}(\mathrm{dual}(\mathfrak{m})) = \mathfrak{m}$, *and if* $\mathfrak{m}$ *is a parameter, then so is* $\mathrm{dual}(\mathfrak{m})$.

It is worth noting that this definition extends the "new simple theory" from [27], used there to produce an example of incomparability in ZFC. That said, the present version is substantially more general and more flexible, both in its set-up and in its incorporation of symmetry, as the next sections will show. [The reader unfamiliar with [27] can safely skip Observation 2.11.]

Observation 2.11. *For every* $f : \omega \to \omega \setminus \{0, 1, 2\}$ *which goes to infinity,* T_f *from* [27] *is equal, up to renaming, to* $T_{\mathfrak{m}}$ *for some basic parameter* $\mathfrak{m}$. *If in addition* $f(k) \geq k+1$, *then in addition* $\mathfrak{m}$ *is a parameter.*

Proof. Using the notation of [27] Definition 2.4, let's check definitions 2.3 and 2.5.

Let $\mathcal{T}_{2,n} = \prod_{\ell<n} f(\ell)$ and let $\mathcal{T}_2 = \bigcup_n \mathcal{T}_{2,n}$.

In order to define $\mathcal{T}_1$, recall that in [27] 2.4, there was a natural tree structure on the left-hand side given as follows. We called $s \subseteq \mathcal{T}_{1,\leq k}$ "k-maximal" if (a) it is a subtree, thus downward closed (closed under initial segment), and (b) it does not

[4] One drawback is that this isn't satisfied by the theories of [27].

contain all immediate successors of any given node. The point is that an element of the left-hand side in a model of T_f determined some such s (by its connections on the right) and that $\subseteq$ gives a natural partial ordering on the set of all s that are k-maximal for some finite k, forming an infinite, finitely branching tree. So, we choose $\mathcal{T}_1 \subseteq {}^{\omega>}\omega$ to be equivalent to this tree (up to renaming) and choose $\mathcal{R}$ so that $\mathcal{R}_k$ holds between $\eta \in \mathcal{T}_{1,k}$ and $\rho \in \mathcal{T}_{2,k}$ if and only if η was (before renaming) the subtree s and s contains ρ. Let ξ be the sequence constantly equal to 1. This completes the specification of $\mathfrak{m}$, so let us check 2.3. Clearly (1), (2), (3), (4), (6) hold. (5) is trivially satisfied as ξ is constantly 1. Likewise, it is straightforward to check that as long as $f(k) \geq k+1$, the fullness and extension conditions of 2.5 follow easily from the use of k-maximal s's. Thus, $\mathfrak{m}$ is a parameter. $\square$

Next we use our template $\mathfrak{m}$ to produce a universal theory and its model completion. Note that this theory is in a different signature, and a priori has no access to the trees and edges mentioned in $\mathfrak{m}$. (For a further discussion, see §3.) $T^0_{\mathfrak{m}}$ will be a theory in an infinite language, defined as the union of $T^0_{\mathfrak{m},k}$ for all finite k.

Remark 2.12. *When the context is clear, below, we will write* $\mathfrak{m} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$ *instead of* $(\mathcal{T}_{\mathfrak{m},1}, \mathcal{T}_{\mathfrak{m},2}, \mathcal{R}_{\mathfrak{m}})$.

In the next definition, we informally think of $\mathcal{Q}$ as being on the left and $\mathcal{P}$ as being on the right.

Definition 2.13. *Given a parameter* $\mathfrak{m}$ *and* $k < \omega$, *define* $T^0_{\mathfrak{m},k}$, *a universal first order theory, as follows. Let* $\tau_k = \tau_{\mathfrak{m},k}$ *denote*[5]

$$\{\mathcal{Q}, \mathcal{P}, Q_\eta, P_\rho : \eta \in \mathcal{T}_{1,\leq k}, \rho \in \mathcal{T}_{2,\leq k}\} \cup \{R\}.$$

Then $T^0_{\mathfrak{m},k}$ *is the universal theory in* $\mathcal{L}(\tau_{\mathfrak{m},k})$ *such that a* τ_k*-model* M *is a model of* $T^0_{\mathfrak{m},k}$ *if and only if:*

(1) $\mathcal{Q}^M$, $\mathcal{P}^M$ *is a partition of* $\mathrm{dom}(M)$. *We identify* $\mathcal{Q}$ *and* $Q_{\langle\rangle}$, $\mathcal{P}$ *and* $P_{\langle\rangle}$.

(2) $\langle Q^M_\eta : \eta \in \mathcal{T}_{1,n}\rangle$ *is a partition of* $\mathcal{Q}^M$ *for each* $n \leq k$ *and this partition satisfies*
$$\eta \trianglelefteq \nu \in \mathcal{T}_{1,\leq k} \text{ implies } Q^M_\eta \supseteq Q^M_\nu.$$

(3) $\langle P^M_\rho : \rho \in \mathcal{T}_{2,n}\rangle$ *is a partition of* $\mathcal{P}^M$ *for each* $n \leq k$ *and this partition satisfies*
$$\rho \trianglelefteq \nu \in \mathcal{T}_{2,\leq k} \text{ implies } P^M_\rho \supseteq P^M_\nu.$$

(4) $R^M \subseteq \{(b,a) : b \in Q^M, a \in P^M$ *and for every* $n < \omega$, *there are* $\eta_1 \in \mathcal{T}_{1,n}$, $\eta_2 \in \mathcal{T}_{2,n}$ *such that* $(\eta_1, \eta_2) \in \mathcal{R}_n$ *and* $(b,a) \in Q^M_{\eta_1} \times P^M_{\eta_2}\}$.

Informally, condition 2.13(4) says there can only be R-edges in M between elements which belong to "leaves" all of whose initial segments of the same height were connected in the template $\mathcal{R}$.

Observation 2.14. $T^0_{\mathfrak{m},k} \subseteq T^0_{\mathfrak{m},k+1}$.

[5]We could have used predicates $P_1, P_2, P_{1,\eta}, P_{2,\eta}$ to emphasize the symmetry and to continue the notation of $\mathcal{T}_1, \mathcal{T}_2$, but chose $\mathcal{P}, \mathcal{Q}$ for readability.

Definition 2.15. *Given a parameter $\mathfrak{m}$ we define $T^0_{\mathfrak{m}}$, a universal first order theory, as follows. The vocabulary is $\tau = \tau_{\mathfrak{m}} = \{\mathcal{Q}, \mathcal{P}, Q_\eta, P_\eta, R : \eta \in \mathcal{T}_1, \nu \in \mathcal{T}_2\}$ where $\mathcal{Q}, \mathcal{P}, Q_\eta, P_\eta$ are unary predicates and R is a binary predicate, and*

$$T^0_{\mathfrak{m}} = \bigcup\{T^0_k : k < \omega\}.$$

Claim 2.16. *For each $k < \omega$, the model completion $T_{\mathfrak{m},k}$ of $T^0_{\mathfrak{m},k}$ exists.*

Proof. $T^0_{\mathfrak{m},k}$ is a universal theory in a finite relational language, and the class of its models has JEP and AP. Suppose we are given any two $M_1, M_2 \models T^0_{\mathfrak{m},k}$. For JEP, we also assume $M_1 \cap M_2 = \emptyset$; for AP, we also assume there is a model $M_0 \models T^0_{\mathfrak{m},k}$ such that $M_0 \subseteq M_\ell$ for $\ell = 1, 2$, $M_1 \cap M_2 = M_0$. Then consider the following model N. The domain of N is $M_1 \cup M_2$, for each unary predicate $X \in \tau_k$, let $X^N = X^{M_1} \cup X^{M_2}$, and let $R^N = R^{M_1} \cup R^{M_2}$. Thus $T_{\mathfrak{m},k}$ exists. $\square$

Claim 2.17. *For every $k_* < \omega$ the following holds: if M is a model of $T_{\mathfrak{m},k_*}$, N is a model of $T_{\mathfrak{m},k_*+1}$, and ψ is a sentence of τ_{k_*} of length $\leq k_*$ (or just such that any subformula has $\leq k_*$ free variables), then $M \models \psi \iff N \models \psi$.*

Proof. Let $\mathcal{F}$ be defined by: $f \in \mathcal{F} = \mathcal{F}_{k_*}$ iff for some $k \leq k_*$ and $a_0, \ldots, a_{k-1} \in M$, $b_0, \ldots, b_{k-1} \in N$, we have that $f = \{(a_\ell, b_\ell) : \ell < k\}$, and for every atomic $\varphi(x_0, \ldots, x_{k-1}) \in \mathcal{L}(\tau_{k_*})$, we have that

$$M \models \varphi[a_0, \ldots, a_{k-1}] \iff N \models \varphi[b_0, \ldots, b_{k-1}].$$

(We could just as well replace "atomic" by "quantifier free.") Thus, $\mathcal{F}$ is a set of partial one to one functions f from M into N such that $|\operatorname{dom}(f)| \leq k_*$, and clearly if $f \in \mathcal{F}$ and $A \subseteq \operatorname{dom}(f)$ then $f \restriction A \in \mathcal{F}$.

We claim that if $f \in \mathcal{F}$, $|\operatorname{dom}(f)| < k_*$ and $a \in M, b \in N$ then there are $a' \in M$, $b' \in N$ such that $f \cup \{(a', b)\} \in \mathcal{F}$ and $f \cup \{(a, b')\} \in \mathcal{F}$. Suppose we are given $f = \{(a_\ell, b_\ell) : \ell < k < k_*\}$ along with a, b. Since the proofs are similar, it will suffice to find b'. Since either $a \in \mathcal{P}^N$ or $a \in \mathcal{Q}^N$, by symmetry (i.e. we can use dual($\mathfrak{m}$)) it suffices to consider the case $a \in Q^N$.[6]

Consider the sequence $\{a_\ell : \ell < k\}$ in M. Each a_ℓ is either in $\mathcal{Q}^M$ or $\mathcal{P}^M$. Renumbering, without loss of generality, there is $\ell_* \leq k$ such that $a_\ell \in \mathcal{P}^M$ for $\ell < \ell_*$ and $a_\ell \in \mathcal{Q}^M$ otherwise (so the corresponding fact is true for the b_ℓ's in N). Also, without loss of generality, the sequence $\langle a_\ell : \ell < \ell_* \rangle$ is without repetition, and $a \notin \{a_\ell : \ell < \ell_*\}$, otherwise it is trivial. Since we have assumed that a, our new element, is in $\mathcal{Q}^M$, let $\eta \in \mathcal{T}_{1,k_*}$ be such that $a \in Q^M_\eta$. Recalling that N has more information, let $\rho_0, \ldots, \rho_{\ell_*-1} \in \mathcal{T}_{2,k_*+1}$ be such that $b_\ell \in P^N_{\rho_\ell}$ for $\ell < \ell_*$. (It

[6]Informally, here is the worry: M is a model of $T_{\mathfrak{m},k_*}$, so the best quantifier-free information we have about the a_ℓ's in M is to know which leaf at level k_* each of them belongs to (i.e. which Q_η or P_ρ for $\eta \in \mathcal{T}_{1,k_*}$ or $\rho \in \mathcal{T}_{2,k_*}$) and whether or not they connect to each other via R. In the model N, the corresponding b_ℓ's have the same quantifier-free τ_{k_*}-type as their counterparts in M, but in N we have an additional level of resolution: we know which leaf at level $k_* + 1$ each b_ℓ belongs to. For example, if $(Q^M_\eta, P^M_\rho, R^M)$ form an infinite bipartite random graph in M, then for any finite set u of elements of Q^M_η there is $a \in P^M_\rho$ R-connecting to all of them. But suppose f had mapped the elements of u to elements of Q^N_η which happened to span $Q^N_{\eta^\frown\langle i \rangle}$ for $i < |\operatorname{ims}_{\mathcal{T}_1}(\eta)|$. Then we could not find a corresponding b' in N. We solve or avoid this simply by limiting the size of sets u in terms of k_* and applying the extension axioms.

follows by our definition of $f \in \mathcal{F}$ that $a_\ell \in P^N_{\rho_\ell \restriction k_*}$ for $\ell < \ell_*$.) It will suffice to find $b' \in Q^N_\eta$ such that

(a) $$(a, a_\ell) \in R^M \iff (b', b_\ell) \in R^N.$$

The inequalities are easy so we ignore them. Note that the axioms for $T^0_{\mathfrak{m},k_*} \subseteq T_{\mathfrak{m},k_*}$ in 2.13(4) imply that

$$(a, a_\ell) \in R^M \implies (\eta, \rho_\ell \restriction k) \in \mathcal{R}_{k_*}.$$

Thus, for equation (a), it will suffice to show that there is some $\eta' \in \mathcal{T}_{1,k_*+1}$ such that $\eta \trianglelefteq \eta'$ and

$$(a, a_\ell) \in R^M \implies (\eta, \rho_\ell) \in \mathcal{R}_{k_*+1}.$$

(It doesn't matter to us here whether the non-edges come from the randomness between leaves or from leaves with no edges between them.) Since $\ell_* < k_*$, this is exactly the content of the relevant extension axiom 2.5. $\square$

Corollary 2.18. *When $\mathfrak{m}$ is a parameter, the sequence $\langle T_{\mathfrak{m},k} : k < \omega \rangle$ converges. Moreover, for every formula $\varphi(\bar{x})$ of $\tau_\mathfrak{m}$, for some quantifier free $\psi(\bar{x})$, for every $k < \omega$ large enough, we have*

$$(\forall \bar{x})(\ \psi(\bar{x}) \equiv \varphi(\bar{x})\ ) \in T_{\mathfrak{m},k}.$$

Conclusion 2.19. *Let $\mathfrak{m}$ be a parameter and $T^0_\mathfrak{m}$ be the universal theory from* 2.15. *Then its model completion $T_\mathfrak{m}$ is well defined, eliminates quantifiers, and is equal to the limit of $\langle T_{\mathfrak{m},k} : k < \omega \rangle$.*

We will continue with a description of the models and types of $T_\mathfrak{m}$ in §4 after some discussion.

3. A dark woods

In this primarily expository section we motivate the new theories of §2 by means of a very simple example (of finite height), followed by some remarks on the general case. The reader may prefer to read it before, after, or in parallel to §2.

We start with a very simple analogue of a parameter, meant to show how the trees and the theory reflect each other. *To make the illustrations easier, the example does not necessarily have all the edges required by the extension axioms; moreover, it is finite!* The initial data records a blueprint; its language is not yet the language of our theory. As we will see, its role is not unlike that of the reduced graph in Szemerédi's regularity lemma.

A. Warm-up.

Suppose we are given two trees and a family of edges. In this example, suppose the trees are:

- $\mathcal{T}_1 = \{\{\emptyset, 0, 1, 00, 01, 02, 10, 11, 12\}, \trianglelefteq_1\}$, where $\trianglelefteq_1$ is the partial order given by initial segment.
- $\mathcal{T}_2 = \{\{\emptyset, 0, 1, 00, 01, 02, 10, 11, 12, 13, 20, 21, 22, 23\}, \trianglelefteq_2\}$.

Let $\mathcal{T}_{i,k}$ be the k-th level of $\mathcal{T}_i$, so e.g. $\mathcal{T}_{1,0} = \{\emptyset\}$, $\mathcal{T}_{1,1} = \{0, 1\}$. Note that if $\eta \in \mathcal{T}_{i,k}$ then η takes values on $0, \ldots, k-1$.

Suppose that the edge family $\mathcal{R}_k \subseteq \mathcal{T}_{1,k} \times \mathcal{T}_{2,k}$ is as follows:

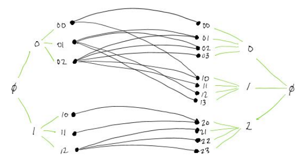
- $\mathcal{R}_0 = \{(\emptyset, \emptyset)\}$, $\mathcal{R}_1 = \{(0,0), (0,1), (1,2)\}$, $\mathcal{R}_2 = \{(00,00), (00,10), (01,01), (01,02), (01,12), (01,13), (02,01), (02,02), (02,10), (02,11), (10,20), (11,20), (12,21), (12,22), (12,23)\}$.

Let $\mathcal{R} = \bigcup_{k \leq 2} \mathcal{R}_k$.

In the following image, black edges are pictures of the corresponding $\mathcal{R}_i$, and the green edges are just for visual clarity, to record the tree structure.

$\mathcal{R}_0$ and $\mathcal{R}_1$:

$\mathcal{R}_2$:



Given this data, let us now define T^0, a universal first order theory:

- its vocabulary is $\tau = \{Q_\eta : \eta \in \mathcal{T}_2\} \cup \{P_\rho : \rho \in \mathcal{T}_1\} \cup \{R\}$ where R is a binary predicate and all other predicates are unary. We may write Q, P for $Q_\emptyset, P_\emptyset$ respectively.
- The axioms first say that the unary predicates encode the structure of the tree. That is, M is a model of T^0 when:
  - $Q^M_\emptyset$ and $P^M_\emptyset$ partition the domain.
  - For each $k < 2$ and $\eta \in \mathcal{T}_{1,k}$, $\{Q^M_{\eta'} : \eta \trianglelefteq_1 \eta' \in \mathcal{T}_{1,k+1}\}$ is a partition of Q^M_η.
  - For each $k < 2$ and $\rho \in \mathcal{T}_{2,k}$, $\{P^M_{\rho'} : \rho \trianglelefteq_2 \rho' \in \mathcal{T}_{2,k+1}\}$ is a partition of P^M_ρ.

  Finally, the axioms say that edges must respect the meta-pattern imposed by $\mathcal{R}$, that is,
  - $R^M \subseteq \{(a,b) : \exists(\eta, \rho) \in \mathcal{R}_2 \text{ s.t. } a \in Q^M_\eta, \ b \in Q^M_\rho\}$.

Notice that T^0 is a universal theory in a finite relational language. It has the joint embedding property and amalgamation: given models M, N, the model whose domain is the disjoint union of $\mathrm{dom}(M) \cap \mathrm{dom}(N)$, $\mathrm{dom}(M) \setminus \mathrm{dom}(N)$, and $\mathrm{dom}(N) \setminus \mathrm{dom}(M)$ is also a model of the theory. As a result, a model completion T exists and is well defined, and complete.

The family $\mathcal{R}$ essentially determines which pairs of predicates Q_η, P_ρ are potentially allowed to have edges between their members. If $(\eta, \rho) \notin \mathcal{R}$, then the infinite bipartite graph $(Q^M_\eta, P^M_\rho, R^M \restriction Q^M_\eta \times P^M_\rho)$ has no edges. If $(\eta, \rho) \in \mathcal{R}$, this graph may have some edges, though it isn't required to by the universal theory T^0.

Remark 3.1. *This example can easily be extended to different choices of pairs of trees of any finite height k along with families of edges $\mathcal{R}_1, \dots, \mathcal{R}_k$ with the appropriate monotonicity properties.*

Discussion 3.2. At this point the reader may wish to read the actual definition of parameter, which involves infinite trees, and notice the mild but useful coherence properties which have been put on the trees, to ensure they are full enough, and to ensure via the left and right extension axioms that the model completion still exists and is well defined and eliminates quantifiers, when the trees are infinite. The reader should also notice the *level function* 2.2 discussed in 2.4, which didn't appear in the warm-up, but will be very useful for us in what follows.

B. Parameters and reduced graphs.

The following notation, which makes sense for any parameter, will simplify our discussion.

Definition 3.3. *Suppose we are given a parameter* $\mathfrak{m}$*, a finite* k*, and nonempty sets* $V \subseteq \mathcal{T}_{1,k}$ *and* $W \subseteq \mathcal{T}_{2,k}$*. Let*

$$H_k(V, W) = (V, W, \mathcal{R}_k \restriction V \times W).$$

Definition 3.4. *Suppose we are given a parameter* $\mathfrak{m}$ *thus* $T_{\mathfrak{m}}$*, a finite* k*, a model* $M \models T_{\mathfrak{m}}$*, and nonempty sets* $V \subseteq \mathcal{T}_{1,k}$ *and* $W \subseteq \mathcal{T}_{2,k}$*. Let*

$$G_k(V, W) = G_k(V, W)[M] = (\bigcup_{\eta \in V} Q^M_\eta, \bigcup_{\eta \in W} P^M_\rho, R \restriction (\bigcup_{\eta \in V} Q^M_\eta \times \bigcup_{\eta \in W} P^M_\rho)).$$

Remark 3.5. Although our warm-up example, being finite, is not strictly speaking a parameter, we can apply this notation to it to say: if $(\eta, \rho) \in \mathcal{R}_2$ [*as the trees have height two, these are leaves*] then $G(\{\eta\}, \{\rho\})$ is an infinite, bipartite random graph. Of course, these are statements in the meta-theory, referring to the data of the trees, which are not axioms of T, though of course any statements to the effect that between certain specific definable sets R forms a bipartite random graph will follow from the axioms of T. What T doesn't have access to is the quantification over e.g. elements of $\mathcal{T}_1, \mathcal{T}_2$ or $\mathcal{R}$. More generally, in our finite warm-up example, whenever V, W are such that $H_2(V, W)$ is a complete bipartite graph, then $G_2(V, W)$ is an infinite, bipartite random graph.

Remark 3.6. Recall, see e.g. [11], that in Szemerédi's regularity lemma for graphs, a given finite graph is partitioned into clusters, so that between most pairs of clusters the edges are distributed ϵ-uniformly. One definition of the reduced graph is the graph with one vertex for each cluster, and with an edge between two points whose associated clusters are ϵ-regular with density ϵ-bounded away from 0 and 1.

In Remark 3.6 the reduced graph doesn't only record the "randomness" of a given pair of clusters, but also entails that there is a certain further genericity in the interaction of more than two clusters. For instance, if three points in the reduced graph form a triangle, we should be able to get (many copies of) any configuration on three vertices spanning the associated clusters in the original graph. One consequence of our construction of parameters $\mathfrak{m}$ and theories $T_{\mathfrak{m}}$ will be that given V, W, the bipartite graphs $H(V, W)$ and $G(V, W)$ retain the natural analogue of this phenomenon for the infinite setting, as we now explain.

C. Our main case: $\mathfrak{m}$ a parameter, $T = T_{\mathfrak{m}}$ from 2.19.

To finish this section of examples, we briefly discuss what models of T look like in our main, infinite case, where $\mathfrak{m}$ is a parameter, the trees both have countable height and $T_{\mathfrak{m}}$ is the model completion given in 2.19.

When the trees become infinite, the structure we analyzed above at level 2 (or similarly at any fixed finite level k in $H_k(V, W)$ or $G_k(V, W)$) carries over in a natural sense to the "leaves," as the next definitions explain. The word "virtual" reflects that the objects are generally not definable, though they may be type-definable. Notice, though, that although what we call "$\mathcal{R}^\infty$" is not definable in 3.7, the edge relation in 3.8 is simply R^M.

Definition 3.7 (Virtual reduced graph)**.** *Let $\mathfrak{m} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$ be a parameter.*

(1) Define $\mathcal{R}^\infty = \{(\rho, \eta) : (\rho, \eta) \in \lim(\mathcal{T}_1) \times \lim(\mathcal{T}_2)$ and $(\rho \restriction k, \eta \restriction k) \in \mathcal{R}_k$ for all $k < \omega\}$.

(2) Then for any nonempty $V \subseteq \lim(\mathcal{T}_1)$ and $W \subseteq \lim(\mathcal{T}_2)$, define the virtual reduced graph

$$H^\infty(V, W)$$

to be the bipartite graph $(V, W, \mathcal{R}^\infty)$.

That is, 3.7 defines a bipartite graph whose vertices are the leaves of $\mathcal{T}_1$ on the left and the leaves of $\mathcal{T}_2$ on the right and where (η, ν) is an edge if and only if $(\eta \restriction n, \nu \restriction n) \in \mathcal{R}_n$ for all $n < \omega$. The notation in (2) gives various induced subgraphs.

Definition 3.8 (Virtual graph)**.** *Continuing in the notation of* 3.7*, suppose we are given any model $M \models T_{\mathfrak{m}}$.*

(1) For any $V \subseteq \lim(\mathcal{T}_1)$, let the expression $Q^\infty_V = Q^\infty_V[M]$ denote

$$\{a \in \mathrm{dom}(M) : \text{ for some } \eta \in V,\ M \models Q_{\eta \restriction k}(a) \text{ for all } k < \omega\}.$$

In particular, for any $\eta \in \lim(\mathcal{T}_1)$, $Q^\infty_{\{\eta\}} = Q^\infty_{\{\eta\}}[M]$ denotes the subset of M realizing the type $\{Q_{\eta \restriction k}(x) : k < \omega\}$.

(2) Likewise for any $W \subseteq \lim(\mathcal{T}_2)$, let the expression $Q^\infty_W = Q^\infty_W[M]$ denote

$$\{b \in \mathrm{dom}(M) : \text{ for some } \rho \in W,\ M \models P_{\eta \restriction k}(b) \text{ for all } k < \omega\}.$$

In particular, for any $\rho \in \lim(\mathcal{T}_2)$, $P^\infty_{\{\rho\}} = P^\infty_{\{\rho\}}[M]$ denotes the subset of M realizing the type $\{P_{\rho \restriction k}(x) : k < \omega\}$.

(3) For any nonempty $V \subseteq \lim(\mathcal{T}_1)$, $W \subseteq \lim(\mathcal{T}_2)$, let the virtual graph

$$G^\infty(V, W) = G^\infty(V, W)[M]$$

be the bipartite graph

$$(Q^\infty_V,\ P^\infty_W,\ R^M \restriction Q^\infty_V \times P^\infty_W).$$

That is, 3.8 defines a bipartite graph from M whose vertices are elements of Q belonging to certain "leaves" on the left and the elements of P belonging to certain other "leaves" on the right, along with the edge relation given by R.

Discussion 3.9. *We defer to §§2, 4 for details and proofs.*

a) When the trees are infinite, the model completion $T_{\mathfrak{m}}$ indeed exists and is well defined, complete, and eliminates quantifiers (see 2.19).

b) The structure of models $M \models T_{\mathfrak{m}}$ is in some sense simple: in the language of 3.7 and 3.8, the point is that for any model $M \models T_{\mathfrak{m}}$, and any nonempty $V \subseteq \lim(\mathcal{T}_1)$ and $W \subseteq \lim(\mathcal{T}_2)$, if $H^\infty(V,W)$ is a complete graph, then $G^\infty(V,W)[M]$ is an infinite bipartite random graph, and if $H^\infty(\{\eta\},\{\rho\})$ is empty, then so is $G^\infty(\{\eta\},\{\rho\})[M]$. (Letting V, W vary, these two facts together are enough to put together the whole picture.)

c) In any $\aleph_1$-saturated model M of $T_{\mathfrak{m}}$, e.g. in a regular ultrapower, for any leaves $\eta \in \lim(\mathcal{T}_1)$, $\rho \in \lim(\mathcal{T}_2)$, the sets $Q^\infty_{\{\eta\}}$, $P^\infty_{\{\rho\}}$ will be infinite, and will have among them the infinite empty or random graph structure just mentioned. We will see in detail in §4 that for λ^+-saturation, we will want each such $Q^\infty_{\{\eta\}}$ and each such $P^\infty_{\{\rho\}}$ to have size at least λ^+, and moreover, for every $V \subseteq \lim(\mathcal{T}_1)$ and $W \subseteq \lim(T_2)$ such that $H^\infty(V,W)$ is a complete graph[7] $G^\infty(V,W)$ is λ^+-saturated as a bipartite random graph, i.e.
  (i) for any two disjoint subsets A, B of P^∞_W of size λ, and any $\eta \in V$, there is $c \in Q^\infty_{\{\eta\}}$ which R-connects to all $a \in A$, no $b \in B$.
  (ii) the parallel reversing V, W and Q, P.

In what follows, we will focus on $\mathfrak{m}$ such that $\mathfrak{m} = \mathrm{dual}(\mathfrak{m})$; so by symmetry, it will be enough to handle one of (i), (ii), and as we will see in 4.8 and 5.7 below, it will generally be enough to realize partial positive R-types.

Where does the potential for widely differing complexity arise? The following very informal discussion may help the reader follow the proof.

Why might these theories interact with ultrapowers in an interesting way? In an ultrapower of a model of $T_{\mathfrak{m}}$, elements which are "on average" part of the same leaf may nonetheless appear, when projected to a given index model, to be in too many different predicates at a given height k, blocking realization of the type in that index model when splitting is constrained. Both the size of allowed splitting at a given height in a given tree (and, by extension, the level functions) come into play, which in turn reflect the degrees of the vertices in the reduced graphs H_k.

Why might different parameters $\mathfrak{m}$, $\mathfrak{n}$ produce theories $T_{\mathfrak{m}}$, $T_{\mathfrak{n}}$ which look different to ultrapowers? The structure of each theory $T_{\mathfrak{m}}$ will reflect its sequence of "reduced graphs," based on the finite bipartite graphs $\mathcal{R}_i = \mathcal{R}_i(\mathfrak{m})$, and the related level function $\xi = \xi(\mathfrak{m})$, which is active at infinitely many $n \in \omega$. When ξ is not active, $\mathcal{R}_{i+1}$ gives essentially no new information beyond $\mathcal{R}_i$. A natural way to vary the $\mathcal{R}_i$'s will be to consider a single fast-growing sequence of sparse graphs $\langle E_i : i < \omega \rangle$, choose many level functions which are independent in a natural sense, and build for each such ξ a theory whose $\mathcal{R}_i$ essentially copies E_i at active levels and copies a complete bipartite graph of the right size at lazy levels. This allows us to vary the sequences of reduced graphs in a very clear way. Remarkably, these differences are detected in a very strong sense both by the theories themselves and by ultrafilters. To prove this will also require an advance in ultrafilter construction.

Remark 3.10. *To make these suggestions precise will, of course, require the rest of the paper; but notice that the construction already suggests many further modifications and interesting future directions, some discussed at the end of the paper.*

[7] What about other Ws? It can't hurt, but won't add anything: see last line of proof of 4.6.

4. Models of $T_{\mathfrak{m}}$

In this section we analyze the types over a model M of $T_{\mathfrak{m}}$, which will help later in dealing with saturation.

Note that we use almost nothing about the level functions in this section; we just need the extension axioms to ensure a minimum increase in the edges. The level functions operate at a different scale in the sense that they control variations in the number of edges well beyond the minimum established by the extension axioms, and will mainly play a role in later sections, where we try to compare theories.

Convention 4.1. *In this section, $\mathfrak{m}$ is an arbitrary but fixed parameter, and M is a model of $T_{\mathfrak{m}}$.*

For the purposes of our analysis, because of the symmetry of $\mathfrak{m}$, it will suffice to deal with types $q(x)$ in one free variable x which describe an element on the left, i.e. $q(x) \vdash \mathcal{Q}(x)$. Note that any such type, being complete, will specify that $Q_{\rho\restriction n}(x)$ for some $\rho \in \lim(\mathcal{T}_1)$ and all $n < \omega$.

Definition 4.2. *For $\rho \in \lim(\mathcal{T}_1)$, we define:*

(1) $\mathcal{S}_\rho = \{\nu :$ for some finite n, $\nu \in \mathcal{T}_{2,n}$ and $(\nu, \rho \restriction n) \in \mathcal{R}_n\}$.
(2) $\lim(\mathcal{S}_\rho) = \{\eta \in \lim(\mathcal{T}_2) : \eta \restriction n \in \mathcal{S}_\rho$ for $n < \omega\}$
$= \{\eta \in \lim(\mathcal{T}_2) : (\rho, \eta) \in \mathcal{R}^\infty\}$.

Observation 4.3. *If $\rho \in \lim(\mathcal{T}_1)$, $\mathcal{S}_\rho$ is a subtree of $\mathcal{T}_2$ with no maximal node.*

Recall our notation from the previous section, in particular:

- recall that for $\eta \in \lim(\mathcal{T}_2)$, $P^\infty_{\{\eta\}} = P^\infty_{\{\eta\}}[M]$ denotes the elements of M which are "in the leaf" corresponding to η, and the corresponding notation for $\rho \in \lim(\mathcal{T}_1)$ is $Q^\infty_{\{\rho\}} = Q^\infty_{\{\rho\}}[M]$.
- likewise P^∞_V, Q^∞_W, which also depend on M.
- recall the virtual reduced graph $H^\infty(V, W)$ from 3.7.
- recall the virtual graph $G^\infty(V, W) = G^\infty(V, W)[M]$ from 3.8.

Observation 4.4. *For any $\rho \in \lim(\mathcal{T}_1)$ and $W \subseteq \lim(\mathcal{T}_2)$ such that $H^\infty(\{\rho\}, W)$ is complete, we have that $\lim(\mathcal{S}_\rho) \supseteq W$, in other words, $\mathcal{S}_\rho$ contains all proper initial segments of elements of W.*

Claim 4.5. *Suppose $\rho \in \lim(\mathcal{T}_1)$ and write $W = \lim(\mathcal{S}_\rho)$. For any*

$$A, B \subseteq P^\infty_W[M] \text{ with } A \cap B = \emptyset$$

the following set of formulas is a non-algebraic partial type of M:

$$\{Q_{\rho\restriction n}(x) : n < \omega\} \cup \{R(x, a) : a \in A\} \cup \{\neg R(x, b) : b \in B\}.$$

Proof. Consider a finite subset, which without loss of generality is of the form:

$$\{Q_{\rho\restriction n}(x) : n < k\} \cup \{R(x, a_0), \dots, R(x, a_{\ell-1})\} \cup \{\neg R(x, b_0), \dots, \neg R(x, b_{r-1})\}.$$

Each of the elements a_i, b_j has a leaf in M: let's suppose that for each $i < \ell$, η_i is such that $M \models P_{\eta_i\restriction n}(a_i)$ for $n < \omega$ and for each $j < r$, ν_j is such that $M \models P_{\nu_j\restriction n}(b_j)$ for $n < \omega$, though these leaves need not necessarily be distinct. By our choice of A, B [that is, by the definition of W], we have that for any finite level, and in particular for k,

$$(\rho \restriction k, \eta_i \restriction k) \in \mathcal{R}_k \text{ and } (\rho \restriction k, \nu_j \restriction k) \in \mathcal{R}_k$$

for $i < \ell$, $j < r$. Thus the following sentence belongs to $T_{\mathfrak{m},k}$:

$$(\exists x)(\bigwedge_{i<\ell} R(x, a_i) \wedge \bigwedge_{j<r} R(x, b_j)).$$

By 2.17, this remains true all the way to $T_{\mathfrak{m}}$. Since this shows an arbitrary finite subset is consistent, we finish the proof. $\square$

Corollary 4.6. *Suppose $\rho \in \lim(\mathcal{T}_1)$ and write $W = \lim(\mathcal{S}_\rho)$. For any $A, B \subseteq M$ with $A \cap B = \emptyset$ we have that*

$$r(x) = \{Q_{\rho \upharpoonright n}(x) : n < \omega\} \cup \{R(x, a) : a \in A\} \cup \{\neg R(x, b) : b \in B\}$$

is a non-algebraic partial type of M if and only if $A \subseteq P^\infty_W[M]$.

Proof. Suppose we denote $A_0 = A \cap P^\infty_W$ and $B_0 = B \cap P^\infty_W$. Let

$$r_0 = \{Q_{\rho \upharpoonright n}(x) : n < \omega\} \cup \{R(x, a) : a \in A_0\ \} \cup \{\neg R(x, b) : b \in B_0\}.$$

Claim 4.5 tells us that r_0 is a partial type.

First consider any elements $b \in B \setminus B_0$. If $b \in Q^M$, then $\neg R(x, b)$ follows by definition as $R^M \subseteq Q^M \times P^M$. If $b \in P^M$, then because M is a model, there is some η such that $b \in P^M_{\eta \upharpoonright n}$ for all $n < \omega$. If $(\rho, \eta) \notin \mathcal{R}^\infty$, then there is some $n < \omega$ for which $(\eta \upharpoonright n, \rho \upharpoonright n) \notin \mathcal{R}_n$, which translates to

$$M \models (\forall x)(\forall y)(Q_{\eta \upharpoonright n}(x) \wedge P_{\rho \upharpoonright n}(y) \implies \neg R(x, y))$$

and so $r_0 \vdash \neg R(x, b)$.

Finally, suppose that $A \setminus A_0$ is nonempty, and let a be any one of its elements. Let η be such that $a \in P^M_{\eta \upharpoonright n}$ for all $n < \omega$. By definition of A_0, $(\rho, \eta) \notin \mathcal{R}^\infty$, so it follows from the previous paragraph that $r_0 \vdash \neg R(x, a)$. Thus r is consistent if and only if $A \setminus A_0 = \emptyset$.

Note that this proof shows that if r is consistent, $r_0 \vdash r$. $\square$

The next claim justifies restricting our saturation arguments to considering types of a very simple form.

Definition 4.7. *We say a model M of $T_{\mathfrak{m}}$ is* weakly λ^+-saturated *when:*

(1) "the leaves are large": for any $\eta \in \lim(\mathcal{T}_1)$, $|\{a \in M : Q^M_{\eta \upharpoonright n}(a)$ for all $n < \omega\}| > \lambda$, and likewise for $\nu \in \lim(\mathcal{T}_2)$.
(2) if $c \in Q^M$ then $\{b : (c, b) \in R^M\} \subseteq P^M$ has cardinality $> \lambda$.
(3) the dual to the previous line: if $b \in P^M$ then $\{c : (c, b) \in R^M\} \subseteq Q^M$ has cardinality $> \lambda$.

Claim 4.8 (A basic form for types)**.** *Suppose M is weakly λ^+-saturated. For any $C \subseteq M$, $|C| \leq \lambda$, and $q \in \mathbf{S}(C)$ such that $q(x) \vdash Q(x)$, there exist ρ, $W \subseteq \lim(\mathcal{S}_\rho)$, $A \subseteq P^\infty_W[M]$, $B \subseteq P^M$ (or even $B \subseteq P^\infty_W[M]$) with $A \cap B = \emptyset$ and $|A| = |B| = \lambda$, such that writing*

$$r(x) = \{Q_{\rho \upharpoonright n}(x) : n < \omega\} \cup \{R(x, a) : a \in A\} \cup \{\neg R(x, b) : b \in B\}.$$

we have $r(x) \subseteq q(x)$ and $r(x) \vdash q(x)$.

Proof. By hypothesis, every "leaf" $Q^\infty_{\{\eta\}}[M]$ or $P^\infty_{\{\rho\}}[M]$ has size $> \lambda$.

To start, let $\rho \in \lim(\mathcal{T}_2)$ be such that $q(x) \vdash Q_{\rho \upharpoonright n}(x)$ for all $n < \omega$, which exists as q is a type. Let $r(x)$ be the largest subset of q which is of the form given, so this determines our $W := \lim(\mathcal{S}_\rho)$, and our initial $A := \{c \in C : q \vdash R(x, c)\} \cap P^\infty_W[M]$, and initial $B := \{c \in C : q \vdash \neg R(x, c)\}$. [By 4.6,these requirements imply $r(x)$ is a partial type.] Since $T_{\mathfrak{m}} = Th(M)$ eliminates quantifiers, it suffices to consider quantifier-free formulas, but we should justify that we do not need any additional information of the following kinds – in other words, by possibly increasing A, B but adding no more than λ elements, we can ensure that any formulas of the following kinds which belong to our original q are implied by $r(x)$.

(a) $\mathcal{Q}(x) \ \in q$

As long as A is nonempty, choose any $a \in A$; then $R(x, a) \vdash \mathcal{Q}(x) \wedge \mathcal{P}(a)$. Since we can add up to λ parameters, recalling 4.7(1), without loss of generality A is nonempty: choose any $a \in P^{\infty}_{W}[M] \setminus B$.

(b) $x \neq c \ \in q$

If $c \in \mathcal{P}^M$, then this follows from $\mathcal{Q}(x)$. So enumerate as $\langle c_\alpha : \alpha < \kappa\rangle$ all elements such that "$x \neq c_\alpha$" appears in q, for some $\kappa \leq \lambda$, and $c \in \mathcal{Q}^M$. By induction on α, we will define two increasing continuous sequences (really one is enough) $\langle A_\alpha : \alpha < \kappa\rangle$ and $\langle B_\alpha : \alpha < \kappa\rangle$ such that $A_0 = A$, $B_0 = B$, $A_\alpha \subseteq P^{\infty}_{W}[M]$, $B_\alpha \subseteq P^M$, and $A_\alpha \cap B_\alpha = \emptyset$. At stage α, choose some $b_\alpha \notin A_\alpha$ such that $M \models R(c_\alpha, b_\alpha)$. We can do this because $R(c_\alpha, x)$ defines an infinite subset of M which is, by hypothesis and Definition 4.7 of size at least λ^+. Define $A_{\alpha+1} = A_\alpha$, $B_{\alpha+1} = B_\alpha \cup \{b_\alpha\}$. At limits take unions. Arriving to $A = A_\kappa$, $B = B_\kappa$, $r(x)$ is still consistent and implies $x \neq c_\alpha$ for all $\alpha < \kappa$, as desired.

(c) $\neg Q_\nu(x) \ \in q$

This follows from our assumption that $\{Q_{\rho\restriction n}(x) : n < \omega\} \in q$.

(d) $\neg R(x, b)$, if $b \in B \setminus P^{\infty}_{W}[M]$

For any $b \notin P^{\infty}_{W}[M]$, there is a finitary reason for the non-membership, i.e. there is $k < \omega$ and $\nu \in \mathcal{T}_{2,k}$ such that $(\rho \restriction k, \nu) \notin \mathcal{R}_k$ and $b \in P^M_\nu$. Then $T_{\mathfrak{m}} \vdash (\forall x)(\forall y)(Q_{\rho\restriction k}(x) \wedge P_\nu(y) \implies \neg R(x, y))$. By part (d), $r(x) \vdash Q_{\rho\restriction k}(x)$, so necessarily $r(x) \vdash \neg R(x, b)$.

Finally, let's check the size of $r(x)$: as there are only countably many possible ν's, in (c) we add at most countably many new formulas to the type; and otherwise we add at most λ many new formulas to the type, so this completes the proof. $\square$

Remark 4.9. *By symmetry of* $\mathfrak{m}$*, the analogue of* 4.8 *is true for types* $p(y)$ *such that* $p(y) \vdash \mathcal{P}(y)$*, and since* $\mathcal{Q}, \mathcal{P}$ *partition* M*, this covers all* 1*-types, which are sufficient for saturation.*

5. Ultrapowers of models of $T_{\mathfrak{m}}$

Convention 5.1. *In this section,* $\mathfrak{m}$ *is an arbitrary but fixed parameter.*

By 4.8, to ensure a model of $T_{\mathfrak{m}}$ with large leaves is λ^+-saturated, it suffices to realize R-types. In our main proofs, we focus on saturating regular ultrapowers. This section gives some basic additional features of the ultrapower case.

Fact 5.2. *Suppose I is an infinite set and $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$. Then for any infinite model N in a countable language, including but not limited to models of $T_{\mathfrak{m}}$:*

1. *the ultrapower $N^I/\mathcal{D}$ is $\aleph_1$-saturated.*
2. *if $N \models T_{\mathfrak{m}}$ then $N^I/\mathcal{D}$ is weakly λ^+-saturated, also:*
   (a) *If $\eta \in \lim(\mathcal{T}_1)$, then $|\{a \in M : M \models Q_{\eta\restriction n}(x)$ for $n < \omega\}| = 2^\lambda$ (though really $> \lambda$ is sufficient for us).*
   (b) *similarly for $\eta \in \lim(\mathcal{T}_1)$.*

   In other words, (a)+(b) say that $M \restriction \{P_\eta, Q_\nu : \eta \in \mathcal{T}_2, \nu \in \mathcal{T}_1\}$ is λ^+-saturated.

Proof. (1) Well known, see for example Chang and Keisler [2] 6.1.1.

(2) Let $N_1 \equiv N$ be λ^+-saturated. Since $\mathcal{D}$ is a regular ultrafilter, we know it is enough to prove this for N_1 (see e.g. Keisler [9] 2.1a), but this is immediate. $\square$

Fact 5.3 (see [14])**.** *Saturation of regular ultrapowers reduces to saturation of φ-types, that is, if M is a model in a countable language and $\mathcal{D}$ is a regular ultrafilter on λ, then $M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if it is λ^+-saturated for φ-types, for every formula φ.*

Corollary 5.4. *If $M_0 \models T_{\mathfrak{m}}$, $\mathcal{D}$ is a regular ultrafilter, and $M = (M_0)^\lambda/\mathcal{D}$, then clause (1) of Definition 4.7 holds for M. If in addition M satisfies the conclusion of 5.6(1) below, then the hypotheses of 4.8 hold for M.*

Convention 5.5. *We will say that a regular ultrafilter $\mathcal{D}$ on a set I, $|I| = \lambda$ is "good for" a countable theory T if for some, equivalently*[8] *every, $M \models T$, the ultrapower $M^I/\mathcal{D}$ is λ^+-saturated.*

Fact 5.6. *Suppose I is an infinite set and $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$. Then for any infinite model N in a countable language, including but not limited to models of $T_{\mathfrak{m}}$:*

1. *Suppose in addition that $\mathcal{D}$ is good for every countable stable theory. Then any infinite definable subset of $N^I/\mathcal{D}$ has size at least λ^+.*
2. *Suppose in addition that $\mathcal{D}$ is good for the theory of the random graph. Then:*
   (a) *for any two disjoint $A, B \subseteq N^I/\mathcal{D}$, there is an internal predicate separating A and B.*
   (b) *$\mathcal{D}$ is good for every countable stable T.*

Proof. (1) In fact, this characterizes $\mathcal{D}$ being good for all stable theories, see [31] Chapter VI, Theorem 5.1(1)-(2) p. 379.

(2) In fact, this characterizes $\mathcal{D}$ being good for the theory of the random graph, see [16] §3 p. 1585. $\square$

Conclusion 5.7 gives a sufficient collection of types to realize in order to saturate regular ultrapowers for self-dual $\mathfrak{m}$ (our main case following 6.16 below).

Conclusion 5.7. *Suppose I is an infinite set, $|I| = \lambda$, and $\mathcal{D}$ is a regular ultrafilter on I which is good for the theory of the random graph. Suppose that $\mathfrak{m} = \mathrm{dual}(\mathfrak{m})$. Let $M_0 \models T_{\mathfrak{m}}$. Then in order to show that $M = (M_0)^I/\mathcal{D}$ is λ^+-saturated, it is sufficient to show that:*

[8] the equivalence is by regularity, see Keisler [9] 2.1a.

$(\star)_{M_0,I,\mathcal{D}}$ *every partial type of M of the form*

$$r(x) = \{Q_\nu(x)\} \cup \{R(x,a) : a \in A\}$$

is realized, where $\nu \in \mathcal{T}_{1,n}$ for some $n < \omega$, $A \subseteq M$ and $|A| \leq \lambda$.

Proof. Case 1. For types including $\mathcal{Q}(x)$, by 5.6(2) and our present assumption, the conclusion of 5.6(2)(b) holds. Hence by 5.4, second sentence, the hypothesis of 4.8 holds. Hence, by Claim 4.8, it suffices to deal with partial types of the form

$$r(x) = \{Q_{\rho\restriction n}(x) : n < \omega\} \cup \{R(x,a) : a \in A\} \cup \{\neg R(x,b) : b \in B\}$$

where $\rho \in \lim(\mathcal{T}_1)$ and for some $W \subseteq \lim(\mathcal{S}_\rho)$, we have $A \subseteq P^\infty_W[M]$ with $|A| = \lambda$, and $B \in P^M \setminus A$ with $|B| = \lambda$.

As saturation of regular ultrapowers reduces to saturation of φ-types, it suffices to consider the case where $\nu \in \mathcal{T}_{1,n}$ for some $n < \omega$, and so to deal with a partial type of the form

$$r(x) = \{Q_\nu(x)\} \cup \{R(x,a) : a \in A\} \cup \{\neg R(x,b) : b \in B\}.$$

Note that the assertion that $r(x)$ is a partial type means that for some ρ' with $\nu \trianglelefteq \rho \in \lim(\mathcal{T}_1)$ and some $W \subseteq \lim(\mathcal{S}_{\rho'})$, we have $A \subseteq P^\infty_W[M]$ with $|A| = \lambda$, and $B \in P^M \setminus A$ with $|B| = \lambda$.

As we are assuming $\mathcal{D}$ is good for the theory of the random graph, by 5.6(2) we can assume there is an internal predicate X separating A and B, so let us justify that it sufficies to realize the positive part of the type. Enumerate r as $\langle R(x,a_\alpha) : \beta < \lambda, \alpha = 2\beta\rangle$ and $\langle \neg R(x,b_\alpha) : \beta < \lambda, \alpha = 2\beta+1\rangle$. Let $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{D}$ be a regularizing family. Let $f : [\lambda]^{<\aleph_0} \to \mathcal{D}$ be the "distribution" given by

$$\sigma \mapsto \{t \in I : M \models (\exists x)\left(P_\nu(x) \wedge \bigwedge_{\alpha\in\sigma \text{ even}} R(x,a_\alpha[t]) \wedge \bigwedge_{\alpha\in\sigma \text{ odd}} \neg R(x,a_\alpha[t])\right)\} \cap \bigcap_{\alpha\in\sigma} X_\alpha.$$

Then it is straightforward to see that r is realized if and only if f has a multiplicative refinement. Let g be the refinement of f given by

$$\sigma \;\mapsto\; f(\sigma) \cap \{t \in I : M \models \bigwedge_{\alpha\in\sigma \text{ even}} X(a_\alpha[t]) \wedge \bigwedge_{\alpha\in\sigma \text{ odd}} \neg(X(a_\alpha[t]))\}.$$

Now let us verify: g is a function with domain $[\lambda]^{<\aleph_0}$ (trivial), range$(g) \subseteq \mathcal{P}(\lambda)$ (trivial), range $g \subseteq \mathcal{D}$ (by the choice of X), g is multiplicative (by its definition), and g refines f (by choice of X and the properties of the random graph).

Case 2. For types including $P(x)$, we use the additional fact that $\mathfrak{m}$ is self-dual, so for any type on the left, there is a type on the right with an identical distribution. One will have a multiplicative refinement (i.e. be realized) if and only if the other does.

As $M \models (\forall x)(P(x) \vee Q(x))$, this finishes the proof that M is λ^+-saturated. $\square$

Remark 5.8. *By 4.6, an equivalent formulation of $(\star)_{M_0,I,\mathcal{D}}$ would have been: "every set of formulas of M of the form $r(x) = \{R(x,a) : a \in A\}$ where for some $\rho \in \lim(\mathcal{T}_1)$ and $W \subseteq \lim(\mathcal{S}_\rho)$, we have $A \subseteq P^\infty_W[M]$ with $|A| \leq \lambda$ is realized."*

Remark 5.9. *Assume* $\mathrm{dual}(\mathfrak{m}) = \mathfrak{m}$, *we have that* $\mathrm{dual}(M_0)$, *defined naturally, is a model of $T_\mathfrak{m}$. If* $\mathrm{dual}(M_0)$ *is isomorphic to M_0, then the ultrapowers of the two models are isomorphic, and so they have the same saturation. But maybe* $\mathrm{dual}(M_0) \not\cong M_0$. *However we know that* $Th(M_0) = Th(\mathrm{dual}(M_0))$ *hence it is well* *known that* $(M_0)^\lambda/\mathcal{D}$, $(\mathrm{dual}(M_0))^\lambda/\mathcal{D}$ *are* L_{∞,λ^+}*-equivalent, hence the argument above holds.*

Discussion 5.10. *If we were not assuming* $\mathfrak{m} = \mathrm{dual}(\mathfrak{m})$*, then we would just need to add the parallel for types containing* $\mathcal{P}(x)$*. We may do this in at least two ways: either we update* $(\star)_{M_0,I,\mathcal{D}}$ *to include the parallel condition for* $\mathcal{P}(x)$ *replacing* $\mathcal{Q}(x)$ *(with the corresponding minor notational changes), or else, we add the condition* $(\star)_{\mathrm{dual}(M_0),I,\mathcal{D}}$*, where* $\mathrm{dual}(M_0)$ *is a model of* $T_{\mathrm{dual}(\mathfrak{m})}$*, since* $\mathcal{Q}$ *in* $\mathrm{dual}(M_0)$ *is the parallel of* $\mathcal{P}$ *in* M_0*.*

6. Sizes

Definition 6.1. *We say the sequence* $\bar{m} \in {}^\omega(\omega \setminus \{0\})$ *is* increasing fast enough*, or for short, that* $\bar{m}$ *is a* fast sequence*, when we can write it as*

(a) $$\langle (m_i, m_i^\circ) : i < \omega \rangle$$

where for each i, $m_i^\circ = \prod_{j<i} m_i$*, and*

(b) $$m_i \geq \left((m_i^\circ)^{i^i} \right)^{4(m_i^\circ)^{i^i}}.$$

Note that the m_i°*'s are uniquely determined by the* m_i*'s, so we may or may not write them, depending on emphasis. To avoid triviality, we assume* $m_0 > 1$*.*

In the next definition, the E_i are graphs, not bipartite graphs. We have chosen to allow self-loops (i.e. (a, a) can be an edge), but this is not crucial.

Definition 6.2. *Let* $\bar{m}$ *be a fast sequence.*

(1) $\bar{E}$ *will denote a* sequence of graphs for $\bar{m}$ *when each* $E_i \subseteq [m_i]^2 \cup \{(a, a) : a \in [m_i]\}$ *and for* $i = 0$ *we have equality.*

(2) $\bar{E}$ *is a* good sequence of graphs for $\bar{m}$*, or just* good for $\bar{m}$*, when in addition, there is some finite* i_* *such that for all* $i \geq i_*$*,*

(i) if $u \subseteq m_i$*,*
$$|u| \leq (m_i^\circ)^{i^i}$$
then u *is "*$\bar{E}$*-small," meaning that that there is* s *such that*
$$(\forall t \in u)(E_i(s, t)).$$

(ii) if $u \subseteq m_i$ *and*
$$|u| \geq \frac{m_i}{(m_i^\circ)^{i^i}}$$
then u *is "*$\bar{E}$*-large," meaning that there is no* s *such that*
$$((\forall t \in u)(E_i(s, t)).$$

We shall omit $\bar{E}$ *in "small" and "large" when it is clear from the context.*

The definition of "small" is used in the proof of existence of a model completion, and the definition of "large" in the proof of non-saturation below.

Convention 6.3. *For the next few lemmas, let* $g : \mathbb{N} \to \mathbb{N}$ *be the function given by*
$$g(i) = 2(m_i^\circ)^{i^i}.$$
(of course, we could have called this $g(m_i)$*).)*

Fact 6.4 (see e.g. Bollobás [1] Corollary 3.14)**.** *Let $\Delta(G)$ denote the maximal degree of G and let G_p denote a graph from $\mathcal{G}_{n,p}$, random graphs on n vertices with edge probability p. If $pn/\log n \to \infty$, then a.e. G_p satisfies*

$$\Delta(G_p) = \{1 + o(1)\}pn.$$

Observation 6.5. *Let g be from 6.3 and suppose*

$$p = p(m_i) = \frac{1}{(m_i)^{\frac{1}{g(i)}}}.$$

Then $p \cdot m_i/\log m_i \to \infty$, so 6.4 applies. In particular, as $i \to \infty$, the proportion of $G \in \mathcal{G}_{m_i,p}$ which have no vertices of "large" degree goes to 1.

Proof. Recalling that $g(i)$ from (6.3) is monotonic and strictly increasing and approaches ∞, and recalling that $p = \frac{1}{(m_i)^{\frac{1}{g(i)}}}$ we have

$$\text{(c)} \qquad \lim_{i\to\infty} \left(\frac{1}{(m_i)^{\frac{1}{g(i)}}}\right)\left(\frac{m_i}{\log m_i}\right) = \lim_{i\to\infty}\left(\frac{m_i^{1-\frac{1}{g(i)}}}{\log m_i}\right) = \infty.$$

So Fact 6.4 applies (actually $g(i) \geq 2$ suffices) and for some fixed constant c, in almost every G_{p,m_i}, every vertex has degree $\leq cpm_i$. Let us verify that this degree is eventually not "large" in the sense of 6.2. For this it would suffice to show that

$$\text{(d)} \qquad \frac{m_i}{(m_i^\circ)^{i^i}} \text{ is quite a bit bigger than } \frac{m_i}{{m_i}^{1/g(i)}}$$

[the left-hand side is the threshold for "large" from 6.2 and the right-hand side is $p \cdot m_i$] and for this it would suffice to show that

$$\text{(e)} \qquad (m_i^\circ)^{i^i} \text{ is quite a bit smaller than } {m_i}^{1/g(i)}.$$

which is ensured by (b) of Definition 6.1 and the definition of g in 6.3. $\square$

Observation 6.6. *For each i, $(m_i^\circ)^{i^i} < (m_i)^{1/4}$.*

Proof. We verify that

$$\text{(f)} \qquad (m_i^\circ)^{i^i} < \left(\left((m_i^\circ)^{i^i}\right)^{4(m_i^\circ)^{i^i}}\right)^{1/4} \leq (m_i)^{1/4}$$

recalling 6.1(b). $\square$

Lemma 6.7. *Let $E_i \subseteq [m_i]^2 \cup \{\{a,a\} : a \in [m_i]\}$ be symmetric and random with probability p from 6.5 for each to be an edge. In such a graph, with probability close to 1 (for us nonzero probability is sufficient):*

(1) for every $u \subseteq m_i$, if

$$|u| \leq (m_i^\circ)^{i^i}$$

then there is s so that $(\forall t \in u)(E_i(s,t))$.

(2) for every $u \subseteq m_i$, if

$$|u| \geq \frac{m_i}{(m_i^\circ)^{i^i}}$$

then there does not exist s so that $(\forall t \in u)(E_i(s,t))$.

Remark 6.8. *For now we will define a graph on m_i vertices with edge relation E_i, allowing self loops. In the translation from E_i to $\mathcal{R}_i$ in 6.12 below, we will use this graph to make a bipartite graph, which will then be symmetric [in the sense that the derived* $\mathfrak{m} = \mathrm{dual}(\mathfrak{m})$] *by construction.*

Proof. Recall $[n]$ denotes the n-element set $\{0, \dots, n-1\}$. We define a probability measure $\mu = \mu_i$ on $\{X : X \subseteq [m_i]^2 \cup \{\{a,a\} : a \in [m_i]\}\ \}$ by flipping a coin $c_{\{a,b\}}$ for each potential edge[9] with probability of heads (=yes) being

$$\text{(g)} \qquad \frac{1}{(m_i)^{\frac{1}{g(i)}}}$$

where $g(i)$ is again from 6.3. Condition (2) will be handled by Observation 6.5, so let us address condition (1). Let us say that a set $v \subseteq [m_i]$ is "covered" if there exists $b \in [m_i]$ such that $(\forall t \in v)(E_i(b,t))$. Clearly it will suffice to show that all sets of size $(m_i^\circ)^{i^i}$ are covered.

Let $\mathcal{E}_i^1$ be the probability that an arbitrary $v \subseteq [m_i]$ of size x is covered (x to be chosen later, but the main case will be $(m_i^\circ)^{i^i}$.) We will bound $\mathcal{E}_i^1$ in a moment, but first note that the probability that some $v \subseteq [m_i]$ of size x is not covered is

$$\text{(h)} \qquad \binom{m_i}{x} (\mathcal{E}_i^1)$$

i.e. the number of ways to choose a v of size x times the probability that a given v is not covered. Now consider

$$\text{(i)} \qquad \mathcal{E}_i^1 = \left(1 - \frac{1}{(m_i)^{x/g(i)}}\right)^{m_i}$$

[the right term represents the chance that each particular b fails to cover v; there are m_i choices for b]. Recalling that $(1 - 1/n)^n$ is well approximated by e for large n, we may rewrite the right side of (i) as

$$\text{(j)} \qquad \left(\left(1 - \frac{1}{(m_i)^{x/g(i)}}\right)^{(m_i)^{x/g(i)}}\right)^{\frac{m_i}{m_i{}^{x/g(i)}}}$$

and then (j) is well approximated by

$$\text{(k)} \qquad \frac{1}{e^{[(m_i)^{1-x/g(i)}]}}.$$

We need equation (j) to be very small, so we need $e^{[(m_i)^{1-x/g(i)}]}$ to be very large, so we need $[(m_i)^{1-x/g(i)}]$ to be very large, so we need $1 - x/g(i)$ to be not too small. For our present calculations, let us verify that it suffices to have $x/g(i) = 1/2$, which is satisfied in our main case when $x = (m_i^\circ)^{i^i}$ from 6.2 and $g(i)$ is from 6.3. In this case,

$$\text{(l)} \qquad \mathcal{E}_i^1 \leq \frac{1}{e^{\sqrt{m_i}}}$$

hence equation (h) has as an upper bound

$$\text{(m)} \qquad (m_i)^x(\mathcal{E}_i^1) = \left(e^{x \ln m_i}\right)\left(\frac{1}{e^{\sqrt{m_i}}}\right) = \frac{1}{e^{\sqrt{m_i} - x \ln m_i}}.$$

[9]note that this edge relation is by definition symmetric.

It is sufficient for us that the exponent be nonnegative and growing; for instance, $x < (m_i)^{1/4}$ suffices for us, and is indeed satisfied in our main case $x = (m_i^\circ)^{i^i}$ recalling 6.6. This completes the proof. $\square$

Conclusion 6.9. *If $\bar{m}$ is a fast sequence, there exists $\bar{E}$ which is good for $\bar{m}$.*

Remark 6.10. *The bounds in* 6.2 *are for definiteness, what we really use is* 6.7-6.9*: $\bar{m}$ grows quickly enough to find a sequence of graphs $\bar{E}$ with a coherent and growing notion of "small" and "large"* (*every small set has an x connected to all of its elements, and no large set has an x connected to all of its elements*)*. Zero-one laws* [32] *suggest much flexibility in choosing such bounds.*

Next, given the three key ingredients $\bar{m}$, $\bar{E}$, and a level function ξ (recall 2.2), we construct a parameter $\mathfrak{m}$ whose sequence of reduced graphs reflects $\bar{E}$ in a natural sense.

Notation 6.11. *For $\bar{m}$ a countable sequence of natural numbers and $\eta \in {}^{\omega>}\omega$ or $\eta \in {}^{\omega}\omega$, the notation $\eta < \bar{m}$ means $\eta(i) < m_i$ for all $i < \mathrm{lgn}(\eta)$.*

Definition 6.12 (The parameters we use)**.** *For any fast sequence $\bar{m} = \langle m_i : i < \omega\rangle$, any $\bar{E} = \langle E_i : i < \omega\rangle$ which is good for $\bar{m}$, and any level function $\xi \in {}^{\omega}2$, we define a parameter $\mathfrak{m} = \mathrm{par}(\bar{m}, \bar{E}, \xi)$ as follows:*

(1) *if $\ell = 1, 2$, $\mathcal{T}_{\mathfrak{m},\ell} = \{\eta : \eta \in {}^{\omega>}\omega, \eta < \bar{m}\}$.*
In particular, for each $\nu \in \mathcal{T}_\ell$, if $\nu \in \mathcal{T}_{\ell,i}$ then $|\mathrm{ims}_{\mathcal{T}_\ell}(\nu)| = m_i$.
(2) *for the next two items, for $i < \omega$, let $E^1_i = E_i$, and let E^0_i be $m_i \times m_i$, the complete graph with self-loops.*
(3) *for $i = 0$, $\mathcal{R}_0 = \{(\emptyset, \emptyset)\}$.*
(4) *for $i + 1$,*

$$\mathcal{R}_{i+1} = \{(\eta_1, \eta_2) : \eta_\ell \in \mathcal{T}_{\ell,i+1},\ (\eta_1 \restriction i, \eta_2 \restriction i) \in \mathcal{R}_i,\ (\eta_1(i), \eta_2(i)) \in E_i^{\xi(i)}\}.$$

Convention 6.13. *In what follows, we use the letters $\mathfrak{m}$ and $\mathfrak{n}$, often with subscripts, for parameters. We will often write e.g.*

$$\mathfrak{m} = \mathfrak{m}[\bar{m}, \bar{E}, \xi] \text{ or } \mathfrak{n} = \mathfrak{n}[\bar{m}, \bar{E}, \xi] \text{ or } \mathfrak{m} = \mathrm{par}[\bar{m}, \bar{E}, \xi]$$

to indicate the dependence of a given parameter on the three inputs.

Observation 6.14. *Suppose $\bar{m}, \bar{E}$ are as above and $\xi_1, \xi_2 \in {}^{\omega}\{0,1\}$ and $\mathfrak{m}_\ell = \mathrm{par}[\bar{m}, \bar{E}, \xi_\ell]$. If we have $\xi_2^{-1}\{1\} \subseteq \xi_1^{-1}\{1\}$, then $\mathcal{R}^{\mathfrak{m}_1} \subseteq \mathcal{R}^{\mathfrak{m}_2}$.*

(Observation 6.14 will be used in §12 below. Informally, if there are more active levels, there are more constraints, so $\mathcal{R}$ has fewer edges.)

Discussion 6.15 (Indexing)**.** So that the key points of the construction in 6.12 are not hidden in the indexing, we review:

(a) $\mathcal{T}_{1,0} = \mathcal{T}_{2,0} = \{\emptyset\}$, and $\mathcal{R}_0 = \{(\emptyset, \emptyset)\}$.
(b) For $i+1$, recall that the elements of $\mathcal{T}_{\ell,i+1}$ are sequences of length $i+1$, i.e. functions η from $\{0, \dots, i\}$ to ω subject to the constraint that $\eta(j) < m_j$ for each $j \leq i$.
(c) $\mathcal{T}_{\ell,1}$ has m_0 nodes; in general, for $k > 0$ $\mathcal{T}_{\ell,k}$ has $\prod_{j<k} m_j = m^\circ_k$ nodes. Also for $k = 0$, $m^\circ_k = 1 = \prod_{j<0} m_j$.
(d) If $\eta \in \mathcal{T}_{\ell,k}$ then η has m_k immediate successors in $\mathcal{T}_{\ell,k+1}$, so it follows that $\mathcal{T}_{\ell,k+1}$ has size $(\prod_{j<k} m_j) \cdot m_k = \prod_{j<k+1} m_j = m^\circ_{k+1}$, as desired.

(e) If $(\eta, \nu) \in \mathcal{T}_{1,k} \times \mathcal{T}_{2,k}$ and $(\eta, \nu) \in \mathcal{R}_k$, then letting $A = \mathrm{ims}_{\mathcal{T}_{1,k+1}}(\eta)$, $B = \mathrm{ims}_{\mathcal{T}_{2,k+1}}(\nu)$, if $k+1$ is an active level, then $(A, B, \mathcal{R}_{k+1} \restriction A \times B)$ is isomorphic as a bipartite graph to (m_k, m_k, E_k). If $k+1$ is a lazy level, it is a complete bipartite graph.

(f) *In a slogan: given two elements of length k connected by $\mathcal{R}_k$, E_k gives us the pattern of edges between their sets of successors, provided those successors are at an active level.* ***Note to the reader:*** *so there is no confusion, we repeat that $\xi(k)$ affects edges at level $k+1$, recalling* 2.4.

Note that 6.12 establishes self-duality essentially for free. (The key point is that each of the $\mathcal{R}_i$'s is symmetric as a bipartite graph; from each $\mathcal{R}_{i+1}$ one can naturally recover $E_i^{\xi(i+1)}$ by identifying each vertex on the left with its parallel on the right.)

Claim 6.16. *If $\mathfrak{m}$ is constructed from any $\bar{m}$, $\bar{E}$ which is good for $\bar{m}$ as in* 6.12, *and ξ which satisfies $\xi(i) = 0$ for $i < i_* = i_*(\bar{E})$ (recalling* 6.2(2)), *then $\mathfrak{m} = \mathrm{dual}(\mathfrak{m})$.*

Proof. Immediate: this is the symmetry of the construction in 6.12. □

Claim 6.17. *If $\mathfrak{m}$ is constructed from any $\bar{m}$, $\bar{E}$, ξ as in* 6.16, *then $\mathfrak{m}$ is a parameter. Moreover, for every $k < \omega$,*

(1) if $\{\rho_i : i < s\} \subseteq \mathcal{T}_{2,k+1}$ and $s \leq (m_k^\circ)^{k^k}$, and $\nu \in \mathcal{T}_{1,k}$ with $(\nu, \rho_i \restriction k) \in \mathcal{R}_k$ for each $i < s$, then there exists an immediate successor $\nu^\frown\langle t\rangle$ of ν such that $(\nu^\frown\langle t\rangle, \rho_i) \in \mathcal{R}_{k+1}$ for each $i < s$.

(2) the parallel condition to (1) holds switching $\mathcal{T}_{2,k+1}$ and $\mathcal{T}_{1,k+1}$.

Proof. We check 2.3 and 2.5. 2.3(1), (2), (3) are obvious from our construction. (4) holds since the degree of a vertex in E_i^a is at least two. (5) is immediate from the construction and (6) does not require verification.

For 2.5, it will be helpful to first prove the "moreover" clauses of our Claim, which greatly strengthen Extension in one aspect. Note that as $\mathfrak{m} = \mathrm{dual}(\mathfrak{m})$ it suffices to prove (1). Consider $s < \omega$ and $\nu \in \mathcal{T}_{1,k}$ and $\rho_0, \dots \rho_{s-1} \in \mathcal{T}_{2,k+1}$ such that $(\nu, \rho_i \restriction k) \in \mathcal{R}_k$ for $i < s$. Write $\rho_i = (\rho_i \restriction k)^\frown\langle \ell_i\rangle$ for each $i < k$. In the graph E_k, $\{\ell_i : i < k\}$ is a small subset of $[m_k]$, so there is some $t \in [m_i]$ such that (t, ℓ_i) is an edge in E_k for every $i < k$. Recall that if $\xi(k) = 1$, for each $i < k$, to form $\mathcal{R}_k$ in 6.12 we simply put a bipartite copy of E_k [always the same E_k] between the immediate successors of ν and the immediate successors of $\rho_i \restriction k$. As a result, for each $i < k$ (we just look one by one), $(\nu^\frown\langle t\rangle, \rho_i \restriction k^\frown\langle \ell_i\rangle) \in \mathcal{R}_{k+1}$. In other words, $\nu^\frown\langle t\rangle$ connects to each ρ_i (again note: even though they may not have an immediate common predecessor). If $\xi(k) = 0$, we use a complete graph instead of E_i, so this is all true a fortiori and there are even many such elements. This proves (1), and also (2).

Finally, we check the extension conditions from 2.5, and as $\mathfrak{m}$ is self-dual, it will suffice to prove one way. Applying the preceding paragraph in the case $s = k$ tells us there is at least one successor of ν connecting to $\rho_0, \dots \rho_{k-1}$. The additional point is to note that there is not just one t but at least $k+1$. When $\xi(k) = 0$, as noted, there will be many such successors, actually all. To see that there are many when $\xi(k) = 1$, by induction on $\ell \leq k$ choose $t_\ell \in [m_k] \setminus \{t_0, \dots, t_{\ell-1}\}$ such that $\bigwedge_{i<k}(\nu^\frown\langle t_\ell\rangle, \rho_i) \in \mathcal{R}_{k+1}$, as follows: note that we could choose ρ such that $j < \ell \implies (\nu^\frown\langle t_j\rangle, \rho) \notin \mathcal{R}_{k+1}$, possible as no vertex has large degree, and apply the previous paragraph to the still small set $\{\rho_i : i < k\} \cup \{\rho\}$ to find $\nu^\frown\langle s\rangle$ which connects to $\rho_0, \ldots, \rho_{k-1}, \rho$, necessarily for $s \neq t_j$ for $j < \ell$, so s can serve as t_ℓ. Continuing in this way, we find the $k+1$ elements for the extension axiom. $\square$

We may now invoke the level functions to build a family of parameters whose active levels are independent in the following precise sense.

Notation 6.18. *Recall that $A \subseteq^* B$ means that $\{a \in A : a \notin B\}$ is finite.*

Fact 6.19. *For any $i_* < \omega$, there is $\Xi = \{\xi_\alpha : \alpha < 2^{\aleph_0}\} \subseteq {}^\omega\{0,1\}$ of size continuum such that:*

if $u \subseteq 2^{\aleph_0}$ is finite and $\beta < 2^{\aleph_0}$ is not from u, then

$$\xi_\beta^{-1}\{1\} \not\subseteq^* \bigcup\{\xi_\alpha^{-1}\{1\} : \alpha \in u\}$$

and moreover $\xi_\alpha(i) = 0$ for all $i < i_*$ and all $\alpha < 2^{\aleph_0}$.

Proof. We can use e.g. the existence of a family $\mathcal{G} = \{g_\alpha : \alpha < 2^{\aleph_0}\} \subseteq {}^\omega\omega$ of continuum many independent functions, see [3] or [31] Appendix, Theorem 1.5(1) p. 656. Recall that this means that each $g \in \mathcal{G}$ is a function from ω to ω and for any finite n, any distinct $\alpha_0, \ldots, \alpha_{n-1} < 2^{\aleph_0}$, and any $t_0, \ldots, t_{n-1} < \omega$ (not necessarily distinct), the set $\{i < \omega : g_{\alpha_0}(i) = t_0 \wedge \cdots \wedge g_{\alpha_{n-1}}(i) = t_{n-1}\} \neq \emptyset$. In particular (as we can play with setting the values of any finitely many other functions) it follows from the definition of independent that for any distinct $\alpha_0, \ldots, \alpha_{n-1}, \beta < 2^{\aleph_0}$ and any $s_0, \ldots, s_{n-1}, t < \omega$, the set

$$\{i < \omega : g_{\alpha_0}(i) = s_0 \ \wedge \cdots \wedge g_{\alpha_{n-1}}(i) = s_{n-1} \ \wedge \ g_\beta(i) = t\}$$

is infinite. For each g_α in our family $\mathcal{G}$, let ξ_α be the function $i \mapsto (g_\alpha(i) \mod 2)$. It follows that $\{\xi_\alpha : \alpha < 2^{\aleph_0}\}$ is as desired.

Since changing the first i values for every g_α in the family to be zero does not alter the independence, we can ensure the last line for any finite i_* given in advance. $\square$

Remark 6.20. *We could have also demanded that the sets $\xi_\alpha^{-1}\{1\}$ are infinite and pairwise almost disjoint.*

Corollary 6.21. *Thus, for any fast sequence $\bar{m}$, any $\bar{E}$ which is good for $\bar{m}$, and $\Xi = \langle \xi_\alpha : \alpha < 2^{\aleph_0} \rangle$ from 6.19 with $i_* = i_*(\bar{E})$, we can define a set*

$$\mathbf{M}_* = \{\mathfrak{m}_\alpha = \mathrm{par}[\bar{m}, \bar{E}, \xi_\alpha] : \alpha < 2^{\aleph_0}\}$$

with no repetition. For any $\mathcal{M} \subseteq \mathbf{M}_*$, define

$$\mathcal{I}_\mathcal{M} = \{v \subseteq \omega : \textit{ for some } \mathfrak{m}_{\alpha_0}, \ldots, \mathfrak{m}_{\alpha_{n-1}} \in \mathcal{M}, \ v \subseteq^* \bigcup\{\xi_{\alpha_\ell}^{-1}\{1\} : \ell < n\}\}.$$

Then if $\mathcal{N} \subseteq \mathbf{M}_* \setminus \mathcal{M}$, we will have that:

(a) if $\mathfrak{m}_\alpha \in \mathcal{M}$, then $\xi_\alpha^{-1}\{1\} \in \mathcal{I}_\mathcal{M}$.
(b) if $\mathfrak{m}_\beta \in \mathcal{N}$, then $\xi_\beta^{-1}\{1\} \neq \emptyset \mod \mathcal{I}_\mathcal{M}$.

Remark 6.22. *Corollary 6.21 summarizes a natural sense in which any two elements, or indeed any two disjoint subsets, of $\mathbf{M}_*$ are orthogonal.*

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7. Possibility patterns and ultrapowers

In what follows we will be interested in analyzing types in regular ultrapowers. The following set-up is especially useful for this. To readers familiar with "separation of variables" in the sense of [19], there is nothing new here; we simply explain that framework and fix some notation.

The first idea is that a regular ultrafilter on λ can be "projected" onto any reasonable Boolean algebra (complete, of size $\leq 2^\lambda$, with the λ^+-c.c.) and studied there. Let us give the definition, then discuss how it can be used.

Definition 7.1 (Regular ultrafilters built from tuples, from [19] Theorem 6.13)**.** *Suppose $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$. We say that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*, \mathbf{j})$ when:*

(1) $\mathcal{D}_0$ is a regular, $|I|^+$-good filter on I
(2) $\mathfrak{B}$ is a Boolean algebra
(3) $\mathcal{D}_*$ is an ultrafilter on $\mathfrak{B}$
(4) $\mathbf{j} : \mathcal{P}(I) \rightarrow \mathfrak{B}$ is a surjective homomorphism such that:
   (a) $\mathcal{D}_0 = \mathbf{j}^{-1}(\{1_\mathfrak{B}\})$
   (b) $\mathcal{D} = \{A \subseteq I : \mathbf{j}(A) \in \mathcal{D}_*\}$.

Since 7.1 is defined with an eye towards Keisler's order, an important feature of this definition is that the problem of realizing types is naturally projected to the Boolean algebra, too, as 7.2 explains.[10]

Definition 7.2. *Continuing in the notation of* 7.1*, suppose that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*, \mathbf{j})$. Consider a complete countable T and $M \models T$. Suppose $N = M^\lambda/\mathcal{D}$ and p is a partial type over $\theta \leq \lambda$ parameters in N, so $p = \langle \varphi_\alpha(x, a_\alpha) : \alpha < \theta \rangle$. For each finite $u \subseteq \theta$, the Los map L sends $u \mapsto B_u$ where*

$$B_u := \{t \in I : M \models (\exists x) \bigwedge_{\alpha \in u} \{\varphi_\alpha(x, a_\alpha[t])\}\}.$$

Let $\mathbf{b}_u = \mathbf{j}(B_u)$. We write $\bar{B} = \langle B_u : u \in [\theta]^{<\aleph_0} \rangle$, and $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0} \rangle$.

Fact 7.3 (Separation of variables theorem, special case, [19] Theorem 6.13)**.** *In the notation of* 7.2*, $\langle B_u : u \in [\theta]^{<\aleph_0} \rangle$ has a multiplicative refinement in $\mathcal{D}$ [so p is realized in N] if and only if $\langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0} \rangle$ has a multiplicative refinement in $\mathcal{D}_*$.*

Given $\mathfrak{B}$ and $\mathcal{D}$, it can be useful to remember an ultrapower it came from.

Notation 7.4. *Given some $\mathcal{D}_*$ and $\mathfrak{B}$, and given a corresponding choice of $\mathcal{D}$, $\mathcal{D}_0$, and M as in 7.2, we may call $M^\lambda/\mathcal{D}$ an "enveloping ultrapower" for $\mathcal{D}_*$ and $\mathfrak{B}$.*

First use. In light of 7.2, a first use of 7.1 is to build regular ultrafilters in interesting ways. That is, 7.2 points out that if we have some Boolean algebra $\mathfrak{B}$ and some ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$, then even if $\mathcal{D}_*$ itself is not regular, we can use $\mathcal{D}_*$ to build a regular $\mathcal{D}$ by first trying to build a regular good filter $\mathcal{D}_0$ so that the quotient Boolean algebra is isomorphic to $\mathfrak{B}$. Then using $\mathcal{D}_*$ and $\mathcal{D}_0$ together we can determine $\mathcal{D}$. Among other things, this opens the door to using all sorts of non-regular ultrafilters in the construction of regular $\mathcal{D}$. The condition on building $\mathcal{D}_0$ is not onerous: such a $\mathcal{D}_0$ can always be built when $\mathfrak{B}$ is complete, $|\mathfrak{B}| \leq 2^\lambda$ and has the λ^+-c.c.

[10]The next definition seems to suggest that in $\varphi(x, y)$, $\ell(y) = 1$. This will be our main case in this paper, but it's not a constraint.

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Possibility patterns. Definition 7.2 highlights sequences $\langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0} \rangle$ of elements of $\mathfrak{B}^+$ which come directly from patterns of types in some enveloping ultrapower. In [19] and subsequent papers, we use *possibility pattern* to mean any sequence $\bar{\mathbf{b}}$ that arises from some $\bar{B}$ in this way. There is a combinatorial definition which doesn't rely on specifying an enveloping ultrapower, see [19] Definition 6.1. In the present paper, we will only need to handle sequences $\bar{\mathbf{b}}$ coming directly from types of a very specific kind, the R-types, so we won't need the full generality of that definition. So while we will often call such sequences "possibility patterns," the reader may substitute a phrase like "$\mathbf{j}$-images of some $\bar{B}$ arising from some positive R-type in some enveloping ultrapower of the theory in question."

Second use. This discussion leads us to a natural second use of separation of variables, central to the proofs below. This is to use 7.1 to repeatedly calibrate the building of a Boolean algebra $\mathfrak{B}$ along with an ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$, as follows. Suppose we are building $\mathcal{D}_*$ and $\mathfrak{B}$ together by induction, and at each stage in the construction we have some Boolean algebra $\mathfrak{B}_\alpha$ and some ultrafilter $\mathcal{D}_\alpha$ on $\mathfrak{B}_\alpha$, and we want to ensure by the time we get to $\mathcal{D}_*$, $\mathfrak{B}$ that all relevant possibility patterns $\bar{\mathbf{b}}$, for a given countable theory T or set $\mathcal{T}$ of countable theories, have multiplicative refinements. At each stage in the construction we'll be handling some such $\bar{\mathbf{b}}$, and at that point we may choose some enveloping ultrapower (for $\mathfrak{B}_\alpha, \mathcal{D}_\alpha$) and work there with a choice of corresponding $\bar{B}$, where the picture may be clearer. In this way we eventually construct $\mathfrak{B}$ and $\mathcal{D}_*$ to handle all relevant possibility patterns. At the end of the construction, one final use of 7.1 connects it to Keisler's order: any regular $\mathcal{D}$ built from this $\mathcal{D}_*$ will be a regular ultrafilter on λ with the property that for any $T \in \mathcal{T}$ and any model $M \models T$, $M^\lambda/\mathcal{D}$ is λ^+-saturated.

Fact 7.5. *If $\bar{\mathbf{b}}$ is a possibility pattern for T in a complete Boolean algebra $\mathfrak{B}$, and $\mathfrak{B} \lessdot \mathfrak{B}'$, then $\bar{\mathbf{b}}$ remains a possibilty pattern for T in $\mathfrak{B}'$.*

We now make some observations specific to our $T_\mathfrak{m}$ (for $\mathfrak{m} \in \mathbf{M}_*$).

Convention 7.6. *Fix $\bar{m}$ a fast sequence, $\bar{E}$ which is good for $\bar{m}$, and a level function ξ, and we consider $\mathfrak{m} = \mathrm{par}[\bar{m}, \bar{E}, \xi]$ in the sense of* 6.12. *Let us also fix an uncountable cardinal λ, a set I with $|I| = \lambda$, a regular ultrafilter $\mathcal{D}$ on I, a model $M \models T_\mathfrak{m}$, and a partial type*

$$q(x) = \{R(x, a_\alpha) : \alpha < \lambda\}$$

of $M^I/\mathcal{D}$. We fix a lifting in advance, i.e. pedantically $a_\alpha \in M^I$, and only $a_\alpha/\mathcal{D} \in M^I/\mathcal{D}$. Note that $q(x) \vdash Q(x)$ by definition of R.

Notation 7.7.

(a) Let $\langle B_u : u \in [\lambda]^{<\aleph_0} \rangle$ be from the Los map, as usual, recalling 7.2.

(b) For $\psi[\bar{a}]$ any formula, possibly with parameters, of $M^I/\mathcal{D}$, let $A[\ \psi[\bar{a}]\ ]$ be defined as

$$\{t \in I : M \models \psi[\bar{a}[t]]\}.$$

Observation 7.8. *$A[\ (\exists x)R(x, a_\alpha)\ ] = B_{\{\alpha\}} = I$, for any $\alpha < \lambda$.*

Proof. The first equality points out some useful redundancy in notation; the second points out that by definition of $T_\mathfrak{m}$ (in particular the extension axioms) no single instance of R can be inconsistent, no matter the value of $a_\alpha[t]$. $\square$

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Claim 7.9. *Suppose $u \in [\lambda]^{<\aleph_0}$. Then for any finite k, the set*

$$B_u = A[\ (\exists x) \bigwedge_{\alpha \in u} R(x, a_\alpha)\ ]$$

is the union, not necessarily disjoint, of the sets

$$A[\ (\exists x)(Q_\rho(x) \wedge \bigwedge_{\alpha \in u} R(x, a_\alpha)\ ]$$

for $\rho \in \mathcal{T}_{1,k}$.

Proof. By the axioms of $T_{\mathfrak{m}}$, for any finite k, $Q(x) \vdash \bigvee_{\rho \in \mathcal{T}_{1,k}} Q_\rho(x)$. (Although this is a partition, of course there may be different x's in different Q_ρ's which work.) $\square$

8. The chain condition

This section begins work on the ultrafilters. To motivate the construction and the problems it has to solve, consider the limitations of the example of [27], the one previous example of incomparability in ZFC. In that paper, as usual, we had considered completions of free Boolean algebras of the form

$$\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$$

where θ was not necessarily countable [recall the notation means: $\mathfrak{B}$ is generated by 2^λ independent partitions each of size μ, where the intersection of fewer than θ elements from different partitions is nonzero]. It was shown there that theories called T_f, distant precursors of our theories here, could be handled by *some* ultrafilter on $\mathfrak{B}$ when $\theta > \aleph_0$, and by *no* ultrafilter when $\theta = \aleph_0$. This suggests that if we want to handle theories of this general form while keeping $\theta = \aleph_0$, *we should use Boolean algebras which are in a strong sense not free.* In our present case, if we want to handle some $T_{\mathfrak{m}}$ while not handling another $T_{\mathfrak{n}}$, for $\mathfrak{m}, \mathfrak{n}$ orthogonal, perhaps we can start the induction with the completion of a free Boolean algebra and keep enough of the freeness for some failure of saturation for $T_{\mathfrak{n}}$ to persist even as we build the Boolean algebra to be gradually less free in a sense relevant to $T_{\mathfrak{m}}$ (adding formal solutions below).

The arbiter of the freeness we need will come in the form of a chain condition which we now explain.[11] Recall:

Definition 8.1. *Let κ be an uncountable regular cardinal. The Boolean algebra $\mathfrak{B}$ has the κ-c.c. when: given $\langle \mathbf{a}_\alpha : \alpha < \kappa \rangle$ a sequence of elements of $\mathfrak{B}^+$, we can find $\alpha \neq \beta < \kappa$ such that $\mathbf{a}_\alpha \cap \mathbf{a}_\beta > 0$.*

Definition 8.2 (The $(\kappa, \mathcal{I}, \bar{m})$-c.c.)**.** *Let κ be an uncountable regular cardinal. Let $\mathcal{I}$ be an ideal on ω extending $[\omega]^{<\aleph_0}$ and $\bar{m}$ a fast sequence. We say that the Boolean algebra $\mathfrak{B}$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. when: given $\langle \mathbf{a}_\alpha : \alpha \in \mathcal{U}_2 \rangle$ with $\mathcal{U}_2 \in [\kappa]^\kappa$ a sequence*[12] *of elements of $\mathfrak{B}^+$, we can find $j < \omega$, $\mathcal{U}_1 \in [\kappa]^\kappa$ and $A \in \mathcal{I}$ such that:*

[11] Although this is not needed for reading the proof, readers who are set theorists may recognize in the chain condition some methods intimately connected to finite support iteration (and may also conjecture that there may be potential for very interesting further interaction here).

[12] or renaming, without loss of generality, $\mathcal{U}_2 = \kappa$.

$\oplus$ *for every $n \in \omega \setminus A$ and every finite $u \subseteq \mathcal{U}_1$ and every $i < n - j$, if*

$$\frac{m_n}{(m_n^\circ)^{n^i}} < |u| \leq m_n$$

then there is some $v \subseteq u$ such that

$$|v| \geq \frac{|u|}{(m_n^\circ)^{n^{i+j}}} \quad \text{and} \quad \bigcap\{\mathbf{a}_\alpha : \alpha \in v\} > 0_{\mathfrak{B}}.$$

Observation 8.3. *If $\mathfrak{B}$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. for some $\mathcal{I}$ and $\bar{m}$, then $\mathfrak{B}$ has the κ-c.c. In particular, if $\kappa = \aleph_1$, then $\mathfrak{B}$ has the c.c.c.*

Proof. Clearly, the condition in 8.2 implies that given any κ nonzero elements of the Boolean algebra, at least two of them must have nonzero intersection. $\square$

Discussion 8.4. Informal discussion of the $(\kappa, \mathcal{I}, \bar{m})$-c.c. For any n, remember from 6.2 that any u such that

$$|u| \geq \frac{m_n}{(m_n^\circ)^{n^n}}$$

is *large*, so if $|u| \geq m_n/(m_n^\circ)^{n^{i+j}}$ for some $i+j < n$, then u is, by monotonicity, still large. In the statement of 8.2, we use $\mathcal{U}_2$ and $\mathcal{U}_1$ for easy quotation later on, but of course without loss of generality (after renumbering) $\mathcal{U}_2 = \kappa$. In this notation, given any sequence of κ nonzero elements, after possibly moving to a subsequence $\mathcal{U}$ of the same size, we can find a "shrinking factor" j so that (after possibly excluding a small set of n's) whenever we have a *really* large u, i.e. $|u| \geq m_n/(m_n^\circ)^{n^i}$ for some i such that $i+j$ is still $< n$, then we can shrink it a little bit (in the denominator the exponent n^i becomes n^{i+j}) to find a subset v which is still large, and all consistent.

The next claim shows that our upgraded c.c. is sometimes easy to satisfy. Recall from 1.7 that $\mathfrak{B}^0_{\alpha,\mu,\aleph_0}$ is the free Boolean algebra generated by α independent partitions each of size μ, and $\mathfrak{B}^1_{\alpha,\mu,\aleph_0}$ is its completion.

Claim 8.5. *Suppose μ is any infinite cardinal, κ is regular and uncountable, α is an ordinal, and $\mu < \kappa \leq \alpha$. Then for any ideal $\mathcal{I}$ on ω extending $[\omega]^{<\aleph_0}$, and any fast sequence $\bar{m}$, $\mathfrak{B} = \mathfrak{B}^1_{\alpha,\mu,\aleph_0}$ has the $(\kappa, \mathcal{I}, \bar{\mathfrak{m}})$-c.c.*

Proof. Recall from 1.8 above that the elements of the form $\mathbf{x}_f$ for $f \in \mathrm{FI}_{\aleph_0}(\alpha)$ are dense in $\mathfrak{B}$. Suppose we are given $\langle \mathbf{a}_\alpha : \alpha < \kappa \rangle$ a sequence of positive elements of $\mathfrak{B}$. First, for each $\alpha < \kappa$, we may choose $f_\alpha \in \mathrm{FI}_{\aleph_0}(\alpha)$ such that $\mathbf{x}_{f_\alpha} \leq \mathbf{a}_\alpha$. For each $\alpha < \kappa$, let $u_\alpha = \mathrm{dom}(f_\alpha)$, so u_α is finite. By the Δ-system lemma 1.10, there is some u_* and $\mathcal{V} \in [\kappa]^\kappa$ such that $u_\alpha \cap u_\beta = u_*$ for $\alpha, \beta \in \mathcal{V}$. Since the range of each f_α is a subset of $\mu < \kappa$, there is $\mathcal{U} \in [\mathcal{V}]^\kappa$ such that $f_\alpha \restriction u_* = f_\beta \restriction u_*$ for $\alpha, \beta \in \mathcal{U}$. Notice this tells us for any finitely many $\alpha_0, \dots, \alpha_{n-1}$ from $\mathcal{U}$, $f = \cup_{i<n} f_i$ is a function, thus $\mathbf{x}_f > 0$. In other words, for any finite $v \subseteq \mathcal{U}$,

$$\bigcap\{\mathbf{a}_\alpha : \alpha \in v\} > 0_{\mathfrak{B}}$$

so condition is trivially satisified [indeed, taking $A = \emptyset$ and $j = 0$] for any n and u, simply by using $v = u$. $\square$

Remark 8.6. *We have stated 8.5 to be compliant with Definition 8.2, but notice the proof would go through for any ideal $\mathcal{I}$ including the trivial ideal $\emptyset$.*

Discussion 8.7. *In the next definition 8.8 note we replace μ by the more general $<\kappa$, which includes the case where κ is a regular limit cardinal, thus weakly inaccessible. This is natural to include, and even to use as $\mathfrak{B}_*$ in 9.1(2), but is not our main case. For our main theorems, the case $\kappa = \mu^+$, and thus using as a base for our induction Boolean algebras of the form $\mathfrak{B}^1_{\alpha,\mu,\aleph_0}$, or even $\mathfrak{B}^1_{\kappa,\aleph_0,\aleph_0}$, will suffice.*

Definition 8.8. *Let $\theta = \operatorname{cof}(\theta) \leq \kappa = \operatorname{cof}(\kappa)$ and let α be an ordinal.*

(1) $\mathcal{F}_{\alpha,<\kappa,\aleph_0} = \{f : \operatorname{dom}(f) \in [\alpha]^{<\theta}$, *and if* $\beta \in \operatorname{dom}(f)$ *then* $f(\beta) < \operatorname{rm}_\kappa(\beta)\}$ *where* $\operatorname{rm}_\kappa(\beta) = \min\{i : \bigvee_\gamma (\beta = \kappa \cdot \gamma + i)\}$ *is the remainder of* β *mod* κ.

(2) $\mathfrak{B}^0_{\alpha,<\kappa,\theta}$ *is the Boolean algebra generated by* $\{\mathbf{x}_f : f \in \mathcal{F}_{\alpha,<\kappa,\theta}\}$ *freely except:*
   (a) $\mathbf{x}_f \leq \mathbf{x}_g$ *when* $g \subseteq f \in \mathcal{F}_{\alpha,<\kappa,\theta}$
   (b) $\mathbf{x}_f \cap \mathbf{x}_g = 0$ *when* $(\exists \beta \in \operatorname{dom}(f) \cap \operatorname{dom}(g))[f(\beta) \neq g(\beta)]$.

(3) *Let* $\mathfrak{B}_{\alpha,<\kappa,\theta} = \mathfrak{B}^1_{\alpha,<\kappa,\theta}$ *be the completion of* $\mathfrak{B}^0_{\alpha,<\kappa,\theta}$.

We verify that Claim 8.5 likewise holds for $\mathfrak{B}_{\alpha,<\kappa,\theta}$.

Claim 8.9. *Suppose κ is regular and uncountable and $\alpha \in \mathrm{Ord}$. Suppose that in addition $(\forall \alpha < \kappa)(\alpha^{<\theta} < \kappa)$. Then $\mathfrak{B}^0_{\alpha,<\kappa,\aleph_0}$ and $\mathfrak{B} = \mathfrak{B}^1_{\alpha,<\kappa,\theta}$ satisfy the κ-c.c., and indeed the $(\kappa, \mathcal{I}, \bar{m})$-c.c. for any $\bar{m}$ and $\mathcal{I}$.*

Proof. Similarly to 8.5, suppose we are given $\langle \mathbf{a}_\alpha : \alpha < \kappa \rangle$ a sequence of positive elements of $\mathfrak{B}$. For each $\alpha < \kappa$, we may choose $f_\alpha \in \mathrm{FI}_\theta(\alpha)$, so $u_\alpha = \operatorname{dom}(f_\alpha) \in [\alpha]^{<\theta}$, such that $\mathbf{x}_{f_\alpha} \leq \mathbf{a}_\alpha$. Assuming that $(\forall \alpha < \kappa)(\alpha^{<\nu} < \kappa)$, the hypotheses of the Δ-system lemma 1.9 apply, and there is some u_* and $\mathcal{V} \in [\kappa]^\kappa$ such that $u_\alpha \cap u_\beta = u_*$ for $\alpha, \beta \in \mathcal{V}$. Let $\gamma = \sup\{\operatorname{rm}_\kappa(\beta) : \beta \in u_*\}$, so $\gamma < \kappa$ as $|u_*| < \aleph_0$. By definition 8.8(1), each $f_\alpha \restriction u_*$ has range $\subseteq \gamma + 1 < \kappa$, so there is $\mathcal{U} \in [\mathcal{V}]^\kappa$ such that $f_\alpha \restriction u_* = f_\beta \restriction u_*$ for $\alpha, \beta \in \mathcal{U}$. So just as before, for any finite $v \subseteq \mathcal{U}$,

$$\bigcap \{\mathbf{a}_\alpha : \alpha \in v\} > 0_{\mathfrak{B}}$$

so condition for the $(\kappa, \mathcal{I}, \bar{m})$-c.c. is trivially satisisfied for any n and u, simply by using $v = u$. $\square$

Convention 8.10. *For Boolean algebras, write that $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$ to mean that $\mathfrak{B}_1$ is a complete subalgebra of $\mathfrak{B}_2$.*

In our inductive construction, the so-called "pattern transfer property" (Definition 8.12 below) will play a key role in ensuring that the $(\kappa, \mathcal{I}, \bar{m})$-c.c. is preserved at limit stages. In some sense, it shows a close connection between a smaller and larger Boolean algebra which is enough to allow the $(\kappa, \mathcal{I}, \bar{m})$-c.c. to transfer from the smaller to the larger one. We first need a definition.

Definition 8.11. *Let $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$ and let $\mathbf{b} \in \mathfrak{B}_2^+$. We say that $\mathbf{a} \in \mathfrak{B}_1^+$ is* below the projection of $\mathbf{b}$, *or* below $\mathbf{b}$ in projection, *in symbols*

$$\mathbf{a} \leq_{\operatorname{proj}_{\mathfrak{B}_2 \mathfrak{B}_1}} \mathbf{b}$$

when for every $\mathbf{c} \in \mathfrak{B}_1^+$ such that $\mathfrak{B}_1 \models \mathbf{c} \leq \mathbf{a}$, we have that $\mathfrak{B}_2 \models \mathbf{c} \cap \mathbf{b} > 0$. When $\mathfrak{B}_2, \mathfrak{B}_1$ are clear from context, we may omit them from the subscript.

Definition 8.12. *Let κ be an uncountable cardinal, $\mathcal{I}$ an ideal on ω extending $[\omega]^{<\aleph_0}$, and $\bar{m}$ a fast sequence. We say that the pair $(\mathfrak{B}_1, \mathfrak{B}_2)$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property when:*

(1) *$\mathfrak{B}_1$, $\mathfrak{B}_2$ are both complete Boolean algebras, and $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$.*

(2) $\mathfrak{B}_1$ *satisfies the* κ*-c.c.*[13]

(3) if $\bar{\mathbf{a}}^2 = \langle \mathbf{a}^2_\alpha : \alpha \in \mathcal{U}_2 \rangle$ *for* $\mathcal{U}_2 \in [\kappa]^\kappa$ *is a sequence of distinct elements of* $\mathfrak{B}^+_2$*, we can find a quadruple* $(j, \mathcal{U}_1, A, \bar{\mathbf{a}}^1)$ *such that:*

(a) $j < \omega$

(b) $\mathcal{U}_1 \in [\mathcal{U}_2]^\kappa$

(c) $A \in \mathcal{I}$

(d) $\bar{\mathbf{a}}^1 = \langle \mathbf{a}^1_\alpha : \alpha < \kappa \rangle$ *is a sequence of distinct elements of* $\mathfrak{B}^+_1$ *(though only* $\bar{\mathbf{a}}^1 \restriction \mathcal{U}_1$ *matters)*

(e) $\alpha < \kappa$ *implies* $\mathbf{a}^1_\alpha \leq_{\text{proj}} \mathbf{a}^2_\alpha$

(f) (i) implies (ii) where:

(i) we are given $n \in \omega \setminus A$, $i + j < n$, $u \subseteq \mathcal{U}_1$*, and* $\mathbf{a} \in \mathfrak{B}^+_1$ *such that* $m_n/(m^\circ_n)^{n^i} < |u| < m_n$ *and*

$$\mathfrak{B}_1 \models \text{“} \mathbf{a} \leq \bigcap_{\alpha \in u} \mathbf{a}^1_\alpha \text{”}$$

(ii) there is v *such that* $v \subseteq u$ *and* $|v| \geq |u|/(m^\circ_n)^{n^{i+j}}$ *and*

$$\mathfrak{B}_2 \models \text{“} \bigcap_{\alpha \in v} \mathbf{a}^2_\alpha \ \cap \ \mathbf{a} > 0.\text{”}$$

Remark 8.13. *One reason* 8.12(3)(f) *is not trivial is that* $\mathbf{a}^1_\alpha \leq_{\text{proj}} \mathbf{a}^2_\alpha$ *does not imply* $\mathbf{a}^1_\alpha \leq \mathbf{a}^2_\alpha$*.*

Observation 8.14. *If* $\mathfrak{B}_1 \lessdot \mathfrak{B}_2 \lessdot \mathfrak{B}_3$ *are complete Boolean algebras and the pairs* $(\mathfrak{B}_1, \mathfrak{B}_2)$ *and* $(\mathfrak{B}_2, \mathfrak{B}_3)$ *have the* $(\kappa, \mathcal{I}, \bar{m})$*-pattern transfer property, then so does* $(\mathfrak{B}_1, \mathfrak{B}_3)$*.*

Proof. We start with $\bar{\mathbf{a}}^3 = \langle \mathbf{a}^3_\alpha : \alpha <\in \mathcal{U}_3 \rangle$, $\mathcal{U}_3 \in [\kappa]^\kappa$ in $\mathfrak{B}_3$. By the assumed property for $(\mathfrak{B}_2, \mathfrak{B}_3)$, we find $j_2, \mathcal{U}_2, A_2$, and a sequence $\langle \mathbf{a}^2_\alpha : \alpha < \mathcal{U}_2 \rangle$ of elements of $\mathfrak{B}^+_2$ satisfying the definition. Then by the assumed property for $(\mathfrak{B}_1, \mathfrak{B}_2)$ applied to $\langle \mathbf{a}^2_\alpha : \alpha \in \mathcal{U}_2 \rangle$, we find $j_1, \mathcal{U}_1, A_1$, and a sequence $\langle \mathbf{a}^1_\alpha : \alpha \in \mathcal{U}_1 \rangle$ of elements of $\mathfrak{B}^+_1$ satisfying the definition. So $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Let $\mathcal{U} = \mathcal{U}_1$ and let $A = A_1 \cup A_2 \cup \{0, 1\}$. Let $j_* = j_1 + j_2 + 1$.

Now we would like to show 8.12(f)(i)$\rightarrow$(ii) when $\mathfrak{B}_2$, $\bar{\mathbf{a}}^2$ there is replaced by $\mathfrak{B}_3$, $\bar{\mathbf{a}}^3$, for the quadruple $(j_*, \mathcal{U}, A, \bar{\mathbf{a}}_1)$. Suppose then that we are given $n \in \omega \setminus A$, $i + j_* < n$, $u \subseteq \mathcal{U}_1$, $\mathbf{a} \in \mathfrak{B}^+_1$ such that $m_n/(m^\circ_n)^{n^i} < |u| < m_n$ and $\mathfrak{B}_1 \models \mathbf{a} \leq \bigcap_{\alpha \in u} \mathbf{a}^1_\alpha$. First we apply the property for $(\mathfrak{B}_2, \mathfrak{B}_3)$. Note that the hypotheses of (f)(i) are all clearly satisfied in this case as $n \in \omega \setminus A$ implies $n \in \omega \setminus A_2$, $u \subseteq \mathcal{U} \subseteq \mathcal{U}_2$ was chosen to have the right size, and $i + j_* < n$ implies $i + j_2 < n$. So by the property for $(\mathfrak{B}_3, \mathfrak{B}_2)$ via the quadruple $(j_2, \mathcal{U}_2, A_2, \bar{\mathbf{a}}_2)$, there is $v' \subseteq u$ with $|v'| \geq |u|/(m^\circ_n)^{n^{i+j_2}}$ and $\mathfrak{B}_2 \models \bigcap_{\alpha \in v'} \mathbf{a}^2_\alpha \ \cap \ \mathbf{a} > 0$. Choose an element $\mathbf{a}'$ of $\mathfrak{B}^+_2$ such that $\mathbf{a}' \leq \bigcap_{\alpha \in v'} \mathbf{a}^2_\alpha \ \cap \ \mathbf{a}$ (possible by the last clause of the previous sentence). Note that $|v'| \geq |u|/(m^\circ_n)^{n^{i+j_2}}$ and $|u| > m_n/(m^\circ_n)^{n^i}$ together imply

$$|v'| > \frac{m_n}{(m^\circ_n)^{n^i}(m^\circ_n)^{n^{i+j_2}}} \geq \frac{m_n}{(m^\circ_n)^{n^{i+j_2+1}}} \ .$$

Next we apply the property for $(\mathfrak{B}_1, \mathfrak{B}_2)$ using the same n and using $i + j_2 + 1$ v', and $\mathbf{a}'$ in place of i, u, $\mathbf{a}$ respectively in (f)(i). Once again, we meet the conditions

[13] note for this definition, we don't ask that $\mathfrak{B}_1$ have the $(\kappa, \mathcal{I}, \bar{m})$-c.c., only the κ-c.c., though in every application in the paper, $\mathfrak{B}_1$ will have the $(\kappa, \mathcal{I}, \bar{m})$-c.c.

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of the quadruple $(j_1, \mathcal{U}_1, A_1, \bar{\mathbf{a}}_1)$, recalling in particular that we chose i so that $i + j_* < n$. So we obtain from (f)(ii) a subset v with $|v| \geq |v'|/(m^\circ_n)^{n^{(i+j_2+1)+j_1}}$ such that $\mathfrak{B}_3 \models \bigcap_{\alpha \in v} \mathbf{a}^3_\alpha \cap \mathbf{a}' > 0$, and since $\mathbf{a}' \leq \mathbf{a}$, $\mathfrak{B}_3 \models \bigcap_{\alpha \in v} \mathbf{a}^3_\alpha \cap \mathbf{a} > 0$. This shows that for $\bar{\mathbf{a}}^3$ the quadruple $(j_* = j_1 + j_2 + 1, \mathcal{U}, A, \bar{\mathbf{a}}^1)$ works, which completes the verification. □

Observation 8.15. *Suppose $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$, $\mathfrak{B}_1$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c., and $(\mathfrak{B}_1, \mathfrak{B}_2)$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property. Then $\mathfrak{B}_2$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c.*

Proof. Suppose we are given $\langle \mathbf{a}^2_\alpha : \alpha \in \mathcal{U}_2 \rangle$, $\mathcal{U}_2 \in [\kappa]^\kappa$ in $\mathfrak{B}_2$. Applying the pattern transfer property, we find $j_1, \mathcal{U}_1, A_1$, and $\langle \mathbf{a}^1_\alpha : \alpha \in \mathcal{U}_1 \rangle$ in $\mathfrak{B}^+_1$ satisfying 8.12. Next, working in $\mathfrak{B}_1$, we apply the $(\kappa, \mathcal{I}, \bar{m})$-c.c. to the sequence $\langle \mathbf{a}^1_\alpha : \alpha < \kappa \rangle$ and find $j_2, \mathcal{U}_2, A_2$ satisfying 8.2. Now let us check that $\mathfrak{B}_2$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c. using $\mathcal{U} = \mathcal{U}_1 \subseteq \mathcal{U}_2$, $A = A_1 \cup A_2$, and $j = j_1 + j_2 + 2$. Suppose we are given $n \in \omega \setminus A$ and a finite $u \subseteq \mathcal{U}$. We know that if $i + j < n$ and

$$\frac{m_n}{(m^\circ_n)^{n^i}} < |u| \leq m_n$$

there is some $v' \subseteq u$ such that in $\mathfrak{B}_1$,

$$|v'| \geq \frac{|u|}{(m^\circ_n)^{n^{i+j_1}}} \geq \frac{m_n}{(m^\circ_n)^{n^{i+j_1+1}}} \text{ and } \mathfrak{B}_1 \models \bigcap \{\mathbf{a}^1_\alpha : \alpha \in v'\} > 0.$$

Now 8.12(f)(i) holds, using v', $i + j_1 + 1$ here for u, i there. (Note: the identity of $\mathbf{a}$ in that equation is not important here; we'll use it later when we deal with omitting types.) So 8.12(f)(ii) tells us there is $v \subseteq v'$ such that $|v| \geq |v'|/(m^\circ_n)^{n^{i+j_1+j_2+1}}$ and $\mathfrak{B}_2 \models \bigcap_{\alpha \in v} \mathbf{a}^2_\alpha > 0$. Checking the size,

$$|v| \geq \frac{|v'|}{(m^\circ_n)^{n^{i+j_1+j_2+1}}} \geq \frac{|u|}{(m^\circ_n)^{n^{i+j_1}} (m^\circ_n)^{n^{i+j_1+j_2+1}}} \geq \frac{|u|}{(m^\circ_n)^{n^{i+j_1+j_2+2}}}.$$

This completes the verification. □

We recall the following fact which we will upgrade to the case of our chain condition in 8.18.

Fact 8.16 (Jech, Corollary 16.10 [8]). *Let $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\beta \subseteq \cdots$ $(\beta < \alpha)$ be a sequence of complete Boolean algebras such that for all $\beta < \gamma$, $\mathfrak{B}_\beta \lessdot \mathfrak{B}_\gamma$, and for each limit ordinal γ, $\bigcup_{\beta<\gamma} B_\beta$ is dense in B_γ. If every B_β satisfies the κ-c.c. then $\bigcup_{\beta<\alpha} B_\beta$ satisfies the κ-c.c.*

Definition 8.17. *We say that $\bar{\mathfrak{B}} = \langle \mathfrak{B}_\gamma : \gamma \leq \delta \rangle$ is a $\lessdot$-increasing continuous sequence of complete Boolean algebras when:*

1. *for $\gamma < \delta$, $\mathfrak{B}_\gamma \lessdot \mathfrak{B}_{\gamma+1}$;*
2. *at limit stages $\bar{\mathfrak{B}}$ is continuous, meaning that if $\gamma < \delta$ is a limit ordinal then $\bigcup \{\mathfrak{B}_\gamma : \gamma < \delta\}$ is a dense subset of $\mathfrak{B}_\delta$. (Requiring that the union is $\lessdot \mathfrak{B}_\delta$ is ok also.)*

Claim 8.18. *Suppose that $\bar{\mathfrak{B}} = \langle \mathfrak{B}_\gamma : \gamma \leq \delta \rangle$ is a $\lessdot$-increasing continuous sequence of complete Boolean algebras, where:*

1. *$\mathfrak{B}_\gamma$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c. for every $\gamma < \delta$.*
2. *if $\gamma < \beta < \delta$ then the pair $(\mathfrak{B}_\gamma, \mathfrak{B}_\beta)$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property.*

Then $\mathfrak{B}_\delta$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c., and moreover, if $\gamma < \delta$ then the pair $(\mathfrak{B}_\gamma, \mathfrak{B}_\delta)$-satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property.

Proof. Fix $\gamma_* < \delta$, and we will show that $(\mathfrak{B}_{\gamma_*}, \mathfrak{B}_\delta)$ has the $(\kappa, \mathcal{I}, \bar{m})$-transfer property, and that therefore by 8.15, $\mathfrak{B}_\delta$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. Recall κ is regular.

Suppose we are given $\langle \mathbf{a}^\delta_\alpha : \alpha < \kappa \rangle$ a sequence of elements of $\mathfrak{B}^+_\delta$. Without loss of generality (by the definition of continuous) each $\mathbf{a}^\delta_\alpha$ is an element of $\bigcup_{\gamma<\delta} \mathfrak{B}_\gamma$ (i.e., each element is already a member of some $\mathfrak{B}_\gamma$).

If $\operatorname{cof}(\delta) \neq \kappa$, there is $\gamma_{**} < \delta$ such that $\mathcal{U} = \{\alpha < \kappa : \mathbf{a}^\delta_\alpha \in \mathfrak{B}_{\gamma_{**}}\}$ has size κ. Since $\mathfrak{B}_{\delta_{**}}$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. by hypothesis, $\mathfrak{B}_\gamma$ will inherit it from the restriction to $\mathcal{U}$. Since without loss of generality $\gamma_{**} > \gamma_*$, and $(\mathfrak{B}_{\gamma_*}, \mathfrak{B}_{\gamma_{**}})$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property by hypothesis, we can similarly see that $(\mathfrak{B}_{\gamma_*}, \mathfrak{B}_\delta)$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property too (by restricting to $\mathcal{U}$).

Suppose then that $\operatorname{cof}(\delta) = \kappa$. In this case fix a strictly increasing continuous sequence $\langle i_\alpha : \alpha < \kappa \rangle$ of ordinals $< \delta$ but above γ_*, with limit δ.

Now for each $\alpha < \kappa$ we choose $\mathbf{b}_\alpha$, as follows. First, there is $\zeta = \zeta(\alpha) \in (\alpha, \kappa)$ such that

$$\mathbf{a}^\delta_\alpha \in \mathfrak{B}_{i_{\zeta(\alpha)}} .$$

As $\alpha < \zeta(\alpha)$, by our assumption

$$\mathfrak{B}_{i_\alpha} \lessdot \mathfrak{B}_{i_{\zeta(\alpha)}}.$$

As $\mathbf{a}^\delta_\alpha \in \mathfrak{B}_{i_{\zeta(\alpha)}}$, by definition of $\lessdot$, we may find a $\mathbf{b}_\alpha \in \mathfrak{B}^+_{i_\alpha}$ which is below the projection of $\mathbf{a}^\delta_\alpha$.

Let $\langle \mathbf{b}_\alpha : \alpha < \kappa \rangle$ be the sequence of nonzero elements defined in this way.

For every limit ordinal $\epsilon < \kappa$, $\bigcup_{\alpha<\epsilon} \mathfrak{B}_{i_\alpha}$ is dense in $\mathfrak{B}_{i_\epsilon}$ by definition of continuous. So for every limit ordinal $\epsilon < \kappa$, there is $\mathbf{c}_\epsilon$ such that $\mathbf{c}_\epsilon \in \bigcup_{\alpha<\epsilon} \mathfrak{B}_{i_\alpha}$, $\mathfrak{B}_{i_\epsilon} \models 0 < \mathbf{c}_\epsilon < \mathbf{b}_\epsilon$, and $\mathbf{c}_\epsilon \in \mathfrak{B}^+_{i_{\rho(\epsilon)}}$ for some $\rho(\epsilon) < \epsilon$. Now the function $\epsilon \mapsto \rho(\epsilon)$ is defined and regressive on the limit ordinals $< \kappa$ so by Fodor's lemma there is $\rho_* < \kappa$ such that $\mathcal{U}_0 = \{\epsilon < \kappa : \rho(\epsilon) < \rho_*\}$ is a stationary subset of κ, recalling that $\kappa = \operatorname{cof}(\kappa) > \aleph_0$ by 8.2.

Let E be a club of κ such that $\epsilon < \beta \in E$ implies $\zeta(\epsilon) < \beta$. Then $\mathcal{U} = E \cap \mathcal{U}_0$ is a stationary subset of κ, hence of size κ. Recall that for any $\alpha \in \mathcal{U}$, $\mathbf{c}_\alpha < \mathbf{b}_\alpha \leq_{\mathrm{proj}} a^\delta_\alpha$. So if $u \in \mathcal{U}$ is finite, for any ρ_*, and indeed for $\gamma > \max u$, by the choice of E, we have that

$$\mathfrak{B}_{i_\gamma} \models \bigcap_{\alpha \in u} \mathbf{c}_\alpha > 0 \implies \mathfrak{B}_\delta \models \bigcap_{\alpha \in u} \mathbf{a}^\delta_\alpha > 0.$$

Clearly, this allows us to transfer the $(\kappa, \mathcal{I}, \bar{m})$-c.c. from $\mathfrak{B}_{i_{\rho_*}}$ to $\mathfrak{B}_\delta$, and it also tells us that $(\mathfrak{B}_{\rho_*}, \mathfrak{B}_\delta)$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property. By construction $\gamma_* < i_{\rho_*}$, and so by hypothesis $(\mathfrak{B}_{\gamma_*}, \mathfrak{B}_{i_{\rho_*}})$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property, so recalling 8.14, $(\mathfrak{B}_{\gamma_*}, \mathfrak{B}_\delta)$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property too, which completes the proof. $\square$

Lemma 8.19. *If $\langle \mathfrak{B}_\gamma : \gamma \leq \delta \rangle$ is $\lessdot$-increasing continuous and $\mathfrak{B}_0$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c. and $(\mathfrak{B}_\gamma, \mathfrak{B}_{\gamma+1})$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property for all $\gamma < \delta$, then $\mathfrak{B}_\delta$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c. and for all $\gamma < \delta$, the pair $(\mathfrak{B}_\gamma, \mathfrak{B}_\delta)$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property.*

Proof. By induction on δ; immediate from the previous claim and 8.14, 8.15. $\square$

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We shall use the property from the proof of 8.18 later on so we phrase it below. It is stronger than 8.12, so does not supercede that definition, but as we have seen it will imply it, and occasionally it will be simpler to show.

Corollary 8.20. *$\mathfrak{B}_2$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c., and even the pair $(\mathfrak{B}_1, \mathfrak{B}_2)$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property, when:*

(a) $\mathfrak{B}_1 \lessdot \mathfrak{B}_2$ are complete Boolean algebras
(b) $\mathfrak{B}_1$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c.
(c) for any $\mathbf{a}_\alpha \in \mathfrak{B}_2^+$ for $\alpha < \kappa$, we can find $\mathcal{U} \in [\kappa]^\kappa$ and $\mathbf{x}_\alpha \in \mathfrak{B}_1^+$ for $\alpha \in \mathcal{U}$ such that: if $u \in [\mathcal{U}]^{<\aleph_0}$ and $\mathfrak{B}_1 \models$ "$\bigcap_{\alpha \in u} \mathbf{x}_\alpha > 0$" <u>then</u> $\mathfrak{B}_2 \models$ "$\bigcap_{\alpha \in u} \mathbf{a}_\alpha > 0$."

9. The c.c. and omitting types

Convention 9.1. *In this section we fix:*

(1) κ, μ, λ infinite cardinals, with κ regular, and $\mu < \kappa \leq \lambda$.
(2) $\mathfrak{B}_* = \mathfrak{B}^1_{\kappa,\mu,\aleph_0}$.
(3) $\bar{m}$ a fast sequence.
(4) $\mathcal{I}$ an ideal on ω extending $[\omega]^{<\aleph_0}$.

Discussion 9.2. *Although we allow the generality of $\mu < \kappa$ with no constraints on their distance, for our main results on incomparability it would suffice to use $\mu = \aleph_0$ and κ an uncountable successor, e.g. $\aleph_1$. In other words, on one hand, to see the differences between these theories we do not need to go out very far, but on the other hand, it is not a phenomenon limited to small sizes, due to the freedom in the construction.*

In this section we connect our chain condition to omitting types. The prototype is [27] Lemma 3.2, which (in our language) amounts to showing non-saturation in the base case, for $\mathfrak{B} = \mathfrak{B}_*$ and any nontrivial ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$.

Lemma 9.3. *Suppose $\mathfrak{B}$ is a complete Boolean algebra, $\mathfrak{B}_* \lessdot \mathfrak{B}$, and $(\mathfrak{B}_*, \mathfrak{B})$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property. Let ξ be any level function such that $\xi^{-1}\{1\} \neq \emptyset \mod \mathcal{I}$. Let $T = T_\mathfrak{n}$ where $\mathfrak{n} = \operatorname{par}[\bar{m}, \bar{E}, \xi]$ for some sequence of graphs $\bar{E}$ which is good for $\bar{m}$. Let $\mathcal{D}$ be any ultrafilter on $\mathfrak{B}$.*

Then $\mathcal{D}$ is not $(\kappa^+, T_\mathfrak{n})$-moral.

Remark 9.4. *The proof of 9.3 below will show that if we fix any nonprincipal ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}_*$ in advance, then for any such $T_\mathfrak{n}$, there is a specific possibility pattern in $(\mathfrak{B}_*, \mathcal{D}_*)$ which fails to have a multiplicative refinement in $(\mathfrak{B}, \mathcal{D})$ for any complete Boolean algebra $\mathfrak{B}$ satisfying the hypotheses of the claim, and any ultrafilter $\mathcal{D}$ on $\mathfrak{B}$ which extends $\mathcal{D}_*$.*

Corollary 9.5. *Recall from 8.5 that $\mathfrak{B}_*$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. So by Observation* 8.15*, it will follow from Lemma* 9.3 *that $\mathfrak{B}$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c.*

Proof of Lemma 9.3. Using the framework of separation of variables, we work in $T_\mathfrak{n}$, recalling that

(a) $$\mathfrak{n} = \operatorname{par}[\bar{m}, \bar{E}, \xi_\mathfrak{n}].$$

and recalling that

(b) $$\bar{m} = \langle m_k : k < \omega \rangle.$$

For each $\rho \in \mathcal{T}_{2,k}$ and $\alpha < \kappa$, we define $\mathbf{a}[\ P_\rho(x_\alpha)\ ] \in \mathfrak{B}_*^+$ by induction on $k < \omega$:

- if $k = 0$, let $\mathbf{a}[\ P_{\langle\ \rangle}(x_\alpha)\ ] = 1_{\mathfrak{B}_*}$.
- for k and $i < m_k$, let $g_{\alpha,k,i}$ be the function with domain $\{\omega\alpha + k\}$ such that $g_{\alpha,k,i}(\omega\alpha + k) = i$. Without loss of generality, assume $\mathbf{x}_{g_{\alpha,k,0}} \in \mathcal{D}_*$. Then define: for $i < m_k - 1$,
$$\mathbf{a}[\ P_{\rho^\frown\langle i\rangle}(x_\alpha)\ ] = \mathbf{a}[\ P_\rho(x_\alpha)\ ] \cap \mathbf{x}_{g_{\alpha,k,i}}.$$
If $i = m_k - 1$, let
$$\mathbf{a}[\ P_{\rho^\frown\langle i\rangle}(x_\alpha)\ ] = \mathbf{a}[\ P_\rho(x_\alpha)\ ]\ -\ \left(\bigcup_{j<i} \mathbf{a}[\ P_{\rho^\frown\langle j\rangle}(x_\alpha)\ ]\right).$$
- Note that $\langle \mathbf{a}[\ P_\rho(x_\alpha)\ ] : \rho \in \mathcal{T}_{2,k}\rangle$ is a partition of $1_{\mathfrak{B}}$.
- for $\alpha < \beta < \kappa$, note
$$\mathbf{a}[\ x_\alpha \neq x_\beta\ ] = 1_{\mathfrak{B}_*}.$$

As $\mathbf{x}_{g_{\alpha,k,0}} \in \mathcal{D}_*$, letting $\langle 0_k \rangle$ denote the constant 0 sequence of length k, we have that for each $\alpha < \kappa$ and each $k < \omega$,
$$\mathbf{a}[\ P_{\langle 0_k\rangle}(x_\alpha)\ ] \in \mathcal{D}_*.$$
For each $u \in [\kappa]^{<\aleph_0}$, define
$$\mathbf{c}_u = \mathbf{a}[\ \exists \bigwedge_{\alpha\in u} R(x, x_\alpha)\ ].$$
It follows from the construction that

(c) $$\bar{\mathbf{c}} = \bar{\mathbf{c}}[\mathfrak{n}] = \langle \mathbf{c}_u : u \in [\kappa]^{<\aleph_0}\rangle = \langle \mathbf{c}_{\mathfrak{n},u} : u \in [\kappa]^{<\aleph_0}\rangle$$

is a possibility pattern in $(\mathfrak{B}_*, \mathcal{D}_*)$, hence in $(\mathfrak{B}, \mathcal{D})$ [moreover, that we could choose appropriate parameters c_α to fill in for x_α in any enveloping ultrapower].

Note to the reader: if we had run this construction for any other $\mathfrak{n}' \in \mathbf{M}_*$, the elements $\mathbf{c}_{\{\alpha\}}$ would be exactly the same (and equal to $1_{\mathfrak{B}}$); but the sets $\mathbf{c}_u$ for $|u| > 1$ could differ depending on $\mathfrak{n}'$. So although this construction will work for any parameter in $\mathbf{M}_*$, it does not necessarily give the same $\bar{\mathbf{c}}$.

Assume for a contradiction that $\mathbf{a}^2 = \langle \mathbf{a}^2_\alpha : \alpha < \kappa\rangle$ is a multiplicative refinement of $\bar{\mathbf{c}}$ in $\mathfrak{B}$, where $\bar{\mathbf{a}}^2 = \langle \mathbf{a}^2_\alpha : \alpha < \kappa\rangle$ is a sequence of members of $\mathfrak{B}^+$. We apply the definition of "$(\mathfrak{B}_*, \mathfrak{B})$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property," hence there is a quadruple $(j, \mathcal{U}_0, A, \bar{\mathbf{a}}^1)$ as there, noting $\bar{\mathbf{a}}^1$ is a sequence of elements of $\mathfrak{B}^+_*$.

Now by the choice of $\mathfrak{B}_*$, for each $\alpha < \kappa$ there is $f_\alpha \in \mathrm{FI}_{\aleph_0}(\kappa, \mu)$, i.e. f_α is a finite function from κ to μ, such that
$$\mathfrak{B}_* \models \mathbf{x}_{f_\alpha} \leq \mathbf{a}^1_\alpha \text{ [hence this holds also in } \mathfrak{B}\text{]}.$$
Since each f_α has finite domain, there is a smallest positive integer k_α such that for every $\beta \in \mathrm{dom}(f_\alpha)$, the remainder of $\beta \mod \omega$ is $< k_\alpha$. So there is $\mathcal{U}_1 \in [\mathcal{U}_0]^\kappa$ and $n \in \omega \setminus A$ so that $k_\alpha = k_\beta < n$ for $\alpha, \beta \in \mathcal{U}_1$. Without loss of generality $j < n$ and $\xi_{\mathfrak{n}}(n) = 1$, possible as $\xi_{\mathfrak{n}}^{-1}\{1\} \neq \emptyset \mod \mathcal{I}$, while $A \in \mathcal{I}$.

For each $\alpha \in \mathcal{U}_1$, the elements $\{\mathbf{a}[P_\nu(x_\alpha)] : \nu \in \mathcal{T}_{2,n}\}$ form a finite, maximal antichain of $\mathfrak{B}_*$, so choose $\nu_\alpha \in \mathcal{T}_{2,n}$ such that $\mathbf{x}_{f_\alpha} \cap \mathbf{a}[P_\nu(x_\alpha)] > 0$. As $\mathcal{T}_{2,n}$ is finite, there is $\mathcal{U}_2 \in [\mathcal{U}_1]^\kappa$ for which ν_α is constant, say equal to ν_*. In other words, letting

$$f^*_\alpha = f_\alpha \cup \bigcup_{k<n} \{(\omega\alpha + k, \nu_*(k))\}$$

we have that

$$\mathfrak{B}_* \models \text{“ } \mathbf{x}_{f^*_\alpha} = \mathbf{x}_{f_\alpha} \cap \mathbf{a}[P_{\nu_*}(x_\alpha)] \ > 0\text{”}.$$

In particular, for later quotation, we note that in $\mathfrak{B}_*$

(d) $$\mathbf{x}_{f^*_\alpha} \leq \mathbf{a}[P_{\nu_*}(x_\alpha)] \text{ and moreover } \mathbf{x}_{f^*_\alpha} \leq \mathbf{x}_{f_\alpha}.$$

Note that for every $\beta \in \mathrm{dom}(f^*_\alpha)$ the remainder of $\beta \mod \omega$ is still $< n$, and it is still the case that $\mathrm{dom}(f^*_\alpha)$ is finite.

For each $\alpha < \kappa$, let $u_\alpha = \mathrm{dom}(f^*_\alpha)$. By the Δ-system lemma 1.10, there is some u_* and $\mathcal{U}_3 \in [\mathcal{U}_2]^\kappa$ such that $u_\alpha \cap u_\beta = u_*$ for $\alpha, \beta \in \mathcal{U}_3$. Since the range of each f^*_α is a finite subset of μ and $\mu^+ \leq \kappa$, there is $\mathcal{U}_4 \in [\mathcal{U}_3]^\kappa$ such that $f^*_\alpha \restriction u_* = f^*_\beta \restriction u_*$ for $\alpha, \beta \in \mathcal{U}_4$. Notice this tells us for any finitely many $\alpha_0, \dots, \alpha_{n-1}$ from $\mathcal{U}_4$, $f = \cup_{i<n} f^*_i$ is a function thus $\mathbf{x}_f > 0$.

To summarize, for any finite n and $\alpha_0, \dots, \alpha_{n-1} \in \mathcal{U}_4$, we have that in $\mathfrak{B}_*$,

(e) $$0 < \left(\bigcap_{\ell<n} \mathbf{x}_{f^*_{\alpha_\ell}} \right) \leq \bigcap_{\ell<n} \mathbf{a}[P_{\nu_*}(x_{\alpha_\ell})].$$

(Recall (d), and moreover $\mathbf{x}_{f^*_{\alpha_0}} \cap \dots \cap \mathbf{x}_{f^*_{\alpha_{n-1}}} > 0$ because $\bigcup_{\ell<n} f^*_{\alpha_\ell}$ is a function.)

Next, note that for every $\alpha \in \mathcal{U}_4$, $\mathrm{dom}(f^*_\alpha) \cap \{\omega\alpha + n\} = \emptyset$, by the remark after equation (d). It follows that by our definition of $\bar{\mathbf{c}}$, for any $\ell < m_n$ [recalling $\bar{m}$ from (a)] and any $\alpha \in \mathcal{U}_4$, we have that in $\mathfrak{B}_*$,

(f) $$\mathbf{x}_{f^*_\alpha} \cap \mathbf{a}[P_{\nu_* {}^\frown \langle \ell \rangle}(x_\alpha)] > 0$$

because the intersection on the left hand side is equal to $\mathbf{x}_{g_{\eta,\ell}}$ (really, equal unless $\ell = m_n - 1$, in which case the left hand side is $\geq \mathbf{x}_{g_{\eta,\ell}}$) where

(g) $$g_{\eta,\ell} = f^*_\alpha \cup \{(\omega\alpha + n, \ell)\}.$$

Recall that we chose n so that $\xi_{\mathbf{n}}(n) = 1$. Let $w \subseteq \mathcal{U}_4$ be such that $|w| = m_n$. By equation (e),

$$\mathbf{y}_0 := \bigcap_{\ell<|w|} \mathbf{x}_{f^*_{\alpha_\ell}} > 0$$

and also recall that for each $\alpha < |w|$,

$$\mathbf{y}_0 \leq \mathbf{a}[P_{\nu_*}(x_\alpha)].$$

Thus, if we enumerate w as $\alpha_0, \dots, \alpha_{|w|-1}$, then in $\mathfrak{B}_*$ (hence also in $\mathfrak{B}$)

(h) $$\mathbf{y}_1 = \bigcap_{\ell<|w|} \mathbf{x}_{f^*_{\alpha_\ell}} \ \cap \ \bigcap_{\ell<|w|} \mathbf{a}[P_{\nu_*}(x_{\alpha_\ell})] \ \cap \ \bigcap_{\ell<|w|} \mathbf{a}[P_{\nu_* {}^\frown \langle \ell \rangle}(x_{\alpha_\ell})] \ > 0.$$

Now we use the choice of the quadruple $(j, \mathcal{U}_0, A, \bar{\mathbf{a}}^1)$ for our given w.

In particular, we apply clause (3)(f) of 8.12 using $n = n$, $i = 1$, $u = w$, and $\mathbf{a} = \mathbf{y}_1 \in \mathfrak{B}_*$. Then indeed

$$m_n/(m^\circ_n) < |w| \leq m_n$$

and (h) translates to tell us that $0 < \mathbf{y}_1 \leq \bigcap_{\alpha \in w} \mathbf{a}^1_\alpha$ in $\mathfrak{B}_*$. So (i) of 8.12(f) holds, and by (ii) of that clause there is v such that $v \subseteq w$ and $|v| \geq |w|/(m^\circ_n)^{n^{1+j}}$ and

$$\mathfrak{B} \models \text{“} \bigcap_{\alpha \in v} \mathbf{a}^2_\alpha \cap \mathbf{y}_1 > 0.\text{”}$$

Let us name this intersection:

$$\mathfrak{B} \models \text{“}\mathbf{y}_2 := \bigcap_{\alpha \in v} \mathbf{a}^2_\alpha \cap \mathbf{y}_1 > 0\text{”}.$$

Recall that $\bar{\mathbf{a}}^2$ is a solution to $\bar{\mathbf{c}}$, so by our definition of multiplicative refinement

(i) $$\mathfrak{B} \models \bigcap_{\alpha \in v} \mathbf{a}^2_\alpha \leq \mathbf{c}_v \quad \text{which tells us that} \quad \mathbf{y}_2 \leq \mathbf{c}_v.$$

However, since $\xi_\mathfrak{n}(n) = 1$, the definition of a type in $T_\mathfrak{n}$ doesn't allow "large" splitting at n, so necessarily in $\mathfrak{B}$

(j) $$\mathbf{c}_v \cap \left(\bigcap_{\ell < |w|} \mathbf{a}[P_{\nu_*}(x_{\alpha_\ell})] \cap \bigcap_{\ell < |w|} \mathbf{a}[P_{\nu_* {}^\frown \langle \ell \rangle}(x_{\alpha_\ell})] \right) = 0.$$

Together (h), (i) and (j) are a contradiction. This shows that $\bar{\mathbf{c}}$ has no multiplicative refinement. $\square$

Remark 9.6. *Why in the present proof do we not use the weaker choice of the pattern transfer condition 8.20 above? We will see in the next section, and in the preservation in the inductive construction.*

Discussion 9.7. *We could apply the first part of the proof to any $\mathfrak{m} \in \mathbf{M}_*$ and get a corresponding $\bar{\mathbf{a}}^1$, but when we fix $\mathbf{M}_*$, $\mathcal{M}, \mathcal{N}$ the cases in $\mathbf{M}_* \setminus \mathcal{N}$ are not useful; we needed the active level for $\mathfrak{n}$ to get a contradiction. In particular for them we will not necessarily have the failure of saturation, even though we can define the possibility pattern.*

In light of this lemma, our strategy in the inductive construction below will be to show that we can inductively extend our given Boolean algebra, say $\mathfrak{B}_\alpha$, by adding formal solutions to some problems (say, problems from theories $T_\mathfrak{m}$ which generate the given $\mathcal{I}$) to obtain $\mathfrak{B}_{\alpha+1}$, while maintaining "$(\mathfrak{B}_*, \mathfrak{B}_{\alpha+1})$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property" to preserve non-saturation for theories $T_\mathfrak{n}$ "orthogonal" to the given $\mathcal{I}$. By transitivity and 8.18, generally it will be enough to ensure this for $(\mathfrak{B}_\alpha, \mathfrak{B}_{\alpha+1})$.

10. The inductive construction

Convention 10.1. *For this section we fix:*

1. *$\bar{m}$, $\bar{E}$, Ξ satisfying the hypotheses of 6.21,*
2. *a set $\mathbf{M}_* = \{\mathfrak{m}_\alpha = \mathrm{par}[\bar{m}, \bar{E}, \xi_\alpha] : \alpha < 2^{\aleph_0}\}$ of parameters as in 6.21,*
3. *$\mathcal{M}, \mathcal{N}$ two nonempty disjoint subsets of $\mathbf{M}_*$.*
4. *$\aleph_0 \leq \mu < \kappa \leq \lambda$, and κ is regular (and uncountable).*
5. *$\mathcal{I}_\mathcal{M}$ the ideal corresponding to $\mathcal{M}$.*
6. *$\mathfrak{B}_* = \mathfrak{B}^1_{\kappa,\mu,\aleph_0}$.*
7. *$\mathcal{D}_*$ an arbitrary but fixed nonprincipal ultrafilter on $\mathfrak{B}_*$.*

Convention 10.2. *For this section, let* $\mathfrak{a}_*$ *denote* $(\mathfrak{B}_*, \mathcal{D}_*)$ *from* 10.1.

Definition 10.3. *Let* AP_0 *be the class of objects* $\mathfrak{a} = (\mathfrak{B}_\mathfrak{a}, \mathcal{D}_\mathfrak{a})$ *where:*

(1) $\mathfrak{B}_\mathfrak{a}$ *is a complete Boolean algebra and* $\mathfrak{B}_* \lessdot \mathfrak{B}_\mathfrak{a}$.
(2) $\mathcal{D}_\mathfrak{a}$ *is an ultrafilter on* $\mathfrak{B}_\mathfrak{a}$ *extending* $\mathcal{D}_*$.

Definition 10.4. *Let* AP *be the class of objects* $\mathfrak{a} = (\mathfrak{B}_\mathfrak{a}, \mathcal{D}_\mathfrak{a}) \in \mathrm{AP}_0$ *such that in addition* $\mathfrak{B}_\mathfrak{a}$ *satisfies the* $(\kappa, \mathcal{I}, \bar{m})$*-c.c.*

Remark 10.5. *Recalling* $\mathfrak{a}_*$ *from* 10.2*, we have that by definition "*$\mathfrak{a}_* \in \mathrm{AP}_0$*".*

Definition 10.6.

(1) *We define a partial order on the elements of* AP_0*:*
$$\mathfrak{a} \leq_{\mathrm{AP}_0} \mathfrak{b}$$
when $\mathfrak{B}_\mathfrak{a} \lessdot \mathfrak{B}_\mathfrak{b}$ *and* $\mathcal{D}_\mathfrak{a} \subseteq \mathcal{D}_\mathfrak{b}$.
(2) $\leq_{\mathrm{AP}}$ *is the following partial order on* AP*:*
$$\mathfrak{a} \leq_{\mathrm{AP}} \mathfrak{b}$$
if and only if:
  (a) $\mathfrak{a} \leq_{\mathrm{AP}_0} \mathfrak{b}$
  (b) $\mathfrak{a}, \mathfrak{b} \in \mathrm{AP}$
  (c) *the pair* $(\mathfrak{B}_\mathfrak{a}, \mathfrak{B}_\mathfrak{b})$ *satisfies the* $(\kappa, \mathcal{I}, \bar{m})$*-pattern transfer property.*

Discussion 10.7. This is a partial order by 8.14 (transitivity for pattern transfer). Recall that to show $\mathfrak{a} \leq_{\mathrm{AP}} \mathfrak{b}$, by 8.15 it suffices to verify that $\mathfrak{a}$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. (i.e., $\mathfrak{a} \in \mathrm{AP}$) and $(\mathfrak{B}_\mathfrak{a}, \mathfrak{B}_\mathfrak{b})$ has the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property.

Definition 10.8. *Call* $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\gamma : \gamma < \gamma_* \rangle$ *a* general construction sequence *when:*

(A) $\mathfrak{b}_0 = \mathfrak{a}_*$*, so* $\mathfrak{b}_0 \in \mathrm{AP}$.
(B) *for* $\gamma < \gamma_*$*,* $\mathfrak{b}_\gamma \in \mathrm{AP}_0$.
(C) *for* $\gamma = \beta + 1 \leq \gamma_*$*,* $\mathbf{b}_\gamma \leq_{\mathrm{AP}_0} \mathbf{b}_{\gamma+1}$.
(D) *for* γ *a nonzero limit ordinal,* $\bigcup_{\beta<\gamma} \mathfrak{B}_{\mathfrak{b}_\beta}$ *is a dense subset of* $\mathfrak{B}_{\mathfrak{b}_\gamma}$ *and* $\mathcal{D}_{\mathfrak{b}_\gamma}$ *is an ultrafilter on* $\mathfrak{B}_{\mathfrak{b}_\gamma}$ *which includes* $\bigcup_{\beta<\gamma} \mathcal{D}_{\mathfrak{b}_\beta}$.

We say the length of $\bar{\mathfrak{b}}$ *is* γ_*.

This definition is justified by:

Claim 10.9. *Suppose* $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\gamma : \gamma \leq \gamma_* \rangle$ *satisfies* $10.8(A) + (C) + (D)$*. Then for each* $\gamma \leq \gamma_*$*, the ultrafilter* $\mathcal{D}_{\mathfrak{b}_\gamma}$ *exists, and for each* $\beta < \gamma \leq \gamma_*$*,* $\mathfrak{B}_{\mathfrak{b}_\beta} \lessdot \mathfrak{B}_{\mathfrak{b}_\gamma}$*. In particular, each* $\mathfrak{b}_\gamma \in \mathrm{AP}_0$*, and for each* $\beta < \gamma \leq \gamma_*$*,*
$$\mathfrak{b}_\beta \leq_{\mathrm{AP}_0} \mathfrak{b}_\gamma.$$

Proof. In other words, from the definition $10.8(A) + (C) + (D)$ we may prove, by induction on $\gamma \leq \gamma_*$, that each $\mathfrak{b}_\gamma \in \mathrm{AP}_0$ and that $\delta < \gamma$ implies $\mathfrak{b}_\delta \leq_{\mathrm{AP}_0} \mathfrak{b}_\gamma$.

For the base case, we know $\mathfrak{b}_0 \in \mathrm{AP}_0$, indeed $\mathfrak{b}_0 \in AP$. For the successor case, apply 10.8(C). Suppose we are at a limit ordinal.

For the ultrafilter: it suffices to check that for limit γ, $\bigcup_{\beta<\gamma} \mathcal{D}_{\mathfrak{b}_\beta}$ has the finite intersection property, which follows from the fact that each $\mathcal{D}_{\mathfrak{b}_\beta}$ is itself a filter.

For the Boolean algebras: suppose $\beta < \gamma \leq \gamma_*$ and $\gamma = \beta + \alpha$ and argue by induction on α. If $\alpha = 0$ this is immediate, if α is a successor also clear. Suppose α is a limit and let $X \subseteq \mathfrak{B}_\beta$. Let $\mathbf{a}_\beta$ be the supremum of X in $\mathfrak{B}_{\mathfrak{b}_\beta}$ and let $\mathbf{a}_\gamma$ be the supremum of X in $\mathfrak{B}_{\mathfrak{b}_\gamma}$. Suppose for a contradiction that in $\mathfrak{B}_{\mathfrak{b}_\gamma}$, $\mathbf{a}_\beta \setminus \mathbf{a}_\gamma = \mathbf{c} > 0$. By definition of general construction sequence, $\bigcup_{\beta<\gamma} \mathfrak{B}_{\mathfrak{b}_\beta}$ is dense in $\mathfrak{B}_{\mathfrak{b}_\gamma}$, so there is $\delta < \gamma$ and $\mathbf{c}_\delta \in \mathfrak{B}^+_{\mathfrak{b}_\delta}$ such that $\mathbf{c}_\delta < \mathbf{c}$. Then in $\mathfrak{B}_{\mathfrak{b}_\delta}$, $(\mathbf{a}_\beta \setminus \mathbf{c}_\delta) \geq \mathbf{x}$ for all $\mathbf{x} \in X$, contradicting the inductive hypothesis. □

Corollary 10.10. *Suppose $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\gamma : \gamma \leq \gamma_* \rangle$ is a general construction sequence. Suppose that for every $\beta < \gamma_*$, the pair $(\mathfrak{B}_{\mathfrak{a}_\beta}, \mathfrak{B}_{\mathfrak{a}_{\beta+1}})$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property. Then each $\mathfrak{b}_\gamma \in \mathrm{AP}$, and indeed for every $\beta < \delta \leq \delta_*$,*

$$\mathfrak{b}_\beta \leq_{\mathrm{AP}} \mathfrak{b}_\delta.$$

Proof. By Claim 8.5, Lemma 8.19, and Claim 10.9. □

Our next definition expresses that we extend $(\mathfrak{B}_\mathfrak{a}, \mathcal{D}_\mathfrak{a})$ in a certain minimal way: by simply adding a formal solution to some possibility pattern $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0} \rangle$ for some theory. For now, the definition is general, allowing the size θ and the theory to vary. We could think about such extensions as simply ensuring an instance of goodness, adding some multiplicative refinement to some monotonic function. The crucial point is that we do this as freely as possible, essentially only requring that the equations in $\mathfrak{B}_\mathfrak{a}$ are still respected, and the new addition $\langle \mathbf{b}^1_\alpha : \alpha < \theta \rangle$ is a formal solution to $\bar{\mathbf{b}}$, i.e., for each $u \in [\theta]^{<\aleph_0}$, $\bigcap_{\alpha \in u} \mathbf{b}^1_\alpha \leq \mathbf{b}_u$.

Recall that a solution is multiplicative, so it suffices to specify $\mathbf{b}^1_u$ for $|u| = 1$, and for clarity we usually drop parentheses, writing $\mathbf{b}^1_\alpha$ instead of $\mathbf{b}^1_{\{\alpha\}}$. As the $\mathbf{b}^1_\alpha$'s generate the multiplicative sequence $\langle \mathbf{b}^1_u : u \in [\theta]^{<\aleph_0} \rangle$, we could also express the new constraint by saying: $\mathbf{b}^1_u \leq \mathbf{b}_u$ for each finite u. Notice that by 10.11(3), we will need to check existence.

Definition 10.11. *Suppose $\mathfrak{a} \in \mathrm{AP}_0$, T is a complete first-order theory, and $\theta \leq \lambda$ is an infinite cardinal. Say that $\mathfrak{b} = (\mathfrak{B}_\mathfrak{b}, \mathcal{D}_\mathfrak{b})$ is a (θ, T)*-extension of *$\mathfrak{a}$ when there exists a possibility pattern $\bar{\mathbf{b}} = \{\mathbf{b}_u : u \in [\theta]^{<\aleph_0}\}$ and $\mathbf{a}[\psi[\bar{x}_u]] \in \mathfrak{B}_\mathfrak{a}$ for $\psi \in \mathcal{L}(\tau_T)$ in $(\mathfrak{B}_\mathfrak{a}, \mathcal{D}_\mathfrak{a})$ for the theory T and:*

(1) *$\mathfrak{B}_\mathfrak{b}$ is the completion of the Boolean algebra $\mathfrak{B}$ generated by the set $\mathcal{Y}_{\mathfrak{a},\mathbf{b}} = \mathfrak{B}_\mathfrak{a}$ along with the set of new elements $\{\mathbf{b}^1_\alpha : \alpha < \theta\}$, freely except for the set of equations $\Gamma_{\mathfrak{a},\mathbf{b}}$ which are:*[14]
   (a) *the equations already in $\mathfrak{B}_\mathfrak{a}$.*
   (b) *for every nonempty finite $u \subseteq \theta$,*

$$\bigcap_{\alpha \in u} \mathbf{b}^1_\alpha \leq \mathbf{b}_u.$$

(2) *Notation: let $\mathbf{b}^1_u := \bigcap_{\alpha \in u} \mathbf{b}^1_\alpha$, (convention: $\mathbf{b}_\emptyset = 1_\mathfrak{B}$). Let $\mathbf{b}^2_u = \bigcup_{\alpha \in u} \mathbf{b}^1_\alpha$.*
(3) *$\mathcal{D}_\mathfrak{b}$ is an ultrafilter on $\mathfrak{B}_\mathfrak{b}$ which agrees with $\mathcal{D}_\mathfrak{a}$ on $\mathfrak{B}_\mathfrak{a}$, and such that $\mathbf{b}^1_\alpha \in \mathcal{D}_\mathfrak{b}$ for all $\alpha < \theta$,* if *such an ultrafilter exists; otherwise not defined.*

We may say $\mathfrak{b}$ is an $(\theta, T, \bar{\mathbf{b}})$-extension of $\mathfrak{a}$ to emphasize that $\bar{\mathbf{b}}$ is the possibility pattern acquiring a solution.

We record the following here though it refers to upcoming proofs:

Observation 10.12. *If $\mathfrak{b} \in \mathrm{AP}$ is a $(\theta, T, \bar{\mathbf{b}})$-extension of $\mathfrak{a} \in \mathrm{AP}$, and:*

(1) *$\theta < \kappa$, or*

[14] i.e. freely except for the equations already in $\mathfrak{B}_\mathfrak{a}$ and the new rules stating that $\bar{\mathbf{b}}^1$ is a formal solution to $\bar{\mathbf{b}}$.

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(2) $T = T_{\mathfrak{m}}$ for $\mathfrak{m} \in \mathcal{M}$ and $\bar{\mathbf{b}}$ is a possibility pattern coming from a positive R-type, or

(3) $T = T_{\mathbf{rg}}$ and $\bar{\mathbf{b}}$ is a possibility pattern coming from a type in positive and negative instances of R,

then

$$|\mathfrak{B}_{\mathfrak{b}}| \leq (|\mathfrak{B}_{\mathfrak{a}}| + \lambda)^{\kappa}.$$

Proof. By the κ-c.c. which follows from "$\mathfrak{a}, \mathfrak{b} \in \mathrm{AP}$", and which is proved in Claim 10.18 for (1), Theorem 10.20 for (2) and Lemma 10.21 for (3). Alternately, we could use $\lambda \geq \theta, \kappa$ and conclude $|\mathfrak{B}_{\mathfrak{b}}| \leq (|\mathfrak{B}_{\mathfrak{a}}| + \lambda)^{\lambda}$, which suffices for our purposes. □

Remark 10.13. *At this point we put no restrictions on the theory; only in the actual construction do we use $T_{\mathfrak{n}}$, $T_{\mathfrak{m}}$ and $T_{\mathbf{rg}}$.*

Claim 10.14. *Suppose $\mathfrak{b}$ is a $(\theta, T, \bar{\mathbf{b}})$-extension of $\mathfrak{a} \in \mathrm{AP}$, for some theory T and some $\theta \leq \lambda$. Then:*

(1) $\mathfrak{B}_{\mathfrak{a}} \subseteq \mathfrak{B}_{\mathfrak{b}}$,

(2) indeed, $\mathfrak{B}_{\mathfrak{a}} \lessdot \mathfrak{B}_{\mathfrak{b}}$.

(3) there exists an ultrafilter $\mathcal{D}$ on $\mathfrak{B}_{\mathfrak{b}}$ which agrees with $\mathcal{D}_{\mathfrak{a}}$ on $\mathfrak{B}_{\mathfrak{a}}$ and contains $\mathbf{b}^1_\alpha$ for all $\alpha < \theta$, hence $\mathcal{D}_{\mathfrak{b}}$ is such an ultrafilter.

Proof. Recall from 10.11 that $\mathfrak{B}_{\mathfrak{b}}$ is the completion of the Boolean algebra generated by $\mathcal{Y} = \mathfrak{B}_{\mathfrak{a}} \cup \{\mathbf{b}^1_\alpha : \alpha < \theta\}$ freely except for the set of equations Γ which include all equations already in $\mathfrak{B}_{\mathfrak{a}}$ along with the conditions that for each finite $u \in [\theta]^{<\aleph_0}$, $\bigcap_{\alpha \in u} \mathfrak{B}^1_\alpha \leq \mathbf{b}_u$.

For each $u \in [\theta]^{<\aleph_0}$, define $h_u : \mathcal{Y} \to \mathfrak{B}_{\mathfrak{a}}$ as follows: $h_u \restriction \mathfrak{B}_{\mathfrak{a}}$ is the identity, $h_u(\mathbf{b}^1_\alpha) = \mathbf{b}_u$ if $\alpha \in u$, and $h_u(\mathbf{b}^1_\alpha) = 0_{\mathfrak{B}_{\mathfrak{b}}}$ if $\alpha \in \theta \setminus u$. Note that h_u respects the equations in Γ.

To see that $\mathfrak{B}_{\mathfrak{a}} \subseteq \mathfrak{B}_{\mathfrak{b}}$, note that in the case $u = \emptyset$ (as the generators are dense in the completion) the map $h_\emptyset$ induces an endomorphism $\hat{h}_u$ from $\mathfrak{B}_{\mathfrak{b}}$ onto $\mathfrak{B}_{\mathfrak{a}}$ which extends the identity map on $\mathfrak{B}_{\mathfrak{a}}$. This proves (a).

Next we work towards showing that $\mathfrak{B}_{\mathfrak{a}}$ is a complete subalgebra of $\mathfrak{B}_{\mathfrak{b}}$. Note that $\mathfrak{B}_{\mathfrak{b}}$ is by definition a complete Boolean algebra. As the generators are dense in the completion, for any $\mathbf{c} \in \mathfrak{B}^+_{\mathfrak{b}}$, there are

(a) $$\mathbf{a} \in \mathfrak{B}^+_{\mathfrak{a}},\ u, u_0, \dots, u_{n-1} \in [\theta]^{<\aleph_0}$$

such that $u_\ell \cap u = \emptyset$ for $\ell < n$ and

$$\mathfrak{B}_{\mathfrak{b}} \models 0 < \mathbf{a} \cap \mathbf{b}^1_u \cap \bigcap_{\ell < n} (-\mathbf{b}^1_{u_\ell}) \leq \mathbf{c}.$$

[Note that $\{\mathbf{b}^1_\alpha : \alpha < \theta\}$ generates a multiplicative sequence: $\mathbf{b}^1_u = \bigcap_{\alpha \in u} \mathbf{b}^1_\alpha$ for any finite $u \subseteq \theta$. So the positive intersection in the inset equation is without loss of generality given by a single u. Without loss of generality we could take the u_ℓ's to be singletons, but note that $-\mathbf{b}^1_{u_1} \cap \mathbf{b}^1_{u_2} \neq -\mathbf{b}^1_{u_1 \cup u_2}$.]

Thus, for $\mathbf{c}, \mathbf{a}, u, u_0, \dots, u_{n-1}$ as in the previous paragraph, the map $\hat{h}_u$ is constant on $\mathfrak{B}_{\mathfrak{a}}$, takes $\mathbf{b}^1_\alpha$ to $\mathbf{b}_\alpha$ for $\alpha \in u$, and $\mathbf{b}^1_\alpha$ to $0_{\mathfrak{B}_{\mathfrak{a}}}$ for $\alpha \in \theta \setminus u$, hence takes $\mathbf{b}^1_u$ to $\mathbf{b}_u$, and each $\mathbf{b}^1_{u_\ell}$ to $0_{\mathfrak{B}_{\mathfrak{a}}}$ (for $\ell < n$).

It follows that for any $\mathbf{c} \in \mathfrak{B}^+_{\mathfrak{b}}$, if $\mathfrak{B}_{\mathfrak{a}} \models 0 < \mathbf{d} \leq \mathbf{a} \cap \mathbf{b}_u$ then $\mathfrak{B}_{\mathfrak{b}} \models 0 < \mathbf{d} \cap \mathbf{c}$. Thus, any maximal antichain of $\mathfrak{B}_{\mathfrak{a}}$ will remain a maximal antichain of $\mathfrak{B}_{\mathfrak{b}}$ (if not, there is some nonzero $\mathbf{c} \in \mathfrak{B}^+_{\mathfrak{b}}$ which does not have nonempty intersection with any element of the antichain; but its associated $\mathbf{a} \cap \mathbf{b}_u$ must, contradiction). This completes the proof of (b).

Finally, to verify (c), it suffices to show that $\mathcal{D}_{\mathfrak{a}} \cup \{\mathbf{b}^1_\alpha : \alpha < \theta\}$ has the finite intersection property in $\mathfrak{B}_{\mathfrak{b}}$, as then it can be extended to an ultrafilter. This follows from the existence of the $\hat{h}_u$'s. (Suppose that for some finite u and some set $\mathbf{a} \in \mathcal{D}_{\mathfrak{a}}$, $\mathbf{a} \cap \bigcap\{\mathbf{b}^1_\alpha : \alpha \in u\} = 0_{\mathfrak{B}_{\mathfrak{b}}}$. Then $\hat{h}_u(\mathbf{b}^1_\alpha) = \mathbf{b}_u$ for each $\alpha \in u$. Recall that $\langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0}\rangle$ was a possibility pattern for $(\mathfrak{B}_{\mathfrak{a}}, \mathcal{D}_{\mathfrak{a}})$, thus a sequence of elements of $\mathcal{D}_{\mathfrak{a}}$; in particular, $\mathbf{b}_u \in \mathcal{D}_{\mathfrak{a}}$, so $\mathfrak{B}_{\mathfrak{a}} \models \mathbf{b}_u \cap \mathbf{a} > 0$, contradiction.) $\square$

We record a simple variant which may be useful for future proofs:

Observation 10.15. *Suppose that $\mathbf{a} \in \mathrm{AP}$ but instead of taking a (θ, T)-extension of $\mathfrak{B}_{\mathfrak{a}}$, we consider $\mathfrak{B}$ which is generated from $\mathfrak{B}_{\mathfrak{a}}$ along with up to θ new antichains each of cardinality $< \kappa$, as freely as possible: that is, for some $h \in {}^\theta\kappa$, $\mathfrak{B}_{\mathfrak{a}} \cup \{\mathbf{c}_{\alpha,\epsilon} : \epsilon < h(\alpha), \alpha < \theta\}$ freely except for the equations already in $\mathfrak{B}_{\mathfrak{a}}$ and the equations saying that for each α, $\{\mathbf{c}_{\alpha,\epsilon} : \epsilon < h(\alpha)\}$ is an antichain. Let $\mathfrak{B}_{\mathfrak{b}}$ be the completion of $\mathfrak{B}$. Then the proof that*

1. *$\mathfrak{B}_{\mathfrak{a}} \subseteq \mathfrak{B}_{\mathfrak{b}}$*
2. *indeed $\mathfrak{B}_{\mathfrak{a}} \lessdot \mathfrak{B}_{\mathfrak{b}}$*
3. *$\mathfrak{B}_{\mathfrak{b}}$ has the κ-c.c.*

is easier than in 10.14, and just as in 10.16, we may conclude $|\mathfrak{B}_{\mathfrak{b}}| \leq (|\mathfrak{B}_{\mathfrak{a}}| + \mu)^{<\kappa}$.

Claim 10.16. *Suppose $\langle \mathfrak{b}_\gamma : \gamma \leq \gamma_* \rangle$ is a general construction sequence.*

1. *If each $\mathfrak{B}_{\mathfrak{b}_\gamma}$ satisfies the κ-c.c., then $|\mathfrak{B}_{\mathfrak{b}_\gamma}| \leq (|\gamma| + 2)^{<\kappa}$.*
2. *If each $\mathfrak{B}_{\mathfrak{b}_\gamma}$ satisfies the λ^+-c.c., then $|\mathfrak{B}_{\mathfrak{b}_{\gamma|}}| \leq (|\gamma| + 2)^{\lambda}$.*

Proof. (1) By induction. For the cardinality inequality, use the κ-c.c., $\kappa \geq \mu^+$.

(2) Apply part (1) with λ^+ playing the role of κ. $\square$

Keeping in mind 10.10, our main task now will be to show that we can preserve the pattern transfer property at successor stages realizing certain specific types for certain specific theories. We will make repeated use of the move in the proof of 10.14, equation (a), giving a useful "normal form" for elements, so we start by summarizing it here.

Note in the next observation that we bound θ by λ, and in particular, there is no problem for θ to be larger than κ (recalling the remark before 1.10).

Observation 10.17. *Suppose $\mathfrak{a} \in \mathrm{AP}$ and $\mathfrak{b}$ is a $(\theta, T, \bar{\mathbf{b}})$-extension of $\mathfrak{a}$ for some $\theta \leq \lambda$. Suppose we are given a sequence $\langle \mathbf{a}^2_\alpha : \alpha < \kappa \rangle$ of elements of $\mathfrak{B}^+_{\mathfrak{b}}$. Then:*

(a) *for each $\alpha < \kappa$, there are $\mathbf{x}_\alpha \in \mathfrak{B}^+_{\mathfrak{a}}$, $u_\alpha, n_\alpha, u_{\alpha,0}, \dots, u_{\alpha,n_\alpha - 1} \in [\theta]^{<\aleph_0}$ such that $u_{\alpha,\ell} \cap u_\alpha = \emptyset$ for $\ell < n$, without loss of generality $\mathbf{x}_\alpha \leq \mathbf{b}_{u_\alpha}$ in $\mathfrak{B}_{\mathfrak{a}}$, and*
$$\mathfrak{B}_{\mathfrak{b}} \models 0 < \mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha} \cap \bigcap_{\ell < n_\alpha} (-\mathbf{b}^1_{u_{\alpha,\ell}}) \leq \mathbf{a}^2_\alpha.$$

(b) *Let $\mathbf{i}_\alpha = (\mathbf{x}_\alpha, u_\alpha, \langle u_{\alpha,\ell} : \ell < n_\alpha \rangle)$ list this information from (a) for each $\alpha < \theta$, and let $w_\alpha = u_\alpha \cup \bigcup_{\ell < n_\alpha} u_{\alpha,\ell}$. By the Δ-system lemma there is $\mathcal{U} \in [\kappa]^\kappa$ such that $\langle w_\alpha : \alpha \in \mathcal{U} \rangle$ is a Δ-system with heart w_*. Without loss of generality for some u_*, n, $\langle u^*_\ell : \ell < n \rangle$, for every $\alpha \in \mathcal{U}$, we have that $n_\alpha = n$, $u_\alpha \cap w_* = u_*$, $u_{\alpha,\ell} \cap w_* = u^*_\ell$.*

(c) *For every $\alpha \in \mathcal{U}$, for $\mathbf{x}_\alpha$ as in part (a), $\mathbf{x}_\alpha \leq_{\mathrm{proj}} \mathbf{a}^2_\alpha$.*

(d) *Suppose $\mathcal{U}$ is from (b) and $X \subseteq \mathcal{U}$ is finite and*

$$\mathfrak{B}_{\mathfrak{b}} \models \bigcap_{\alpha \in X} (\mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha}) > 0.$$

Then also

$$\mathfrak{B}_{\mathfrak{b}} \models \bigcap_{\alpha \in X} \left(\mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha} \cap \bigcap_{\ell < n_\alpha} (-\mathbf{b}^1_{u_{\alpha,\ell}}) \right) > 0$$

(*i.e., checking for positive intersections we may safely ignore complements*).

Proof. For part (a), the generators are dense in the completion, and as mentioned in the proof of 10.14, we can gather the intersection of elements of the form $\mathbf{b}^1_\beta$ into a single $\mathbf{b}^1_{u_\alpha}$. Since $\mathbf{x}_\alpha \cap \mathbf{b}_u > 0$ in $\mathfrak{B}_{\mathfrak{a}}$, there is no harm in assuming $\mathbf{x}_\alpha \leq \mathbf{b}_u$.

Part (b) is by the definition of Δ-system, recalling that κ is regular.

To verify $\mathbf{x}_\alpha \leq_{\text{proj}} \mathbf{a}^2_\alpha$ for (c), it suffices to show that there is an endomorphism f from $\mathfrak{B}_{\mathfrak{b}}$ onto $\mathfrak{B}_{\mathfrak{a}}$ such that $f(\mathbf{a}^2_\alpha) \geq \mathbf{x}_\alpha$. Consider $\hat{h}_{u_\alpha} : \mathfrak{B}_{\mathfrak{b}} \to \mathfrak{B}_{\mathfrak{a}}$ defined in the proof of 10.14. Then $\hat{h}_{u_\alpha}(\mathbf{x}_\alpha) = \mathbf{x}_\alpha$, $\hat{h}_{u_\alpha}(\mathbf{b}^1_{u_\alpha}) = \mathbf{b}_{u_\alpha}$, and by the disjointness conditions of the Δ-system, $\hat{h}_{u_\alpha}(\mathbf{b}^1_{u_{\alpha,\ell}}) = 0_{\mathfrak{B}_{\mathfrak{a}}}$ for all $\ell < n$, so $\hat{h}_{u_\alpha}(\mathbf{a}^2_\alpha) \geq \mathbf{x}_\alpha$.

Part (d) is similar. Let $\mathbf{y} = \bigcap_{\alpha \in x} (\mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha})$. It suffices to show there is an endomorphism f from $\mathfrak{B}_{\mathfrak{b}}$ onto $\mathfrak{B}_{\mathfrak{a}}$ such that $f(\mathbf{y}) > 0$ but $f(\mathbf{b}^1_{u_{\alpha,\ell}}) = 0_{\mathfrak{B}_{\mathfrak{a}}}$ for all $\alpha \in X, \ell < n$. Let $u = \bigcup_{\alpha \in X} u_\alpha$ and consider $\hat{h}_u : \mathfrak{B}_{\mathfrak{b}} \to \mathfrak{B}_{\mathfrak{a}}$. Then $\hat{h}_u(\mathbf{x}_\alpha) = \mathbf{x}_\alpha$ for $\alpha \in X$, and $\hat{h}_u(\mathbf{b}^1_\beta) = \mathbf{b}_u$ for all $\alpha \in X$ and $\beta \in u_\alpha$, so $\hat{h}_u(\mathbf{b}^1_{u_\alpha}) = \mathbf{b}_u$ for $\alpha \in X$, thus $\hat{h}_u(\mathbf{y}) = \mathbf{b}_u > 0$. The effect of the Δ-system ensures that $u_{\alpha,\ell} \cap u = \emptyset$ for $\alpha \in X, \ell < n$, so $\hat{h}_u(\mathbf{b}^1_\beta) = 0_{\mathfrak{B}_{\mathfrak{a}}}$ for $\beta \in u_{\alpha,\ell}$, $\alpha \in X, \ell < n$, as desired. $\square$

Our next claim says essentially that if $\theta < \kappa$ then there is no problem realizing any (T, θ)-type for any T. Thus, we can arrange for our final ultrafilters to be κ-good, even though they will be far from κ^+-good.

Claim 10.18. *Assume $\mathfrak{a} \in \mathrm{AP}$, $\mathfrak{b}$ is a $(\theta, T, \bar{\mathbf{b}})$-extension of $\mathfrak{a}$, where $\theta < \kappa$. Then $(\mathfrak{B}_{\mathfrak{a}}, \mathfrak{B}_{\mathfrak{b}})$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property.*

Remark 10.19. *Note that in 10.18, $\theta < \kappa$. The proof will also show: it suffices to suppose that $\kappa \leq \theta \leq \lambda$ but for some $\mu < \kappa$ we have $(\forall u \in [\theta]^{<\aleph_0})[\ \mathfrak{B}_{\mathfrak{a}} \models \mathbf{b}_u = \bigcap \{\mathbf{b}_{\{\alpha\}} : \alpha \in (u \setminus \mu)\} \cap \mathbf{b}_{u \cap \mu}\ ]$.*

Proof of 10.18. This proof and the proof of 10.20 share a picture, so even though the current proof is quite simple, we go slowly to motivate here the second proof and to emphasize the one point where they are different.

Let $\langle \mathbf{a}^2_\alpha : \alpha < \kappa \rangle$ be given, with each $\mathbf{a}^2_\alpha \in \mathfrak{B}^+_{\mathfrak{b}}$. By 10.17(a) there are $\mathbf{x}_\alpha \in \mathfrak{B}^+_{\mathfrak{a}}$, $u_\alpha, n_\alpha, u_{\alpha,0}, \ldots, u_{\alpha,n_\alpha - 1} \in [\theta]^{<\aleph_0}$ such that $u_{\alpha,\ell} \cap u_\alpha = \emptyset$ for $\ell < n$ and

$$\mathfrak{B}_{\mathfrak{b}} \models 0 < \mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha} \cap \bigcap_{\ell < n_\alpha} (-\mathbf{b}^1_{u_{\alpha,\ell}}) \leq \mathbf{a}^2_\alpha.$$

By 10.17(b) there is $\mathcal{U} \in [\kappa]^\kappa$ such that $\langle w_\alpha : \alpha \in \mathcal{U} \rangle$ is a Δ-system with heart w_*, so for some u_*, n, $\langle u^*_\ell : \ell < n \rangle$, for every $\alpha \in \mathcal{U}$: $n_\alpha = n$, $u_\alpha \cap w_* = u_*$, $u_{\alpha,\ell} \cap w_* = u^*_\ell$.

Since we have assumed $\theta < \kappa$, without loss of generality

(a) $$\langle (u_\alpha \cap \mu, \langle u_{\alpha,\ell} \cap \mu : \ell < n \rangle) : \alpha \in \mathcal{U} \rangle$$

is constant, and say equal to $(u_\oplus, \langle u_{\oplus,\ell} : \ell < n\rangle)$. What we have really found is $\mathcal{U} \in [\kappa]^\kappa$ on which the sequence $\langle (u_\alpha, \langle u_{\alpha,\ell} : \ell < n\rangle) : \alpha \in \mathcal{U}\rangle$ is constant. Let

(b) $$\mathbf{a}^1_\alpha = \mathbf{x}_\alpha \cap \mathbf{b}_{u_\oplus} = \mathbf{x}_\alpha \text{ for each } \alpha \in \mathcal{U}$$

recalling that each $\mathbf{x}_\alpha \leq \mathbf{b}_{u_\alpha}$. Then each $\mathbf{a}^1_\alpha \leq_{\text{proj}} \mathbf{a}^2_\alpha$ (by 10.17(c)).

Let us verify 8.12(3)(f) holds for $A = \omega$, $j = 0$. The statements about what will suffice remain true later, but in the present proof, we'll see $v = u$ already works.

Fix for awhile some $n \in \omega \setminus A$ and some finite u and nonzero $\mathbf{a}$ such that $m_n/(m^\circ_n)^{n^i} < |u| < m_n$ and

(c) $$\mathfrak{B}_\mathfrak{a} \models 0 < \mathbf{a} \leq \bigcap_{\alpha \in u} \mathbf{a}^1_\alpha.$$

To fulfill the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer, by 10.17(d) it is enough to show that for some $v \subseteq u$ of the correct size,

(d) $$\mathfrak{B}_\mathfrak{b} \models \bigcap_{\alpha \in v} (\mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha}) \cap \mathbf{a} > 0.$$

Recall that in 10.17(a) we ensured that $\mathbf{x}_\alpha \leq \mathbf{b}_{u_\alpha}$. Observe that for (d) would suffice to show that for some $w \subseteq u$ of the correct size, writing $W = \bigcup_{\alpha \in w} u_\alpha$,

(e) $$\bigcap_{\alpha \in w} \mathbf{x}_\alpha \leq \mathbf{b}_W$$

because if so, we would have that $\hat{h}_W$ from the proof of 10.14 is an endomorphism from $\mathfrak{B}_\mathfrak{b}$ onto $\mathfrak{B}_\mathfrak{a}$ which takes each $\mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha}$ to $\mathbf{x}_\alpha \cap \mathbf{b}_W$.

In our present case, the sequence $\langle u_\alpha : \alpha \in u\rangle$ is constant and equal to $u_\oplus$. So when $w = u$, $W = u_\oplus$ and (e) is immediately true by (b). This completes the proof of the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer.

To see that Remark 10.19 is sufficient, suppose we are given such a μ, and in equation (a) replace θ by μ and then let $\mathbf{a}^1_\alpha = \mathbf{x}_\alpha \cap \mathbf{b}_{u_\oplus} \cap \bigcap_{\alpha \in u_\alpha \setminus \mu} \mathbf{b}_\alpha$ for $\alpha \in \mathcal{U}_1$, and verify that $\langle \mathbf{a}^1_\alpha : \alpha \in \mathcal{U}\rangle$ is as required, for $j = 1$ (or 0). $\square$

Theorem 10.20. *Assume $\mathfrak{a} \in \mathrm{AP}$ and $\mathfrak{b}$ is a $(\theta, T, \bar{\mathbf{b}})$-extension of $\mathfrak{a}$ where $\theta \leq \lambda$, $T = T_\mathfrak{m}$ for some $\mathfrak{m} \in \mathcal{M}$, and $\bar{\mathbf{b}}$ is a possibility pattern arising from a a type of the form $\{Q_{\rho_*}(x)\} \wedge \{R(x, a_\alpha) : \alpha < \theta\}$ for some $\rho_* \in \mathcal{T}_1$ of finite height. Then the pair $(\mathfrak{B}_\mathfrak{a}, \mathfrak{B}_\mathfrak{b})$ satisfies the $(\kappa, \mathcal{I}, \bar{\mathfrak{m}})$-pattern transfer property.*

Proof. By hypothesis our $T_\mathfrak{m}$ is given by some

$$\mathfrak{m} = \mathfrak{m}(\xi, \bar{E}, \bar{m}) \in \mathcal{M}.$$

The picture of this proof is quite similar to 10.18, except that here we will have to show that we can handle the larger size θ by restricting to the more transparent case of R-types in our distinguished theory $T_\mathfrak{m}$.

Recall from 10.11 that in this set-up we will refer to the following objects: $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0}\rangle$, $\mathbf{b}_u \in \mathfrak{B}_\mathfrak{a}$, $\mathbf{a}[\psi[\bar{y}_{[u]}] \in \mathfrak{B}_\mathfrak{a}$ from the given problem, and $\mathbf{b}^1_\alpha, \mathbf{b}^1_u, \mathbf{b}^2_u$ in $\mathfrak{B}_\mathfrak{b}$ from the formal solution.

Let $\langle \mathbf{a}^2_\alpha : \alpha < \kappa\rangle$ be given, with each $\mathbf{a}^2_\alpha \in \mathfrak{B}^+_\mathfrak{b}$. By 10.17(a) there are $\mathbf{x}_\alpha \in \mathfrak{B}^+_\mathfrak{a}$, $u_\alpha, m_\alpha, u_{\alpha,0}, \dots, u_{\alpha,m_\alpha - 1} \in [\theta]^{<\aleph_0}$ such that $u_{\alpha,\ell} \not\subseteq u_\alpha$ for $\ell < m$ and

(a) $$\mathfrak{B}_\mathfrak{b} \models 0 < \mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha} \cap \bigcap_{\ell < m_\alpha} (-\mathbf{b}^1_{u_{\alpha,\ell}}) \leq \mathbf{a}^2_\alpha.$$

As $\mathbf{b}^1_{u_\alpha} \leq \mathbf{b}_\alpha$, $\mathbf{b}_\alpha \in \mathfrak{B}_{\mathfrak{a}}$, without loss of generality

(b) $$\mathbf{x}_\alpha \leq \mathbf{b}_{u_\alpha}$$

and let us define $w_\alpha = \bigcup_{\ell < m_\alpha} u_{\alpha,\ell} \cup u_\alpha$. Recall from 10.17(a) that for each α, $\mathbf{x}_\alpha \leq \mathbf{b}_{u_\alpha}$, so in particular, in $\mathfrak{B}_{\mathfrak{a}}$,

(c) $$\mathbf{x}_\alpha \leq \mathbf{a}[Q_{\rho_*}(x)].$$

A point which did not appear in 10.18: for every $\alpha < \kappa$ and quantifier-free atomic formula $\psi \in \{x_\alpha = x_\beta, \mathcal{P}(x_\alpha), \mathcal{Q}(x_\beta) : \alpha, \beta \in w_\alpha\}$, without loss of generality we have

(d) $$\mathbf{x}_\alpha \leq \mathbf{a}[\psi] \text{ or } \mathbf{x}_\alpha \cap \mathbf{a}[\psi] = \emptyset.$$

It follows easily that when $\beta \in u_\alpha$,

(e) $$\mathbf{x}_\alpha \leq \mathbf{a}[\mathcal{P}(x_\beta)].$$

Now by 10.17(b) there is $\mathcal{U} \in [\kappa]^\kappa$ such that $\langle w_\alpha : \alpha \in \mathcal{U} \rangle$ is a Δ-system with heart w_*, and for some u_*, m, $\langle u^*_\ell : \ell < m \rangle$, for every $\alpha \in \mathcal{U}$: $m_\alpha = m$, $u_\alpha \cap w_* = u_*$, $u_{\alpha,\ell} \cap w_* = u^*_\ell$. Let

(f) $$\mathbf{a}^1_\alpha = \mathbf{x}_\alpha \text{ for each } \alpha \in \mathcal{U}.$$

Then for each α we have $\mathbf{a}^1_\alpha \leq_{\mathrm{proj}} \mathbf{a}^2_\alpha$ (by 10.17(c)).

As $\mathfrak{a} \in \mathrm{AP}$, we know that the $(\kappa, \mathcal{I}, \bar{m})$-c.c. holds for $\mathfrak{B}_{\mathfrak{a}}$, that is, given the sequence of elements $\langle \mathbf{x}_\alpha : \alpha < \kappa \rangle$, we can find $j_1 < \omega$, $\mathcal{U}_1 \in [\kappa]^\kappa$, and $A_1 \in \mathcal{I}$ such that $\oplus$ of 8.2 holds.

Let us verify 8.12(3)(f) holds for this $\mathcal{U}_1$, $j = j_1 + 2$, $A = A_1 \cup \{\ell : \ell \leq j\} \cup \xi^{-1}\{1\}$ [recall ξ is the level function for T from the second line of the proof] and $\langle \mathbf{x}_\alpha : \alpha \in \mathcal{U}_1 \rangle$ [which we could have called $\langle \mathbf{a}^1_\alpha : \alpha < \kappa \rangle$, but we retain this notation for transparency].

Fix for awhile some n such that

(g) $$n \in \omega \setminus A \setminus m \setminus \lg n(\rho_*)$$

[recall m is the uniform size of u_α for $\alpha \in \mathcal{U}_1 \subseteq \mathcal{U}$] and some finite u' and $i \leq n - j$ and nonzero $\mathbf{a}$ such that[15]

(h) $$m_n/(m^\circ_n)^{n^i} < |u| < m_n \text{ and } \mathfrak{B}_{\mathfrak{a}} \models 0 < \mathbf{a} \leq \bigcap_{\alpha \in u} \mathbf{x}_\alpha.$$

Recall that, just as in the proof of 10.18, by 10.17(d) it is enough to show that for some $v \subseteq u$ with $|v| \geq |u'|/(m^\circ_n)^{n^{i+j}}$

(i) $$\mathfrak{B}_{\mathfrak{b}} \models \bigcap_{\alpha \in v} (\mathbf{x}_\alpha \cap \mathbf{b}^1_{u_\alpha}) \cap \mathbf{a} > 0.$$

Without loss of generality, really by equation (e), we can decrease $\mathbf{a}$ from (h), so that if $\gamma \in \bigcup\{u_\alpha : \alpha \in u\}$ then for some $\eta_\gamma \in \mathcal{T}_{1,n}$, $\mathbf{a} \leq \mathbf{a}[P_{\eta_\gamma}(x_\gamma)]$. [Informally, for each relevant formula $R(x, x_\gamma)$, we decide which branch x_γ belongs to at level n.] Let $\langle \gamma(\alpha, \ell) : \ell < m \rangle$ list u_α in order, necessarily without repetitions. As $|\mathcal{T}_{1,n}| = |\mathcal{T}_{2,n}| = m^\circ_n$, there are at most $(m^\circ_n)^m$ possible choices that a sequence of m elements may make regarding branch membership at level n: thus, there is a single sequence $\langle \nu_\ell : \ell < m \rangle$ of members of $\mathcal{T}_{2,n}$ such that the set

$$w' = \{\alpha \in u' : \text{ if } \ell < m \text{ then } \eta_{\gamma(\alpha,\ell)} = \nu_\ell\}$$

[15] here we use u', v', etc to not conflict with u, v, w used in the Δ-system earlier in the proof.

has $\geq |u'|/(m_n^\circ)^m \geq |u'|/(m_n^\circ)^n$ members. [Informally, for each $\alpha \in w'$, the sequence of branches to which the x_γ's belong as γ varies in u_α, is the same.] We compute:

$$|w'| \geq |u'|/(m_n^\circ)^n \geq m_n/(m_n^\circ)^{n^i \cdot n} = m_n/(m_n^\circ)^{n^{i+1}}$$

so w' is indeed of the right size. Let $W = \{\gamma : \gamma \in u_\alpha \text{ for some } \alpha \in w'\}$. Aiming for (h) with $v = w'$ it would suffice to show that for $W = \bigcup_{\alpha \in w'} u_\alpha$,

(j) $$\mathbf{a} \leq \mathbf{b}_W$$

where note that

(k) $$|W| \leq m \cdot |w'| < m \cdot \frac{m_n}{(m_n^\circ)^{n^{i+j}}} \leq \frac{m_n}{(m_n^\circ)^{i+2}}.$$

In other words, we are asking whether the set of formulas

(l) $$\{Q_{\rho_*}(x) \wedge R(x, x_{\gamma(\alpha,\ell)}) \wedge \mathcal{P}_{\eta_\ell}(x_{\gamma(\alpha,\ell)}) : \alpha \in w', \ell < m\}$$

is a partial type for $T_{\mathfrak{m}}$. Since $\mathbf{a} \leq \mathbf{b}_{u_\alpha}$ for each $\alpha \in w'$ (and in particular, for one of them), we know that

(m) $$\{Q_{\rho_*}(x) \wedge R(x, x_{\gamma(\alpha,\ell)}) \wedge \mathcal{P}_{\eta_\ell}(x_{\gamma(\alpha,\ell)}) : \ell < m\}$$

is a partial type for $T_{\mathfrak{m}}$. This means that for some $\rho \in \mathcal{T}_{1,n}$ such that $\rho_* \trianglelefteq \rho$ (recalling $n > \lg(\rho_*)$ by (g)) we have that

(n) $$\{\rho\} \times \{\nu_\ell : \ell < m\} \subseteq \mathcal{R}_n.$$

Since $\xi(n) = 0$, recalling 2.2, our construction ensures that every edge in $\mathcal{R}_n$ between two nodes of height n blows up to a complete bipartite graph between their immediate successors in $\mathcal{R}_{n+1}$, so there is no new incompatibility. Let us now justify that (A) implies (B) where:

- (A) $N \models T_{\mathfrak{m}}$, $|W| \leq m_n/(m_n^\circ)^{i+2}$, $i+2 \leq n$, for each $\gamma \in W$ the element $a_\gamma \in N$ realizes $\{\mathcal{P}_{\eta_\gamma \restriction k}(x) : k < \omega\}$, and there is $\rho \in \mathcal{T}_{1,n}$ such that $(\rho, \eta_\gamma) \in \mathcal{R}_n$ for each $\gamma \in W$.
- (B) there is $\varrho \in \lim(\mathcal{T}_1)$ such that $\rho \trianglelefteq \varrho$ and $(\varrho, \eta_\gamma) \in \mathcal{R}_\infty$ for every $\gamma \in W$, i.e. there is $\varrho \in \lim(\mathcal{T}_1)$ such that for any $k \in [n, \omega)$, we have $(\varrho \restriction k, \eta_\gamma \restriction k) \in \mathcal{R}_k$.

Note that the data of (A) tells us that each a_γ is in the leaf $\{P_{\gamma \restriction k}(x) : k < \omega\}$.

We choose $\varrho \restriction k$ by induction on $k \geq n$. The case of $k = n$ is given by (A). For $k = n+1$ we just use the fact that $\xi(n) = 0$: recalling 6.15, let ϱ_{n+2} be any element of $\mathcal{T}_{1,n+2}$ such that $\rho \trianglelefteq \varrho_{n+2}$ [we know some such element exists] and then $(\varrho_{n+2}, \eta_\gamma \restriction (n+2)) \in \mathcal{R}_{n+2}$ for all $\gamma \in W$. For $k > n+1$ and $\xi(k) = 0$ or 1, the set $\{\eta_\gamma \restriction k : \gamma \in W\}$ is by now "small" in the sense of 6.2 [note that $m_k^\circ > m_{n+1}^\circ > m_n > |W|$] so we can apply Claim 6.17(1) [which in our notation says that given $\{\eta_\gamma : \gamma \in W\} \subseteq \mathcal{T}_{2,k+1}$ and $|W| \leq (m_k^\circ)^{k^k}$, and given that our inductive hypothesis holds, there exists a successor ϱ_{k+1} of ϱ_k such that $(\varrho_{k+1}, \eta_\gamma \restriction (k+1)) \in \mathcal{R}_{k+1}$ for each $\gamma \in W$]. This completes the induction, and so completes the proof. $\square$

For our third and final case of realizing types, in the case of the random graph, the theory is simple enough that we will construct the refinement directly. The proof is like [22] Theorem 3.2, though there are still many differences. The point of the proof is mainly a matter of keeping track of equality and sorting the objects by hand, so to speak.

Lemma 10.21. *Suppose $\mathfrak{a} \in \mathrm{AP}$ and let $\theta \leq \lambda$. Suppose $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\theta]^{<\aleph_0} \rangle$ is a $(\theta, T_{\mathbf{rg}})$-problem coming from a type $\{R(x, a_\alpha)^{\mathbf{t}(\alpha)} : \alpha < \theta, \mathbf{t} : \theta \to \{0,1\}\}$, and $\langle \mathbf{a}[y_\alpha = y_\beta] : \alpha, \beta < \theta \rangle$ are "truth values" in $\mathfrak{B}_\mathfrak{a}$ of "$a_\alpha = a_\beta$" as usual.*

Then there is $\mathfrak{b} \in \mathrm{AP}$ such that

(1) $\mathfrak{a} \leq_{\mathrm{AP}} \mathfrak{b}$, and
(2) there is a solution of $\bar{\mathbf{b}}$ in $\mathfrak{b}$, i.e. there are $\mathbf{b}^1_\alpha \in \mathcal{D}_\mathfrak{b}$ for $\alpha < \theta$ such that $\mathfrak{B}_\mathfrak{b} \models \bigcap \{\mathbf{b}^1_\alpha : \alpha \in u\} \leq \mathbf{b}_u$ for $u \in [\theta]^{<\aleph_0}$.

Proof. First, we point out what "$\bar{\mathbf{b}}$ is a problem in $\mathfrak{B}_\mathfrak{b}$ for the theory of the random graph" tells us. For each $u \in [\theta]^{<\aleph_0}$, we have that

(a) $$\mathbf{b}_u = \bigcap \{\mathbf{a}[a_\alpha \neq a_\beta] : \alpha, \beta \in u, \mathbf{t}(\alpha) \neq \mathbf{t}(\beta)\}.$$

[This just translates the fact that in the enveloping ultrapower, at a given index $i \in I$, $\{R(x, a_\alpha[i])^{\mathbf{t}(\alpha)} : \alpha \in u\}$ is consistent when $\mathbf{t}(\alpha) \neq \mathbf{t}(\beta) \implies \mathbf{a}_\alpha[i] \neq \mathbf{a}_\beta[i]$.] By definition, each $\mathbf{b}_u \in \mathcal{D}_\mathfrak{a}$, so in particular each $\mathbf{b}_u > 0_{\mathfrak{B}_\mathfrak{a}}$. In keeping with our earlier notation, we will drop parentheses on singletons, and so write

(b) $$\mathbf{b}_\alpha \text{ for } \mathbf{b}_{\{\alpha\}}.$$

Second, a crucial point in [22] which we repeat here is the following. For each $\alpha < \theta$, call $\mathbf{x} \in \mathfrak{B}^+_\mathfrak{a}$ *collapsed* for α if for some $\beta \leq \alpha$, $0 < \mathbf{x} \leq \mathbf{a}[a_\alpha = a_\beta]$ but for all $\gamma < \beta$, $\mathbf{x} \cap \mathbf{a}[a_\alpha = a_\gamma] = 0_{\mathfrak{B}_\mathfrak{a}}$. For every $\alpha < \theta$ and $\mathbf{a} \in \mathfrak{B}^+_\mathfrak{a}$ there is $\mathbf{x}$ with $0 < \mathbf{x} \leq \mathbf{a}$ which is collapsed for α, because the ordinals are well ordered. So for each $\alpha < \theta$, we try to choose

$$(\mathbf{a}_{\alpha,\epsilon}, \beta_{\alpha,\epsilon})$$

by induction on $\epsilon < \kappa$, such that:

- each $(\mathbf{a}_{\alpha,\epsilon}, \beta_{\alpha,\epsilon})$ is as above
- $\zeta < \epsilon \implies \mathbf{a}_{\alpha,\epsilon} \cap \mathbf{a}_{\alpha,\zeta} = 0$.

As $\mathfrak{B}$ satisfies the κ-c.c. we stop at some $\epsilon = \epsilon_\alpha < \kappa$. Renaming and renumbering, $\epsilon_\alpha = \mu_\alpha$, for some cardinal $\mu_\alpha < \kappa$. So we have built

(c) $$\langle (\mathbf{a}_{\alpha,\epsilon}, \beta_{\alpha,\epsilon}) : \epsilon < \mu_\alpha \rangle \text{ where } \langle \mathbf{a}_{\alpha,\epsilon} : \epsilon < \mu_\alpha \rangle \text{ is a maximal antichain}$$

and for each $\epsilon < \mu_\alpha$, $\beta_{\alpha,\epsilon}$ is the unique $\beta \leq \alpha$ such that $\mathbf{a}_{\alpha,\epsilon} \leq \mathbf{a}[a_\alpha = a_\beta]$ and for all $\gamma < \beta$, $\mathbf{a}_{\alpha,\epsilon} \cap \mathbf{a}[a_\alpha = a_\beta] = 0$. Without loss of generality, we may assume that in this antichain, $\epsilon < \epsilon'$ implies $\beta_{\alpha,\epsilon} \neq \beta_{\alpha,\epsilon'}$: if not, combine all elements of the antichain with the same β and renumber.

On the relation of these antichains to each other, observe that it follows from the definition of "collapsed" and the construction of (c) that if $\gamma = \beta_{\alpha,\epsilon}$ there is one (and only one) $\delta < \mu_\gamma$ such that

(d) $$\mathbf{a}_{\alpha,\epsilon} \leq \mathbf{a}_{\gamma,\delta}$$

namely the unique δ for which $\beta_{\gamma,\delta} = \beta_{\alpha,\epsilon}$. We will often refer to this, so let us give it a name: for $\alpha < \theta$, $\epsilon < \mu_\alpha$ let

(e) $$\xi_{\alpha,\epsilon} < \mu_{\beta_{\alpha,\epsilon}}$$

be the unique element so that

$$\mathbf{a}_{\alpha,\epsilon} \leq \mathbf{a}_{\beta_{\alpha,\epsilon}, \xi_{\alpha,\epsilon}}.$$

To summarize, we may think of the elements of (c) as

(f) $$\langle (\mathbf{a}_{\alpha,\epsilon}, \beta_{\alpha,\epsilon}, \xi_{\alpha,\epsilon}) : \epsilon < \mu_\alpha \rangle.$$

where informally, this information records that on $\mathbf{a}_{\alpha,\epsilon}$, " a_α collapses to a_β" where $\beta = \beta_{\alpha,\epsilon}$, and $\xi = \xi_{\alpha,\epsilon}$ tells us $\mathbf{a}_{\alpha,\epsilon}$ is below the ξ-th element of the antichain for β.

Let $\mathcal{F}_0$ be the set of finite functions f such that $\mathrm{dom}(f)$ is a finite subset of θ and $\alpha \in \mathrm{dom}(f) \implies f(\alpha) < \mu_\alpha$. For each $f \in \mathcal{F}_0$, let

$$\mathbf{a}_f = \bigcap \{\mathbf{a}_{\alpha,f(\alpha)} : \alpha \in \mathrm{dom}(f)\} \tag{g}$$

where $\mathbf{a}_{\alpha,f(\alpha)}$ denotes the appropriate element from the antichain in equation (c) (so a priori $\mathbf{a}_f$ may be zero).

Let $\mathcal{F} = \{f \in \mathcal{F}_0 : \mathbf{a}_f \in \mathfrak{B}^+_{\mathfrak{a}} \text{ and } \mathbf{a}_f \leq \mathbf{b}_{\mathrm{dom}(f)}\}$. Let us justify that this set is nonempty by showing something stronger: for any finite $u \subseteq \theta$ there is $f \in \mathcal{F}$ with $\mathrm{dom}(f) = u$ such that $0 < \mathbf{a}_f \leq \mathbf{b}_u$. Fix $u \in [\theta]^{<\aleph_0}$. Because u is finite, $\mathbf{b}_u > 0$ and each $\langle \mathbf{a}_{\alpha,\epsilon} : \epsilon < \mu_\alpha \rangle$ is a maximal antichain, there is a finite function g with domain u and $\alpha \in \mathrm{dom}(g) \implies g(\alpha) < \mu_\alpha$, such that

$$\bigcap_{\alpha \in u} \mathbf{a}_{\alpha,g(\alpha)} \cap \mathbf{b}_u > 0.$$

Remembering the definition of $\mathbf{b}_u$ in equation (a) [it just depends on collisions], and that each of the $\mathbf{a}_{\alpha,g(\alpha)}$'s are collapsed for the respective α's, in fact

$$\bigcap_{\alpha \in u} \mathbf{a}_{\alpha,g(\alpha)} \cap \mathbf{b}_u > 0 \implies \bigcap_{\alpha \in u} \mathbf{a}_{\alpha,g(\alpha)} \leq \mathbf{b}_u$$

as desired.

In what follows it will be convenient to have enriched functions which code both u and the collisions of u, and for this we need a few more definitions.

For each $f \in \mathcal{F}$ let $f_\oplus$ be the following finite function with domain $\subseteq \theta$ such that $\alpha \in \mathrm{dom}(f_\oplus)$ implies $f_\oplus(\alpha) < \mu_\alpha$. Let

$$\mathrm{dom}(f_\oplus) = \mathrm{dom}(f) \cup \{\beta_{\alpha,f(\alpha)} : \alpha \in \mathrm{dom}(f)\}.$$

Let $f_\oplus$ agree with f on $\mathrm{dom}(f)$. For each $\gamma \in \mathrm{dom}(f_\oplus \setminus f)$, let $\alpha \in \mathrm{dom}(f)$ be such that $\gamma = \beta_{\alpha,f(\alpha)}$, and let $f(\gamma)$ be $\xi_{\alpha,f(\alpha)}$ from equation (e) above, so we have

$$\mathbf{a}_{\alpha,f(\alpha)} \leq \mathbf{a}_{\beta_{\alpha,f(\alpha)},\xi_{\alpha,f(\alpha)}} = \mathbf{a}_{\beta_{\alpha,f(\alpha)},f_\oplus(\gamma)}.$$

To check that $f_\oplus$ is well defined, we repeatedly apply transitivity of equality and $f \in \mathcal{F}$. Suppose $\gamma \in \mathrm{dom}(f_\oplus \setminus f)$ could arise in two different ways, as $\beta_{\alpha,\epsilon}$ or as $\beta_{\alpha',\epsilon'}$. This says two different elements have the same collapse, but either way, recalling (d) and $f \in \mathcal{F}$, $f_\oplus(\gamma)$ will be the unique $\delta < \mu_\gamma$ such that $\beta_{\gamma,\delta} = \gamma$. If $\gamma \in \mathrm{dom}(f_\oplus \setminus f)$ is equal to some $\alpha \in \mathrm{dom}(f)$, $f_\oplus(\gamma) = f(\alpha)$ will be the unique $\delta < \mu_\alpha$ such that $\beta_{\alpha,\delta} = \alpha$, so again the function is well defined, as $f \in \mathcal{F}$.

For any such $f_\oplus$, let $\bar{\zeta}(f_\oplus)$ list its domain in increasing order. We define

$$\bar{\mathfrak{s}} = \bar{\mathfrak{s}}(f_\oplus) \tag{h}$$

to be the following sequence of truth values:

- $\mathrm{dom}(\bar{\mathfrak{s}}) = \mathrm{dom}(f_\oplus)$
- if $\gamma \in \mathrm{dom}(f)$ then $s_\gamma = \mathbf{t}_\gamma$
- if $\gamma \in \mathrm{dom}(f_\oplus) \setminus \mathrm{dom}(f)$ and $\gamma = \beta_{\alpha,f(\alpha)}$ then $s_\gamma = \mathbf{t}_\alpha$.

[Informally, we record the truth values associated to the original element before the collapse.] As $f_\oplus$ is a function, $\bar{\mathfrak{s}}$ is well defined. Note also that for each $\alpha < \theta$, $S_\alpha = \{\epsilon : \mathbf{a}_{\alpha,\epsilon} \leq \mathbf{b}_\alpha\} = \mu_\alpha$, because $\mathbf{b}_\alpha = 1$ [recalling our notation in equation (b)]. For explanatory purposes, we retain the notation S_α (rather than just μ_α) below.

We now define the Boolean algebra $\mathfrak{B}_{\mathfrak{b}}$. First, let $\mathfrak{B}^0_{\mathfrak{b}}$ be the Boolean algebra generated by

$$\mathcal{X} = \mathfrak{B}_{\mathfrak{a}} \cup \{\mathbf{c}_{\alpha,\epsilon} : \alpha < \theta, \epsilon < \mu_\alpha\}$$

freely except:

(i) the equations in $\mathfrak{B}_{\mathfrak{a}}$
(ii) $\mathbf{c}_{\alpha,\epsilon} \cap \mathbf{c}_{\alpha,\zeta} = 0$ when $\epsilon < \zeta < \mu_\alpha$
(iii) $\mathbf{c}_{\alpha_1,\epsilon_1} \cap \mathbf{c}_{\alpha_2,\epsilon_2} = 0$ when
$\mathbf{t}_{\alpha_1} \neq \mathbf{t}_{\alpha_2}$, $\beta_{\alpha_1,\epsilon_1} = \beta_{\alpha_2,\epsilon_2}$, $\xi_{\alpha_1,\epsilon_1} = \xi_{\alpha_2,\epsilon_2}$, $\epsilon_1 \in S_{\alpha_1}$, $\epsilon_2 \in S_{\alpha_2}$.

Let us verify that without loss of generality $\mathfrak{B}_{\mathfrak{a}} \subseteq \mathfrak{B}^0_{\mathfrak{b}}$. Let h be the identity on $\mathfrak{B}_{\mathfrak{a}}$ and map $\mathbf{c}_{\alpha,\epsilon}$ (for $\alpha < \theta$, $\epsilon < \mu_\alpha$) to $0_{\mathfrak{B}_{\mathfrak{a}}}$. So h is a function from $\mathcal{X}$ onto $\mathfrak{B}_{\mathfrak{a}}$ respecting the equations, hence has an extension $\hat{h}$, a homomorphism from $\mathfrak{B}^0_{\mathfrak{b}}$ into $\mathfrak{B}_{\mathfrak{a}}$ but $\hat{h} \restriction \mathfrak{B}_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{B}_{\mathfrak{a}}}$, so we are done.

Let us verify that $\mathfrak{B}_{\mathfrak{a}} \lessdot \mathfrak{B}^0_{\mathfrak{b}}$. Let $\mathbf{c} \in (\mathfrak{B}^0_{\mathfrak{b}})^+$ and we shall find a projection from $\mathfrak{B}^0_{\mathfrak{b}}$ onto $\mathfrak{B}_{\mathfrak{a}}$ mapping $\mathbf{c}$ to a positive element. We can replace $\mathbf{c}$ by any $\mathbf{c}' \leq \mathbf{c}$ which is not zero. As each $\{\mathbf{c}_{\alpha,\epsilon} : \epsilon < \mu_\alpha\}$ is an antichain, without loss of generality for some $\mathbf{a} \in \mathfrak{B}^+_{\mathfrak{a}}$, disjoint finite $u, v \subseteq \theta$, for a function f with finite domain $u \subseteq \theta$ such that $f(\alpha) < \mu_\alpha$ for $\alpha \in u$, and finite $w_\beta \subseteq \mu_\beta$ for $\beta \in v$, we have

(i) $$0 < \bigcap\{\mathbf{c}_{\alpha,f(\alpha)} : \alpha \in u\} \cap \bigcap\{-\mathbf{c}_{\beta,\epsilon} : \beta \in v, \epsilon \in w_\beta\} \cap \mathbf{a} \leq \mathbf{c}$$

where necessarily $\{(\alpha, f(\alpha)) : \alpha \in u\} \cap \{(\beta,\epsilon) : \beta \in v, \epsilon \in w_\beta\} = \emptyset$, and if $\alpha_1 \neq \alpha_2 \in u$ and $f(\alpha_1) = f(\alpha_2)$ then either $\beta_{\alpha_1,f(\alpha_1)} \neq \beta_{\alpha_2,f(\alpha_2)}$ or $\xi_{\alpha_1,f(\alpha_1)} \neq \xi_{\alpha_2,f(\alpha_2)}$. [If not, $\mathbf{c}_{\alpha_1,f(\alpha_1)} \cap \mathbf{c}_{\alpha_2,f(\alpha_2)} = 0$, which contradicts the intersection being positive.] This time we define $h : \mathcal{X} \rightarrow \mathfrak{B}_{\mathfrak{a}}$ so that h is the identity on $\mathfrak{B}_{\mathfrak{a}}$, $h(\mathbf{c}_{\alpha,f(\alpha)}) = \mathbf{a}$ for $\alpha \in u$, and $h(\mathbf{c}_{\alpha,\epsilon}) = 0_{\mathfrak{B}_{\mathfrak{a}}}$ for $(\alpha,\epsilon) \notin \{(\alpha, f(\alpha)) : \alpha \in u\}$. By the note after equation (i), h respects the equations in (iii). So h extends to a homomorphism $\hat{h}$ from $\mathfrak{B}^0_{\mathfrak{b}}$ onto $\mathfrak{B}_{\mathfrak{a}}$ which is the identity on $\mathfrak{B}_{\mathfrak{a}}$ and sends $\mathbf{c}$ to a positive element.

Let $\mathfrak{B}_{\mathfrak{b}}$ be the completion of $\mathfrak{B}^0_{\mathfrak{b}}$.

Now we define our multiplicative refinement.

(j) $$\text{For } \alpha < \theta,\ \epsilon \in S_\alpha \text{ let } \mathbf{b}^1_{\alpha,\epsilon} = \mathbf{a}_{\alpha,\epsilon} \cap \mathbf{c}_{\alpha,\epsilon}.$$

Then for every $\alpha < \theta$ let

(k) $$\mathbf{b}^1_\alpha = \bigcup\{\mathbf{b}^1_{\alpha,\epsilon} : \epsilon \in S_\alpha\}.$$

We have to check that

(l) $$u \in [\theta]^{<\aleph_0} \implies \bigcap_{\alpha \in u} \mathbf{b}^1_\alpha \leq \mathbf{b}_u$$

and also that

(m) $$\mathbf{b}^1_\alpha \in \mathcal{D} \text{ for suitable } \mathcal{D}.$$

First we check equation (l). Fix $u \in [\theta]^{<\aleph_0}$ and we will check that $\bigcap_{\alpha \in u} \mathbf{b}^1_\alpha \leq \mathbf{b}_u$ Without loss of generality $u = \{\alpha_1, \alpha_2\}$, $\alpha_1 \neq \alpha_2$. [If $u = \{\alpha\}$ use the choice of S_α, and if $|u| > 2$ remember the definition of $\mathbf{b}_u$ depends on consistency of pairs.] So it suffices to prove for $(\epsilon_1, \epsilon_2) \in S_{\alpha_1} \times S_{\alpha_2}$ that

$$\mathbf{b}^1_{\alpha_1,\epsilon_1} \cap \mathbf{b}^1_{\alpha_2,\epsilon_2} \leq \mathbf{b}_{\{\alpha_1,\alpha_2\}}.$$

There are four cases.

Case 1. $\mathbf{t}_{\alpha_1} = \mathbf{t}_{\alpha_2}$. Then $\mathbf{b}_{\{\alpha_1,\alpha_2\}} = 1_{\mathfrak{B}}$ so the inequality holds trivially. [16]

Case 2. $\mathbf{t}_{\alpha_1} \neq \mathbf{t}_{\alpha_2}$, $\beta_{\alpha_1,\epsilon_1} \neq \beta_{\alpha_2,\epsilon_2}$. In this case $\mathbf{a}_{\alpha_1,\epsilon_1} \cap \mathbf{a}_{\alpha_2,\epsilon_2} \leq \mathbf{a}[x_{\alpha_1} \neq x_{\alpha_2}]$ which suffices.

Case 3. $\mathbf{t}_{\alpha_1} \neq \mathbf{t}_{\alpha_2}$, $\beta_{\alpha_1,\epsilon_1} = \beta_{\alpha_2,\epsilon_2}$ (call it β) and $\xi_{\alpha_1,\epsilon_1} \neq \xi_{\alpha_2,\epsilon_2}$. Recalling that $\langle \mathbf{a}_{\beta,\epsilon} : \epsilon < \mu_\beta \rangle$ is an antichain,

$$\mathbf{b}_{\alpha_1,\epsilon_1} \cap \mathbf{b}_{\alpha_2,\epsilon_2} \leq \mathbf{a}_{\alpha_1,\epsilon_1} \cap \mathbf{a}_{\alpha_2,\epsilon_2} \leq \mathbf{a}_{\beta,\xi_{\alpha_1,\epsilon_1}} \cap \mathbf{a}_{\beta,\xi_{\alpha_2,\epsilon_2}} = 0_{\mathfrak{B}}$$

which suffices.

Case 4. $\mathbf{t}_{\alpha_1} \neq \mathbf{t}_{\alpha_2}$, $\beta_{\alpha_1,\epsilon_1} = \beta_{\alpha_2,\epsilon_2}$ (call it β) and $\xi_{\alpha_1,\epsilon_1} = \xi_{\alpha_2,\epsilon_2}$ (call it ξ). So

$$\mathbf{b}_{\alpha_1,\epsilon_1} \cap \mathbf{b}_{\alpha_2,\epsilon_2} \leq \mathbf{c}_{\alpha_1,\epsilon_1} \cap \mathbf{c}_{\alpha_2,\epsilon_2} = 0_{\mathfrak{B}}$$

by clause (iii) in the definition of $\mathfrak{B}_{\mathfrak{b}}$.

This completes the verification of equation (l).

Next we verify equation (m). For this we should show there is $\mathcal{D}$ an ultrafilter on $\mathfrak{B}_{\mathfrak{b}}$ extending $\mathcal{D}_{\mathfrak{a}} \cup \{\mathbf{b}^1_\alpha : \alpha < \theta\}$. As in 10.14, it suffices to prove that given a finite $u \subseteq \theta$ and $\mathbf{d} \in \mathcal{D}_{\mathfrak{a}}$ then

$$\mathbf{b}^1_u = \bigcap\{\mathbf{b}^1_\alpha : \alpha \in u\} \text{ is not disjoint to } \mathbf{d}.$$

As $\mathbf{b}_u \in \mathcal{D}_{\mathfrak{a}}$, $\mathbf{d}_1 := \mathbf{b}_u \cap \mathbf{d} > 0$. As $\langle \mathbf{a}_{\alpha,\epsilon} : \epsilon < \mu_\alpha \rangle$ is a maximal antichain of $\mathfrak{B}_{\mathfrak{a}}$, by induction on $\alpha < |u|$ we can choose f so that

- $f \in \prod_{\alpha \in u} \mu_\alpha$
- $\mathbf{d}_2 = \bigcap\{\mathbf{a}_{\alpha,f(\alpha)} : \alpha \in u\} \cap \mathbf{b}_u \cap \mathbf{d}_1 > 0$

Now $\mathrm{dom}(f) = u$ and $\mathbf{a}_f \cap \mathbf{b}_u = \mathbf{b}_{\mathrm{dom}(f)}$ by construction, so indeed $\mathbf{a}_f \leq \mathbf{b}_u$ (recall our notation in equation (g)), so $f \in \mathcal{F}$. Since $\mathbf{d}_2 \leq \mathbf{a}_f \leq \mathbf{b}_u$, by choice of f and definition of $\mathbf{b}_u$, for any $\alpha_1, \alpha_2 \in u$ we have that $\alpha_1 \neq \alpha_2$ implies either $\mathbf{t}_1 = \mathbf{t}_2$ or $\beta_{\alpha_1,f(\alpha_1)} \neq \beta_{\alpha_2,f(\alpha_2)}$. It follows that the corresponding equations in the definition of $\mathfrak{B}^0_{\mathfrak{b}}$ are respected, so

$$\mathbf{d}_2 \cap \{\mathbf{c}_{\alpha,f(\alpha)} : \alpha \in u\} > 0.$$

Now define $h : \mathcal{X} \to \mathfrak{B}_{\mathfrak{a}}$ so that h is the identity on $\mathfrak{B}_{\mathfrak{a}}$, $h(\mathbf{c}_{\alpha,f(\alpha)}) = \mathbf{d}_2$ for $\alpha \in u$, and $h(\mathbf{c}_{\alpha,\epsilon}) = 0_{\mathfrak{B}_{\mathfrak{a}}}$ for $(\alpha,\epsilon) \notin \{(\alpha, f(\alpha)) : \alpha \in u\}$. So h extends to a homomorphism $\hat{h}$ from $\mathfrak{B}^0_{\mathfrak{b}}$ onto $\mathfrak{B}_{\mathfrak{a}}$ which is the identity on $\mathfrak{B}_{\mathfrak{a}}$ and sends $\mathbf{d}$ to a positive element.

It remains to verify that the pair $(\mathfrak{B}_{\mathfrak{a}}, \mathfrak{B}_{\mathfrak{b}})$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property. Following 8.20, it suffices to prove that given a sequence $\langle \mathbf{d}^2_\zeta : \zeta < \kappa \rangle$ of elements of $\mathfrak{B}^+_{\mathfrak{b}}$, we can find $\mathcal{U} \in [\kappa]^\kappa$ and a sequence $\langle \mathbf{d}^1_\zeta : \zeta \in \mathcal{U} \rangle$ of elements of $\mathfrak{B}^+_{\mathfrak{a}}$ such that: if $u \in [\mathcal{U}]^{<\aleph_0}$ and $\mathfrak{B}_{\mathfrak{a}} \models$ "$\bigcap_{\zeta \in u} \mathbf{d}^1_\zeta > 0$" then $\mathfrak{B}_{\mathfrak{b}} \models$ "$\bigcap_{\zeta \in u} \mathbf{d}^2_\zeta > 0$."

First, we simplify by replacing each $\mathbf{d}^2_\zeta$ by a possibly smaller nonzero element which has a normal form. For every $\zeta < \kappa$, just as in equation (i) we choose $(\mathbf{a}_\zeta, u_\zeta, f_\zeta, v_\zeta, \bar{w}_\zeta, \mathbf{d}'_\zeta)$ such that $\mathbf{a}_\zeta \in \mathfrak{B}^+_{\mathfrak{a}}$, u_ζ and v_ζ are disjoint finite subsets of θ, for $\bar{w}_\zeta = \langle w_{\zeta,\beta} : \beta \in v_\zeta \rangle$ each $w_{\zeta,\beta}$ is a finite subset of μ_β, and

(n) $$0 < \mathbf{d}'_\zeta := \bigcap\{\mathbf{c}_{\alpha,f_\zeta(\alpha)} : \alpha \in u_\zeta\} \cap \bigcap\{-\mathbf{c}_{\beta,\epsilon} : \beta \in v_\zeta, \epsilon \in w_{\zeta,\beta}\} \cap \mathbf{a}_\zeta \leq \mathbf{d}^2_\zeta.$$

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[16] *Case 1.* We connect to both a_{α_1} and a_{α_2} or neither, so there is no conflict. *Case 2.* The truth values are different but they collapse to different elements, so there is no conflict. *Case 3.* The truth values are different and they collapse to the same β, but are contained in different parts of the antichain for β. This is unlikely, but in any case causes no conflict. *Case 4.* The truth values are different, they collapse to the same β, and are contained in the ξ-th piece of the antichain for β. By definition the $\mathbf{c}$'s in this case formally do not intersect, so there is no conflict.

We may assume $\mathbf{d}'_\zeta \leq \mathbf{b}_{u_\zeta}$ or $\mathbf{d}'_\zeta \cap \mathbf{b}_{u_\zeta} = 0_{\mathfrak{B}_\mathfrak{b}}$. Let $u^\oplus_\zeta = u_\zeta \cup \{\beta_{\alpha, f_\zeta(\alpha)} : \alpha \in u_\zeta\}$, and let $\bar{\mathfrak{s}}_\zeta = \bar{\mathfrak{s}}(f_\zeta)$ be the sequence of truth values from (h).

By the Δ-system lemma and the regularity of κ, on some $\mathcal{U} \in [\kappa]^\kappa$ we may find quite a bit of uniformity:

- *either* for every $\zeta \in \mathcal{U}$, $\mathbf{d}'_\zeta \leq \mathbf{b}_{u_\zeta}$, *or* for every $\zeta \in \mathcal{U}$, $\mathbf{d}'_\zeta \cap \mathbf{b}_{u_\zeta} = 0_{\mathfrak{B}_\mathfrak{b}}$.
- $\langle u^\oplus_\zeta : \zeta \in \mathcal{U}\rangle$ is a Δ-system with heart $u^\oplus_*$
- $\langle v_\zeta : \zeta \in \mathcal{U}\rangle$ is a Δ-system with heart v_*
- $v_\zeta \cap u^\oplus_* = \emptyset$, $u^*_\zeta \cap v_* = \emptyset$ for $\zeta \in \mathcal{U}$
- $\zeta \in \mathcal{U} \implies u_\zeta \cap u^*_* = u_*$ and $\bar{\mathfrak{s}}_\zeta \restriction u^\oplus_* = \bar{s}_*$
- $\zeta \in \mathcal{U} \implies f^*_\zeta \restriction u_* = f_*$.

Let $\mathbf{d}^1_\zeta = \mathbf{a}_\zeta$ for $\zeta \in \mathcal{U}$. It follows that for any finite $u \subseteq \mathcal{U}$, if $\bigcap\{\mathbf{d}^1_\zeta : \zeta \in u\} > 0_{\mathfrak{B}_\mathfrak{a}}$ then then also $\bigcap\{\mathbf{d}'_\zeta : \zeta \in u\} > 0_{\mathfrak{B}_\mathfrak{a}}$. [Recalling (i), (ii), (iii) from the definition of $\mathfrak{B}^0_\mathfrak{b}$ above, here the uniformity of of $u^\oplus_*$ and $\bar{s}_*$ mean that this intersection won't contain two incompatible elements of $\mathfrak{B}^0_\mathfrak{b} \setminus \mathfrak{B}_\mathfrak{a}$.] Hence $\bigcap\{\mathbf{d}^2_\zeta : \zeta \in u\} > 0$, as desired.

This completes the proof. $\square$

Recall that κ is a regular uncountable cardinal and $\lambda \geq \kappa$.

Conclusion 10.22. *There exists a regular ultrafilter $\mathcal{D}$ on λ such that:*

(1) $\mathcal{D}$ is κ-good, i.e. $(\mathbb{N}, <)^\lambda/\mathcal{D}$ is κ-saturated.[17]

(2) if $\mathfrak{m} \in \mathcal{M}$ then $\mathcal{D}$ is $(\lambda^+, T_\mathfrak{m})$-good.

(3) if $\mathfrak{n} \in \mathcal{N}$ then $\mathcal{D}$ is not $(\kappa^+, T_\mathfrak{n})$-good.

Proof. Let $\langle S_\gamma : \gamma < 2^\lambda\rangle$ partition 2^λ into sets each of cardinality 2^λ with $\gamma \leq \min(S_\gamma)$. We choose by induction on $\alpha \leq 2^\lambda$ not only $\mathfrak{b}_\alpha$ but also $\bar{f}_\alpha$ such that:

(1) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\alpha : \alpha \leq 2^\lambda\rangle$ is a $\leq_{\mathrm{AP}}$-increasing continuous general construction sequence, so by definiton and our work above this will mean:
  (a) $\mathfrak{b}_0 = \mathfrak{a}_*$,
  (b) each $\mathfrak{b}_\alpha \in \mathrm{AP}$,
  (c) $\alpha < \beta \implies (\mathfrak{B}_\mathfrak{a}, \mathfrak{B}_\mathfrak{b})$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-pattern transfer property,
  (d) each $|\mathfrak{B}_{\mathfrak{b}_\alpha}| \leq 2^\lambda$,
  (e) each $\mathfrak{B}_{\mathfrak{b}_\alpha}$ satisfies the $(\kappa, \mathcal{I}, \bar{m})$-c.c.

(2) $\bar{f}_\alpha = \langle f_{\alpha,\gamma} : \gamma \in S_\alpha\rangle$

(3) $\bar{f}_\alpha$ lists $\mathcal{F}_\alpha = \{f : f$ is a function from $[\theta]^{<\aleph_0}$ into $\mathfrak{B}_{\mathfrak{b}_\alpha}$, even into $\mathcal{D}_{\mathfrak{b}_\alpha}$, which is monotone for some $\theta = \theta_f \leq \lambda\ \}$.

(4) Now if $\alpha = \gamma + 1$ and $\gamma \in S_\beta$, so $\beta < \gamma$, and range$(f_{\alpha,\gamma}) \subseteq \mathcal{D}_{\mathfrak{b}_\gamma}$, and *if* maintaining the restriction in (1) we can choose $\mathfrak{b}_\alpha$ such that $f_{\alpha,\gamma}$ has a multiplicative refinement g_γ with range $\subseteq \mathcal{D}_{\mathfrak{b}_\alpha}$, *then* there is such a refinement [i.e., then we do so]. Otherwise, we do nothing.

  Alternately, in this step we could say that if $f_{\alpha,\gamma}$ is a possibility pattern for a theory $T = T_{f_{\alpha,\gamma}}$ such that one of the following occurs:
  - $\theta_{f_{\alpha,\gamma}} < \kappa$,
  - $T_{f_{\alpha,\gamma}}$ is $T_\mathfrak{m}$ for $\mathfrak{m} \in \mathcal{M}$, and the possibility pattern in question comes from a positive R-type,
  - $T_{f_{\alpha,\gamma}}$ is $T_{\mathbf{rg}}$ and the possibility pattern comes from a type in positive and negative instances of R.

[17] Equivalently, for any model M with $|\tau_M| < \kappa$, $M^\lambda/\mathcal{D}$ is κ-saturated.

then we solve it using 10.18, 10.20, or 10.21, otherwise we do nothing. Note that 10.18, 10.20, 10.21 show that at the very least, these three kinds of types will be handled if we take the first alternative.

Having built our Boolean algebra and ultrafilter, we choose the data of separation of variables $\mathcal{D}_0, \mathbf{j}$ to go with this $\mathfrak{B}_\mathfrak{a}$ and $\mathcal{D}_\mathfrak{a}$ in the sense of 7.1, which gives us the regular ultrafilter desired, recalling Theorem 7.3. Note that by our analysis of ultrapower types in 5.7, these R-types suffice for saturation for $T_\mathfrak{m}$, and by 9.3 the fact that our Boolean algebra $\mathfrak{B}_\mathfrak{a}$ has the $(\kappa, \mathcal{I}, \bar{m})$-c.c. suffices for non-saturation of $T_\mathfrak{n}$ for $\mathfrak{n} \in \mathcal{N}$. $\square$

Corollary 10.23. *If $\mathcal{D}$ is from 10.22 and $\mathfrak{m}_\ell \in \mathcal{M}$ for $\ell < \omega$, then $T_* = \sum_{\ell<\omega} T_{\mathfrak{m}_\ell}$ is a complete countable simple theory, see* 11.1, *and $\mathcal{D}$ is (λ^+, T_*)-good.*

11. Main results and the big picture

In this section we summarize the main consequences of our construction for Keisler's order.

Recall that we can easily form new countable theories as the "disjoint union," or sum, of up to countably many countable theories. More formally:

Definition 11.1. *Recall $T = \sum T_n$ when without loss of generality $\langle \tau(T_n) : n < \omega \rangle$ are pairwise disjoint and have only predicates, and we have countably many new unary predicates $\{P_n : n < \omega\}$ with $\{P_n : n < \omega\} \cap \tau(T_n) = \emptyset$ for each n. Then $M \models T$ iff $\langle P_n^M : n < \omega \rangle$ are pairwise disjont and $(M \restriction P_n^M) \restriction \tau(T_n) \models T_n$, and if $R \in \tau(T_n)$ then $R^M \subseteq {}^{\mathrm{arity}(R)}(P_n^M)$.*

Recall that $u \in [X]^{\leq \aleph_0}$ means that u is an at most countable subset of X. Given any fast sequence and a family of independent level functions, our construction above gives a set of continuum many theories which can be thought of as Keisler-independent in the following strong sense. Suppose from the basic theories we form all possible "small composite theories" (countable, complete theories formed in the natural way as disjoint unions of at most countably many theories from our original set). Then our construction has shown that even the composite theories interact as freely as possible in the sense of Keisler's order, reflecting only the interaction in their indices, as the next theorem makes precise.

Theorem 11.2. *We can find $\bar{T}$ such that:*

(1) $\bar{T} = \langle T_u : u \in [2^{\aleph_0}]^{\leq \aleph_0} \rangle$
(2) T_u is a complete first order countable simple theory with trivial forking
(3) $T_u \trianglelefteq T_v$ if and only if $u \subseteq v$, for $u, v \in [2^{\aleph_0}]^{\leq \aleph_0}$.
(4) if $W \subseteq 2^{\aleph_0}$, $\aleph_0 \leq \mu < \mu^+ \leq \kappa = \mathrm{cof}(\kappa) \leq \lambda$ then there is a regular ultrafilter $\mathcal{D}$ on λ such that:
  (a) $\mathcal{D}$ is κ-good
  (b) if $u \in [W]^{\leq \aleph_0}$ then $\mathcal{D}$ is (λ^+, T_u)-good
  (c) if $u \in [2^{\aleph_0}]^{\leq \aleph_0}$, $u \not\subseteq W$ then $\mathcal{D}$ is not (κ^+, T_u)-good.

Proof. Let $\bar{m}$ be a fast sequence, let $\bar{E}$ and Ξ satisfy the hypotheses of 6.21, and let $\mathbf{M}_* = \{\mathfrak{m}_\alpha = \mathrm{par}[\bar{m}, \bar{E}, \xi_\alpha] : \alpha < 2^{\aleph_0}\}$ be as defined in 6.21. Let W play the role of $\mathcal{M} \subseteq \mathbf{M}_*$. Then by our construction there is a regular ultrafilter $\mathcal{D}$ on any infinite λ as in (4), satisfying also (4)(a), such that $\mathcal{D}$ is good for every $T_\mathfrak{m}$, $\mathfrak{m} \in \mathcal{M}$, and $\mathcal{D}$ is not good for any $T_{\mathfrak{n}}$, $\mathfrak{n} \in \mathbf{M}_* \setminus \mathcal{M}$. Note that any sum T_u of theories $\{T_{\mathfrak{m}_\alpha} : \alpha \in u\}$ is complete and remains simple with trivial forking. Clearly (3) and (4)(b), (c) follow. $\square$

As immediate consequences, we obtain several "more quotable" theorems.

Theorem 11.3. *There exist a perfect set of incomparable theories in Keisler's order, in ZFC, already within the class of countable simple unstable theories whose only forking comes from equality.*

Proof. Use any set of countably many pairwise incomparable theories from Theorem 11.2 to label the nodes of an infinite binary branching tree, and consider the set of theories which correspond to disjoint unions of theories along a given branch. $\square$

Theorem 11.4. *In the (countable, complete) simple non low theories,*

(1) there is a chain of cardinality continuum, and
(2) there is an antichain of cardinality continuum

in Keisler's order.

Proof. Enumerate a countable subset of the theories from Theorem 11.2 as $\langle T_q : q \in \mathbb{Q}\rangle$. Let $\langle C_\beta = (C^0_\beta, C^1_\beta) : \beta < 2^{\aleph_0}\rangle$ enumerate the continuum many cuts of the rationals. Let T_β be the theory which is the disjoint union of $\{T_q : q \in C^0_\beta\}$. Then clearly $\alpha \leq \beta$ implies $T_\alpha \trianglelefteq T_\beta$. As the rationals are dense in the reals, and our family of theories are pairwise incomparable, if $\alpha < \beta$ then $T_\alpha \triangleleft T_\beta$. This gives us a chain of cardinality continuum.

For an antichain of cardinality $\mathfrak{c}$, use all the theories from Theorem 11.2. $\square$

Remark 11.5. *Even in this frame the theories involved seem simple.*

Theorem 11.6. *There exist continuum many complete, countable, simple theories whose only forking comes from equality, and which are pairwise incomparable in Keisler's order.*

Proof. Consider all u of size 1 in Theorem 11.2, i.e., in the notation of that proof, consider $\{T_{\mathfrak{m}_\alpha} : \alpha < 2^{\aleph_0}\}$. $\square$

Conclusion 11.7. *Keisler's order is not at all simple, indeed the partial order on the family of subsets of $\mathbb{N}$ is embeddable into it.*

In fact, with more work we can upgrade 11.7 to the following a priori much stronger result. We include the proof in the next section.

Theorem 11.8. $\mathcal{P}(\omega)/\operatorname{fin}$ *embeds into Keisler's order.*

Proof. See page 63, below. $\square$

The theorem whose proof concludes the next section explains:

Theorem 11.9. *There is a family of parameters $\{\mathfrak{m}[A] : A \subseteq \omega\}$ such that each $T_{\mathfrak{m}[A]}$ is countable, complete, simple, and low, and the following are equivalent for any $\lambda \geq 2^{\aleph_0}$ and any a set $\mathcal{X} \subseteq \mathcal{P}(\omega)$:*

(1) There exists a regular ultrafilter $\mathcal{D}$ on λ such that $\mathcal{X} = \{A \subseteq \omega : \mathcal{D}$ is $(\lambda^+, T_{\mathfrak{m}[A]})$-good $\}$.
(2) $\mathcal{X} \supseteq [\omega]^{<\aleph_0}$ is an ideal.

Proof. See page 63, below. □

Discussion 11.10. In light of our main theorems, it would be natural to consider strengthenings of Keisler's order, for instance, by restricting consideration to ultrafilters on λ which are λ-good [but of course not necessarily λ^+-good]. However, our construction can be carried out even under such a definition, as we now spell out.

Definition 11.11. *Let the* strong Keisler order *mean that $T_1 \trianglelefteq^s T_2$ if and only if for every infinite λ, for every regular ultrafilter on λ* which is λ-good*, for every $M_1 \models T_1$, and every $M_2 \models T_2$, if $(M_2)^\lambda/\mathcal{D}$ is λ^+-good, then $(M_1)^\lambda/\mathcal{D}$ is λ^+-good.*

Theorem 11.12. *There exists a family $\{\mathcal{T}_{\mathfrak{m}_\alpha} : \alpha < 2^{\aleph_0}\}$ of continuum many complete countable simple theories, with forking coming only from equality, such that for any $u, v \subseteq 2^{\aleph_0}$ and u, v and most countable, letting T_u, T_v denote the "disjoint unions" of $\{T_{\mathfrak{m}_\alpha} : \alpha \in u\}$, $\{T_{\mathfrak{m}_\beta} : \beta \in v\}$ respectively, we have that $T_u \trianglelefteq^s T_v$ in the strong Keisler order if and only if $u \subseteq v$.*

Proof. This is a simplified version of Theorem 11.2 in the case $\mu < \mu^+ = \kappa = \lambda$. □

Since we know that Keisler's order reduces to the study of φ-types, it was asked whether there exists a function associating to each complete countable theory T a formula φ_T of the language of T such that for any infinite λ, any regular ultrafilter $\mathcal{D}$ on λ, and any model $M \models T$, the ultrapower $M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if it is λ^+-saturated for φ_T-types. Call such a function *an assignment of formulas to theories witnessing Keisler's order.* However, the results of the present paper show no formula assignment can exist.

Corollary 11.13. *There cannot exist an assignment of formulas to theories witnessing Keisler's order.*

Proof. Continuing in the notation of Theorem 11.2, let $\langle \mathfrak{m}_n : n < \omega \rangle$ be any countable sequence of distinct elements of $\mathcal{M}$ and let $T = \sum T_{\mathfrak{m}_n}$ be their disjoint union. Following 11.1, we may assume that without loss of generality $\{\tau(T_{\mathfrak{m}_n}) : n < \omega\}$ are pairwise disjoint and have only predicates. Assume for a contradiction that there exists a formula φ_T such that for any infinite λ, any regular ultrafilter $\mathcal{D}$ on λ, and any model $M \models T$, we would have $M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if it is λ^+-saturated for φ_T-types. Fix M. The formula φ_T is, being finite, is a formula of the language of $\bigcup\{\tau_{T_{\mathfrak{m}_n}} : n \in v\}$ for some finite $v \subseteq \omega$. By Theorem 11.2, there is a regular ultrafilter $\mathcal{D}$ on any uncountable λ which is λ^+-good for $\{T_{\mathfrak{m}_n} : n \in v\}$ and not λ^+-good for $\{T_{\mathfrak{m}_\ell} : \ell \in \omega \setminus v\}$. So in $M^\lambda/\mathcal{D}$ we realize all φ_T-types over sets of size λ, but omit some other ψ-type over some set of size λ. □

Discussion 11.14. *However, it may be interesting to give a model-theoretic characterization of the theories T for which such a φ_T exists. This includes all stable theories* (*choose any formula with the finite cover property if one exists, if not choose any formula* [31] *VI.5.2*)*, and all theories with SOP$_2$* [20].

12. Embedding $\mathcal{P}(\omega)/\operatorname{fin}$

In this section we prove Theorem 11.8: Keisler's order embeds $\mathcal{P}(\omega)/\operatorname{fin}$, and Theorem 11.9.

Convention 12.1. *For this section, fix some fast sequence $\bar{m}$, and a sequence of graphs $\bar{E}$ which is good for it, just as in 6.12 above.*

Convention 12.2. *In this section, for each $A \subseteq \omega$, let $\mathfrak{m}[A]$ be the parameter $\operatorname{par}[\bar{m}, \bar{E}, \xi]$ where $\xi : \omega \to \{0,1\}$ satisfies $\xi^{-1}\{1\} = A$. Let $T_{\mathfrak{m}[A]}$ be its associated theory.*

Our proof will show that $T_{\mathfrak{m}[A]} \trianglelefteq T_{\mathfrak{m}[B]}$ if and only if $A \subseteq^* B$. In fact, the proof will also show this for the interpretability order $\trianglelefteq^*$, not just Keisler's order $\trianglelefteq$ [for one implication this is stronger, not for if and only if].

It will be useful to make explicit a property of parameters which held in our main case above, and allowed them to essentially just depend on a sequence of graphs.[18] Note that 12.3(1)(c) says that we can write, for each level k, a graph $\mathcal{S}_k$ such that the connections in $\mathcal{R}$ depend only on these graphs.

Definition 12.3.

(1) *We say $\mathfrak{m}$ is forgetful when*
  - (a) *$\mathfrak{m}$ is self-dual for transparency*[19]
  - (b) *for each $\mathbf{i} = 1,2$ and k there is $m^{\mathbf{i}}_k$ such that for all $\eta \in \mathcal{T}_{\mathbf{i},k}$,*
    $(\forall j)(\eta^\frown\langle j\rangle \in \mathcal{T}_{\mathbf{i},k+1} \iff j < m^{\mathbf{i}}_k)$
  - (c) *for each k there is $S_k \subseteq m^1_k \times m^2_k$ such that if $(\eta,\nu) \in \mathcal{T}_{1,k} \times \mathcal{T}_{2,k}$ then $(\eta,\nu) \in \mathcal{R}^{\mathfrak{m}}_k$ if and only if $(\forall \ell < k)(\eta(\ell),\nu(\ell)) \in S_\ell)$.*

(2) *So if $\mathfrak{m}$ is forgetful and $k < \omega$, then $\mathfrak{n} = \mathfrak{m}\restriction_{\geq k}$ is defined by:*
  $m^{\mathbf{i}}_n[\mathfrak{n}] = m^{\mathbf{i}}_{n+k}[\mathfrak{m}]$*, and this does not depend on* $\mathbf{i}$
  $\mathcal{T}_{\mathbf{i},n}[\mathfrak{n}] = \prod_{i<n} m^{\mathbf{i}}_{k+i}[\mathfrak{m}]$*, and*
  $S_n[\mathfrak{n}] = S_{k+n}[\mathfrak{m}]$*, which are each symmetric.*

Informally, all nodes at level k have the same number of immediate successors, and when extending edges to the next level (which recall only happens between elements whose restrictions are connected at all earlier levels) only the last value matters:

Remark 12.4. 12.3(1)(*b*) *implies that if we are given $\eta \in \mathcal{T}_{1,k}$ and $\rho \in \mathcal{T}_{2,k}$ and $\eta' \in \mathcal{T}_{1,k}$ and $\rho' \in \mathcal{T}_{2,k}$, and $(\eta,\rho) \in \mathcal{R}$ and $(\eta',\rho') \in \mathcal{R}$, then for any $i < m^1_{k+1}$ and $j < m^2_{k+1}$,*

$$(\eta^\frown\langle i\rangle, \rho^\frown\langle j\rangle) \in \mathcal{R} \iff (\eta'{}^\frown\langle i\rangle, \rho'{}^\frown\langle j\rangle) \in \mathcal{R}.$$

So the parameter depends in the natural way on a sequence of graphs. The parameters used in the main arguments above satisfied this definition. The following easy consequence will also be useful.

Claim 12.5. *Suppose $\mathfrak{m}$ is a forgetful parameter, $\eta_* = \eta_0{}^\frown\langle i\rangle^\frown\eta_\infty \in \lim(\mathcal{T}^{\mathfrak{m}}_1)$, and $\rho_* = \rho_0{}^\frown\langle j\rangle^\frown\rho_\infty \in \lim(\mathcal{T}^{\mathfrak{m}}_2)$, where $\operatorname{lgn}(\eta_0) = \operatorname{lgn}(\rho_0) < \omega$. Suppose that for*

[18] In Definition 12.3, whether $(\eta^\frown\langle\ell_1\rangle, \rho^\frown\langle\ell_2\rangle)$ is an edge in $\mathcal{R}$ depends only on the fact that $(\eta,\rho) \in \mathcal{R}$ and on ℓ_1, ℓ_2. There is no "memory" coming from the path in the tree which affects the graph used at this stage (just the level matters).

[19] so we don't really need m^1 and m^2 in item (b)

every $s < \omega$, $(\eta_* \restriction s,\ \rho_* \restriction s) \in \mathcal{R}^{\mathfrak{m}}$. If we replace i, j by i', j' respectively so that $(\eta_0{}^\frown\langle i'\rangle,\ \rho_0{}^\frown\langle j'\rangle) \in \mathcal{R}^{\mathfrak{m}}$, then it remains true that for every $s < \omega$,

$$(\eta_0{}^\frown\langle i'\rangle^\frown\eta_\infty \ \restriction s,\ \rho_0{}^\frown\langle j'\rangle^\frown\rho_\infty \restriction s) \in \mathcal{R}^{\mathfrak{m}}.$$

Proof. By definition of forgetful. $\square$

Next, let $\mathfrak{m}$ be any forgetful parameter. We define a useful class of restricted models, starting with the simplest case.

Definition 12.6. *For any $\mathfrak{m}$ and $M \models T^0_{\mathfrak{m}}$, $k < \omega$, and $(\eta, \rho) \in \mathcal{T}^{\mathfrak{m}}_{1,k} \times \mathcal{T}^{\mathfrak{m}}_{2,k}$,*

(A) we define $\mathfrak{n} = \mathfrak{m} \restriction (\eta, \rho)$ by:
- *(a) $\mathcal{T}^{\mathfrak{n}}_1 = \{\nu : \eta^\frown\nu \in \mathcal{T}^{\mathfrak{m}}_1\}$*
- *(b) $\mathcal{T}^{\mathfrak{n}}_2 = \{\nu : \rho^\frown\nu \in \mathcal{T}^{\mathfrak{m}}_2\}$*
- *(c) $\mathcal{R}^{\mathfrak{n}}_k = \{(\nu_1, \nu_2) : (\eta^\frown\nu_1, \rho^\frown\nu_2) \in \mathcal{R}^{\mathfrak{m}}_k\}$*

(B) we define $N = N[\eta, \rho, M]$ as the following $\tau(T_{\mathfrak{n}})$-model:
- *(a) the universe is $Q^M_\eta \cup P^M_\rho$.*
- *(b) $Q^N_\nu = Q^M_{\eta^\frown\nu}$ for $\eta^\frown\nu \in \mathcal{T}^{\mathfrak{m}}_1$*
- *(c) $P^N_\nu = P^M_{\eta^\frown\nu}$ for $\eta^\frown\nu \in \mathcal{T}^{\mathfrak{m}}_2$*
- *(d) $\mathcal{R}^N = \mathcal{R}^M \restriction (Q^M_\eta \times P^M_\rho)$.*

Claim 12.7. *Assume $\mathfrak{n} = \mathfrak{m} \restriction (\eta, \rho)$ where $(\eta, \rho) \in \mathcal{R}^{\mathfrak{m}}_k$, and $N = N[\eta, \rho, M]$. Then:*

(1) If $M \models T^0_{\mathfrak{m}}$, then $N \models T^0_{\mathfrak{n}}$.
(2) If $M \models T_{\mathfrak{m}}$, then:
- *(a) $N \models T_{\mathfrak{n}}$, and*
- *(b) if $\bar{a}, \bar{b} \in {}^nN$ then $\mathrm{tp}(\bar{a}, \emptyset, M) = \mathrm{tp}(\bar{b}, \emptyset, M)$ if and only if $\mathrm{tp}(\bar{a}, \emptyset, N) = \mathrm{tp}(\bar{b}, \emptyset, N)$.*

(3) If M is a κ-saturated model of $T_{\mathfrak{m}}$, then N is a κ-saturated model of $T_{\mathfrak{n}}$.

Proof. (1) is by the definitions. (2) (a) is easy. (2)(b) is by the elimination of quantifiers. (3) follows from (2)(b). (Or see 12.12 below.) $\square$

Claim 12.8. *Assume $\mathfrak{m}$ is forgetful and $k < \omega$.*

(Claim) If $(\eta, \rho) \in \mathcal{R}_{\mathfrak{m},k}$ and $(\eta', \rho') \in \mathcal{R}_{\mathfrak{m},k}$ then $\mathfrak{m} \restriction (\eta, \rho) = \mathfrak{m} \restriction (\eta', \rho')$.
(Defn) Let $\mathfrak{n} = \mathfrak{m} \restriction_{\geq k}$ be $\mathfrak{m} \restriction (\eta, \rho)$ for any (some) $(\eta, \rho) \in \mathcal{R}_{\mathfrak{m},k}$.

Proof. By the definition of forgetful. $\square$

Now we generalize 12.6, but there is a subtlety. Suppose we are given $M \models T_{\mathfrak{n}}$ (or just $M \models T^0_{\mathfrak{n}}$) and $k < \omega$, $\Lambda_1 \subseteq \mathcal{T}_{1,k}$, $\Lambda_2 \subseteq \mathcal{T}_{2,k}$ such that $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}$ (i.e. a complete bipartite graph in the template at level k). We define $N = N[\Lambda_1, \Lambda_2, M]$ <u>not</u> by looking at the multi-rooted subtrees of $\mathcal{T}_1$, $\mathcal{T}_2$ which start in Λ_1 or Λ_2 respectively, and then restricting M to predicates coming from these multi-rooted subtrees in the natural sense. <u>Rather</u>, we form a new model "by gluing" (or "by stacking on top of") all the subtrees with roots in Λ_1 to make the rooted tree on the left, and stacking up all the subtrees with roots in Λ_2 to make the rooted tree on the right. So $Q^N_{\langle\rangle}$ is the union $\bigcup\{Q^M_\eta : \eta \in \Lambda_1\}$ and $P^N_{\langle\rangle}$ is the union $\bigcup\{P^M_\rho : \rho \in \Lambda_2\}$. In general, Q^N_ν is the union of $\bigcup\{Q^M_{\eta^\frown\nu} : \eta \in \Lambda_1\}$, and P^N_ν is the union of $\bigcup\{P^M_{\rho^\frown\nu} : \rho \in \Lambda_2\}$. The edges in the model stay the same: $R^N = R^M \restriction Q^N_{\langle\rangle} \times P^N_{\langle\rangle}$ [note: roman R in the model, not $\mathcal{R}$ in the template; for more, see 12.12]. This works well because $\mathfrak{m}$ is forgetful, so the subtrees sit exactly on top of each other, with the same branching and the same information when it comes to $\mathcal{R}$.

We will be most interested in cases of the form $N = N[\{\eta\}, \Lambda, M]$ where η is a singleton and $\Lambda = \{\rho \in \mathcal{T}_2 : (\eta, \rho) \in \mathcal{R}\}$ is the set of all neighbors of η, or the parallel $N = N[\Lambda, \{\rho\}, M]$, though we give the general definition.

Definition 12.9. *Assume $\mathfrak{m}$ is forgetful, $M \models T_{\mathfrak{m}}$, $k < \omega$, $\emptyset \neq \Lambda_\ell \subseteq \mathcal{T}^{\mathfrak{m}}_{\ell,k}$ for $\ell = 1, 2$ and $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}^{\mathfrak{m}}_k$.*

We define $N = N[\Lambda_1, \Lambda_2, M]$ as the following $\tau(\mathfrak{m}\restriction_{\geq k})$-model:
- *universe: $\bigcup\{Q^M_\eta \cup P^M_\rho : (\eta, \rho) \in \Lambda_1 \times \Lambda_2\}$*
- *$Q^N_\nu = \bigcup\{Q^M_{\eta^\frown\nu} : \eta \in \Lambda_1\}$, so $\mathcal{Q}^N = Q^N_{\langle\rangle}$.*
- *$P^N_\nu = \bigcup\{P^M_{\eta^\frown\nu} : \eta \in \Lambda_1\}$, so $\mathcal{P}^N = P^N_{\langle\rangle}$.*
- *$R^N = R^M \restriction \mathcal{Q}^N \times \mathcal{P}^N$.*

Since we will mainly use the case where either Λ_1 or Λ_2 has cardinality 1, we spell out this special case.

Observation 12.10. *Let $\mathfrak{m}$ be a forgetful parameter. For $M \models T_{\mathfrak{m}}$,*
1. *if $\Lambda_1 = \{\eta\}$ and $\Lambda_2 \subseteq \{\rho \in \mathcal{T}_{2,k} : (\eta, \rho) \in \mathcal{R}\}$, then our model has*
   - *universe: $Q^M_\eta \cup \bigcup\{P^M_\rho : \rho \in \Lambda_2\}$*
   - *$Q^N_\nu = Q^M_{\eta^\frown\nu}$.*
   - *$P^N_\nu = \bigcup\{P^M_{\eta^\frown\nu} : \eta \in \Lambda\}$.*
   - *$R^N = R^M \restriction \mathcal{Q}^N \times \mathcal{P}^N$, i.e. we retain all existing edges between elements which make it into the domain of the new model.*
2. *we have the parallel in the case where $\Lambda_2 = \{\rho\}$ and $\Lambda_1 \subseteq \{\eta \in \mathcal{T}_{1,k} : (\eta, \rho) \in \mathcal{R}\}$.*

Convention 12.11. *We may write $N[\eta, \Lambda, M]$ instead of $N[\{\eta\}, \Lambda, M]$ and likewise we may write $N[\Lambda, \rho, M]$ instead of $N[\Lambda, \{\rho\}, M]$.*

Comparing Definition 12.6 and Definition 12.9, the reader will notice that in 12.9 we don't define a new parameter $\mathfrak{n} = \mathfrak{n} \restriction (\Lambda_1, \Lambda_2)$. This is because, by forgetfulness, we already have an appropriate parameter, namely: in 12.10, the structure $N = N[\Lambda_1, \Lambda_2, M]$ is a model of $T^0_{\mathfrak{n}}$ where $\mathfrak{n} = \mathfrak{m}\restriction_{\geq k}$.

Claim 12.12. *Assume $\mathfrak{m}$ is forgetful. Assume (Λ_1, Λ_2) are as in 12.10, $N = N[\Lambda_1, \Lambda_2, M]$ and $\mathfrak{n} = \mathfrak{m}\restriction_{\geq k}$. Then:*
1. *If $M \models T^0_{\mathfrak{m}}$, then $N \models T^0_{\mathfrak{n}}$.*
2. *If $M \models T_{\mathfrak{m}}$, then:*
   - *(a) $N \models T_{\mathfrak{n}}$, and*
   - *(b) if $\bar{a}, \bar{b} \in {}^nN$ and $\mathrm{tp}(\bar{a}, \emptyset, M) = \mathrm{tp}(\bar{b}, \emptyset, M)$, then $\mathrm{tp}(\bar{a}, \emptyset, N) = \mathrm{tp}(\bar{b}, \emptyset, N)$.*
3. *If M is a κ-saturated model of $T_{\mathfrak{m}}$, then N is a κ-saturated model of $T_{\mathfrak{n}}$.*
4. *If M is a κ-special model of $T_{\mathfrak{m}}$, then N is a κ-special model of $T_{\mathfrak{n}}$.*

Proof. Easy, but as this is central we elaborate.

(1) There are two parts to check: the unary predicates match the structure of the tree, and the edges R indeed reflect the instructions of $\mathcal{R}^{\mathfrak{n}}$. Since $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}^{\mathfrak{m}}$, and $\mathfrak{m}$ is forgetful, $\mathfrak{n} = \mathfrak{m}\restriction_{\geq k}$ is well defined. Fix for a moment $\nu_1 \in \mathcal{T}^{\mathfrak{n}}_{1,\ell}$, $\nu_2 \in \mathcal{T}^{\mathfrak{n}}_{2,\ell}$. Then whenever $(\nu_1, \nu_2) \notin \mathcal{R}^{\mathfrak{n}}$, $T^0_{\mathfrak{n}}$ implies $(\forall x \in Q_{\nu_1})(\forall y \in P_{\nu_2})(\neg R(x, y))$. So let us check: in the model N,

$$Q^N_{\nu_1} = \bigcup\{Q^M_{\eta^\frown\nu_1} : \eta \in \Lambda_1\}$$

$$P^N_{\nu_2} = \bigcup\{P^M_{\eta^\frown\nu_1} : \eta \in \Lambda_2\}.$$

Suppose $(\nu_1, \nu_2) \notin \mathcal{R}_{\mathfrak{n},\ell}$. Then (as $\mathfrak{m} \restriction_{\geq k}$) is well defined) for any choice of $(\eta, \rho) \in \Lambda_1 \times \Lambda_2$, $(\eta^\frown\nu_1, \rho^\frown\nu_2) \notin \mathcal{R}_{\mathfrak{m},k+\ell}$. Since for all $a \in Q^N_{\nu_1}$, and all $b \in P^N_{\nu_2}$, there are $\eta \in \Lambda_1$ and $\rho \in \Lambda_2$ such that $a \in Q^M_{\eta^\frown\nu_1}$ and $b \in P^M_{\rho^\frown\nu_2}$, $M \models \neg R(a, b)$, so as N changes no edges among the elements in its domain, also $N \models \neg R(a, b)$. So $(\forall x \in Q_{\nu_1})(\forall y \in P_{\nu_2})(\neg R(x, y))$ holds in N. In this way we can verify the universal axioms of $T^0_{\mathfrak{n}}$.

(2) Since $T_{\mathfrak{n}}$ eliminates quantifiers, it suffices to check truth of relevant formulas following the strategy of the quantifier elimination argument 2.17, which is straightforward.

(3) Without loss of generality M is κ-saturated. Convention: given a leaf, say $\rho \in \lim(\mathcal{T}^n_2)$, let us write P^M_ρ to abbreviate $\bigcap\{P^M_{\rho\restriction n} : n < \omega\}$, and the parallel for $\lim(\mathcal{T}^n_1)$. Sometimes we will repeat this for emphasis. As $T_{\mathfrak{n}}$ has elimination of quantifiers, it suffices to verify that:

> $(\star)_1$ if $\nu \in \lim(\mathcal{T}^{\mathfrak{n}}_1)$, $X = \{\rho \in \lim(\mathcal{T}^{\mathfrak{n}}_2) : \bigwedge_m (\nu \restriction m, \rho \restriction m) \in \mathcal{R}^{\mathfrak{n}}_m\}$, and $A, B \subseteq \bigcup\{P^N_\rho : \rho \in X\}$ are disjoint and $|A|+|B| < \kappa$, then for some $c \in Q^N_\nu = \bigcap_n Q^N_{\nu\restriction n}$ we have that $(c, a) \in R^N$ for all $a \in A$, and $(c, b) \notin R^N$ for all $b \in B$.

as well as $(\star)_2$ the dual reversing the trees.

In the case where A, B are finite, this is the content of redoing the elimination of quantifiers as mentioned in (2).

As we are using Λ_1, Λ_2, symmetry holds and it suffices to prove $(\star)_1$. In particular it suffices to prove that for some $\eta \in \Lambda_1$ there is $c \in \bigcap\{Q^M_{\eta^\frown\nu\restriction n} : n < \omega\}$ such that $(c, a) \in R^M$ for all $a \in A$, and $(c, b) \notin R^M$ for all $b \in B$. Thus we move the problem to the model M.

In M, the sets A and B clearly remain disjoint. Let us locate the a's and b's. Since our N was a model of $T_{\mathfrak{n}}$, for each $a \in A$, there is $\nu_a \in \lim(\mathcal{T}^{\mathfrak{n}}_2)$ such that $a \in P^N_{\nu_a}$, and likewise for each $b \in B$, there is $\nu_b \in \lim(\mathcal{T}^{\mathfrak{n}}_2)$ such that $b \in P^N_{\nu_b}$. Therefore in M, for each $a \in A$ there is $\rho_a \in \Lambda_2$ such that $a \in P^M_{\rho_a{}^\frown\nu_a}$, and likewise for each $b \in B$ there is $\rho_b \in \Lambda_2$ such that $b \in P^M_{\rho_b{}^\frown\nu_b}$, though the positive part is what is most important. Choose any $\eta \in \Lambda_1$. Since $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}^{\mathfrak{m}}$, and by definition of X, we conclude that $\bigwedge_n(\eta^\frown\nu \restriction n,\ \rho_a{}^\frown\nu_a \restriction n) \in \mathcal{R}^{\mathfrak{m}}_n$. So the type $p(x)$ saying that $\{R(x, a) : a \in A\} \cup \{\neg R(x, b) : b \in B\} \cup \{P_{\eta^\frown\nu\restriction n}(x) : n < \omega\}$ is indeed a consistent partial type in M, and therefore there is such a $c \in |M|$ satisfying $(c, a) \in R^M$ for all $a \in A$, and $(c, b) \notin R^M$ for all $b \in B$, by saturation of M. Moreover, by definition of $p(x)$, it must be that $c \in P^M_\eta$, thus $c \in |N|$ by construction. This completes the proof.

(4) Similarly to (3). $\square$

Claim 12.13. *Let $\mathcal{D}$ be a regular ultrafilter on I, $|I| = \lambda$. Suppose $\mathfrak{m}$ is a forgetful parameter (thus self-dual) and $k < \omega$. Suppose $\mathfrak{n} = \mathfrak{m} \restriction_{\geq k}$. Then the following are equivalent:*

(1) $(M_{\mathfrak{m}})^I/\mathcal{D}$ is λ^+-saturated, for some, equivalently every, $M_{\mathfrak{m}} \models T_{\mathfrak{m}}$.
(2) $(M_{\mathfrak{n}})^I/\mathcal{D}$ is λ^+-saturated, for some, equivalently every, $M_{\mathfrak{n}} \models T_{\mathfrak{n}}$.

In other words, $T_{\mathfrak{m}}$ and $T_{\mathfrak{n}}$ are equivalent in Keisler's order.[20]

Proof. First note: for each given η, Λ, clearly we can interpret $N[\eta, \Lambda, M]$ in the model M. So as ultrapowers commute with reducts, $(N[\eta, \Lambda, M])^I/\mathcal{D}$ is canonically isomorphic to $N[\eta, \Lambda, M^I/\mathcal{D}]$. The parallel holds for $N[\Lambda, \eta, M]$, though since $\mathfrak{m}$ is forgetful therefore symmetric, it suffices to consider the first case.

(1) implies (2): As the ultrafilter is regular, it won't matter which model of $T_{\mathfrak{m}}$ or $T_{\mathfrak{n}}$ we choose (see Keisler [9] 2.1a), so we may as well choose $\eta \in \mathcal{T}^{\mathfrak{m}}_{1,k}$, $\rho \in \mathcal{T}^{\mathfrak{m}}_{2,k}$ and $M_{\mathfrak{n}} = N[\eta, \rho, M_{\mathfrak{m}}]$. Now apply 12.12 and the fact that ultrapowers commute with reducts.

(2) implies (1): Suppose (2). Then $\mathcal{D}$ is good for the theory of the random graph, because the theory of $M_{\mathfrak{n}}$ will be unstable and the theory of the random graph is minimum in Keisler's order among the unstable theories.

Let $M^*_{\mathfrak{m}} = (M_{\mathfrak{m}})^I/\mathcal{D}$. By Conclusion 5.7, it suffices to show that every partial type of $M^*_{\mathfrak{m}}$ of the form

$$r(x) = \{Q_\nu(x)\} \cup \{R(x,a) : a \in A\}$$

is realized, where $\nu \in \mathcal{T}_1$, $A \subseteq M^*_{\mathfrak{m}}$ and $|A| \leq \lambda$. So ν has finite length, without loss of generality $\mathrm{lgn}(\nu) \geq k$, though of course $> k$ is allowed. Let $\eta = \nu \restriction k$.

Since r is consistent, we may choose η_* such that:

(a) $\eta^\frown\eta_* \in \lim(\mathcal{T}^{\mathfrak{m}}_1)$, and also $\nu \trianglelefteq \eta^\frown\eta_*$. [That is, we extend ν to a branch, but we write it as its restruction to k plus the continuation.]

(b) and moreover the larger type

$$r_*(x) = \{Q_{\eta^\frown\eta_* \restriction \ell}(x) : \ell < \omega\} \cup \{R(x,a) : a \in A\}$$

is still consistent.

Moreover, for each $a \in A$, we may choose $\nu_a \in \mathcal{T}_{2,k}$ and ρ_a with $\nu_a{}^\frown\rho_a \in \lim(\mathcal{T}^{\mathfrak{n}}_2)$ such that for each $a \in A$, $M^*_{\mathfrak{m}} \models \{P_{\nu_a{}^\frown\rho_a \restriction \ell}(a) : \ell < \omega\}$. [Again we identify the branch of each $a \in A$, and write it as its restriction to k plus the continuation.]

By the fact that r_* is a [partial] type, note that

(c) if we write $\Lambda = \{\rho \in \mathcal{T}^{\mathfrak{m}}_{2,k} : (\eta, \rho) \in \mathcal{R}^{\mathfrak{m}}\}$, then $\nu_a \in \Lambda$ for each $a \in A$.

(d) moreover, for each $a \in A$ and each $\ell < \omega$,

$$(\eta^\frown\eta_* \restriction \ell,\ \nu_a{}^\frown\rho_a \restriction \ell) \in \mathcal{R}^{\mathfrak{m}}.$$

Let $M_{\mathfrak{n}} = N[\eta, \Lambda, M_{\mathfrak{m}}]$. Then, recalling the beginning of the proof, we have that $M^*_{\mathfrak{n}} = N[\eta, \Lambda, M^*_{\mathfrak{m}}] = (\ N[\eta, \Lambda, M_{\mathfrak{m}}]\ )^I/\mathcal{D}$.

By (c), we have that $A \subseteq M^*_{\mathfrak{n}}$ [i.e. $A \subseteq \mathrm{dom}(M^*_{\mathfrak{n}})$].

By (d) and the definition of forgetful, we have that $(\eta_* \restriction \ell, \rho_a \restriction \ell) \in \mathcal{R}^{\mathfrak{n}}$ for each $a \in A$ and $\ell < \omega$.

Thus $q_*(x) = \{Q_{\eta_* \restriction \ell}(x) : \ell < \omega\} \cup \{R(x,a) : a \in A\}$ is a partial type in $M^*_{\mathfrak{m}}$. By our hypothesis (2), $M^*_{\mathfrak{n}}$ is λ^+-saturated, so q_* is realized, say by b. By construction, $\mathrm{dom}(M^*_{\mathfrak{n}}) \subseteq \mathrm{dom}(M^*_{\mathfrak{m}})$, so $b \in M^*_{\mathfrak{m}}$, indeed $M^*_{\mathfrak{m}} \models Q_\eta(b)$. Moreover, since ν (from the definition of $r(x)$ above) is an initial segment of $\eta^\frown\eta_*$, $M^*_{\mathfrak{m}} \models Q_\nu(b)$. This shows that b realizes $r(x)$ in $M^*_{\mathfrak{m}}$, which completes the proof. $\square$

The proofs just given show:

Conclusion 12.14. *Let $\mathfrak{m}$ be a forgetful parameter.*

[20] We pedantically avoid using the letter N for a model in this proof to avoid confusion with the operation $N[\Lambda_1, \Lambda_2, M]$.

(1) *Suppose $M = T_{\mathfrak{m}}$. M is λ^+-saturated if and only if for every pair (Λ_1, Λ_2) such that $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}^{\mathfrak{m}}$ and either $|\Lambda_1| = 1$ or $|\Lambda_2| = 1$, the model $N = N[\Lambda_1, \Lambda_2, M]$ is λ^+-saturated.*
(2) *Let $k < \omega$ and let $\mathfrak{n} = \mathfrak{m} \restriction_{\geq k}$. Then $T_{\mathfrak{m}}$, $T_{\mathfrak{n}}$ are equivalent in Keisler's order.*

The above shows that without loss of generality $A \subseteq B$, so Claim 12.15, which we now state, will follow from 12.24 below.

Claim 12.15. *Suppose $A \subseteq^* B$. Then $T_{\mathfrak{m}[A]} \trianglelefteq T_{\mathfrak{m}[B]}$.*

Proof: this will follow from 12.24 below.

Recall that $\mathcal{Q} = Q_{\langle\rangle}$, $\mathcal{P} = P_{\langle\rangle}$. Write $\mathcal{T}_{\mathbf{i}}^A$ for $\mathcal{T}_{\mathbf{i}}^{\mathfrak{m}[A]}$, $\mathcal{T}_{\mathbf{i}}^B$ for $\mathcal{T}_{\mathbf{i}}^{\mathfrak{m}[B]}$ and $\mathbf{i} = 1, 2$, and $\mathcal{R}^A = \mathcal{R}^{\mathfrak{m}[A]}$, $\mathcal{R}^B = \mathcal{R}^{\mathfrak{m}[B]}$.

Observation 12.16. *As $A \subseteq B$, we have that:*

- $\mathcal{T}_{\mathbf{i}}^A = \mathcal{T}_{\mathbf{i}}^B$ *for* $\mathbf{i} = 1, 2$.
- $\mathcal{R}^B \subseteq \mathcal{R}^A$.

Convention 12.17. *Notation:*

(1) *If $a \in \mathcal{P}^M$, saying that "ρ is the leaf of a" means that ρ is the unique element of $\lim(\mathcal{T}_2^M)$ such that $M \models P_{\rho\restriction\ell}(a)$ for $\ell < \omega$, and similarly for $a \in \mathcal{Q}^M$ and $\eta \in \lim(\mathcal{T}_1^M)$.*
(2) *Write $(\eta, \rho) \in \mathcal{R}_\infty^A$ to mean that $\eta \in \lim(\mathcal{T}_1^A)$, $\rho \in \lim(\mathcal{T}_2^A)$ and for all $\ell < \omega$, $(\eta \restriction \ell, \rho \restriction \ell) \in \mathcal{R}^A$.*

Observation 12.18. *Suppose $n < \omega$, $Y \subseteq \mathcal{P}^{M_A}$ is finite, $\eta \in \mathcal{T}_1^A$. For $a \in Y$ let $\rho_a \in \lim(\mathcal{T}_2^A)$ be the leaf of a. The following are equivalent:*

(1) *$M_A \models (\exists x) \bigwedge_{a\in Y}(\ Q_\eta(x) \wedge R(x, a)\ )$*
(2) *there is $\nu \in \lim(\mathcal{T}_1^A)$, $\eta \trianglelefteq \nu$ such that $(\nu, \rho_a) \in \mathcal{R}_\infty^A$ for each $a \in Y$.*

Observation 12.19 (Discussion/Observation)**.** Suppose we are in a model $M_A \models T_A$, $\eta \in \mathcal{T}_1^A$ [so $\lg n(\eta) < \omega$], and we have a finite set of formulas [*not necessarily a partial type*] of the form

$$\{Q_\eta(x)\} \cup \{R(x, a) : a \in X\}$$

which we may write for uniformity as a sequence, possibly with repetitions:

$$p(x) = \langle Q_\eta(x) \wedge R(x, a) : a \in X\rangle.$$

For each $a \in X$, let $\rho_a \in \lim(\mathcal{T}_2^A)$ be the leaf of a. Then the "pattern" of this set of formulas in M_A is captured by the data of: for which $\sigma \subseteq X$ is it the case that there exists $\eta_* \in \lim(\mathcal{T}_1^A)$ such that (i) $\eta \trianglelefteq \eta_*$ and (ii) $(\eta_*, \rho_a) \in \mathcal{R}_\infty^A$ for all $a \in X$. Call this set

$$\Sigma = \Sigma_{p,X,M_A} \subseteq \mathcal{P}(X).$$

Discussion 12.20. *One obstruction to the natural try at $A \subseteq B$ implies $T_A \trianglelefteq T_B$ is: Consider $M_A \models T_A$, $M_B \models T_B$, $\mathcal{D}$ a regular ultrafilter on I, $M_A^* = (M_A)^I/\mathcal{D}$, $M_B^* = (M_B)^I/\mathcal{D}$. Now we might like to show that given a certain type of M_A^*, we can copy it to a type of M_B^* and apply saturation of the second to solve the first. A natural way to copy is to assign finitely many formulas of p to each index $t \in I$, using the regularizing family, and then copy this finite pattern from M_A to M_B. The natural way to copy patterns is to use the same leaves, but even though $\mathcal{T}_{\mathbf{i}}^A = \mathcal{T}_{\mathbf{i}}^B$, the same leaves may give a different pattern in M_B as $\mathcal{R}^A \supsetneq \mathcal{R}^B$.*

Instead, we show that given M_A and a finite pattern there, we can first modify the leaves involved a little without changing the pattern (informally, we make sure there is no 'spreading out' on $k \in \omega \setminus A$), and second, that any such modified set of leaves behaves as desired in $\mathcal{R}^B$. This allows the plan to go through.

Claim 12.21. *Recalling that $\mathfrak{m}[A]$ and $\mathfrak{m}[B]$ are forgetful, suppose we are given $\langle Q_\eta(x) \wedge R(x,a) : a \in X\rangle$ in M_A where:*

- $\eta \in \mathcal{T}_1^A$
- *X finite,*
- *for each $a \in X$, $M_A \models (\exists x)(\ Q_\eta(x) \wedge R(x,a)\ )$*
- thus *we are given $\langle \rho_a : a \in X\rangle$ where $\rho_a \in \lim(\mathcal{T}_2^A)$ is the leaf of a.*

Let $\langle \rho^*_a : a \in X\rangle \in {}^{|X|}\lim(\mathcal{T}_1^B)$ satisfy the following two conditions:

(a) for $k \in A$ and $a \in X$, $\rho^*_a(k) = \rho_a(k)$
(b) for $k \in \omega \setminus A$, for all $a \in X$, $\rho^*_a(k) = 0$

Then for some $\nu \in \mathcal{T}_1^B$ with $\mathrm{lgn}(\nu) = \mathrm{lgn}(\eta)$, for every $\sigma \subseteq X$, the following are equivalent:

(1) there exists $\eta_\sigma \in \lim(\mathcal{T}_1^A)$, $\eta \trianglelefteq \eta_\sigma$ and $(\eta_\sigma, \rho_a) \in \mathcal{R}^A_\infty$ for every $a \in \sigma$
(2) there exists $\nu_\sigma \in \lim(\mathcal{T}_1^B)$, $\nu \trianglelefteq \nu_\sigma$ and $(\nu_\sigma, \rho^*_a) \in \mathcal{R}^B_\infty$ for every $a \in \sigma$.

Informally, *"the pattern remains unchanged and transfers to M_B".*

Remark 12.22. *It follows from Claim 12.21 and Observation 12.18 that if we choose distinct elements $b_a \in |M_B|$ for $a \in X$ so that for each $a \in X$, ρ^*_a is the leaf of b_a in M_B, then for any $\sigma \subseteq X$, the following are equivalent:*

$(\star)_1$ $M_A \models (\exists x) \bigwedge_{a\in\sigma}(\ Q_\eta(x) \wedge R(x,a)\ )$.
$(\star)_2$ $M_B \models (\exists x) \bigwedge_{a\in\sigma}(\ Q_\nu(x) \wedge R(x,b_a)\ )$.

Proof of Claim 12.21. Note: the construction in the proof will give a ν which depends only on η. In particular: for each $k < \mathrm{lgn}(\eta)$, define $\nu(k)$ by:

- if $k \in A$, $\nu(k) = \eta(k)$
- if $k \in \omega \setminus A$, $\nu(k) = \min\{i < m_k : (i,0) \in E^k\}$ which exists by the definition of "small" in the construction of the graphs $\bar{E}$.

Note that if $\mathrm{lgn}(\eta) \subseteq A$, then this gives $\nu = \eta$.

(1) implies (2). By (1) there is such an η_σ. Define $\nu_\sigma \in \lim(\mathcal{T}_1^B)$ [recalling $\mathcal{T}_1^B = \mathcal{T}_1^A$] as follows: for each $k < \omega$, define $\nu_\sigma(k)$ by:

(i) if $k \in A$, $\nu_\sigma(k) = \eta_\sigma(k)$
(ii) if $k \in \omega \setminus A$, $\nu_\sigma(k) = \min\{i < m_k : (i,0) \in E^k\}$ which exists by the definition of "small" in the construction of the graphs $\bar{E}$.

Then recall $\eta \trianglelefteq \eta_\sigma$ hence $\nu \trianglelefteq \nu_\sigma$. For each $a \in \sigma$, we will prove by induction on $\ell < \omega$ that

$$(\star)_\ell \qquad (\eta_\sigma \restriction \ell,\ \rho_a \restriction \ell) \in \mathcal{R}^A \implies (\nu_\sigma \restriction \ell,\ \rho^*_a \restriction \ell) \in \mathcal{R}^B.$$

For $\ell = 0$ this is trivially true as $(\langle\rangle, \langle\rangle) \in \mathcal{R}^A \cap \mathcal{R}^B$. Suppose $(\star)_\ell$ holds. If $\ell \in A$, then $\nu_\sigma(\ell) = \eta_\sigma(\ell)$ and $\rho_a(\ell) = \rho^*_a(\ell)$, so by inductive hypothesis and forgetfulness, $(\star)_{\ell+1}$ is immediate. If $\ell \notin A$, then by inductive hypothesis and (ii),

$$(\ (\nu_\sigma \restriction \ell)^\frown\langle \nu_\sigma(\ell)\rangle,\ (\rho^*_a \restriction \ell)^\frown\langle 0\rangle) \in \mathcal{R}^B$$

i.e. $(\nu_\sigma \restriction \ell+1, \rho^*_a \restriction \ell+1) \in \mathcal{R}^B$, so $(\star)_{\ell+1}$ holds.

This proves $(\star)_{\ell+1}$ and so proves this direction.

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(2) implies (1). Suppose there is such a ν_σ. Write $\nu_\sigma = \nu^\frown\nu_\infty$. Let $\eta_\sigma = \eta^\frown\nu_\infty$. So clearly $\eta \trianglelefteq \eta_\sigma$. For each $a \in \sigma$, we will prove by induction on $\ell < \omega$ that

$$\oplus_\ell \qquad (\nu_\sigma \restriction \ell,\ \rho^*_a \restriction \ell) \in \mathcal{R}^B \implies (\eta_\sigma \restriction \ell,\ \rho_a \restriction \ell) \in \mathcal{R}^A.$$

For $\ell \leq \lg(\eta)$ this is trivially true as it follows from the assumptions of the Claim (third bullet point) that $(\eta, \rho_a \restriction \lg(\eta)) \in \mathcal{R}^A$ for each $a \in X$, so also for each $a \in \sigma$. Suppose $\oplus_\ell$ holds and $\ell \geq \lg(\eta)$. If $\ell \in A$, then $\eta_\sigma(\ell) = \nu_\sigma(\ell)$ and $\rho_a(\ell) = \rho^*_a(\ell)$, so $\oplus_{\ell+1}$ follows by inductive hypothesis and forgetfulness. If $\ell \in \omega \setminus A$, then ℓ is a lazy level for A, so $\oplus_{\ell+1}$ is immediate from $\oplus_\ell$. This proves $\oplus_{\ell+1}$, which finishes this direction and so finishes the proof. $\square$

Lemma 12.23. *Let $\mathcal{D}$ be a regular ultrafilter on I, $|I| = \lambda$. Let $M^*_A = (M_A)^I/\mathcal{D}$. Let $M^*_B = (M_B)^I/\mathcal{D}$. Suppose $p(x) = \{Q_\eta(x)\} \cup \{R(x,a) : a \in X\}$ is a partial type of M_A, with $\eta \in \mathcal{T}^A_1$ and $|X| \leq \lambda$. There is a partial type $q(x) = \{Q_\eta(x)\} \cup \{R(x,b_a) : a \in X\}$ of M^*_B, such that if $q(x)$ is realized in M^*_B, then $p(x)$ is realized in M^*_A.*

Proof. Let $\{X_a : a \in X\} \subseteq \mathcal{D}$ be a regularizing family for $\mathcal{D}$. Enumerate p as $\langle Q_\eta(x) \wedge R(x,a) : a \in X\rangle$, *so here X may be infinite.* Let $f : [X]^{<\aleph_0} \to \mathcal{D}$ be the map given by

$$u \mapsto \{t \in I : M_A \models (\exists x) \bigwedge_{a \in u} (Q_\eta(x) \wedge R(x,a))\} \cap \bigcap_{a \in u} X_a.$$

In particular, for each $t \in I$, $X_t = \{a \in X : t \in f(\{a\})\}$ is finite.

We define the parameters for q (what will be the corresponding type in M^*_B) coordinatewise. For each $t \in I$, apply Claim 12.21 to the following sequence of formulas of M_A:

$$p_t(x) = \langle Q_\eta(x) \wedge R(x, a[t]) : a \in X_t\rangle.$$

Let ν_t and $\langle \rho^*_{t,a} : a \in X_t\rangle$ be as returned by that Claim. Observe that we have ν_t defined for every $t \in I$, but $\{\nu_t : t \in I\}$ is finite, so for some $J \in \mathcal{D}$, $\langle \nu_t : t \in J\rangle$ is constant. [Alternately, by the first sentence of the proof of Claim 12.21, ν_t depends only on η, so in fact we obtain the same ν_t for every $t \in I$.] So we call it ν.

For each $a \in X_t$, choose $b_{t,a}$ to be any element of $P^{M_B}_{\rho^*_{t,a}}$, without loss of generality distinct from any of the previous choices. For $a \in X \setminus X_t$, choose $b_{t,a}$ to be any element of $P^{M_B}_\emptyset$. Recalling 12.22, we have that for any $\sigma \subseteq X$, the following are equivalent:

(1) $M_A \models (\exists x) \bigwedge_{a \in \sigma} (\ Q_\eta(x) \wedge R(x, a[t])\ )$
(2) $M_B \models (\exists x) \bigwedge_{a \in \sigma} (\ Q_\nu(x) \wedge R(x, b_{t,a})\ )$

For each $a \in X$, define $b_a = \langle b_{t,a} : t \in I\rangle/\mathcal{D}$, so $b_a \in M^*_B$. Consider

$$q(x) = \langle Q_\nu(x) \wedge R(x, b_a) : a \in X\rangle.$$

Then, since q has the same 'pattern' as p at each index $t \in I$ under the function f, and p is a partial type in M^*_A, we have that $q(x)$ is also a partial type of M^*_B, and moreover, if q is realized in M^*_B then also p is realized in M^*_A. $\square$

Corollary 12.24. *If $A \subseteq B$, then $T_A \trianglelefteq T_B$.*

We include a slightly more general result for $\trianglelefteq^*$, the interpretability order. For the reader interested in Keisler's order, this is not essential to our main arguments.

Lemma 12.25. *Suppose $\mathfrak{m}_1, \mathfrak{m}_2$ are forgetful and $\mathfrak{m}_1 \restriction_{\geq k} = \mathfrak{n} = \mathfrak{m}_2 \restriction_{\geq k}$. Then $T_{\mathfrak{m}_1}$, $T_{\mathfrak{n}}$, $T_{\mathfrak{m}_2}$ are $\trianglelefteq^*$-equivalent, thus $\trianglelefteq$-equivalent.*

Proof. Let M_1, M, M_2 be models of $T_{\mathfrak{m}_1}$, $T_{\mathfrak{n}}$, $T_{\mathfrak{m}_2}$ respectively such that all three are special of cardinality $\mu = \beth_\delta$, δ limit. Let χ be such that $M_1, M, M_2 \in \mathcal{H}(\chi)$. Let $\mathfrak{B}_*$ be the model $(\mathcal{H}(\chi), \in, M_1, M, M_2)$ or an expansion of it. Note first that for $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}^{\mathfrak{m}_1}_k$, M is isomorphic to $N[\Lambda_1, \Lambda_2, M_1]$ (because M_1 is a special model of $T_{\mathfrak{m}_1}$, so unique in its cardinality) hence $N[\Lambda_1, \Lambda_2, M_1]$ is a model of $T_{\mathfrak{m}_1 \restriction_{\geq k}} = T_{\mathfrak{n}}$, hence $N[\Lambda_1, \Lambda_2, M_1]$ is special of cardinality μ (as it is interpreted in M_1). The parallel holds replacing M_1 by M_2. These isomorphisms are recorded by $\mathfrak{B}_*$. Hence they remain in any $\mathfrak{B} \equiv \mathfrak{B}_*$. That is,

$$\mathfrak{B}_* \models \; M \cong N[\Lambda_1, \Lambda_2, M_{\mathbf{i}}]$$

for $\mathbf{i} = 1, 2$, and Λ_1, Λ_2 as above. Moreover, for any $\mathfrak{B} \equiv \mathfrak{B}_*$,

$$\mathfrak{B} \models \; \text{“}M \cong N[\Lambda_1, \Lambda_2, M_{\mathbf{i}}]\text{”}$$

(as M, M_1, M_2 are definable elements). Note also that

$$(N[\Lambda_1, \Lambda_2, M])^{\mathfrak{B}} = N[\Lambda_1, \Lambda_2, M_{\mathbf{i}}^{\mathfrak{B}}], \;\; \mathbf{i} = 1, 2$$

by absoluteness. (Pedantically, if $\mathfrak{B}$ is not well-founded, the set $\{P_\eta : \mathfrak{B} \models P_\eta \in \tau(M_{\mathbf{i}})\}$ may have non-standard elements, but no harm.) Thus, for $\mathbf{i} = 1, 2$,

$$N[\Lambda_1, \Lambda_2, M_{\mathbf{i}}^{\mathfrak{B}}] \text{ is a special model of } T_{\mathfrak{n}} \text{ of cardinality } \mu \text{ isomorphic to } M^{\mathfrak{B}}$$

and so by transitivity, as both are isomorphic to $M^{\mathfrak{B}}$,

$$N[\Lambda_1, \Lambda_2, M_1^{\mathfrak{B}}] \cong N[\Lambda_1, \Lambda_2, M_2^{\mathfrak{B}}].$$

So $Th(\mathfrak{B}_*)$ witnesses that $T_{\mathfrak{m}_1}, T_{\mathfrak{n}}, T_{\mathfrak{m}_2}$ are $\trianglelefteq^*$-equivalent. Why? $M_{\mathbf{i}}^{\mathfrak{B}}$ is κ-saturated if and only if all the models of the form $N[\Lambda_1, \Lambda_2, M_{\mathbf{i}}^{\mathfrak{B}}]$ for $\Lambda_1 \times \Lambda_2 \subseteq \mathcal{R}^{\mathfrak{m}_{\mathbf{i}}}_k$ are κ-saturated, if and only if $M^{\mathfrak{B}}$ is κ-saturated, which suffices. $\square$

Claim 12.26. *Suppose $B \setminus A$ is infinite. Then $\neg(T_{\mathfrak{m}[B]} \trianglelefteq T_{\mathfrak{m}[A]})$.*

Proof. Just as in the main argument above.

[Discussion: Consider the chain condition Definition 8.2 with the cosmetic difference that we write A, B instead of using level functions: really, we could define $\xi_{\mathfrak{m}[A]}$ to be 1 if $n \notin A$ and 0 otherwise, and $\xi_{\mathfrak{m}[B]}$ to be 1 if $n \notin B$ and 0 otherwise, and then quote. When we dealt in the main argument with independent parameters, what we really used here from independence was the two halves: $A \setminus B$ is infinite, $B \setminus A$ is infinite. Here we only need one of those halves. Let $\mathcal{I}$ be the ideal generated by $\{A\}$. Since we are assuming $B \setminus A$ is infinite, this ideal does not cover all of $\mathcal{P}(\omega)$. Then for any $n \in \omega \setminus A$, the non-saturation proof goes through identically for $T_{\mathfrak{m}[B]}$: the chain condition will concentrate too many conditions at level n for $T_{\mathfrak{m}[B]}$ to be able to satisfy them. On the other hand, this will not affect $T_{\mathfrak{m}[A]}$ as all conditions assigned to level n are compatible, even a large set (and by the time we reach level $n+1$, the set which is large in the sense of level n is small in the sense of level $n+1$).] $\square$

So we arrive at:

Theorem 12.27. *We can find T_A for $A \subseteq \omega$ such that $T_A \trianglelefteq T_B$ if and only if $A \subseteq B$ mod finite.*

Proof. We use the family $\{T_{\mathfrak{m}[A]} : A \subseteq \omega\}$ defined above. If $A \subseteq B$ mod finite, apply 12.15. On the other hand, if $B \setminus A$ is infinite, apply Lemma 12.26.

Thus for $A, B \subseteq \omega$,

$$T_{\mathfrak{m}[A]} \trianglelefteq T_{\mathfrak{m}[B]} \text{ if and only if } A \subseteq^* B.$$

This is what we hoped to prove. □

In fact we have shown more:

Conclusion 12.28. *Suppose we consider the family $\{\mathfrak{m}[A] : A \subseteq \omega\}$ of parameters defined at the beginning of this section. Then:*

(1) $A \subseteq^* B \implies T_{\mathfrak{m}[A]} \trianglelefteq T_{\mathfrak{m}[B]}$, indeed $T_{\mathfrak{m}[A]} \trianglelefteq^* T_{\mathfrak{m}[B]}$
(2) $|B \setminus A| = \aleph_0 \implies \neg(T_{\mathfrak{m}[B]} \trianglelefteq T_{\mathfrak{m}[A]})$, thus a fortiori $\neg(T_{\mathfrak{m}[A]} \trianglelefteq^* T_{\mathfrak{m}[B]})$.

This completes the proof of Theorem 11.8.

To motivate the second theorem of this section, remember that when we were partitioning $\mathbf{M}_*$ into $\mathcal{M}$ and $\mathcal{N}$ in the previous section, we were essentially choosing a partition of independent subsets of ω. We may ask about which partitions are possible of *all* families of subsets of ω. The next theorem answers this question: it is 11.9 above, which we restate for convenience.

Theorem (Theorem 11.9). *There is a family of parameters $\{\mathfrak{m}[A] : A \subseteq \omega\}$ such that each $T_{\mathfrak{m}[A]}$ is countable, complete, simple, and low, and the following are equivalent for any $\lambda \geq 2^{\aleph_0}$ and any a set $\mathcal{X} \subseteq \mathcal{P}(\omega)$:*

(1) There exists a regular ultrafilter $\mathcal{D}$ on λ such that $\mathcal{X} = \{A \subseteq \omega : \mathcal{D}$ is $(\lambda^+, T_{\mathfrak{m}[A]})$-good $\}$.
(2) $\mathcal{X} \supseteq [\omega]^{<\aleph_0}$ is an ideal.

Proof of Theorem 11.9. Fix $\mathcal{X} \subseteq \mathcal{P}(\omega)$, and fix $\lambda \geq 2^{\aleph_0}$.

(2) $\rightarrow$ (1): Immediate from the earlier construction: simply choose the ideal $\mathcal{I}$ in the chain condition 8.2 to be our $\mathcal{X}$.

(1) $\rightarrow$ (2): Suppose $\mathcal{X}$ is an ideal on ω which extends the finite sets. If $A \in \mathcal{X}$ and $B \subseteq^* A$, then $\mathcal{D}$ is $(\lambda^+, T_{\mathfrak{m}[B]})$-good by 12.15 above. So it suffices to show that if $A \in \mathcal{X}$ and $B \in \mathcal{X}$, then $A \cup B$ is in $\mathcal{X}$. In other words, we shall fix $A, B \subseteq \omega$, and assume that $\mathcal{D}$ is a regular ultrafilter on $|I|$, $|I| = \lambda \geq 2^{\aleph_0}$ which is $(\lambda^+, T_{\mathfrak{m}[A]})$-good and $(\lambda^+, T_{\mathfrak{m}[B]})$-good, and we shall show that $\mathcal{D}$ is $(\lambda^+, T_{\mathfrak{m}[A\cup B]})$-good.[21]

Since $T_{\mathfrak{m}[A]}$ and $T_{\mathfrak{m}[B]}$ are both simple unstable, it follows that $\mathcal{D}$ is good for the theory of the random graph. Choose M_A, M_B, $M_{A\cup B}$ to be $\aleph_1$-saturated models of $T_{\mathfrak{m}[A]}$, $T_{\mathfrak{m}[B]}$, and $T_{\mathfrak{m}[A\cup B]}$ respectively. Let $M^*_A, M^*_B, M^*_{A\cup B}$ be the respective ultrapowers using $\mathcal{D}$. As usual, to show that $M^*_{A\cup B}$ is λ^+-saturated, it suffices to prove that all partial types of the form

$$r(x) = \{Q_\eta(x) \cup \{R(x, c) : c \in C\}$$

are realized, where $\eta \in \mathcal{T}_1^{\mathfrak{m}[A\cup B]}$ and $|C| \leq \lambda$. Fix such an r. Without loss of generality, we will assume $|C| = \lambda$.

[21]Informally, if $\mathcal{D}$ can handle types coming from trees where the levels in A are active, and trees where the levels in B are active, then it can handle types coming from trees where the levels in $A \cup B$ are active.

Let $\{Y_c : c \in C\} \subseteq \mathcal{D}$ be a regularizing family for $\mathcal{D}$. [22] Let

$$d : [C]^{<\aleph_0} \to \mathcal{D}$$

be the map given by:

$$u \mapsto \{t \in I : M_{A\cup B} \models \exists x \bigwedge_{c\in u} (Q_\eta(x) \wedge R(x, c[t]))\ \} \cap \bigcap_{c\in u} Y_c.$$

Note that d is monotonic ($u \subseteq v$ implies $d(u) \supseteq d(v)$), and for each $t \in I$, the set $C_t = \{c \in C : t \in d(\{c\})\}$ is finite.

For each t and $c \in C$, there is a leaf $\rho_{c,t} \in \lim(\mathcal{T}_2^{\mathfrak{m}[A]}) = \lim(\mathcal{T}_2^{\mathfrak{m}[B]})$ such that $M \models P_{\rho_{c,t}\restriction \ell}\ (c[t])$ for all $\ell < \omega$. For each $t \in I$ and each $c \in C$ choose[23] $a_{c,t}$ to be any element of M_A such that $M_A \models P_{\rho_{c,t}\restriction \ell}\ (a_{c,t})$ and choose $b_{c,t}$ to be any element of M_B such that $M_B \models P_{\rho_{c,t}\restriction \ell}\ (b_{c,t})$ for all $\ell < \omega$. Let $a_c = \langle a_{c,t} : t \in I\rangle/\mathcal{D} \in M^*_A$, and let $b_c = \langle b_{c,t} : t \in I\rangle/\mathcal{D} \in M^*/B$. [Without loss of generality, $a_c[t] = a_{c,t}$ and $b_c[t] = b_{c,t}$.] Consider

$$r_A = \{Q_\eta(x)\} \cup \{R(x, a_c) : c \in C\}$$

and consider

$$r_B = \{Q_\eta(x)\} \cup \{R(x, b_c) : c \in C\}.$$

Observe that $r_A(x)$ is a partial type in M^*_A since $\mathcal{R}^{\mathfrak{m}[A]} \supseteq \mathcal{R}^{\mathfrak{m}[A\cup B]}$, and likewise $r_B(x)$ is a partial type in M^*_B since $\mathcal{R}^{\mathfrak{m}[B]} \supseteq \mathcal{R}^{\mathfrak{m}[A\cup B]}$. By our assumption on $\mathcal{D}$, r_A is realized, say by $a_* \in M^*_A$, and r_B is also realized, say by $b_* \in M^*_B$. Let us define $d_* : [C]^{<\aleph_0} \to \mathcal{D}$ to be the refinement of d given by:

$$d_*(u) = d(u) \cap \{t \in I : M_A \models R(a_*[t], a_{c,t})\} \cap \{t \in I : M_B \models R(b_*[t], b_{c,t})\}.$$

Thus, for each $t \in I$, we may define $C^*_t = \{c \in C : t \in d_*(\{c\})\}$. It follows from the definition of d_* that for each $t \in I$, $C^*_t \subseteq C_t$, *thus* C^*_t is finite; moreover, for each $t \in I$,

(a) $$M_A \models (\exists x)\left(Q_\eta(x) \wedge \bigwedge_{c\in C^*_t} R(x, a_{c,t})\right)$$

[in particular $a_*[t]$ is such a witness] and likewise

(b) $$M_B \models (\exists x)\left(Q_\eta(x) \wedge \bigwedge_{c\in C^*_t} R(x, b_{c,t})\right)$$

[in particular $b_*[t]$ is such a witness]. Fix $t \in I$. We now aim to prove:

(c) $$M_{A\cup B} \models (\exists x)\left(Q_\eta(x) \wedge \bigwedge_{c\in C^*_t} R(x, c[t]))\right).$$

Recall that

(i) $\mathcal{T}_1^{\mathfrak{m}[A\cup B]} = \mathcal{T}_1^{\mathfrak{m}[A]} = \mathcal{T}_1^{\mathfrak{m}[B]}$

(ii) $\mathcal{R}^{\mathfrak{m}[A\cup B]}$ is contained in each of $\mathcal{R}^{\mathfrak{m}[A]}$ and $\mathcal{R}^{\mathfrak{m}[B]}$.

[22] i.e. any element of the family belongs to $\mathcal{D}$, but the intersection of any infinitely many elements of this family is empty – exists by definition of regular ultrafilter.

[23] One may follow these instructions for $t \in I$ and $c \in C_t$, and otherwise choose arbitrarily.

(iii) for each $\ell < \omega$, for some, equivalently every, choice of[24] $(\eta, \rho) \in \mathcal{R}_\ell^{\mathfrak{m}[A\cup B]}$, $(\eta_A, \rho_A) \in \mathcal{R}_\ell^{\mathfrak{m}[A]}$, $(\eta_B, \rho_B) \in \mathcal{R}_\ell^{\mathfrak{m}[B]}$ we have that: if $\ell \in A$, then for all $i, j < m_\ell$,

$(\eta^\frown\langle i\rangle, \rho^\frown\langle j\rangle) \in \mathcal{R}^{\mathfrak{m}[A\cup B]}$ if and only if $(\eta_A{}^\frown\langle i\rangle, \rho_A{}^\frown\langle j\rangle) \in \mathcal{R}^{\mathfrak{m}[A]}$

and if $\ell \in B$, then for all $i, j < m_\ell$,

$(\eta^\frown\langle i\rangle, \rho^\frown\langle j\rangle) \in \mathcal{R}^{\mathfrak{m}[A\cup B]}$ if and only if $(\eta_B{}^\frown\langle i\rangle, \rho_B{}^\frown\langle j\rangle) \in \mathcal{R}^{\mathfrak{m}[B]}$

and otherwise, if $\ell \in \omega \setminus A \cup B$, for all $i, j < m_\ell$,

$$(\eta^\frown\langle i\rangle, \rho^\frown\langle j\rangle) \in \mathcal{R}^{\mathfrak{m}[A\cup B]}.$$

[Of course if $\ell \in A\cap B$ then the two relevant conditions hold simultaneously.]

Recall that for each $c \in C_t^*$ we had defined its leaf $\rho_{c,t}$. It would suffice to prove that there is $\eta_* \in \lim(\mathcal{T}_1^{\mathfrak{m}[A\cup B]})$ so that

(d) $$(\eta_* \restriction s, \rho_{c,t} \restriction s) \in \mathcal{R}^{\mathfrak{m}[A\cup B]} \text{ for all } c \in C_t^* \text{ and all } s < \omega.$$

Let η_a be the leaf of $a_*[t]$, i.e. the unique element of $\lim(\mathcal{T}_1^{\mathfrak{m}[A]})$ such that $M_A \models Q_{\eta_a\restriction\ell}(a_*[t])$ for all $\ell < \omega$, and let η_b be the leaf of $b_*[t]$, i.e. the unique element of $\lim(\mathcal{T}_1^{\mathfrak{m}[B]})$ such that $M_B \models Q_{\eta_b\restriction\ell}(b_*[t])$ for all $\ell < \omega$. So necessarily $\eta \trianglelefteq \eta_a$ and $\eta \trianglelefteq \eta_b$. Suppose first that $\eta_a = \eta_b$. Let $\eta_* \in \lim(\mathcal{T}_1^{\mathfrak{m}[A\cup B]})$ be given by $\eta_* = \eta_a = \eta_b$. This η_* satisfies (d), as is easy to verify by inductively applying (iii) above.

If not, suppose that there is some $\mathbf{i} < \omega$ minimal for the property that[25] we have $\eta_a^{\mathbf{i}} \in \lim(\mathcal{T}_1^{\mathfrak{m}[A]})$, $\eta_b^{\mathbf{i}} \in \lim(\mathcal{T}_1^{\mathfrak{m}[B]})$ such that:

(1) $\eta_a^{\mathbf{i}} \restriction \mathbf{i} = \eta_b^{\mathbf{i}} \restriction \mathbf{i}$.
(2) $(\eta_a^{\mathbf{i}} \restriction s,\ \rho_{c,t} \restriction s) \in \mathcal{R}^{\mathfrak{m}[A]}$ for each $c \in C_t^*$
(3) $(\eta_b^{\mathbf{i}} \restriction s,\ \rho_{c,t} \restriction s) \in \mathcal{R}^{\mathfrak{m}[B]}$ for each $c \in C_t^*$

and let us prove that we can define $\eta_a^{\mathbf{i}+1}$, $\eta_b^{\mathbf{i}+1}$ so that $\eta_a^{\mathbf{i}} \restriction \mathbf{i} \trianglelefteq \eta_a^{\mathbf{i}+1}$, $\eta_b^{\mathbf{i}} \restriction \mathbf{i} \trianglelefteq \eta_b^{\mathbf{i}+1}$, and properties (1),(2),(3) hold with $\mathbf{i}+1$ in place of $\mathbf{i}$. (By continuing this process one therefore eventually obtains two equivalent sequences.)

Write $\eta_a = \eta_{a,0}{}^\frown\langle i_a\rangle^\frown\eta_{a,\infty}$, and $\eta_b = \eta_{b,0}{}^\frown\langle i_b\rangle^\frown\eta_{b,\infty}$, where $\mathrm{lgn}(\eta_{a,0}) = \mathrm{lgn}(\eta_{b,0}) = \mathbf{i}$. By definition, $\eta_{a_0} = \eta_{b,0}$ and for every $c \in C_t^*$,

$$(\eta_{a,0}, \rho_{c,t}) \in \mathcal{R}^{\mathfrak{m}[A]} \cap \mathcal{R}^{\mathfrak{m}[B]}.$$

There are two cases.

(Case 1) $\mathbf{i} \notin B$. In this case, $\mathbf{i}$ is not an active level for B, so we define $\eta^{\imath+1}i_a = \eta_a^{\mathbf{i}}$, and define $\eta_b^{\mathbf{i}+1} = \eta_{b,0}{}^\frown\langle i_a\rangle^\frown\eta_{b,\infty}$, i.e. replace i_b by i_a. [Since we defined $\mathfrak{m}[A]$, $\mathfrak{m}[B]$ using the same background sequence of graphs, it doesn't matter whether $\mathbf{i} \in B$ or not, recalling Remark 12.4. So applying Claim 12.5, we conclude that $(\eta_b^{\mathbf{i}+1} \restriction s, \rho_{c,t} \restriction s) \in \mathcal{R}^{\mathfrak{m}[B]}$ for each $c \in C_t^*$.

(Case 2) $\mathbf{i} \notin A$. In this case, since $\mathbf{i}$ is not an active level for A, define $\eta_b^{\mathbf{i}+1} = \eta_b$, and define $\eta_a^{\mathbf{i}+1} = \eta_{a,0}{}^\frown\langle i_b\rangle^\frown\eta_{a,\infty}$, i.e. replace i_a by i_b, and again use Remark 12.4 (if necessary) and Claim 12.5.

In this way we eventually construct two equal sequences, so η_* is well defined, so (d) is satisfied, and as this was sufficient to prove (c), we are done. $\square$

[24] note there is always some such choice, by the extension axioms

[25] notice that η_a, η_b satisfy these conditions for $\mathbf{i} = 0$

13. Further discussion and open questions

In the late sixties when Keisler's order was defined, it was natural to conjecture that it had a small finite number of classes (see the introduction to [25]). Though it was quickly understood that the order might give an interesting calibration of theories (see [9] and also [29]), it long remained reasonable to believe that the order's power to give model theoretic information would be tied to its simplicity. We are now at a surprising mathematical juncture, where the order has become very complicated, but without losing its tight connection to and calibration of model-theoretic structure. To communicate some of our excitement, we include a broad list of questions.

A. Saturated models of simple theories. Determining Keisler's order on the stable theories required developing the stability theory to prove a characterization of the saturated models of stable theories (see [31] Theorem III.3.10 and [24] Question 10.4): essentially, that for a model of a complete countable stable theory to be λ^+-saturated it suffices that it is $\aleph_1$-saturated and that every maximal indiscernible set is large. [The theorem is stronger: $\aleph_1$-saturated is really $\kappa(T)$-saturated and the theory need not be countable; for us, regular ultrapowers of models of countable theories are $\aleph_1$-saturated, so this statement suffices.] What, in simple unstable theories, are the right analogues of maximal indiscernible sets?

Problem 13.1. *In light of the results of this paper, formulate a plausible conjecture of a characterization of λ^+-saturated models of simple theories.*

B. Variants of the construction.

Discussion 13.2. Our main construction fixes $\bar{m}$, $\bar{E}$, and Ξ; varying these inputs one would have different theories.

Discussion 13.3. We have written the present construction for a single fast sequence $\bar{m}$ and a family of independent level functions. This was a decision to make the structure of the ideal clearer, among other things. But we might also have written the construction, without level functions, simply for continuum many sequences growing at very different rates (which is, in some sense, what the level functions were formally coding). Looking from this second point of view may give a different perspective on how growth rates of finite families affect model theoretic structure.

C. Interactions with forking.

Question 13.4. *Can the continuum many incomparable classes be reproduced, in ZFC, within the simple non low theories? Within the non simple theories?*

Discussion 13.5. *We may also ask the parallel question for $\trianglelefteq^*$.*

For computability theorists, a natural question may be:

Question 13.6. *Is the structure of the Turing degrees embeddable into Keisler's order?*

E. Questions about ultrafilter construction. An important part of the argument above is constructing ultrafilters, and it may be fruitful to further investigate methods from iterated forcing.

Discussion 13.7. Recall that one way of measuring "size" of a regularizing family $\{X_\alpha : \alpha < \lambda\}$ in an ultrafilter $\mathcal{D}$ on λ is to look at the sequence of integers $\{n_t : t \in \lambda\}$ where $n_t = \{\alpha < \lambda : t \in X_\alpha\}$. Say that a regularizing family is *below* a nonstandard integer if its size is. Flexible ultrafilters are those having a regularizing family below any nonstandard integer [16]. Each of the ultrafilters we build here, by virtue of its connection to certain integer sequences, has a certain amount of flexibility appropriate to those sequences. There remain very interesting open questions about the extent to which flexibility (which is equivalent to "OK") may be separated from goodness, such as Dow's 1985 question, for references and recent work see e.g. Problems 3.5 and 3.6 of [26]. It may be interesting to investigate whether the new family of filters built here, of apparently intermediate flexiblity, sheds light on the landscape around these questions, as our methods suggest further ways of engineering the relation of "sizes" of filters and of sequences.

F. The minimal simple class, and the maximal class. Recall that the theory $T_{\mathbf{rg}}$ of the random graph is minimum among the unstable theories in Keisler's order. There is a set-theoretic characterization of its class (i.e., there is a necessary and sufficient condition for regular ultrafilters to be good for $T_{\mathbf{rg}}$), but to date there is no model-theoretic characterization, indeed no model theoretic characterization of *any* unstable equivalence class. A natural place to begin is:

Problem 13.8. *Give a model theoretic characterization of the class of the theory of the random graph in Keisler's order.*

Any reasonable list of open problems on Keisler's order should recall the parallel of 13.8, one of the major questions on the table: to give a model theoretic characterization of the maximal class in Keisler's order, and under $\trianglelefteq^*$ without instances of GCH. See [20].

G. Variants of Keisler's order. It is a very interesting and natural question to consider what less fine variants of Keisler's order may show about the structure of simple theories, and whether such variants may be found whose number of equivalence classes is finite. We plan to say more about this in future papers.

H. Building blocks of simple theories.

Discussion 13.9. These theories we have built appear quite different from the theories witnessing the infinite descending chain in Keisler's order, which were sums of certain generic n-free k-hypergraphs, studied originally by Hrushovski [7] (for [25], we used the case $n = k+1$). Such theories may be thought of as encoding "pure amalgamation problems."

Indeed, we originally built the precursor to the present theories in [27] to witness Keisler-incomparability with the $T_{k+1,k}$'s. The role of the new theory in [27], this precursor of the $T_{\mathbf{m}}$'s, was in some sense to replace a certain canonical non-low theory in the known, non-ZFC incomparability arguments [35], [21]. We might describe these theories as containing enough of a finite approximation to forking to retain incomparability, but without actually forking. We verified in 2.11 above that the theory of [27] fits in the present framework, though the background Boolean algebras in the two papers are quite different. What does this picture tell us about the building blocks of complexity in simple theories? What interesting non-trivial interactions may occur within simplicity between the weak avatars of forking (the uniform inconsistency along various finite quotient sets) in the $T_{\mathfrak{m}}$'s, and the inconsistency arising from amalgamation in the $T_{n,k}$'s?

I. Hypergraph regularity. Not unrelated to Problem 13.1, the above discussion of graphs and hypergraphs suggests the following speculative question. The theories we have built in the present paper are really fundamentally graphs (layered across predicates). The key relation is a binary relation, and the key underlying densities are densities of bipartite graphs. Is uniform incomparability across a family necessarily a graph (binary) phenomenon? Recalling that the hypergraph analogues of Szemerédi phenomena are known [6], [30], [33] we may ask:

Question 13.10. *Is there a true "hypergraph analogue" of our construction? For instance, can one construct a family of simple theories whose only forking comes from equality, by analogy to what we have done here, which reflect in some fundamental way the densities of certain families of finite 3-uniform hypergraphs, and which themselves form a higher layer of uniform incomparability phenomena in Keisler's order which is not explained by their restrictions to graphs?*

Understanding in either direction may significantly change our understanding of dividing lines in simple theories.

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